

# Multi-Error-Correcting Amplitude Damping Codes

Runyao Duan<sup>\*†</sup>, Markus Grassl<sup>‡</sup>, Zhengfeng Ji<sup>§</sup>, and Bei Zeng<sup>¶</sup>

<sup>\*</sup>Centre for Quantum Computation and Intelligent Systems (QCIS),

Faculty of Engineering and Information Technology, University of Technology, Sydney, NSW 2007, Australia

<sup>†</sup>State Key Laboratory of Intelligent Technology and Systems, Tsinghua National Laboratory for Information Science and Technology, Department of Computer Science and Technology, Tsinghua University, Beijing 100084, China

<sup>‡</sup>Centre for Quantum Technologies, National University of Singapore, Singapore 117543, Singapore

<sup>§</sup>Perimeter Institute for Theoretical Physics, Waterloo, ON, N2L2Y5, Canada

<sup>¶</sup>Institute for Quantum Computing and the Department of Combinatorics and Optimization, University of Waterloo, Waterloo, ON, N2L3G1, Canada

**Abstract**—We construct new families of multi-error-correcting quantum codes for the amplitude damping channel. Our key observation is that, with proper encoding, two uses of the amplitude damping channel simulate a quantum erasure channel. This allows us to use concatenated codes with quantum erasure-correcting codes as outer codes for correcting multiple amplitude damping errors. Our new codes are degenerate stabilizer codes and have parameters which are better than the amplitude damping codes obtained by any previously known construction.

**Index Terms**—Amplitude damping channel, quantum error correction, concatenated quantum codes, quantum erasure code.

## I. INTRODUCTION

In most of works on quantum error correction, it is assumed that the errors to be corrected are completely random, with no knowledge other than that they affect different qubits independently [22], [9]. Or, equivalently, this is to assume that the Pauli-type errors  $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ,  $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ , and  $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ , happen with equal probability  $p_x = p_y = p_z = p/3$ . The quantum channel described by this kind of noise is called depolarizing channel  $\mathcal{E}_{DP}$ .

The most general physical operations (or quantum channels) allowed by quantum mechanics are completely positive, trace preserving linear maps which can be represented in the following Kraus decomposition form:

$$\mathcal{N}(\rho) = \sum_k A_k \rho A_k^\dagger, \quad (1)$$

where  $A_k$  are called Kraus operators of the quantum channel  $\mathcal{N}$  and satisfy the completeness condition  $\sum_k A_k^\dagger A_k = \mathbb{1}$ . In this language of quantum channels, the depolarizing channel  $\mathcal{E}_{DP}$  with error parameter  $p$  acting on any one-qubit quantum state  $\rho \in \mathbb{C}^{2 \times 2}$  as

$$\mathcal{E}_{DP}(\rho) = (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z), \quad (2)$$

so the Kraus operators for the depolarizing channel are the Pauli matrices together with identity.

However, if further information about an error process is available, more efficient codes can be designed. Indeed in many physical systems, the types of noise are likely to be unbalanced between amplitude ( $X$ -type) errors and phase ( $Z$ -type) errors. Recently a lot of attention has been put into

designing codes for this situation and in studying their fault tolerance properties [1], [7], [8], [15], [23]. All those works deal with error models which are still described by Kraus operators that are Pauli matrices (Pauli Kraus operators), but the  $X$ - and  $Y$ -errors happen with equal probability  $p_x = p_y$ , which might be different from the probability  $p_z$  that a  $Z$ -error happens. The quantum channels described by this kind of noise are called asymmetric channels  $\mathcal{E}_{AS}$  acting on any one-qubit quantum state  $\rho$  as

$$\mathcal{E}_{AS}(\rho) = (1 - (2p_x + p_z))\rho + p_x(X\rho X + Y\rho Y) + p_z Z\rho Z. \quad (3)$$

The choice  $p_x = p_y$  is related to a physically realistic error model including amplitude damping (AD) noise and phase damping noise [22]. The Kraus operators for AD noise with damping rate  $\gamma$  are

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}. \quad (4)$$

Note that

$$A_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix} = \frac{\sqrt{\gamma}}{2}(X + iY) \quad \text{and} \\ A_1^\dagger = \frac{\sqrt{\gamma}}{2}(X - iY).$$

Hence the linear span of the operators  $A_1$  and  $A_1^\dagger$  equals the linear span of  $X$  and  $Y$ . If the system is at finite temperature, the Kraus operator  $A_1^\dagger$  will appear in the noise model [22]. Thus, if the code is capable of correcting  $t$   $X$ - and  $t$   $Y$ -errors, it can also correct  $t$   $A_1$ - and  $t$   $A_1^\dagger$ -errors.

It was observed that when the temperature of a physical system is zero or very low, the error  $A_1^\dagger$  is actually negligible [22]. For simplicity, we further ignore the phase damping error (which is characterized by the Pauli operator  $Z$ ). Then the error model is fully characterized by  $A_0$  and  $A_1$ . In this work, we will focus on this quantum channel with only amplitude damping noise, i.e. the AD channel  $\mathcal{E}_{AD}$ , with only two Kraus operators given by Eq. (4). The AD channel is the simplest nonunital channel whose Kraus operators cannot be described by Pauli operations. The AD channel is a quantum analogue of the classical  $\mathcal{Z}$ -channel which transmits 0 faithfully, but maps

1 to either 0 or 1 [26]. For the AD channel we only need to deal with the error  $A_1^\dagger$  (a quantum analogue of the error  $1 \rightarrow 0$ ), but not with  $A_1^\dagger$  (a quantum analogue of the error  $0 \rightarrow 1$ ). So asking to be able to correct both  $X$ - and  $Y$ -errors is a less efficient way for constructing quantum codes for the AD channel.

Since the error model is not described by Pauli Kraus operators, the task of constructing good error-correcting codes becomes very challenging. The known techniques dealing with Pauli errors cannot be applied or result in codes with bad parameters. Several new techniques for the construction of codes which are adapted to this type of noise with non-Pauli Kraus operators, and the AD channel in particular, have been developed [6], [8], [18], [19], [26]. After years' effort, systematic methods for constructing high performance single-error-correcting codes have been found [18], [26]. However, all these methods fail to construct good AD codes correcting multi-errors.

In this paper we present a method for finding families of codes correcting multi-amplitude-damping errors. Our construction is based on the observation that with respect to a simple encoding two uses of the amplitude damping channel simulate a quantum erasure channel. This allows us to apply a concatenated coding scheme with quantum erasure-correcting codes as outer codes, resulting in codes correcting multi-amplitude-damping errors. Our new codes are degenerate stabilizer codes which have better parameters than the codes given by any previously known construction.

## II. CORRECTING AMPLITUDE DAMPING ERRORS

A quantum error-correcting code  $Q$  is a subspace of  $(\mathbb{C}^2)^{\otimes n}$ , the space of  $n$  qubits. For a  $K$ -dimensional code space spanned by the orthonormal set  $|\psi_i\rangle$ ,  $i = 1, \dots, K$  and a set of errors  $\mathcal{E}$  there is a physical operation correcting all elements  $E_\mu \in \mathcal{E}$  if the error correction conditions [3], [16] are satisfied:

$$\forall_{ij, \mu\nu} \langle \psi_i | E_\mu^\dagger E_\nu | \psi_j \rangle = C_{\mu\nu} \delta_{ij}, \quad (5)$$

where  $C_{\mu\nu}$  depends only on  $\mu$  and  $\nu$ . If the matrix  $(C_{\mu\nu})$  has full rank the code is said to be nondegenerate, otherwise it is degenerate.

For the AD channel, if  $\gamma$  is small, we would like to correct the leading order errors that occur during amplitude damping. Setting  $A = X + iY$  and  $B = I - Z$ , we have

$$A_1 = \frac{\sqrt{\gamma}}{2} A \quad \text{and} \quad A_0 = I - \frac{\gamma}{4} B + O(\gamma^2). \quad (6)$$

It has been shown that in order to improve the fidelity of the transmission through an amplitude damping channel from  $1 - \gamma$  to  $1 - \gamma^t$ , it is sufficient to satisfy the error-detection conditions for  $2t$   $A$ -errors and  $t$   $B$ -errors [9, Section 8.7]. We will say that such a code corrects  $t$  amplitude damping errors since it improves the fidelity, to leading order, just as much as a true  $t$ -error-correcting code would for the same channel.

Stabilizer codes are a large kind of quantum codes which contain many good quantum codes [9], [22]. A stabilizer code with  $n$  qubits encoding  $k$  qubits is of distance  $d$  if all errors of

weight at most  $d - 1$  (i.e., operators acting nontrivially on less than  $d$  individual qubits) can be detected or have no effect on  $Q$ , and we denote the parameters of  $Q$  by  $[[n, k, d]]$ . We say an  $[[n, k]]$  stabilizer code is a  $t$ -code if it corrects  $t$  AD-errors. For comparison with stabilizer codes, we say an  $[[n, k, d]]$   $t$ -code is good if  $2t + 1 > d$  for the best possible  $[[n, k, d]]$  code; or,  $n < n'$  for the best possible  $[[n', k, 2t + 1]]$  code; or,  $k > k'$  for the best possible  $[[n, k', 2t + 1]]$  code.

The first AD code given by Leung et al. [19] is a  $[[4, 1]]$  1-code, i.e., correcting a single AD-error. Basis vectors of the code are

$$\begin{aligned} |0\rangle_L &= \frac{1}{\sqrt{2}} (|0000\rangle + |1111\rangle) \\ |1\rangle_L &= \frac{1}{\sqrt{2}} (|0011\rangle + |1100\rangle). \end{aligned} \quad (7)$$

Using only 4 qubits, this 1-code is better than the  $[[5, 1, 3]]$  code, a quantum code correcting an arbitrary single-qubit error and encoding one qubit using the minimal number of qubits [3], [17].

Following the work by Leung et al. [19], several constructions for 1-codes have been proposed [8], [9], [18], [26], including some high performance 1-codes. However, very little is known about good multi-error-correcting AD codes. It turns out that none of the methods known for constructing good 1-codes can be directly generalized to  $t$ -codes with  $t > 1$ .

Gottesman [9, Section 8.7] has shown that Shor's nine-qubit code [25]

$$\begin{aligned} |0\rangle_L &= \frac{1}{2\sqrt{2}} (|000\rangle + |111\rangle)^{\otimes 3} \\ |1\rangle_L &= \frac{1}{2\sqrt{2}} (|000\rangle - |111\rangle)^{\otimes 3} \end{aligned} \quad (8)$$

can correct two AD-errors, despite the fact that it can correct only a single general error. It is the best known 2-code and it is better than the  $[[11, 1, 5]]$  code [9], the best two-error-correcting stabilizer code encoding one qubit [10].

It is interesting to note that the 1-code given by Eq. (7) can be rewritten in another basis as

$$\begin{aligned} |+\rangle_L &= \frac{1}{\sqrt{2}} (|0\rangle_L + |1\rangle_L) = \frac{1}{2} (|00\rangle + |11\rangle)^{\otimes 2} \\ |-\rangle_L &= \frac{1}{\sqrt{2}} (|0\rangle_L - |1\rangle_L) = \frac{1}{2} (|00\rangle - |11\rangle)^{\otimes 2}, \end{aligned} \quad (9)$$

which is of a similar form as Eq. (8).

Therefore, we can generalize the constructions of Eqs. (9) and (8) to  $t$ -codes with basis

$$\begin{aligned} |0\rangle_L &= 2^{-\frac{t+1}{2}} \left( \underbrace{|0 \dots 0\rangle}_{t+1} + \underbrace{|1 \dots 1\rangle}_{t+1} \right)^{\otimes (t+1)} \\ |1\rangle_L &= 2^{-\frac{t+1}{2}} \left( \underbrace{|0 \dots 0\rangle}_{t+1} - \underbrace{|1 \dots 1\rangle}_{t+1} \right)^{\otimes (t+1)}. \end{aligned} \quad (10)$$

However, these  $[[n^2, 1, n]]$  so-called Bacon-Shor code [2], [25] correcting  $t = n - 1$  AD-errors scale badly when  $n$  is large. For instance, there exists a  $[[25, 1, 9]]$  code and a  $[[29, 1, 11]]$  code [10].

Note that these  $[[n^2, 1]]$  codes are of Calderbank-Shor-Steane (CSS) type [5], [27]. They are also degenerate: for instance, a  $Z$ -error acting on the first qubit or the second qubit has the same effect on the code.

In general, CSS codes can be used to construct codes for the AD channel [9, Section 8.7]:

**Proposition 1** *An  $[[n, k]]$  CSS code of  $X$ -distance  $2t + 1$  and  $Z$ -distance  $t + 1$  is an  $[[n, k]]$   $t$ -code.*

In the first column of Table I we provide bounds on the length  $n$  of codes for the AD channel encoding one or two qubits derived from CSS codes with given  $Z$ - and  $X$ -distances  $t + 1$  and  $2t + 1$ , respectively. The lower bounds have been derived using linear programming techniques [24]. The upper bound is based on CSS codes constructed from the database of best known linear codes [4], [10].

In the fifth column we give upper and lower bounds on the length  $n'$  such that an  $[[n', k, t + 1]]$  code may exist. In the last column, we list the bounds on the length of  $t$ -code from Theorem 1. The data for columns  $n'$  and  $2m$  is taken from [10]).

$n$	$k$	$t + 1$	$2t + 1$	$n'$	$2m$
12–13	1	3	5	11	10
19–20	1	4	7	17	20
25–30	1	5	9	23–25	22
33–41	1	6	11	29	32
39–54	1	7	13	35–43	34
47–70	1	8	15	41–53	44–48
53–79	1	9	17	47–61	46–50
–89	1	10	19	53–81	56
–105	1	11	21	59–85	58
14–17	2	3	5	14	16
20–27	2	4	7	20–23	20
27–37	2	5	9	26–27	28
34–45	2	6	11	32–41	32
41–62	2	7	13	38–51	40–46
–71	2	8	15	44–59	44–52
–87	2	9	17	50–78	52–54
–102	2	10	19	56–83	56–56
–110	2	11	21	62–104	64–82

TABLE I  
BOUNDS ON THE LENGTH  $n$  OF AN  $[[n, 1]]$   $t$ -CODE DERIVED FROM CSS CODES, TOGETHER WITH THE BOUNDS ON THE LENGTH  $n'$  OF A STABILIZER CODE  $[[n', 1, 2t + 1]]$  AND THE LENGTH  $2m$  OF AN  $[[2m, 1]]$   $t$ -CODE FROM THEOREM 1.

It can be seen from Table I that the construction of AD codes based on CSS codes unlikely gives good AD codes. But as it is unknown whether these bounds for  $n$  and  $n'$  given in this table can be achieved, we do not have the definite answer. This problem will be addressed in future research.

### III. AD CODE BASED ON QUANTUM ERASURE CODES

As discussed in Sec. II, no good method is known for constructing good multi-error-correcting AD codes. In this section we provide a construction which systematically gives high performance  $t$ -codes with  $t > 1$ . The construction uses concatenated quantum codes with an inner and an outer quantum code. After decoding the inner quantum code, the

effective channel is a quantum erasure channel. We start by proving the following lemma.

**Lemma 1** *Using the quantum dual-rail code  $\mathcal{Q}_i$  which encodes a single qubit into two qubits, given by*

$$|0\rangle_L = |01\rangle, \quad |1\rangle_L = |10\rangle, \quad (11)$$

*two uses of the AD channel simulate a quantum erasure channel.*

*Proof:* For any state  $\rho$  of the code  $\mathcal{Q}_1$ , we observe that

$$\mathcal{E}_{AD}^{\otimes 2}(\rho) = (1 - \gamma)\rho + \gamma(|00\rangle\langle 00|). \quad (12)$$

The state  $|00\rangle$  is orthogonal to the code  $\mathcal{Q}_1$ . Using a measurement that either projects on  $\mathcal{Q}_1$  or its orthogonal complement, it can be detected whether an AD error occurred or not. Hence we obtain a quantum erasure channel with erasure symbol  $|00\rangle$ . ■

**Remark 1** *It can easily be shown that with respect to the dual-rail code  $\{01, 10\}$ , two uses of the  $Z$ -channel simulate a classical erasure channel with erasure symbol 00 (see, e.g. [21]). Lemma 1 is a quantum analogue of this fact, yet Lemma 1 is nontrivial due to the Kraus operator  $A_0$ , which introduces some relative phase error between  $|0\rangle$  and  $|1\rangle$  that has no classical analogue.*

Lemma 1 allows us to use quantum erasure-correcting codes as outer codes for correcting multiple amplitude damping errors. It is known that an  $[[m, k, d]]$  quantum code corrects  $d - 1$  erasure errors [9], [11], [22]. Our main result is given by the following theorem.

**Theorem 1** *If there exists an  $[[m, k, d]]$  quantum code, then there is a  $[[2m, k]]$  code correcting  $t = d - 1$  amplitude damping errors.*

*Proof:* Let  $\mathcal{Q}$  be the concatenated code with the inner code  $\mathcal{Q}_1$  given Eq. (11) and the outer code  $\mathcal{Q}_2$  with parameters  $[[m, k, d]]$ . The code  $\mathcal{Q}_2$  corrects  $d - 1$  erasure errors. A single AD-error on each block of the inner code creates an erasure error for the outer code. The position of the error is indicated by the erasure state  $|00\rangle$ . Hence the outer codes takes care of  $d - 1$  AD-errors acting on different blocks. Two errors acting on the same block annihilate the state, such that the quantum error correction condition given by Eq. (5) is naturally satisfied. Hence  $\mathcal{Q}$  is a  $[[2m, k]]$  AD code correcting  $t = d - 1$  amplitude damping errors. ■

**Remark 2** *It is interesting to compare our construction with the corresponding classical case, where concatenation with the dual-rail code  $\{01, 10\}$  as inner code and an  $[m, k, d]$  erasure-correcting code as outer code yields an  $[2m, k]$   $(d - 1)$ -code for the  $Z$ -channel. However, this  $(d - 1)$ -code is in general not good because simply repeating each codeword of an  $[m, k, d]$  classical code will straightforwardly give a  $[2m, k, 2d]$  code correcting  $d - 1$  arbitrary errors. In the*

quantum case, however, the existence of an  $[[m, k, d]]$  stabilizer code does not necessarily lead to a  $[[2m, k, 2d]]$  stabilizer code.

In Table II, we compare the  $t$ -codes from our construction with the known upper and lower bounds on the minimum distance of stabilizer codes from [10]. We fix the number of logical qubits  $k$  and the number  $t$  of correctable AD-errors within the range  $k = 1, \dots, 6$  and  $t = 1, \dots, 10$ . The length  $n = 2m$  of the code is derived from the shortest known stabilizer code with parameters  $[[m, k, t+1]]$  from [10]. Hence the first three columns gives the parameters of each line in the table corresponds to an  $[[n, k]]$   $t$ -code. The fourth column provides  $2t + 1$ , which is the distance that is required for an  $[[n, k]]$  code to be capable to correct  $t$  arbitrary errors. The last column gives the lower and upper bounds on the distance  $d$  of a  $[[n, k, d]]$  stabilizer code from [10]. Hence all  $t$ -codes with  $2t + 1 > d$  are better than the stabilizer codes with the same length and dimension. With the exception of small parameters, many of our codes outperform the known—or even the best possible—corresponding stabilizer codes correcting  $t$  arbitrary errors. Note that any improvement of the lower bound on the distance  $d$  of a stabilizer code implies some improvement for  $t$ -codes as well.

Note that all the  $t$ -codes listed in the table are degenerate stabilizer codes obtained by concatenation of a stabilizer code as outer code and the quantum dual-rail code  $\mathcal{Q}_1$  given by Eq. (11) as inner code. In order to compute the stabilizer of the concatenated code, note that the inner code  $\mathcal{Q}_1$  is stabilized by  $-ZZ$ , and has logical operators  $\bar{X} = XX$  and  $\bar{Z} = ZI$ . As an example, we compute the stabilizer for the  $[[10, 1]]$  2-code.

**Example 1** A  $[[10, 1]]$  2-code can be derived from the  $[[5, 1, 3]]$  code with stabilizer generated by:

$$\begin{aligned} g_1 &= X Z Z X I \\ g_2 &= I X Z Z X \\ g_3 &= X I X Z Z \\ g_4 &= Z X I X Z \end{aligned} \quad (13)$$

The stabilizer of the  $[[10, 1]]$  2-code is obtained by replacing the operators in Eq. (13) by the logical operators of  $\mathcal{Q}_1$  and adding the stabilizer for each block of the inner code:

$$\begin{aligned} g'_1 &= X X Z I Z I X X I I \\ g'_2 &= I I X X Z I Z I X X \\ g'_3 &= X X I I X X Z I Z I \\ g'_4 &= Z I X X I I X X Z I \\ g'_5 &= -Z Z I I I I I I I I \\ g'_6 &= -I I Z Z I I I I I I \\ g'_7 &= -I I I I Z Z I I I I \\ g'_8 &= -I I I I I I Z Z I I \\ g'_9 &= -I I I I I I I I Z Z \end{aligned} \quad (14)$$

As a degenerate stabilizer code, this code has parameters  $[[10, 1, 4]]$ . As a 2-code, this code is not as good as Shor's nine-qubit code given in Eq. (8), but still better than the shortest

stabilizer code  $[[11, 1, 5]]$  encoding one qubit and correcting two arbitrary errors.

However, the  $[[22, 1]]$  4-code given in Table II is better than the  $[[25, 1]]$  4-code given in Eq. (10), the degenerate  $[[25, 1, 9]]$  code constructed from concatenating two  $[[5, 1, 3]]$  codes, and even the putative stabilizer code  $[[22, 1, 8]]$ .

From the last column in Table I we see that, with the exception when both parameters  $t$  and  $k$  are small, the codes from our construction are better than the  $t$ -codes derived from CSS codes.

#### IV. POSSIBLE GENERALIZATIONS

One possible generalization of our construction is to chose a different inner code. For instance, we can take the inner code as the following quantum code  $\mathcal{Q}'_1$  which encodes one qutrit into three qubits:

$$|0\rangle_L = |001\rangle, \quad |1\rangle_L = |010\rangle, \quad |2\rangle_L = |100\rangle. \quad (15)$$

For any state  $\rho$  of the code  $\mathcal{Q}'_1$ , we observe that

$$\mathcal{E}_{AD}^{\otimes 3}(\rho) = (1 - \gamma)\rho + \gamma(|000\rangle\langle 000|), \quad (16)$$

hence the effective channel is a qutrit quantum erasure channel where the state  $|000\rangle$  indicates an erasure.

Since the inner code  $\mathcal{Q}'_1$  is of dimension 3, the outer code  $\mathcal{Q}'_2$  must be chosen from quantum codes constructed from qutrits rather than qubits, i.e.  $\mathcal{Q}'_2$  is a subspace of  $(\mathbb{C}^3)^{\otimes m}$ . Using a  $[[m, k, d]]_3$  quantum code  $\mathcal{Q}'_2$  (where the subscript 3 indicates that this is a qutrit code), the concatenated code  $\mathcal{Q}$  with inner code  $\mathcal{Q}_1$  and outer code  $\mathcal{Q}_2$  is an AD code correcting  $t = d - 1$  AD errors, with length  $3m$  and encoding a space of dimension  $3^k$ . In general, quantum code of length  $n$  and dimension  $K$  is denoted by  $((n, K))$ , so this construction yields a  $((3m, 3^k))$   $(d - 1)$ -code.

For instance, an  $[[8, 2, 4]]_3$  outer code (see [14], [20]) gives a  $((24, 9))$  AD code correcting 3 AD errors. This is better than the parameters  $[[24, 3, 7-8]]$  of a stabilizer code (cf. [10]), but worse than the  $[[24, 4]]$  3-code given in Table II. It is not yet clear whether this or other generalizations based on concatenation using codes for the erasure channel yield better AD codes than those obtained from the quantum dual-rail codes.

#### V. CONCLUSIONS

We have constructed families of good multi-error-correcting quantum codes for the amplitude damping channel based on code concatenation and quantum erasure-correcting codes. As the rate of our codes can never exceed the rate  $1/2$  of the inner code, other methods—possibly generalized concatenation of quantum codes [12], [13]—have to be used in order to construct high-rate AD codes. However, our method provides the first systematic construction for good multi-error-correcting AD codes. We hope that our method shade lights on constructing good quantum codes adapted for other non-Pauli channels beyond the AD channel, and further understanding on the role that degenerate codes play in quantum coding theory.

$n$	$k$	$t$	$2t+1$	$d$	$n$	$k$	$t$	$2t+1$	$d$	$n$	$k$	$t$	$2t+1$	$d$
8	1	1	3	3	12	3	1	3	4	16	5	1	3	4-5
10	1	2	5	4	16	3	2	5	5	22	5	2	5	6-7
20	1	3	7	7	24	3	3	7	7-8	28	5	3	7	7-9
22	1	4	9	7-8	30	3	4	9	9-10	36	5	4	9	8-11
32	1	5	11	11	40	3	5	11	10-13	42	5	5	11	9-13
34	1	6	13	11-12	48	3	6	13	11-16	50	5	6	13	11-16
48	1	7	15	13-17	52	3	7	15	13-17	60	5	7	15	13-19
50	1	8	17	13-17	54	3	8	17	13-18	78	5	8	17	15-25
56	1	9	19	15-19	72	3	9	19	15-24	86	5	9	19	18-28
58	1	10	21	15-20	82	3	10	21	18-27	98	5	10	21	19-32
8	2	1	3	3	12	4	1	3	4	16	6	1	3	4
16	2	2	5	6	20	4	2	5	6	24	6	2	5	6-7
20	2	3	7	6-7	24	4	3	7	6-8	28	6	3	7	6-8
28	2	4	9	10	32	4	4	9	8-10	36	6	4	9	8-11
32	2	5	11	10-11	40	4	5	11	10-13	48	6	5	11	10-15
46	2	6	13	12-16	50	4	6	13	12-16	58	6	6	13	12-19
52	2	7	15	14-18	52	4	7	15	12-17	64	6	7	15	14-21
54	2	8	17	14-18	70	4	8	17	15-23	84	6	8	17	17-27
56	2	9	19	14-19	80	4	9	19	16-26	92	6	9	19	18-29
82	2	10	21	18-28	96	4	10	21	18-31	104	6	10	21	19-33

TABLE II  
COMPARISON OF OUR  $[[n, k]]$   $t$ -CODES AND THE BOUNDS ON THE MINIMUM DISTANCE  $d$  OF A STABILIZER CODE  $[[n, k, d]]$ .

#### ACKNOWLEDGMENTS

We thank Daniel Gottesman and Peter Shor for helpful discussions. RD is partly supported by QCIS, University of Technology, Sydney, and the NSF of China (Grant Nos. 60736011 and 60702080). BZ is supported by NSERC and QuantumWorks. Centre for Quantum Technologies is a Research Centre of Excellence funded by Ministry of Education and National Research Foundation of Singapore. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research & Innovation.

#### REFERENCES

- [1] P. Aliferis and J. Preskill. Fault-tolerant quantum computation against biased noise. *Physical Review A* 78(5):052331, 2008.
- [2] D. Bacon. Operator quantum error-correcting subsystems for self-correcting quantum memories. *Physical Review A*, 73(1):012340, 2006.
- [3] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters. Mixed state entanglement and quantum error correction. *Physical Review A*, 54(5):3824-3851, 1996.
- [4] W. Bosma, J. J. Cannon, and C. Playoust. The Magma Algebra System I: The User Language. *Journal of Symbolic Computation*, 24(3-4):235-265, 1997.
- [5] A. R. Calderbank and P. W. Shor. Good quantum error-correcting codes exist. *Physical Review A*, 54(2):1098-1106, 1996.
- [6] I. L. Chuang, D. W. Leung, and Y. Yamamoto. Bosonic quantum codes for amplitude damping. *Physical Review A*, 56(2):1114-1125, 1997.
- [7] Z. W. E. Evans, A. M. Stephens, J. H. Cole, and L. C. L. Hollenberg. Error correction optimisation in the presence of  $x/z$  asymmetry. *arXiv:0709.3875*, 2007.
- [8] A. S. Fletcher, P. W. Shor, and M. Z. Win. Channel-adapted quantum error correction for the amplitude damping channel. *IEEE Transactions on Information Theory*, 54(12):5705-5718, 2008.
- [9] D. Gottesman. *Stabilizer Codes and Quantum Error Correction*. PhD thesis, California Institute of Technology, Pasadena, USA, 1997.
- [10] M. Grassl. Tables of quantum error-correcting codes, available on-line at <http://www.codetables.de>
- [11] M. Grassl, T. Beth, and T. Pellizzari. Codes for the quantum erasure channel. *Physical Review A*, 56(1):33-38, 1997.
- [12] M. Grassl, P. W. Shor, G. Smith, J. A. Smolin, and B. Zeng. Generalized Concatenated Quantum Codes. *Physical Review A* 79(5):050306(R), 2009.
- [13] M. Grassl, P. W. Shor, and B. Zeng. Generalized Concatenation for Quantum Codes. In *Proceedings of the 2009 IEEE International Symposium on Information Theory*, pp. 95-3-957, 2009.
- [14] D. Hu, W. Tang, M. Zhao, Q. Chen, S. Yu, and C. H. Oh. Graphical nonbinary quantum error-correcting codes. *Physical Review A*, 78(1):012306, 2008.
- [15] L. Ioffe and M. Mezard. Asymmetric quantum error-correcting codes. *Physical Review A*, 75(3):032345, 2007.
- [16] E. Knill and R. Laflamme. Theory of quantum error-correcting codes. *Physical Review A*, 55(2):900-911, 1997.
- [17] R. Laflamme, C. Miquel, J. P. Paz, and W. H. Zurek. Perfect quantum error correcting code. *Physical Review Letters*, 77(1):198-201, 1996.
- [18] R. Lang and P. W. Shor. Nonadditive quantum error correcting codes adapted to the amplitude damping channel. *arXiv:0712.2586*, 2007.
- [19] D. W. Leung, M. A. Nielsen, I. L. Chuang, and Y. Yamamoto. Approximate quantum error correction can lead to better codes. *Physical Review A*, 56(4):2567-2573, 1997.
- [20] S. Y. Looi, L. Yu, V. Gheorghiu, and R. B. Griffiths. Quantum error correcting codes using qudit graph states. *Physical Review A*, 78(4):042303, 2008.
- [21] J. L. Massey. Zero Error. Lecture at *Information Theory Winter School 2007*, La Colle sur Loup, France, 2007. Available on-line at [http://itwinterschool07.eurecom.fr/Tutorials/Massey\\_Zero\\_error.pdf](http://itwinterschool07.eurecom.fr/Tutorials/Massey_Zero_error.pdf)
- [22] M. Nielsen and I. Chuang. *Quantum computation and quantum information*. Cambridge University Press, Cambridge, England, 2000.
- [23] P. K. Sarvepalli, M. Rötteler, and A. Klappenecker. Asymmetric quantum LDPC codes. In *Proceedings of the 2008 IEEE International Symposium on Information Theory*, pp. 305-309, 2008.
- [24] P. K. Sarvepalli, A. Klappenecker, and M. Rötteler. Asymmetric quantum codes: constructions, bounds, and performance. *Proceedings of the Royal Society London, Series A*, 465(2105):1645-1672, 2009.
- [25] P. W. Shor. Scheme for reducing decoherence in quantum computer memory. *Physical Review A*, 52(4):2493-2496, 1995.
- [26] P. W. Shor, G. Smith, J. A. Smolin, and B. Zeng. High performance single-error-correcting quantum codes for the amplitude damping channel. *arXiv:0907.5149*, 2009.
- [27] A. Steane. Multiple particle interference and quantum error correction. *Proceedings of the Royal Society London, Series A*, 452(1954):2551-2577, 1996.