

Reconsidering unique information: Towards a multivariate information decomposition

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Abstract—The information that two random variables Y, Z contain about a third random variable X can have aspects of *shared information* (contained in both Y and Z), of *complementary information* (only available from (Y, Z) together) and of *unique information* (contained exclusively in either Y or Z). Here, we study measures \widetilde{SI} of shared, \widetilde{UI} unique and \widetilde{CI} complementary information introduced by Bertschinger *et al.* [1] which are motivated from a decision theoretic perspective. We find that in most cases the intuitive rule that *more variables contain more information* applies, with the exception that \widetilde{SI} and \widetilde{CI} information are not monotone in the target variable X . Additionally, we show that it is not possible to extend the bivariate information decomposition into \widetilde{SI} , \widetilde{UI} and \widetilde{CI} to a non-negative decomposition on the partial information lattice of Williams and Beer [2]. Nevertheless, the quantities \widetilde{UI} , \widetilde{SI} and \widetilde{CI} have a well-defined interpretation, even in the multivariate setting.

I. INTRODUCTION

Consider three random variables X, Y, Z with finite state spaces. Suppose that we are interested in the value of X , but we can only observe Y or Z . If the tuple (Y, Z) is not independent of X , then the values of Y or Z or both of them contain information about X . The information about X contained in the tuple (Y, Z) can be distributed in different ways. For example, it may happen that Y contains information about X , but Z does not, or vice versa. In this case, it would suffice to observe only one of the two variables Y, Z , namely the one containing the information. It may also happen, that both Y and Z contain different information, so it would be worthwhile to observe both of the variables. If both Y and Z contain the same information about X , we could chose to observe either Y or Z . Finally, it is possible that neither Y nor Z taken for itself contains any information about X , but together they contain information about X . This effect is called *synergy*, and it occurs, for example, if all variables X, Y, Z are binary, and $X = Y \text{ XOR } Z$. In general, all effects may be present at the same time. That is, the information that (Y, Z) has about X is a combination of *shared information* $SI(X : Y; Z)$ (information contained both in Y and in Z), *unique information* $UI(X : Y \setminus Z)$ and $UI(X : Z \setminus Y)$ (information that only one of Y and Z has) and *synergistic* or *complementary information* $CI(X : Y; Z)$ (information that can only be retrieved when considering Y and Z together)¹.

¹It is often assumed that these three types of information are everything there is, but one may ask, of course, whether there are further types of information.

Many people have tried to make these ideas precise and quantify the amount of unique information, shared information or complementary information. In particular, neuro-scientists have struggled for a long time to come up with a suitable measure of synergy; see [4], [5] and references therein. A promising conceptual point of view was taken in [2] by Williams and Beer, who developed the framework of the *partial information lattice* to define a decomposition of the mutual information into non-negative parts with a well-defined interpretation. Their work prompted a series of other papers trying to improve these results [6], [3], [7]. We recall the definition of the partial information lattice in Section III.

In this paper we build on the bivariate information decomposition defined in [1], which is defined as follows: Let Δ be the set of all joint distributions of X, Y and Z , and for fixed $P \in \Delta$ let Δ_P be the subset of Δ that consists of all distributions $Q \in \Delta$ that have the same marginal distributions on the pairs (X, Y) and (X, Z) , i.e. $Q(X = x, Y = y) = P(X = x, Y = y)$ and $Q(X = x, Z = z) = P(X = x, Z = z)$ for all possible values x, y, z . Then we define

$$\begin{aligned}\widetilde{UI}(X : Y \setminus Z) &= \min_{Q \in \Delta_P} MI_Q(X : Y | Z), \\ \widetilde{UI}(X : Z \setminus Y) &= \min_{Q \in \Delta_P} MI_Q(X : Z | Y), \\ \widetilde{SI}(X : Y; Z) &= \max_{Q \in \Delta_P} CoI_Q(X; Y; Z), \\ \widetilde{CI}(X : Y; Z) &= MI(X : (Y, Z)) \\ &\quad - \min_{Q \in \Delta_P} MI_Q(X : (Y, Z)),\end{aligned}$$

where MI denotes the mutual information, CoI the coinformation (see Section II below), and the index Q in MI_Q or CoI_Q indicates that the corresponding information-theoretic quantity should be computed with respect to the joint distribution Q , as opposed to the “true underlying distribution” P . As shown in [1], these four quantities are non-negative, and

$$\begin{aligned}MI(X : (Y, Z)) &= \widetilde{SI}(X : Y; Z) + \widetilde{UI}(X : Y \setminus Z) \\ &\quad + \widetilde{UI}(X : Z \setminus Y) + \widetilde{CI}(X : Y; Z), \\ (1) \quad MI(X : Y) &= \widetilde{SI}(X : Y; Z) + \widetilde{UI}(X : Y \setminus Z), \\ MI(X : Z) &= \widetilde{SI}(X : Y; Z) + \widetilde{UI}(X : Z \setminus Y).\end{aligned}$$

Moreover, it was argued in [1] that $\widetilde{SI}(X : Y; Z)$ can be considered as a measure of shared information, $\widetilde{CI}(X : Y; Z)$ as a measure of complementary information, and $\widetilde{UI}(X : Y \setminus Z)$ and $\widetilde{UI}(X : Z \setminus Y)$ as measures of unique information.

This interpretation can be justified by the following result, which is a translation of some of the results of [1]:

Theorem 1. *Let $SI(X : Y; Z)$, $UI(X : Y \setminus Z)$, $UI(X : Z \setminus Y)$ and $CI(X : Y; Z)$ be non-negative functions on Δ satisfying an information decomposition of the form (1), and assume that the following holds:*

- 1) *For any $P \in \Delta$, the maps $Q \mapsto UI_Q(X : Y \setminus Z)$ and $Q \mapsto UI_Q(X : Z \setminus Y)$ are constant on Δ_P .*
- 2) *For any $P \in \Delta$ there exists $Q \in \Delta_P$ with $CI_Q(X : Y; Z) = 0$.*

Then $SI = \widetilde{SI}$, $UI = \widetilde{UI}$ and $CI = \widetilde{CI}$ on Δ .

Condition 1) says that the amount of unique information depends only on the marginal distributions of the pairs (X, Y) and (X, Z) formalizing the idea that unique information can be extracted from Y and Z alone independent of their joint distribution. Condition 2) states that the presence or absence of synergistic information cannot be decided from the marginal distributions alone. See [1] for a discussion of these properties.

In the present paper we ask how these results can be extended to the case of more variables. The first question is how the general structure of the decomposition should look like. As stated above, a conceptional answer to this question is given by the PI lattice of Williams and Beer. However, as we will show in Section III, the bivariate decomposition into the functions \widetilde{SI} , \widetilde{UI} and \widetilde{CI} cannot be extended to this framework. The problem is that \widetilde{SI} satisfies the equality

$$(\textit{identity axiom}) \quad \widetilde{SI}((Y, Z) : Y; Z) = MI(Y : Z),$$

which was introduced in [3]. Theorem 2 states that no non-negative information decomposition according to the PI lattice can satisfy the identity axiom. Therefore, if there is a multivariate decomposition of $MI(X : (Y_1, \dots, Y_n))$ that generalizes the information decomposition into \widetilde{SI} , \widetilde{UI} and \widetilde{CI} in a consistent way, then it cannot be a partial information decomposition.

Even without a consistent multivariate information decomposition the functions \widetilde{SI} , \widetilde{UI} and \widetilde{CI} can be used in the context of several variables by partitioning the variables. For example, the quantity

$$\widetilde{UI}(X : Y \setminus (Z_1, \dots, Z_n))$$

should quantify the amount of information that only Y knows about X , but that none of the Z_i has, and that also none of the combinations of the Z_i has. In Section IV we investigate what happens if we enlarge one of the arguments of the functions \widetilde{SI} , \widetilde{UI} and \widetilde{CI} . In particular, we ask whether the functions increase or decrease in this case.

As shown in Section IV-A, \widetilde{UI} behaves quite reasonable in this setting: \widetilde{UI} satisfies

$$\begin{aligned} \widetilde{UI}(X : Y \setminus (Z, Z')) &\leq \widetilde{UI}(X : Y \setminus Z), \\ \widetilde{UI}(X : (Y, Y') \setminus Z) &\geq \widetilde{UI}(X : Y \setminus Z), \\ \widetilde{UI}((X, X') : Y \setminus Z) &\geq \widetilde{UI}(X : Y \setminus Z). \end{aligned}$$

Moreover, in Section IV-B we show that $\widetilde{SI}(X : (Y, Y'); Z) \geq \widetilde{SI}(X : Y; Z)$. On the other hand, there is no monotonic

relation between $\widetilde{SI}((X, X') : Y; Z)$ and $\widetilde{SI}(X : Y; Z)$. In particular, \widetilde{SI} does not satisfy the following inequality, which was called *left monotonicity* in [7]:

$$SI((X, X') : Y; Z) \geq SI(X : Y; Z).$$

Hence, enlarging X may transform shared information into unique information. Finally, in Section IV-C we show that there is no monotonic relation between $\widetilde{CI}(X : (Y, Y'); Z)$ and $\widetilde{CI}(X : Y; Z)$, since the addition of Y' may turn complementary information into shared information. Moreover, there is no monotonic relation between $\widetilde{CI}((X, X') : Y; Z)$ and $\widetilde{CI}(X : Y; Z)$ either. Therefore, enlarging X may transform complementary information into unique information. We interpret our results in the concluding Section V.

II. MUTUAL INFORMATION AND COINFORMATION

The mutual information is defined by

$$MI(X : Y) = H(X) + H(Y) - H(X, Y),$$

where $H(X) = -\sum_x p(X=x) \log p(X=x)$ denotes the Shannon entropy. See [8] for an interpretation and further properties of MI . The mutual information satisfies the chain rule

$$MI(X : (Y, Z)) = MI(X : Y) + MI(X : Z|Y).$$

This identity can be derived from the entropy chain rule

$$H(X, Y) = H(Y) + H(X|Y).$$

Here, the conditional entropy and conditional mutual information are defined as follows: For any value y of Y with $p(Y=y) > 0$, let $H(X|Y=y)$ and $MI(X : Z|Y=y)$ be the entropy and mutual information of random variables distributed according to the conditional distributions $p(X=x|Y=y)$ and $p(X=x, Z=z|Y=y)$. Then

$$H(X|Y) = \sum_y p(Y=y) H(X|Y=y)$$

$$\text{and } MI(X : Z|Y) = \sum_y p(Y=y) MI(X : Z|Y=y).$$

Chain rules are very important in information theory, and they also play an important role in the proofs in this paper. Therefore, it would be nice if the quantities in an information decomposition would satisfy a chain rule. Unfortunately, as discussed in [7], this is not the case in any of the information decompositions proposed so far.

The chain rule and non-negativity imply that $MI(X : (Y, Z)) \geq MI(X : Y)$. This expresses the fact that ‘‘more variables contain more information.’’

The coinformation of three random variables is defined as

$$CoI(X; Y; Z) = MI(X : Y) - MI(X : Y|Z).$$

Expanding $CoI(X; Y; Z)$ one sees that the coinformation is symmetric in its three arguments. Moreover, the coinformation satisfies the chain rule

$$CoI(X; (Y, Y'); Z) = CoI(X; Y; Z) + CoI(X; Y'; Z|Y).$$

However, since the coinformation is not non-negative, in general, it does not increase if one of the variables is enlarged.

From (1) one can deduce

$$CoI(X; Y; Z) = \widetilde{SI}(X : Y; Z) - \widetilde{CI}(X : Y; Z).$$

This expresses the wellknown fact that a positive coinformation is a sign of redundancy, while a negative coinformation indicates synergy.

III. THE PARTIAL INFORMATION LATTICE AND THE IDENTITY AXIOM

In this section we briefly recall the ideas behind the partial information (PI) lattice by Williams and Beer. For details we refer to [2]. The PI lattice is a framework to define information decompositions of arbitrarily many random variables. Unfortunately, as we will show in Theorem 2, a non-negative decomposition of the mutual information according to the PI lattice is not possible with the identity axiom.

Consider $n+1$ variables X, Y_1, \dots, Y_n . We want to study in which way the information that Y_1, \dots, Y_n contain *about* X is distributed over the different combinations of the Y_i . For each subset $A \subseteq \{Y_1, \dots, Y_n\}$, the amount of information contained in A is equal to the mutual information $MI(X : A)$ (where A is interpreted as a random vector). Different subsets $A_1, \dots, A_k \subseteq \{Y_1, \dots, Y_n\}$ may share information, i.e. they may carry redundant information. What we are looking for is a function $I_{\cap}(X : A_1; \dots; A_k)$ to quantify this shared information. Williams and Beer propose that this function should satisfy the following axioms:

- $I_{\cap}(X : A_1; \dots; A_k)$ is symmetric under permutations of A_1, \dots, A_k . (*symmetry*)
- $I_{\cap}(X : A_1) = MI(X : A_1)$. (*self-redundancy*)
- $I_{\cap}(X : A_1; \dots; A_k; A_{k+1}) \leq I_{\cap}(X : A_1; \dots; A_k)$, with equality if $A_i \subseteq A_{k+1}$ for some $i \leq k$. (*monotonicity*)

Any function $I_{\cap}(X : A_1; \dots; A_k)$ that satisfies these axioms is determined from its values on the *antichains*; that is, on the families $\{A_1, \dots, A_k\}$ with $A_i \not\subseteq A_j$ for all $i \neq j$. The antichains of subsets of $\{Y_1, \dots, Y_n\}$ form a lattice with respect to the partial order

$$\begin{aligned} \{A_1, \dots, A_k\} \preceq \{B_1, \dots, B_l\} \\ \iff \text{for each } B_j \text{ there is } A_i \text{ with } A_i \subseteq B_j. \end{aligned}$$

This lattice is called the *partial information (PI) lattice* in this context. According to the Williams-Beer-axioms, $I_{\cap}(X : \cdot)$ is a monotone function on this lattice. The PI lattice for $n = 3$ is depicted in Fig. 1.

Let $A_1, \dots, A_k, A_{k+1} \subseteq \{Y_1, \dots, Y_n\}$. The idea behind the monotonicity axiom is, of course, not only that the amount of redundant information in A_1, \dots, A_k, A_{k+1} is less than the amount of redundant information in A_1, \dots, A_k (when measured in bits), but that, in fact, the redundancy in A_1, \dots, A_k, A_{k+1} really is *a part of* the redundancy in A_1, \dots, A_k . Similarly, in the case that $A_k \subseteq A_{k+1}$, not only should the two amounts of redundant information agree, but they should really refer to *the same information*. Therefore, in general, the difference

$$I_{\cap}(X : A_1; \dots; A_k) - I_{\cap}(X : A_1; \dots; A_k; A_{k+1})$$

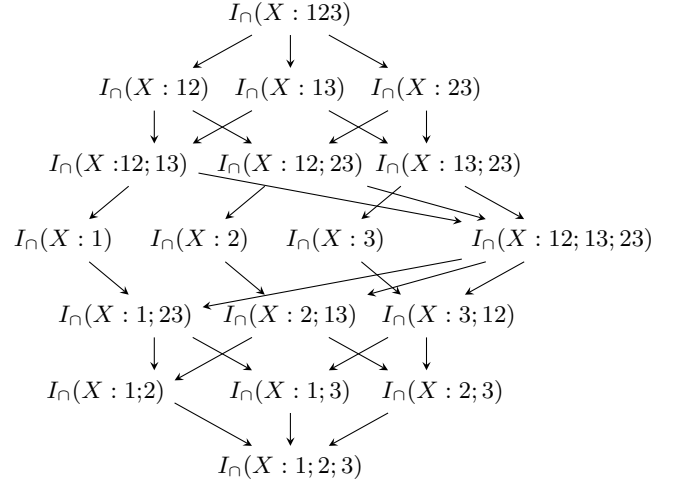


Fig. 1. The PI lattice for three variables. For brevity, the sets A_i are abbreviated by the indices of their elements; that is, $\{Y_1, Y_2\}$ is abbreviated by 12, and so on.

should measure the amount of information that is shared by A_1, \dots, A_k , but that is not contained in A_{k+1} .

Suppose that there exists a function $I_{\partial}(X : A_1; \dots; A_k)$ defined for any antichain $\{A_1, \dots, A_k\}$ that measure the amount of information contained in $I_{\cap}(X : A_1; \dots; A_k)$ that is not contained in any of those terms $I_{\cap}(X : B_1; \dots; B_l)$ where the antichain $\{B_1, \dots, B_l\} \prec \{A_1, \dots, A_k\}$. Then, if any information can be classified according to where, e.g. in which subset, it is available for the first time, e.g. it cannot be obtained from any smaller subset, the following identity should hold:

$$I_{\cap}(X : A_1; \dots; A_k) = \sum_{\{B_1, \dots, B_l\} \preceq \{A_1, \dots, A_k\}} I_{\partial}(X : A_1; \dots; A_k).$$

As shown in [2], this relation defines $I_{\partial}(X : A_1; \dots; A_k)$ uniquely using the Möbius inversion on the PI lattice. In general, however, the Möbius inversion does not yield a non-negative function. The property that I_{∂} is non-negative is called *local positivity* in [7]. Using an idea from the same paper we now show that local positivity contradicts the identity axiom mentioned in the introduction.

Theorem 2. *There are no functions I_{\cap}, I_{∂} that satisfy the Williams-Beer-axioms, local positivity and the identity axiom.*

Proof: Suppose to the contrary that such functions do exist. Consider the case $n = 3$, where Y_1, Y_2 are independent uniformly distributed binary random variables, and where $Y_3 = Y_1 \text{ XOR } Y_2$. Moreover, let $X = (Y_1, Y_2, Y_3)$. By the identity property, $I_{\cap}(\{Y_i, Y_j\} : Y_i; Y_j) = MI(Y_i : Y_j) = 0$ bit for any $i \neq j$. Observe that any pair of the variables $\{Y_1, Y_2, Y_3\}$ determines the third random variable. Therefore, X is just a relabeling of the state space $\{Y_i, Y_j\}$ for any $i \neq j$, and we obtain $I_{\cap}(X : Y_i; Y_j) = I_{\cap}(\{Y_i, Y_j\} : Y_i; Y_j) = 0$ bit. By monotonicity, $I_{\cap}(X : Y_1; Y_2; Y_3) = 0$ bit, and so $I_{\cap}(X : \cdot)$ and $I_{\partial}(X : \cdot)$ vanish on the lower two levels of the PI lattice (Fig. 1). On the next level, if $\{i, j, k\} = \{1, 2, 3\}$,

then by identity $I_{\cap}(X : Y_i; \{Y_j, Y_k\}) = I_{\cap}(\{Y_i, Y_j, Y_k\} : Y_i; \{Y_j, Y_k\}) = MI(Y_i : \{Y_j, Y_k\}) = 1$ bit, and so $I_{\partial}(X : Y_i; \{Y_j, Y_k\}) = 1$ bit. On the other hand, $I_{\cap}(X : \{Y_1, Y_2\}; \{Y_1, Y_3\}; \{Y_2, Y_3\}) \leq MI(X : \{Y_1, Y_2, Y_3\}) = 2$ bit by monotonicity, and so

$$\begin{aligned} I_{\partial}(X : \{Y_1, Y_2\}; \{Y_1, Y_3\}; \{Y_2, Y_3\}) \\ &= I_{\cap}(X : \{Y_1, Y_2\}; \{Y_1, Y_3\}; \{Y_2, Y_3\}) \\ &\quad - \sum_{\{i,j,k\}=\{1,2,3\}} I_{\partial}(X : Y_i; \{Y_j, Y_k\}) \\ &\leq 2 \text{ bit} - 3 \text{ bit} = -1 \text{ bit}. \end{aligned}$$

This contradiction concludes the proof. \blacksquare

IV. \widetilde{UI} , \widetilde{SI} AND \widetilde{CI} IN THE MULTIVARIATE SETTING

In this section we study what happens to the functions \widetilde{UI} , \widetilde{SI} and \widetilde{CI} when one of their arguments is enlarged.

A. The unique information

- Lemma 3.** 1) $\widetilde{UI}(X : Y \setminus (Z, Z')) \leq \widetilde{UI}(X : Y \setminus Z)$.
 2) $\widetilde{UI}(X : (Y, Y') \setminus Z) \geq \widetilde{UI}(X : Y \setminus Z)$.
 3) $\widetilde{UI}((X, X') : Y \setminus Z) \geq \widetilde{UI}(X : Y \setminus Z)$.

Proof: First we prove 1). Let P be the joint distribution of X, Y, Z , and let P' be the joint distribution of X, Y, Z, Z' . By definition, P is a marginal of P' . Let $Q \in \Delta_P$, and let

$$\begin{aligned} Q'(x, y, z, z') &:= \frac{Q(x, y, z)P'(x, z, z')}{P(x, z)} \\ &= Q(x, y, z)P'(z'|x, z) \end{aligned}$$

if $P(x, z) > 0$ and $Q'(x, y, z, z') = 0$ else. Then $Q' \in \Delta_{P'}$. Moreover, Q is the (X, Y, Z) -marginal of Q' , and Z' is independent of Y given X and Z with respect to Q' . Therefore,

$$\begin{aligned} MI_{Q'}(X : Y|Z, Z') \\ &= MI_{Q'}(X, Z' : Y|Z) - MI_{Q'}(Z' : Y|Z) \\ &\leq MI_{Q'}(X, Z' : Y|Z) \\ &= MI_{Q'}(X : Y|Z) + MI_{Q'}(Z' : Y|X, Z) \\ &= MI_{Q'}(X : Y|Z) = MI_Q(X : Y|Z). \end{aligned}$$

The statement follows by taking the minimum over $Q \in \Delta_P$.

Statements 2. and 3. can be proved together. Consider five random variables X, X', Y, Y', Z with joint distribution P' , and let P be the (X, Y, Z) -marginal of P' . Let $Q' \in \Delta_{P'}$, and let Q be the (X, Y, Z) -marginal of Q' . Then $Q \in \Delta_P$. Moreover,

$$\begin{aligned} MI_{Q'}((X, X') : (Y, Y')|Z) &\geq MI_{Q'}(X : Y|Z) \\ &= MI_Q(X : Y|Z). \end{aligned}$$

Taking the minimum for $Q' \in \Delta_{P'}$ implies

$$\widetilde{UI}((X, X') : (Y, Y') \setminus Z) \geq \widetilde{UI}(X : Y \setminus Z).$$

Statements 2. and 3. follow by setting either X' or Y' to a constant random variable. \blacksquare

B. The shared information

Lemma 4. $\widetilde{SI}(X : (Y, Y'); Z) \geq \widetilde{SI}(X : Y; Z)$.

Proof: Let P' be the joint distribution of X, Y, Y', Z , and let P be the (X, Y, Z) -marginals of P' . For any $Q \in \Delta_P$ define a probability distribution Q' by

$$Q'(x, y, y', z) := \begin{cases} \frac{Q(x, y, z)P'(x, y, y')}{P(x, y)}, & \text{if } P(x, y) > 0, \\ 0, & \text{else.} \end{cases}$$

Then $Q' \in \Delta_{P'}$, and Y' and Z are conditionally independent given X and Y with respect to Q' . Observe that $CoI_{Q'}(X, Y, Z) = CoI_Q(X, Y, Z)$ and

$$\begin{aligned} CoI_{Q'}(X, Y', Z|Y) \\ &= MI_{Q'}(Y', Z|Y) - MI_{Q'}(Y', Z|X, Y) \\ &= MI_{Q'}(Y', Z|Y) \geq 0. \end{aligned}$$

Hence, the chain rule of the coinformation implies that $CoI_{Q'}(X, (Y, Y'), Z) \geq CoI_Q(X, Y, Z)$. The statement follows by maximizing $Q \in \Delta_P$. \blacksquare

Should there be a relation between $SI((X, X') : Y; Z)$ and $SI(X : Y; Z)$?

$$SI((X, X') : Y; Z) \geq SI(X : Y; Z)$$

is called *left monotonicity*. As observed in [7], none of the measures of shared information proposed so far satisfies left monotonicity.

\widetilde{SI} also violates left monotonicity. Basically, the identity axiom makes it difficult to satisfy left monotonicity. Consider two independent binary random variables X, Y and let $Z = X \text{ AND } Y$. Even though X and Y are independent, one can argue that they share information about Z . For example, if X and Y are both zero, then both X and Y can deduce that $Z = 0$. And indeed, in this example, $\widetilde{SI}(Z : X; Y) \approx 0.311$ bit [7], and also other proposed information decompositions yield a non-zero shared information [3]. Therefore,

$$\begin{aligned} \widetilde{SI}(Z : X; Y) &> 0 = MI(X : Y) \\ &= \widetilde{SI}((X, Y) : X; Y) = \widetilde{SI}((Z, X, Y) : X; Y). \end{aligned}$$

As observed in [7], a chain rule for the shared information of the form

$$SI((X, X') : Y; Z) = SI(X : Y; Z) + SI(X' : Y; Z|X)$$

would imply left monotonicity. Therefore, \widetilde{SI} does not satisfy a chain rule.

C. The complementary information

Should there be a relation between $CI(X : (Y, Y'); Z)$ and $CI(X : Y; Z)$? Since “more random variables contain more information,” it is easy to find examples where more random variables contain more complementary information,” that is $\widetilde{CI}(X : (Y, Y'); Z) > \widetilde{CI}(X : Y; Z)$. For example, let Y', Y and Z be independent uniformly distributed binary random variables and $X = Y' \text{ XOR } Z$. In this example Y and Z know nothing about X , but Y' and Z together determine X , and so

$$\begin{aligned} 1 \text{ bit} = \widetilde{CI}(X : Y'; Z) &= \widetilde{CI}(X : (Y, Y'); Z) \\ &> 0 \text{ bit} = \widetilde{CI}(X : Y; Z). \end{aligned}$$

On the other hand, there are examples where $\widetilde{CI}(X : (Y, Y'); Z) < \widetilde{CI}(X : Y; Z)$. The reason is that further information may transform synergistic information into redundant information. For example, if $X = Y \text{ XOR } Z$, then

$$1 \text{ bit} = \widetilde{CI}(X : Y; Z) > 0 \text{ bit} = \widetilde{CI}(X : (Y, Z); Z).$$

Neither is there a simple relation between $\widetilde{CI}((X, X') : Y; Z)$ and $\widetilde{CI}(X : Y; Z)$. The argument is similar as for the shared information. In fact, for any pair (Y, Z) of random variables, the identity axiom implies $\widetilde{CI}((Y, Z) : Y; Z) = 0 \text{ bit}$ [1]. Consider again the case that $X = Y \text{ XOR } Z$. As random variables, the triple (X, Y, Z) is equivalent to the pair (Y, Z) . Therefore,

$$1 \text{ bit} = \widetilde{CI}(X : Y; Z) > 0 \text{ bit} \\ = \widetilde{CI}((Y, Z) : Y; Z) = \widetilde{CI}((X, Y, Z) : Y; Z).$$

So the left monotonicity for the synergy is violated again as a consequence of the identity axiom. As above, this implies that \widetilde{CI} does not satisfy a chain rule of the form

$$CI((X, X') : Y; Z) = CI(X : Y; Z) + CI(X' : Y; Z|X).$$

V. CONCLUSIONS

We have seen that \widetilde{UI} behaves according to our intuition if one of its arguments is replaced by a “larger random variable.” Moreover, \widetilde{SI} increases, if one of its right arguments is enlarged. On the other hand, there is no monotone relation for the left argument in \widetilde{SI} , and for \widetilde{CI} there is no monotone relation at all. In these last cases, information is transformed in some way. For example, if the inequality $\widetilde{CI}(X : (Y, Y'); Z) < \widetilde{CI}(X : Y; Z)$ holds, then the addition of Y' transforms synergistic information into redundant information.

Let us look again at the example that demonstrates that \widetilde{SI} violates left monotonicity. In the operational interpretation of [1] this has the following interpretation: If $Z = X \text{ AND } Y$, then the two conditional distributions $p(Z = z|X = x)$ and $p(Z = z|Y = y)$ are identical. Therefore, if X or Y can be used in a decision task which reward depends on Z , none of the two random variables performs better than the other; none of them has an advantage, and so none of them has unique information about Z . On the other hand, X and Y do know different aspects about the random vector (X, Y) , and depending on whether a reward function depends more on X or on Y , they perform differently. Therefore, each of them carries unique information about (X, Y) . Intuitively, one could argue that combining the information in X, Y via the AND function has transformed unique into shared information.

As stated above, the fact that \widetilde{SI} and \widetilde{CI} do not satisfy left monotonicity is related to the identity axiom. For the complementary information this relation is strict: Any measure of complementary information that comes from a bivariate information decomposition of the form (1), that satisfies the identity axiom and that is positive in the XOR-example violates left monotonicity, as the argument in Section IV-C shows. For the shared information this relation is more subtle: Identity and left monotonicity do not directly contradict each

other, but whenever X is a function of Y and Z they imply the strong inequality $SI(X : Y; Z) \leq MI(Y : Z)$.

In Section III we have shown that the identity axiom contradicts a non-negative decomposition according to the PI lattice for $n \geq 3$. Therefore, if we want to extend the bivariate information decomposition into \widetilde{UI} , \widetilde{SI} and \widetilde{CI} to more variables, then this multivariate information decomposition must have a form that is different from the PI lattice. In particular, it is an open question which terms such an information decomposition should have.

Even if the structure of such a decomposition is presently unknown, we can interpret the bivariate quantities \widetilde{UI} , \widetilde{SI} and \widetilde{CI} in this context. For example, the quantity $MI(X : Y_1, \dots, Y_k) - \sum_{i=1}^k \widetilde{UI}(X : Y_i \setminus Y_1, \dots, Y_{i-1}, Y_{i+1}, \dots, Y_k)$ has the natural interpretation as “the union of all information that is either synergistic or shared for some combination of variables.” Hence we conjecture that this difference should be non-negative.

The conjecture would follow from the inequality

$$\widetilde{UI}(X : Y \setminus Z, W) + \widetilde{UI}(X : Z \setminus Y, W) \\ \leq \widetilde{UI}(X : (Y, Z) \setminus W).$$

This inequality states that the unique information contained in a pair of variables is larger than the sum of the unique informations of the single variables. The difference between the right hand side and the left hand side should be due to synergistic effects. Proving (or disproving) the conjecture and this inequality would be a large step towards a better understanding of the function \widetilde{UI} .

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