

Practical sample-and-hold stabilization of nonlinear systems under approximate optimizers

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Abstract—It is a known fact that not all controllable systems can be asymptotically stabilized by a continuous static feedback. Several approaches have been developed throughout the last decades, including time-varying, dynamical and even discontinuous feedbacks. In the latter case, the sample-and-hold framework is widely used, in which the control input is held constant during sampling periods. Consequently, only practical stability can be achieved at best. Existing approaches often require solving optimization problems for finding stabilizing control actions exactly. In practice, each optimization routine has a finite accuracy which might influence the state convergence. This work shows, what bounds on optimization accuracy are required to achieve prescribed stability margins. Simulation studies support the claim that optimization accuracy has high influence on the state convergence.

Index Terms—Stability of nonlinear systems, optimization

I. INTRODUCTION

IT is a classical result due to [2] which demonstrates that not every dynamical system can be stabilized by a continuous static (dependent on the state only) feedback. Since then, comprehensive work has been done in search for alternative solutions to this issue, including discontinuous feedbacks [1], [4], [6]–[9], [15]. This work falls specifically into the framework of sample-and-hold stabilization where the control actions are held constant during specially chosen sample time periods. In this case only practical stability can be achieved at best. A concrete implementation of this setup used inf-convolutions of the given (generally non-smooth) CLF to compute its specific proximal subgradients [4]. Using these, the control actions were calculated by computing optimizers for the CLF decay condition. In most of these techniques, in general, one has to deal with non-linear optimization problems. Unfortunately, each optimization routine has a finite accuracy which leads to approximate optimizers at best. Consequently, stability properties might be compromised. Whereas robustness properties with regards to system, input and measurement disturbance of CLF are thoroughly studied [3], [4], [11], [12], [14], [16], additional attention should be paid to the uncertainty related to optimization inaccuracy. Specifically, in inf-convolution-based feedback designs (see, e.g., [4]), stabilizing control actions are computed using certain proximal subgradients. Under non-exact optimization, the property of being a proximal subgradient may be lost (details in Section III). The new result of this work is a theorem on practical stabilization by an inf-convolution-based feedback under approximate optimizers. Simulation studies of Section VI show significant influence of optimization accuracy on the system performance. Section II presents some basics

of the sample-and-hold framework. Section III summarizes the contributions, whereas Section IV presents the theoretical results. Discussion, importance of the made assumptions and relation to various types of uncertainty can be found in Section V.

Notation: $\overline{\text{co}}(D)$ denotes the closed convex hull of a set D , $\mathcal{B}_R(x)$ denotes a closed ball in \mathbb{R}^n centered at x . If the center is the origin, the notation is simplified to just \mathcal{B}_R . The scalar product is denoted by $\langle \bullet, \bullet \rangle$.

II. PRELIMINARIES

Consider the following dynamical system:

$$\dot{x} = f(x, u). \quad (\text{Sys})$$

Here, $x \in \mathbb{R}^n$ and $u \in \mathcal{U} \subseteq \mathbb{R}^m$ denote the system state and input, respectively. The vector field $f : \mathbb{R}^n \times \mathcal{U} \rightarrow \mathbb{R}^n$ is assumed to satisfy the following local Lipschitz condition:

$$\begin{aligned} \forall c \in \mathbb{R}^n, r > 0, \forall x, y \in \{z : \|z - c\| \leq r\}, \forall u \in \mathcal{U} \\ \|f(x, u) - f(y, u)\| \leq L_f(c, r) \cdot \|x - y\|. \end{aligned} \quad (\text{Lip})$$

In general, one seeks a static feedback of the form $u = \kappa(x)$. In the sample-and-hold setup, the control is held constant during sampling periods of length δ as follows:

$$\dot{x} = f(x, u_k), t \in [k\delta, (k+1)\delta], u_k \equiv \kappa(x(k\delta)). \quad (\text{SH})$$

It should be noted that the results in this work can be straightforwardly generalized to non-periodic partitions of the time axis with a specification of the maximal step size of δ , but, for simplicity, sampling is assumed periodic here. The control goal is to *practically stabilize* (SH) in the sense that, for any given $R > r > 0$, there is a sufficiently small $\delta > 0$ such that any trajectory starting in the ball \mathcal{B}_R is bounded and eventually enters the ball \mathcal{B}_r within a time T that depends uniformly on R, r and the trajectory stays there after T .

Stabilization is achieved by utilizing a locally Lipschitz-continuous, proper and positive-definite CLF $V : \mathbb{R}^n \rightarrow \mathbb{R}$ for which there exists a continuous function $w : \mathbb{R}^n \rightarrow \mathbb{R}, x \neq 0 \implies w(x) > 0$ satisfying the *decay condition*: for any compact set $\mathbb{X} \subseteq \mathbb{R}^n$, there exists a compact set $\mathcal{U}_{\mathbb{X}} \subseteq \mathcal{U}$ such that

$$\forall x \in \mathbb{X} \quad \inf_{\vartheta \in \overline{\text{co}}(f(x, \mathcal{U}_{\mathbb{X}}))} D_{\vartheta} V(x) \leq -w(x). \quad (\text{Dec})$$

Here, $D_{\vartheta} V(x)$ denotes the *lower directional Dini derivative*, defined as follows [4]:

$$D_{\vartheta} V(x) \triangleq \liminf_{\mu \rightarrow 0^+} \frac{V(x + \mu\vartheta) - V(x)}{\mu}. \quad (\text{Dini})$$

Existence of a CLF with the above property is guaranteed for *asymptotically controllable* systems [17]. To actually practically stabilize the system in the sense of sample-and-hold as per (SH), one needs to compute the control actions $u_k, k \in \mathbb{N}$ from the given CLF V . Several techniques exist for this matter. For example, Clarke et al. [4] used *inf-convolutions* of V , which, for a given $0 < \alpha < 1$, are defined as follows:

$$V_\alpha(x) := \inf_{y \in \mathbb{R}^n} (V(y) + \frac{1}{2\alpha^2} \|y - x\|^2). \quad (1)$$

Clarke et al. [4] then used a minimizer $y_\alpha(x)$ of (1) for each given x to determine the vector

$$\zeta_\alpha(x) := \frac{x - y_\alpha(x)}{\alpha^2} \quad (2)$$

which also happens to be a particular *proximal subgradient* of V at the respective $y_\alpha(x)$ as follows:

$$V(z) \geq V(y_\alpha(x)) + \langle \zeta_\alpha(x), z - y_\alpha(x) \rangle - \frac{1}{2\alpha^2} \|z - y_\alpha(x)\|^2 \quad (3)$$

for every $z \in \mathbb{R}^n$ (for a comprehensive study on non-smooth analysis and various notions involved in this work, the reader should refer, e. g., to [5]). Suppose that $y_\alpha(x)$ lies within some compact set \mathbb{Y} . Then, the control actions are determined by finding minimizers for

$$\inf_{u \in \mathbb{U}_Y} \langle \zeta_\alpha(x), f(y_\alpha(x), u) \rangle \quad (4)$$

so as to satisfy the decay condition (Dec) using the fact that, for any proximal subgradient ζ of V at x and any vector ϑ , it holds that

$$\langle \zeta, \vartheta \rangle \leq D_\vartheta V(x). \quad (5)$$

Another approach is based on a technique called *Dini aiming* [11], [12], with a recent application to the non-holonomic integrator [1], by solving special optimization problems involving the CLF and the system dynamics model f . Both the inf-convolution and Dini aiming methods require solving optimization problems with the respective analyzes done under the assumption that the optimization problems be solved exactly (for details, please refer, e. g., to [1], [4]).

In the current work, it is investigated if practical stabilization can be achieved if only approximate minimizers for (1) and (4) can be found. The scope of the current work is focused on the inf-convolution technique as the basis, whereas the developed methodology could be used for Dini aiming in a future study. The summary is given in the next section.

III. WORK AIM AND CONTRIBUTIONS

The stabilizing feedback by inf-convolution amounts to solving the optimization problem (1) first, then (4) at every sample state x . Such a feedback will be called *InfC-feedback* from now on. The main concern of this work is to investigate robustness properties of the CLF under an approximate InfC-feedback in the following sense: for a fixed x , (1) and (4) are only solved approximately. That is, an approximate $y_\alpha^\varepsilon(x)$ is found so that:

$$V(y_\alpha^\varepsilon(x)) + \frac{1}{2\alpha^2} \|y_\alpha^\varepsilon(x) - x\|^2 \leq V_\alpha(x) + \varepsilon_x, \quad (6)$$

where $\varepsilon_x > 0$ characterizes the accuracy of the numerical optimization routine and, in general, may depend on x .

Consequently, an approximate control action κ_x^η is found so as to satisfy:

$$\langle \zeta_\alpha^\varepsilon(x), f(y_\alpha^\varepsilon(x), \kappa_x^\eta) \rangle \leq \inf_{u \in \mathbb{U}_Y} \langle \zeta_\alpha^\varepsilon(x), f(y_\alpha^\varepsilon(x), u) \rangle + \eta_x, \quad (7)$$

where

$$\zeta_\alpha^\varepsilon(x) := \frac{x - y_\alpha^\varepsilon(x)}{\alpha^2} \quad (8)$$

is a *proximal ε_x -subgradient*, and η_x , like ε_x , is related to the optimization accuracy. The following notion summarizes practical stabilization under approximate optimizers:

Definition 1: An InfC-feedback is said to practically stabilize (Sys) in the sense of sample-and-hold (SH) under approximate optimizers if, for any data $R > r > 0$, there is a sufficiently small $\delta > 0$ such that any closed-loop trajectory $x(t), t \geq 0, x(0) \in \mathcal{B}_R$ is bounded and enters and stays in the ball \mathcal{B}_r within a time T depending uniformly on the data R, r , provided that, at every sample state $x_k := x(\delta k), k \in \mathbb{N}$, the accuracies ε_{x_k} and η_{x_k} in (6) and (7), accordingly, are sufficiently small. The **major problem** is that, in general, $\zeta_\alpha^\varepsilon(x)$ is **not even a proximal subgradient** of V at $y_\alpha^\varepsilon(x)$. As a result, the relation (5) does not apply for $\zeta_\alpha^\varepsilon(x)$ in place of ζ . It might have been attractive to use some continuity properties of proximal subdifferentials, i. e., the sets of all proximal subgradients. Continuity properties of ordinary (not proximal) subdifferentials were studied by [10], [18]. However, these results cannot be directly applied to proximal subdifferentials. A proximal subdifferential may happen to be empty, but there still may exist proximal ε -subgradients. To overcome the described difficulties, the following is assumed in the current work:

Assumption 1: The CLF V is globally lower Dini-differentiable and the \liminf in (Dini) is locally uniform. That is, for any compact sets $\mathbb{Y}, \mathbb{F} \in \mathbb{R}^n$, and any $\nu > 0$, there exists $\mu > 0$ such that, for any $y \in \mathbb{Y}, \vartheta \in \mathbb{F}, 0 < \mu' \leq \mu$, it holds that $\left| \frac{V(y + \mu' \vartheta) - V(y)}{\mu'} - D_\vartheta V(y) \right| \leq \nu$.

The main **contribution** of this work is to investigate practical stabilization under approximate optimizers in the sense of Definition 1 under Assumption 1. The main result is summarized in the next section. Assumption 1 as well as further robustness considerations are discussed in Section V.

IV. THEORETICAL RESULTS

The following theorem summarizes the major result on practical stabilization by InfC-feedbacks under approximate optimizers.

Theorem 1: Consider the control system in the sample-and-hold format (SH). Assume that f satisfies the Lipschitz condition (Lip) and there exists a CLF V with a decay condition (Dec) satisfying Assumption 1. Then, the system (SH) can be practically stabilized by an InfC-feedback in the sense of (SH) under approximate optimizers.

Proof: The proof is divided into three steps. The first step deals with necessary preparations and parameter determination. In the second step, a relaxed decay condition of the inf-convolution of the CLF is shown. The actual decay is demonstrated in the third step along with the estimation of the reaching time T . The proof uses technical Lemmas 1, 2 which are found in the appendix.

Step 1. Preliminaries

Let $0 < r < R$ be the radii of the target and starting ball, respectively. First, two non-decreasing functions, ρ_V and λ_V , with the properties

$$\begin{aligned} \forall x \in \mathbb{R}^n, r, v > 0 \quad V(x) \leq \rho_V(r) &\implies \|x\| \leq r, \\ V(x) \geq v &\implies \lambda_V(v) \leq \|x\| \end{aligned}$$

are constructed. Due to Lemma 3.5 in [13], there exist two class \mathcal{K} functions α_1, α_2 with the properties:

$$\forall x \in \mathbb{R}^n \quad \alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|).$$

Then, simply taking $\rho_V(r) := \alpha_1(r)$, it holds that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_1(r) \implies \|x\| \leq r.$$

For $\lambda_V(r)$, take α_2^{-1} which is non-decreasing as well. Then,

$$\alpha_2(\|x\|) \geq V(x) \geq v \implies \|x\| \geq \alpha_2^{-1}(v).$$

In the forthcoming step, the following bounds of V in terms of its inf-convolutions using Lemma 2 from Appendix will be used:

$$V_\alpha(x) \leq V(x) \leq V_\alpha(x) + \varepsilon_1. \quad (9)$$

Here, ε_1 is a positive number that will be determined later. With this in mind, extend the use of $\rho_V(r)$ and $\lambda_V(r)$ to V_α as follows:

$$\begin{aligned} V_\alpha(x) \leq \rho_V(r) - \varepsilon_1 &\implies V(x) \leq \rho_V(r) \implies \|x\| \leq r, \\ V_\alpha(x) \geq v &\implies V(x) \geq v \implies \|x\| \geq \lambda_V(v). \end{aligned}$$

Set $\bar{V} := \sup_{\|x\| \leq R} V(x)$. Choose $R^* > R$ with the corresponding $V^* := \sup_{\|x\| \leq R^*} V(x)$ such that $\Theta := \rho_V(R^*) > \bar{V}$. Further, set $v^* := \rho_V(r)$ and $r^* := \lambda_V\left(\frac{v^*}{4}\right)$. Observe that $r^* \leq r$. To see this, assume, on contrary, that $r^* > r$. Since V is continuous, there exists x such that $\frac{v^*}{4} \leq V(x) \leq v^*$. By the virtue of ρ_V , it follows that $\|x\| \leq r$, whereas $\|x\| \geq \lambda_V\left(\frac{v^*}{4}\right) > r$, a contradiction. Now, let \mathbb{U}^* be the compact set corresponding to the ball $\mathcal{B}_{R^* + \sqrt{2V^*}}$ in the decay condition (Dec):

$$\forall x \in \mathcal{B}_{R^* + \sqrt{2V^*}} \quad \inf_{\vartheta \in \text{co}(f(x, \mathbb{U}^*))} D_\vartheta V(x) \leq -w(x).$$

Let L_f be the Lipschitz constant of f as per (Lip) on the ball $\mathcal{B}_{R^* + \sqrt{2V^*}}$. Set

$$\bar{f} := \sup_{\substack{x \in \mathcal{B}_{R^* + \sqrt{2V^*}} \\ u \in \mathbb{U}^*}} f(x, u)$$

and determine the minimal decay rate as follows:

$$\bar{w} := \inf_{\frac{r^*}{2} \leq \|x\| \leq R^* + \sqrt{2V^*}} w(x).$$

Finally, let ω_V be the continuity modulus of V on $\mathcal{B}_{R^* + \sqrt{2V^*}}$. That is, for all $\varepsilon_1 > 0$, and $x, y \in \mathcal{B}_{R^* + \sqrt{2V^*}}$, it holds that

$$\|x - y\| \leq \omega_V(\varepsilon_1) \implies |V(x) - V(y)| \leq \varepsilon_1.$$

Step 2. Relaxed decay condition

A relaxed decay condition is established by addressing the scalar product $\langle \zeta_\alpha^{\varepsilon^2}(x), f(x, \kappa_x^\eta) \rangle$, where $\zeta_\alpha^{\varepsilon^2}(x)$ will be

a proximal ε_x^2 -subgradient with ε_x determined later on sample-wise at $x_k, k \in N$, and κ_x^η will be the control action determined as an approximate optimizer of $\inf_{u \in \mathbb{U}^*} \langle \zeta_\alpha^{\varepsilon^2}(x), f(x, u) \rangle$ with the yet-to-be-determined accuracy η_x .

Using the Lipschitz constant L_f of f , it holds, for all x with $r^* \leq \|x\| \leq R^*$, that:

$$\begin{aligned} \langle \zeta_\alpha^{\varepsilon^2}(x), f(x, \kappa_x^\eta) \rangle &\leq \eta_x \\ + \underbrace{\inf_{u \in \mathbb{U}^*} \langle \zeta_\alpha^{\varepsilon^2}(x), f(y_\alpha^{\varepsilon^2}(x), u) \rangle}_{\textcircled{1}} &+ L_f \underbrace{\|\zeta_\alpha^{\varepsilon^2}(x)\| \cdot \|y_\alpha^{\varepsilon^2}(x) - x\|}_{\textcircled{2}}, \end{aligned}$$

where $y_\alpha^{\varepsilon^2}(x)$ is an ε_x^2 -minimizer for the respective inf-convolution (1). Notice η_x in the right-hand side which indicates that the control actions κ_x^η are found only in an approximate format.

Now, examine the term $\textcircled{1}$. This is the term which the approximate proximal subgradient condition in Step 3 will be applied to. But first, it requires a fixed α . Therefore, ε will be determined later when all the conditions on α are established. Regarding the term $\textcircled{2}$, observe that, by definitions of $y_\alpha^{\varepsilon^2}(x)$ and $\zeta_\alpha^{\varepsilon^2}(x)$ according to (6), (8), respectively, and Lemma 1 from Appendix,

$$\begin{aligned} \|\zeta_\alpha^{\varepsilon^2}(x)\| \cdot \|y_\alpha^{\varepsilon^2}(x) - x\| &= \frac{1}{\alpha^2} \|y_\alpha^{\varepsilon^2}(x) - x\|^2 \\ &\leq 2(V(x) - V(y_\alpha^{\varepsilon^2})) + 2\varepsilon_x^2. \end{aligned}$$

If α is chosen to satisfy the following condition:

$$\sqrt{2V^*}\alpha \leq \min \left\{ 1, \frac{r^*}{2}, \omega_V(\varepsilon_1) \right\}, \quad (10)$$

where ε_1 is a positive number yet to be determined, then, it holds that

$$\begin{aligned} \langle \zeta_\alpha^{\varepsilon^2}(x), f(x, \kappa_x^\eta) \rangle &\leq \inf_{u \in \mathbb{U}^*} \langle \zeta_\alpha^{\varepsilon^2}(x), f(y_\alpha^{\varepsilon^2}(x), u) \rangle \\ &\quad + 2L_f(\varepsilon_1 + \varepsilon_x^2) + \eta_x. \end{aligned} \quad (11)$$

Also, observe that $\frac{r^*}{2} \leq \|y_\alpha^{\varepsilon^2}(x)\| \leq R^* + \sqrt{2V^*}$ and (9) holds. The parameters $\varepsilon_x, \varepsilon_1, \eta_x$ in (11) will be determined in Step 3.

Step 3. Decay

First, it follows directly from the definitions, that a variant of ‘‘Taylor expansion’’ for V_α holds in an approximate format, i.e., for all $h, \vartheta \in \mathbb{R}^n$,

$$V_\alpha(x + h\vartheta) \leq V_\alpha(x) + h \langle \zeta_\alpha^{\varepsilon^2}(x), \vartheta \rangle + \frac{h^2 \|\vartheta\|^2}{2\alpha^2} + \varepsilon_x^2. \quad (12)$$

Proceed by induction over the sample time periods. Suppose that the trajectory of (SH) exists locally at the time period $[k\delta, (k+1)\delta]$. Assume $V_\alpha(x_k) \leq \bar{V}$. To show that the trajectory exists on the entire sample time period, observe that, due to (9), for $t \in [k\delta, (k+1)\delta]$,

$$V(x(t)) \leq V_\alpha(x(t)) + \varepsilon_1 \leq \bar{V} + \varepsilon_1. \quad (13)$$

Therefore, ε_1 can be chosen so as to satisfy the following condition $V(x(t)) \leq \Theta$. Consequently, $x(t) \in \mathcal{B}_{R^*}$ and thus the overshoot is bounded. In the following, it is shown that $V_\alpha(x_k)$ may only decay down to a prescribed limit sample-wise, i.e., for $k \in \mathbb{N}$ $V_\alpha(x_{k+1}) < V_\alpha(x_k)$ until $V_\alpha(x_k) \leq v^*$.

Therefore, boundedness of the trajectory on each sample period is secured. The following cases are now possible.

Case 1: $V_\alpha(x_k) \geq \frac{v^*}{2}$. Use the ‘‘Taylor expansion’’ (12) to deduce, for any $t \in [k\delta, (k+1)\delta]$, $\Delta t := t - k\delta$, that

$$V_\alpha(x(t)) - V_\alpha(x_k) = V_\alpha(x_k + \delta F_k) - V_\alpha(x_k) \leq \delta \langle \zeta_\alpha^{\varepsilon^2}(x_k), F_k \rangle + \frac{\delta^2 \|F_k\|^2}{2\alpha^2} + \varepsilon_{x_k}^2, \quad (14)$$

where F_k is defined by the integral form of the trajectory:

$$x(t) = x_k + \int_{k\delta}^t f(x(\tau), \kappa_{x_k}^\eta) d\tau = x_k + \delta \underbrace{\left(\frac{1}{\delta} \int_{k\delta}^t f(x(\tau), \kappa_{x_k}^\eta) d\tau \right)}_{=: F_k}$$

and, with the property $\|F_k\| \leq \frac{1}{\delta} \Delta t \bar{f}$, F_k can be re-expressed as

$$F_k = \frac{\Delta t}{\delta} f(x_k, \kappa_{x_k}^\eta) + \underbrace{\frac{1}{\delta} \int_{k\delta}^t (f(x(\tau), \kappa_{x_k}^\eta) - f(x_k, \kappa_{x_k}^\eta)) d\tau}_{=: A}$$

Now, since $\|A\| \leq \frac{\Delta t^2}{\delta} L_f \bar{f}$, the scalar product $\langle \zeta_\alpha^{\varepsilon^2}(x_k), F_k \rangle$, is bounded as follows:

$$\begin{aligned} \langle \zeta_\alpha^{\varepsilon^2}(x_k), F_k \rangle &\leq \frac{\Delta t}{\delta} \langle \zeta_\alpha^{\varepsilon^2}(x_k), f(x_k, \kappa_{x_k}^\eta) \rangle + \|\zeta_\alpha^{\varepsilon^2}(x_k)\| \frac{\Delta t^2}{\delta} L_f \bar{f} \leq \\ &\frac{\Delta t}{\delta} \left(\inf_{u \in \mathbb{U}^*} \langle \zeta_\alpha^{\varepsilon^2}(x_k), f(y_\alpha^{\varepsilon^2}(x_k), u) \rangle + 2L_f(\varepsilon_1 + \varepsilon_{x_k}^2) + \eta_{x_k} \right) \\ &+ \frac{\sqrt{2V^*}}{\alpha} \frac{\Delta t^2}{\delta} L_f \bar{f}, \end{aligned}$$

which follows from the relaxed decay condition (11). Therefore,

$$\begin{aligned} V_\alpha(x(t)) - V_\alpha(x_k) &\leq \Delta t \left(\inf_{u \in \mathbb{U}^*} \langle \zeta_\alpha^{\varepsilon^2}(x_k), f(y_\alpha^{\varepsilon^2}(x_k), u) \rangle \right. \\ &\left. + 2L_f(\varepsilon_1 + \varepsilon_{x_k}^2) + \eta_{x_k} + \frac{\sqrt{2V^*}}{\alpha} \Delta t L_f \bar{f} + \frac{\Delta t \bar{f}^2}{2\alpha^2} \right) + \varepsilon_{x_k}^2. \end{aligned} \quad (15)$$

In particular, for $t = (k+1)\delta$, observe that

$$\begin{aligned} V_\alpha(x_{k+1}) - V_\alpha(x_k) &\leq \delta \left(\inf_{u \in \mathbb{U}^*} \langle \zeta_\alpha^{\varepsilon^2}(x_k), f(y_\alpha^{\varepsilon^2}(x_k), u) \rangle \right. \\ &\left. + 2L_f(\varepsilon_1 + \varepsilon_{x_k}^2) + \eta_{x_k} + \frac{\sqrt{2V^*}}{\alpha} \delta L_f \bar{f} + \frac{\delta \bar{f}^2}{2\alpha^2} \right) + \varepsilon_{x_k}^2. \end{aligned} \quad (16)$$

Case 2: $V_\alpha(x_k) \leq \frac{3v^*}{4}$. If the sample period size δ satisfies $\delta \bar{f} \leq \omega_V(\varepsilon_2)$ for some $\varepsilon_2 > 0$, then

$$V_\alpha(x(t)) \leq V_\alpha(x_k) + \varepsilon_2. \quad (17)$$

Choosing $\varepsilon_2 \leq \frac{v^*}{8}$ guarantees that $V_\alpha(x(t)) \leq \frac{7v^*}{8}$. Now, recall the condition (9) where ε_1 needed to be determined. Taking into account (13), ε_1 can be determined so as to satisfy the condition $V(x(t)) \leq v^*$ which also implies $\|x(t)\| \leq r$.

Finally, the necessary parameters can be determined to establish the decay. So far, certain bounds on $\varepsilon_1, \varepsilon_2$ have already been established. Recalling (16), constrain $\delta < 1$ and ε_1 also via $2L_f \varepsilon_1 \leq \frac{\bar{w}}{20}$. Notice that this indirectly constrains

α and ε_{x_k} as well by Lemma 2 from Appendix. From now on, α is fixed. Bound the sample step size δ as follows:

$$\eta_{x_k} \leq \frac{\bar{w}}{20}, \quad \frac{\delta \bar{f}^2}{2\alpha^2} \leq \frac{\bar{w}}{20}, \quad \delta \frac{\bar{w}}{2} \leq \frac{v^*}{4}. \quad (18)$$

Force δ to additionally satisfy $\frac{\sqrt{2V^*}}{\alpha} L_f \bar{f} \delta \leq \frac{\bar{w}}{20}$. From now on, δ is considered fixed. Constrain ε further by the conditions:

$$\varepsilon_{x_k}^2 \leq \delta \frac{\bar{w}}{20}, \quad 2L_f \varepsilon_{x_k}^2 \leq \frac{\bar{w}}{20}. \quad (19)$$

Now, the most crucial part, optimization accuracy for the subgradients is addressed. In the following, it is shown how to find a further bound on $\varepsilon_{x_k}^2$ such that the next relation holds:

$$\inf_{u \in \mathbb{U}^*} \langle \zeta_\alpha^{\varepsilon^2}(x_k), f(y_\alpha^{\varepsilon^2}(x_k), u) \rangle \leq -\frac{3\bar{w}}{4}. \quad (20)$$

Condition on approximate proximal subgradients

First, observe that, for any $z \in \mathbb{R}^n$, the following holds:

$$\begin{aligned} V(z) &\geq V(y_\alpha^{\varepsilon^2}(x_k)) + \langle \zeta_\alpha^{\varepsilon^2}(x_k), z - y_\alpha^{\varepsilon^2}(x_k) \rangle \\ &\quad - \frac{1}{2\alpha^2} \|z - y_\alpha^{\varepsilon^2}(x_k)\|^2 - \varepsilon_{x_k}^2. \end{aligned}$$

In particular, for any $\vartheta \in \mathbb{R}^n$, the following bound applies:

$$\begin{aligned} V(y_\alpha^{\varepsilon^2}(x_k) + \varepsilon \vartheta) &\geq V(y_\alpha^{\varepsilon^2}(x_k)) + \varepsilon_{x_k} \langle \zeta_\alpha^{\varepsilon^2}(x_k), \vartheta \rangle \\ &\quad - \frac{1}{2\alpha^2} \varepsilon_{x_k}^2 \|\vartheta\|^2 - \varepsilon_{x_k}^2. \end{aligned}$$

Then, it follows that $\langle \zeta_\alpha^{\varepsilon^2}(x), \vartheta \rangle$ is bounded from above by

$$\frac{V(y_\alpha^{\varepsilon^2}(x_k) + \varepsilon_{x_k} \vartheta) - V(y_\alpha^{\varepsilon^2}(x_k))}{\varepsilon_{x_k}} + \frac{1}{2\alpha^2} \varepsilon_{x_k} \|\vartheta\|^2 + \varepsilon_{x_k}. \quad (21)$$

By Assumption 1, for all $y \in \mathbb{Y}_k^* := \{y \in \mathbb{R}^n : \|y - x_k\| \leq (2\bar{V})^{1/2} \alpha\}$, $\vartheta \in \overline{\text{co}}(f(\mathbb{Y}_k^*, \mathbb{U}^*))$, there is a $\mu > 0$ such that

$$\left| \frac{V(y) + \mu' \vartheta - V(y)}{\mu'} - D_\vartheta V(y) \right| \leq \frac{\bar{w}}{5} \quad (22)$$

holds for any $0 < \mu' \leq \mu$. Notice that any approximate optimizer $y_\alpha^{\varepsilon^2}(x_k)$ lies in \mathbb{Y}_k^* disregarding $\varepsilon_{x_k}^2$ by Lemma 1 (see Appendix). Set $\varepsilon_{x_k}^2$ not greater than μ and conclude that

$$\left| \frac{V(y_\alpha^{\varepsilon^2}(x_k) + \varepsilon_{x_k}^2 \vartheta) - V(y_\alpha^{\varepsilon^2}(x_k))}{\varepsilon_{x_k}^2} - D_\vartheta V(y_\alpha^{\varepsilon^2}(x_k)) \right| \leq \frac{\bar{w}}{5} \quad (23)$$

holds for any $\vartheta \in \overline{\text{co}}(f(y_\alpha^{\varepsilon^2}(x_k), \mathbb{U}^*))$. Remember that μ depends on x_k and, therefore, $\varepsilon_{x_k}^2$ does so as well. Putting together the bound (21) and relation (23) yields

$$\begin{aligned} \langle \zeta_\alpha^{\varepsilon^2}(x_k), f(y_\alpha^{\varepsilon^2}(x_k), u) \rangle &\leq D_\vartheta V(y_\alpha^{\varepsilon^2}(x_k), u) V(y_\alpha^{\varepsilon^2}(x_k)) + \frac{\bar{w}}{5} \\ &\quad + \frac{\varepsilon_{x_k}}{2\alpha^2} \|f(y_\alpha^{\varepsilon^2}(x_k), u)\|^2 + \varepsilon_{x_k} \end{aligned}$$

for any $u \in \mathbb{U}^*$. Consequently, it holds that

$$\begin{aligned} \inf_{\vartheta \in \overline{\text{co}}(f(y_\alpha^{\varepsilon^2}(x_k), \mathbb{U}^*))} \langle \zeta_\alpha^{\varepsilon^2}(x_k), \vartheta \rangle &\leq \\ \inf_{\vartheta \in \overline{\text{co}}(f(y_\alpha^{\varepsilon^2}(x_k), \mathbb{U}^*))} D_\vartheta V(y_\alpha^{\varepsilon^2}(x_k)) + \frac{\bar{w}}{5} + \frac{\varepsilon_{x_k}}{2\alpha^2} \bar{f}^2 + \varepsilon_{x_k} &\leq \\ -\frac{4\bar{w}}{5} + \frac{\varepsilon_{x_k}}{2\alpha^2} \bar{f}^2 + \varepsilon_{x_k}. \end{aligned}$$

Therefore, the following bound

$$\varepsilon_{x_k} \leq \frac{\bar{w} \alpha^2}{10(\bar{f}^2 + 2\alpha^2)} \quad (24)$$

ensures the condition

$$\inf_{\vartheta \in \overline{\text{co}}(f(y_\alpha^{\varepsilon^2}(x_k), \mathbb{U}^*))} \langle \zeta_\alpha^{\varepsilon^2}(x_k), \vartheta \rangle \leq -\frac{3\bar{w}}{4}.$$

By [12, Lemma 1], it also holds that

$$\inf_{u \in \mathbb{U}^*} \langle \zeta_{\alpha}^{\varepsilon^2}(x_k), f(y_{\alpha}^{\varepsilon^2}(x_k), u) \rangle \leq -\frac{3\bar{w}}{4}, \quad (25)$$

which is the required condition.

Reaching time

Putting all the relevant constraints together yields an inter-sample decay rate of $\delta \frac{\bar{w}}{2}$ on V_{α} . The time T of reaching *Case 2* can be determined given by $T_{\alpha} = 4 \frac{\bar{V} - \frac{v^*}{2}}{\bar{w}}$. As soon as *Case 2* is reached, two subcases in the consequent sampling periods are possible.

Subcase 2.1: $\frac{v^*}{2} \leq V_{\alpha}(x_k) \leq v^*$. *Subcase 2.2:* $V_{\alpha}(x_k) \leq \frac{3v^*}{4}$. Being in *Subcase 2.1*, V_{α} can either stay in it in the next sample period or jump to *Subcase 2.2* since the decay condition holds. Being in *Subcase 2.2*, V_{α} can either stay in it or move to *Case 2*, from which, again, V_{α} can only move to one of the subcases. In both subcases, as well as in *Case 2*, the trajectory stays in the target ball. ■

Remark 1: Explicit bounds on $\delta, \alpha, \eta_{x_k}, \varepsilon_{x_k}$ are all in principle computable from various conditions given in the proof – in particular, via (18)–(20), (23), (24) etc.

Remark 2: The derived result is semi-global in the sense of [4]. In case of a local CLF with the decay property (Dec) where some compact set Ω is used instead of \mathbb{R}^n , the only difference is that the starting ball \mathcal{B}_R must be restricted so that the corresponding $\mathcal{B}_{R^* + \sqrt{2V^*}}$ is within Ω .

Remark 3: Stabilization can also be achieved if Assumption 1 does not hold globally, i. e., if \mathbb{Y} may merely belong to some subset \mathbb{A} of \mathbb{R}^n such that $\mathbb{A}_0 := \mathbb{R}^n \setminus \mathbb{A}$ has measure zero, as long as, for each $x_k \notin \mathcal{B}_{r^*/2}$, the set \mathbb{Y}_k^* does not lie within some set $\mathbb{A}_{0_{\chi}} := \{y \in \mathbb{R}^n : \|y - \mathbb{A}_0\| \leq \chi\}$, where $\chi > 0$ may be arbitrary. Otherwise, Assumption 1 can be relaxed to the following: for any $\delta' > 0$, any compact sets $\mathbb{Y}, \mathbb{F} \subset \mathbb{R}^n$, any $\nu, \sigma > 0$, there exist $\chi > 0$, a set $\mathbb{Y}' \subseteq \mathbb{Y}$ and $\mu > 0$ such that

- 1) for all $y \in \mathbb{Y}'$, all $\vartheta \in \mathbb{F}$ and all $0 < \mu' \leq \mu$, it holds that $\left| \frac{V(y + \mu'\vartheta) - V(y)}{\mu'} - D_{\vartheta}V(y) \right| \leq \nu$;
- 2) there exists $M \in \mathbb{N}$, such that, for all $\tilde{y} \in \tilde{\mathbb{Y}} := (\mathbb{Y} \setminus \mathbb{Y}')_{\chi}$, there exists $\{\pi_i\}_{i=1}^M \in \mathbb{U}^M$ such that the trajectory of (SH) with $\delta = \delta'$, the initial condition \tilde{y} and under the control sequence $\{\pi_i\}_{i=1}^M$ satisfies either of the following:
 - i) $x(\delta'M) \in \mathbb{Y} \setminus \tilde{\mathbb{Y}}$ and $V(x(\delta'M)) \leq V(\tilde{y}) + \sigma$,
 - ii) $V(x(\delta'M)) < V(\tilde{y})$.

In this case, setting $\sigma \leq \delta\bar{w}/4, \alpha \leq \chi/\sqrt{2\bar{V}}$, the local controller $\{\pi_i\}_{i=1}^M$ may be invoked whenever $\|x_k - \mathbb{Y} \setminus \mathbb{Y}'\| \leq \chi$.

V. DISCUSSION

Theorem 1 demonstrates robustness properties of a given CLF against non-exact optimization in the InfC-feedback represented via the problems (1) and (4). Further types of uncertainties can be introduced into the setup without compromising the practical stability provided that they are (essentially) bounded in a certain way. For instance, an uncertainty of the type $\dot{x} = f(x, u) + g(t)$ can be addressed mainly by taking care of the ‘‘Taylor expansion’’ (14), where a respective term of the form $\int_{k\delta}^t g(\tau) d\tau$ appears, and bounding g sufficiently along the lines of [4]. An actuator uncertainty of the type $\dot{x} = f(x, u + d(t))$ can be converted into a one of the type

$\dot{x} = f(x, u) + g(t)$ using continuity of f . In this sense, an approximate InfC-feedback in the spirit of Definition 1 can be made robust against the said uncertainties. A natural question is whether the optimization accuracies η_x and ε_x can be ‘‘merged’’ with the uncertainty $g(t)$. Whereas it is relatively simple with η_x , introducing ε_x into $g(t)$ would require special attention. The problem is that, in presence of optimization inaccuracy, as pointed out earlier, a qualitative obstruction, that is loss of the property of being a proximal subgradient and, subsequently, failing to satisfy (5), may occur. The parameter ε_x occurs at many places in the proof of Theorem 1, but mainly in the **condition on approximate proximal subgradients**. To derive a necessary bound on g so as to address the optimization inaccuracy would require tracking back the derivations in the said fragments of the theorem. Therefore, the present result should be seen as a complementary one to the works based on InfC-feedbacks such as in [4]. As other types of discontinuous feedbacks, an approximate InfC-feedback is still vulnerable under a measurement error of the type $\dot{x} = f(x, (u(x + d(t))))$. In this regard, the presented result might be merged with the one in [14] and [16, Theorem E]. The first one suggests an internal tracking controller to amend the main one, whereas the latter uses, besides an upper, a lower bound on the sampling interval to compensate for measurement noise. Without Assumption 1, no locally uniform bound on ε_x could be derived in Theorem 1. In absence of local uniformness in the sense of Assumption 1 or its relaxation in Remark 3, the accuracy ε_x would, in particular, depend on the direction, which is actually defined afterwards using the said accuracy, a circular reasoning. Notice an important difference to [4] where no such an assumption was necessary. Still, Assumption 1 is merely sufficient, whereas its necessity must be further investigated. Finally, whereas δ is an absolutely crucial parameter in the analysis, the accuracy bounds ε_x and η_x should be paid high attention as well.

VI. SIMULATION STUDY

As the basis for the simulation study, practical stabilization of the nonholonomic integrator

$$(\dot{x}_1, \dot{x}_2, \dot{x}_3)^{\top} = (u_1, u_2, x_1u_2 - x_2u_1)^{\top}$$

is used. The nonholonomic integrator can be regarded as a three-wheel robot with a speed and steering control and serves thus as an important basis of a class of systems describing wheeled machines. The purpose of the current study is not to compare the InfC-feedbacks to other stabilization techniques, but rather to demonstrate the effects of the optimization accuracy on stability properties. It was shown in [1] that the following function is a global CLF for the nonholonomic integrator:

$$V(x) = x_1^2 + x_2^2 + 2x_3^2 - 2|x_3|\sqrt{x_1^2 + x_2^2},$$

under the constraint $u \in [-1 \ 1]^2$. It can be verified that V satisfies Assumption 1 everywhere except for $\{x \in \mathbb{R}^n : x_3 = 0\}_{\chi} \cup \{x \in \mathbb{R}^n : x_1 = 0 \wedge x_2 = 0\}_{\chi}$ for any arbitrary, but fixed, $\chi > 0$. In the current simulation result, the trajectory stayed apart from $\{x \in \mathbb{R}^n : x_3 = 0\} \cup \{x \in \mathbb{R}^n : x_1 = 0 \wedge x_2 = 0\}$ (see Fig. 1). A sampling time $\delta = 0.005$ and an initial condition $x_0 = (1, 0.5, -0.1)^{\top}$ are assumed. The parameter α is set

equal 0.1. The optimization problems (1), (7) of the InfC-feedback are solved until the accuracies ε_x^2 and, respectively, η_x are achieved. For simplicity, ε_x^2 and η_x are set equal. Therefore, only η_x is specified from now on. Fig. 1 shows the norm of the system trajectory under three different values of η_x . It can be

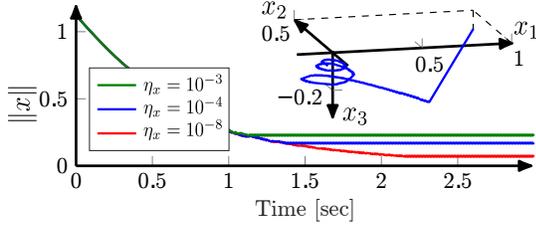


Fig. 1. State norm under various η_x and state trajectory for $\eta_x = 10^{-8}$ observed that η_x significantly influences the stability margins, especially the vicinity into which the state trajectory converges. Fig. 2 shows the respective behaviors of V . A particular input signal for the case with $\eta_x = 10^{-8}$ is shown in the right-upper corner of that figure. One can see a characteristic chattering of

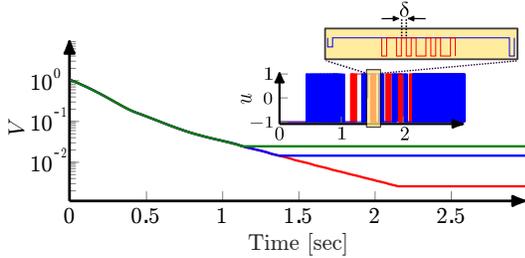


Fig. 2. Behavior of V . The input is shown for $\eta_x = 10^{-8}$ (u_1 , u_2) the control which was also observed by [1]. Future investigation is required for other techniques, such as Dini aiming, and also for further relaxations of Assumption 1.

VII. CONCLUSION

This work is concerned with practical stabilization of nonlinear dynamical systems in the sample-and-hold framework. The key new result is an analysis of practical stability under approximate optimizers for various optimization problems. Bounds on optimization accuracy to achieve prescribed stability margins are derived. A simulation study showed significant effects of optimization accuracy on stability properties.

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APPENDIX

Lemma 1: For all $x \in \mathcal{B}_R$ for some $R > 0$, $0 < \alpha < 1$, and any $\varepsilon > 0$, there exists an ε -minimizer $y_\alpha^\varepsilon(x)$ for (1) satisfying:

$$\|y_\alpha^\varepsilon(x) - x\| \leq \sqrt{2\bar{V}\alpha},$$

where $\bar{V} := \sup_{\|x\| \leq R} V(x)$.

Proof: Letting $R' := (2\bar{V})^{1/2}\alpha$ yields the following:

$$\inf_{\|x-y\| \leq R'} (V(y) + \frac{1}{2\alpha^2}\|y-x\|^2) \leq V(x) \leq \bar{V}.$$

On the other hand, for any $R'' > R'$, it holds that

$$\inf_{R' \leq \|x-y\| \leq R''} (V(y) + \frac{1}{2\alpha^2}\|y-x\|^2) \geq \frac{1}{2\alpha^2}R'^2 \geq \bar{V}.$$

Therefore,

$$\inf_{y \in \mathbb{R}^n} (V(y) + \frac{1}{2\alpha^2}\|y-x\|^2) = \inf_{\|x-y\| \leq R'} (V(y) + \frac{1}{2\alpha^2}\|y-x\|^2)$$

and the conclusion follows. ■

Lemma 2: With the conditions of Lemma 1, for any $\varepsilon_1 > 0$, an $\varepsilon > 0$ and $0 < \alpha < 1$ for the approximate minimizers $y_\alpha^\varepsilon(x)$ can be chosen so as to satisfy:

$$V_\alpha(x) \leq V(x) \leq V_\alpha(x) + \varepsilon_1.$$

Proof: The first inequality $V_\alpha(x) \leq V(x)$ follows directly from (1). As for the second, Lemma 1 implies $\|y_\alpha^\varepsilon(x) - x\| \leq (2\bar{V})^{1/2}\alpha$. Let $(2\bar{V})^{1/2}\alpha \leq \omega_V(\frac{\varepsilon_1}{2})$. Choose $\varepsilon < \frac{\varepsilon_1}{2}$. It follows from (6), that $V(y_\alpha^\varepsilon) \leq V_\alpha(x) \geq +\varepsilon$. By the continuity of V , it holds that $V(x) \leq V(y_\alpha^\varepsilon) + \frac{\varepsilon_1}{2}$. Putting this together gives the result. ■