The Explicit Dynamic Model and Inertial Parameters of the PUMA 560 Arm †

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Abstract

To provide COSMOS, a dynamic model based manipulator control system, with an improved dynamic model, a PUMA 560 arm was disassembled; the inertial properties of the individual links were measured; and an explicit model incorporating all of the non-zero measured parameters was derived. The explicit model of the PUMA arm has been obtained with a derivation procedure comprised of several heuristic rules for simplification. A simplified model, abbreviated from the full explicit model with a 1% significance criterion, can be evaluated with 305 calculations, one fifth the number required by the recursive Newton-Euler method. The procedure used to derive the model is laid out; the measured inertial parameters are presented, and the model is included in an appendix.

1. Introduction

The Implementation of dynamic control systems for manipulators has been hampered because the models are difficult to derive and computationally expensive, and because the needed parameters of the manipulator are generally unavailable. Recursive methods for computing the dynamic forces have been available for several years [Luh, Walker and Paul 1980a; Hollerbach 1980]. Several authors have proposed and simulated the use of RNE in control systems [Luh, Walker and Paul 1980b; Kim and Shin 1985]; and [Valavanis, Leahy and Sardsi 1985] have used the RNE to control a PUMA - 600 arm. The RNE algorithm has also found use in the computation of forward dynamics for simulation [Walker and Orin 1982; Koozekanani et al. 1983], and nominal trajectory control [Vukobratović and Kirćanski 1984]. The RNE meets the need for calculation of dynamic forces in these applications, but does not offer several advantages available provided by an explicit model. The explicit model allows of the calculation decomposition based on a significance criterion or other criteria, and provides a more direct solution for dynamic simulation.

The tremendous size of an explicit dynamic model is the greatest barrier to its realization. Correspondingly, a considerable portion of the effort spent investigating dynamic models for control has been directed toward efficient formulation and automatic generation of the manipulator equations of motion. Programs for automatic generation of manipulator dynamics are reported in [Liégeois et al. 1976; Megahed and Renaud 1982; Cesareo, F. Nicolò and S. Nicosia 1984 ; Murray and Neuman 1984; Renaud 1984; Aldon and Liégeois 1984; Aldon et al. 1985]. The size of the models generated by these programs varies widely; and there is little consensus on the question of whether the explicit models can be made sufficiently compact to be used for control. Aldon and Liégeois [1984] present an algorithm for obtaining efficient dynamic models; but none-the-less recomend the use of recursive algorithms for real time control, claiming that the complete results are too complicated for real-time control of robots.

As we show, explicit dynamic models of manipulators that are more computationally efficient than the alternative recursive algorithms can be obtained. The computational cost of the RNE algorithm, the full explicit PUMA model, and the explicit PUMA model abbreviated with a 1% significance criterion are presented in Table 1. The method presented here for factoring the dynamic equations has yielded a dynamic model of the PUMA 560 arm that requires 1165 calculations (739 multiplications and 426 additions), 25% fewer than the 1560 calculations required by the 6 dof RNE. With the application of a 1% sensitivity criterion, the explicit model can be evaluated with one fifth the count of calculations required by the recursive algorithm. Furthermore, this formulation of the explicit model is not optimally compact; factorizations that were discovered and employed during the model derivation have been expanded out to present explicit expressions for each component of the dynamic model. Renaud and Burdick both report automatic generation of 6 dof manipulator models that are more compact than that presented here [Renaud 1984; Burdick 1985]. Their models incorporate nested factorizations, which were not used here.

Table 1. Calculations Required to Compute the Forces of Motion by 3 Methods.

Method	Calculations
Recursive Newton-Euler	1560
Evaluation of the Full Explicit PUMA Model	1165
Evaluation of the Abbreviated Explicit PUMA Model	305

The count of 1165 calculations for the full PUMA model is the total required to evaluate the model presented in the appendix and equation (1) below. This total and other totals presented do not include the calculations required to evaluate the sines and cosines.

2. Derivation of the Dynamic Model

The dynamic model used for this analysis follows from [Liégeois et al. 1976] . It is:

† Financial support for the first author has been provided by Hewlett Packard Co., through their Faculty Development Program. Partial support was provided by NSF under contract MEA 80-19628, and by DARPA through the Intelligent Task Automation Project, managed by the Air Force Materials Laboratory, under a subcontract to Honeywell, contract F 33615-82-C-5092.

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$$A(\mathbf{q})\ddot{\mathbf{q}} + B(\mathbf{q})[\dot{\mathbf{q}}\dot{\mathbf{q}}] + C(\mathbf{q})[\dot{\mathbf{q}}^2] + \mathbf{g}(\mathbf{q}) = \mathbf{\Gamma}; \tag{1}$$

where $A(\mathbf{q})$ is the n × n kinetic energy matrix;
$B(\mathbf{q})$ is the n × n(n-1)/2 matrix of Coriolis torques;
$C(\mathbf{q})$ is the n × n matrix of centrifugal torques;
$\mathbf{g}(\mathbf{q})$ is the n-vector of gravity torques;
$\ddot{\mathbf{q}}$ is the n-vector of accelerations;
$\mathbf{\Gamma}$ is the generalized joint force vector.

The symbols $[\dot{\mathbf{q}}\dot{\mathbf{q}}]$ and $[\dot{\mathbf{q}}^2]$ are notation for the n(-1)/2-vector of velocity products and the n-vector of squared velocities. $[\dot{\mathbf{q}}\dot{\mathbf{q}}]$ and $[\dot{\mathbf{q}}^2]$ are given by:

$$[\dot{\mathbf{q}}\dot{\mathbf{q}}] = [\dot{q}_1\dot{q}_2,\ \dot{q}_1\dot{q}_3\ \dots\ \dot{q}_1\dot{q}_n, \dot{q}_2\dot{q}_3, \dot{q}_2\dot{q}_4\ \dots\ \dot{q}_{n-2}\dot{q}_n, \dot{q}_{n-1}\dot{q}_n]^T,$$
$$[\dot{\mathbf{q}}^2] = [\dot{q}_1^2,\ \dot{q}_2^2\ \dots\ \dot{q}_n^2]^T.$$

The procedure used to derive the dynamic model entails four steps:

1. Symbolic Generation of the kinetic energy matrix and gravity vector elements by performing the summations of either Lagrange's or the Gibbs-Alembert formulation.
2. Simplification of the kinetic energy matrix elements by combining inertia constants that multiply common variable expressions.
3. Expression of the Coriolis and centrifugal matrix elements in terms of partial derivatives of kinetic energy matrix elements; and reduction of these expressions with four relations that hold on these partial derivatives.
4. Formation of the needed partial derivatives, expansion of the Coriolis and centrifugal matrix elements in terms of the derivatives, and simplification by combining inertia constants as in 2.

The first step was carried out with a LISP program, named EMDEG, which symbolically generates the dynamic model of an articulated mechanism. EMDEG employs Kane's dynamic formulation [Kane 1968], and produced a result comparable in form and size to that of ARM [Murry and Neuman 1984]. Three simplifying assumptions were made for this analysis: the rigid body assumption; link 6 has been assumed to be symmetric, that is $I_{xx} = I_{yy}$; and only the mass moments of inertia are considered, that is I_{xx}, I_{yy} and I_{zz}. The original output of EMDEG, including Coriolis and centrifugal terms, required 15,000 multiplications and 3,500 additions. This step might also have been performed with the momentum theorem method used in [Izaguirre and Paul 1985].

In the second step of this procedure, the kinetic energy matrix elements are simplified by combining inertia constants that multiply common variable expressions. This is the greatest source of computational efficiency. Looking to the dynamic model of a 3 dof manipulator presented in [Murry and Neuman 1984], we see that the kinetic energy matrix element a_{11} is given by:

$$\begin{aligned} a_{11} = {} & J_{3xx}\,cos^2(\theta_2+\theta_3) + J_{3yy}\,sin^2(\theta_2+\theta_3) + J_{3zz} + d_3^2m_3 \\ & +2\ M_{3x}a_2\ cos(\theta_2)cos(\theta_2+\theta_3)\ + a_2^2m_3\ cos^2(\theta_2) \\ & +2\ M_{3x}a_3\ cos^2(\theta_2+\theta_3) + a_3^2m_3\ cos^2(\theta_2+\theta_3) \\ & +2\ a_2a_3m_3\ cos(\theta_2)cos(\theta_2+\theta_3)\ \ + J_{2yy}\ sin^2(\theta_2) \\ & +J_{2xx}\ cos^2(\theta_2) + 2\ d_2d_3m_3 + 2\ M_{2x}a_2\ cos^2(\theta_2) \\ & +a_2^2m_2\ cos^2(\theta_2) + d_3^2m_3 + d_2^2m_2 + J_{2zz} + J_{1xx} + J_{1zz} \end{aligned} \tag{2}$$

Calculations required: 37 multiplications, 18 additions.

By combining inertial constants with common variable terms and expanding $sin^2(\theta_2)$ into $(1 - cos^2(\theta_2))$, equation (2) can be reduced to:

$$\begin{aligned} a_{11} = I_1 + I_2\ cos^2(\theta_2) + I_3\ cos(\theta_2)cos(\theta_2+\theta_3) \\ +I_4\ cos^2(\theta_2+\theta_3) \end{aligned} \tag{3}$$

Calculations required: 3 multiplications, 3 additions.

where $I_1 = d_3^2m_3 + d_2^2m_3 + 2\ d_2d_3m_3 + d_2^2m_2 + J_{3yy} + J_{3zz} + J_{2zz} + J_{2yy} + J_{1xx} + J_{1zz}$; etc.

Creating I_1 through I_4, which are constants of the mechanism, leads to a reduction from 35 to 3 multiplications and from 18 to 3 additions. Computing the constant I_1 involves 18 calculations. Since the simple parameters required for the calculation of I_1 are the input to the RNE, the RNE will effectively carry out the calculation of I_1 on every pass, producing considerable unnecessary computation. Thirty four lumped constants are needed by the full PUMA model, 8 fewer than the count of 42 simple parameters required to describe the arm.

In the third step the elements of the Coriolis matrix, b_{ij}, and of the centrifugal matrix, c_{ij}, are written in terms of the Christoffel symbols of the first kind [Corben and Stehle 1950; Liégeois et al. 1976]* giving:

$$b_{ij} = 2\ \beta^{i,kl} \tag{4}$$

$$c_{ij} = \beta^{i,jj} \tag{5}$$

where $(\dot{q}_k * \dot{q}_l)$ is the j^{th} velocity product in the $[\dot{\mathbf{q}}\dot{\mathbf{q}}]$ vector, and where

$$\beta^{i,jk} = \frac{1}{2}\left(\frac{\partial a_{ij}}{\partial q_k} + \frac{\partial a_{ik}}{\partial q_j} - \frac{\partial a_{jk}}{\partial q_i}\right) \tag{6}$$

is the Christoffel symbol.

The number of unique non-zero Christoffel symbols required by the PUMA model can be reduced from 126 to 39 with four equations that hold on the derivatives of the kinetic energy matrix elements. The first two equations are general; the last two are specific to the PUMA 560. The equations are:

$$\frac{\partial A_{ij}}{\partial q_k} = \frac{\partial A_{ji}}{\partial q_k} \qquad \forall_{i,j,k} \tag{7}$$

$$\frac{\partial A_{ij}}{\partial q_k} = 0 \qquad \forall_{i\geq k, j\geq k} \tag{8}$$

$$\frac{\partial A_{ij}}{\partial q_6} = 0 \qquad \forall_{i,j} \tag{9}$$

$$\frac{\partial A_{13}}{\partial q_2} = \frac{\partial A_{13}}{\partial q_3} = \frac{\partial A_{12}}{\partial q_3} \tag{10}$$

The reduction of Equation (7) arises from the symmetry of the kinetic energy matrix. Equation (8) obtains because the kinetic energy imparted by the velocity of a joint is independent of the configuration of the prior joints. Equation (9) results from the symmetry of the sixth and terminal link of the PUMA arm. And equation (10) holds because the second and third axes of the PUMA arm are parallel. Of the reduction from 126 to 39 unique Christoffel symbols, 61 eliminations are obtained with the general equations, 14 more with (9) and a further 12 with (10).

Step four requires differentiating the mass matrix elements with respect to the configuration variables. The means to carry out differentiation automatically have been available for some

* The French authors seem to assume the use of Cristoffel symbols, while the American authors seem unaware of them. Corben and Stehle, in the 1950 edition of their text, derive the results required here; but the derivation is largely omitted from their 1960 edition.

time [Liégeois et al. 1976; MIT Mathlab Group 1983]. Only the derivatives required after the simplification of step 3 need to be formed. Of the 126 derivatives possible when n=6, 46 are required by the model of the PUMA arm. After the needed derivatives are formed and expanded into the Christoffel symbols, inertial constants that multiply common variable expressions are again combined.

Our method of model derivation is able to simplify to managable form the complex sum-of-product expressions that are produced by symbolically carrying out the summations of Lagrange's equations. Simplification is in general a non-deterministic task that grows very rapidly with the number of terms in an equation; but the procedure presented is deterministic, with a cost that grows most rapidly as p^2, where p is the number of sum-of-product expressions in the largest individual kinetic energy matrix element. Our procedure has the virtue of producing explicit expressions for each component of the dynamic model: a result that is very useful for design analysis and that allows straight forward simplification by application of a sensitivity criterion.

Steps 2 through 4 of the above procedure were carried out by hand, requiring five weeks of rather tedious labor. To discover errors, the explicit solution was numerically checked against the RNE algorithm, extended to give B and C matrix elements individually in a manner similar to that of Walker and Orin. Over a range of configurations, the explicit solution of the PUMA dynamics agrees exactly with the RNE calculation. It is instructive to observe that the RNE algorithm was coded in 5 hours, 2% of the time required to develop the full explicit model.

3. Several Advantages Obtained from Decomposition of the Explicit Model

The explicit solution of PUMA dynamics shows two structural properties that can be used to advantage: a tremendous range between the largest and the smallest contributing terms within most equations, and the depend solely upon configuration of the A, B and C matrix elements. Using the measured PUMA parameters an abbreviated dynamic model has been formed. This model is derived from the full PUMA model by eliminating all terms that are less than 1% as great as the greatest term within the same equation, or less than 0.1% as great as the largest constant term applicable to the same joint. All of the elements of the A, B and C matrices are retained: the significance test is applied on an equation by equation basis. The reduction in required calculations achieved via the significance test is roughly a factor of four, as shown in Table 1 above.

Observing that the A, B and C matrix elements depend only on configuration, it is possible to decompose the calculation into configuration dependent and velocity or acceleration dependent components. Because configuration changes more slowly than velocity or acceleration, the configuration dependent components may be computed at a slower rate [Khatib 1985; Izaguirre and Paul 1985]. Shown in Table 2 is the evaluation rate of the PUMA 560 dynamics that can be achieved with 100,000 floating point operations per second, the approximate speed of a PDP-11. In the first case the entire model is recomputed in each pass; in the second case the A, B and C matrix elements are computed only once for every four iterations of the multiplication by velocity and acceleration vectors. This partitioning of the dynamic calculation reduces the pace of computing the configuration dependent terms by one third; but increases the pace of computing the velocity and acceleration dependent terms by a factor of two and one half. The advantage of this decomposition applies equally well to the calculation of forward dynamics for simulation, where tesselation is the step size rather than servo interval and the cost is run time rather than bounded computing power.

Table 2. PUMA 560 Dynamic Model Evaluation Rate Attainable with 100k FLOPS.

Method	Rate of Evaluation of Configuration Dependent Terms	Rate of Computation of Torque
Evaluation of the Full Model Each Iteration	78 hz	78 hz
Evaluation of the Configuration Dependent Terms once during every four Evaluations of the Velocity and Acceleration Dependent Terms	50 hz	200 hz

A final decomposition to be considered is that for multiprocessing, an issue likely to become more important. The recursive formulations are well suited to pipeline computation, but poorly suited to multiprocessor computation. For the recursive algorithms, the number of calculations that can be performed by cooperating processors is small in relation to the volume of communication that is required. Using an explicit model the blocks of parallel computation can be made much larger, and the ratio of computation to communication correspondingly higher. The decomposition into configuration dependent and velocity or acceleration dependent components is particularly suitable for multiprocessing and has been implemented at the Stanford Artificial Intelligence Laboratory [Khatib 1985].

4. The Utility of an Explicit Model for Dynamic Simulation

Walker and Orin have demonstrated the use of the RNE algorithm in the calculation of forward dynamics for simulation. By taking advantage of the symmetry of the kinetic energy matrix they have reduced the model order that must be considered in successive applications of the RNE [Walker and Orin 1982]. The RNE algorithm has also been used to compute dynamics for simulation in fields outside of robotics [Benati et al. 1980; Koozekanani et al. 1983]. Presented in table 3 are the number of calculations required to compute the elements of the kinetic energy matrix using Walker and Orin's method, using the full PUMA model, using the simplified model reported in [Izaguirre and Paul 1985], and using the abbreviated (1% significance criterion) model. The analytic models all show a tremendous advantage over the RNE algorithm.

Table 3. Calculations Required to determine the Kinetic Energy Matrix Elements for a PUMA 560 Arm.

Method	Calculations
Walker and Orin	2737
Full Explicit Model	278
Izaguirre and Paul Simplified Model	58
Abbreviated Explicit Model	25

5. Measurement of the PUMA 560 Dynamic Parameters

The link parameters required to calculate the elements of A, B, C and $\mathbf{g}$ in equation (1) are mass, location of the center of gravity and the terms of the inertia dyadic. The wrist, link three and link two of a PUMA 560 arm were detached in order to measure these parameters. The mass of each component was determined with a beam balance; the center of gravity was located by balancing each link on a knife edge, once orthogonal to each axis; and the diagonal terms of the inertia dyadic were measured with a two wire suspension.

The motor and drive mechanism at each joint contributes to the inertia about that joint an amount equal to the inertia of the rotating pieces magnified by the gear ratio squared. The drives and reduction gears were not removed from the links, so the total motor and drive contribution at each joint was determined by an identification method. This contribution is considered separately from the I_{zz} term of the link itself because the motor and drive inertia seen through the reduction gear does not contribute to the inertial forces at the other joints in the arm. The motors were left installed in links two and three when the inertia of these links were measured, so the effect of their mass as the supporting links move is correctly considered. The gyroscopic forces imparted by the rotating motor armatures is neglected in the model, but the data presented below include armature inertia and gear ratios, so these forces can be determined.

The parameters of the wrist links were not directly measured. The wrist itself was not disassembled. But the needed parameters were estimated using measurements of the wrist mass and the external dimensions of the individual links. To obtain the inertial terms, the wrist links were modeled as thin shells.

Measurement of Rotational Inertia

The two wire suspension shown in Figure 1 was used to measure the I_{xx}, I_{yy} and I_{zz} parameters of links two and three *. With this arrangement a rotational pendulum is created about an axis parallel to and halfway between the suspension wires. The link's center of gravity must lie on this axis. The two wire suspension method of measuring the rotational inertia requires knowledge of parameters that are easily measured: the mass

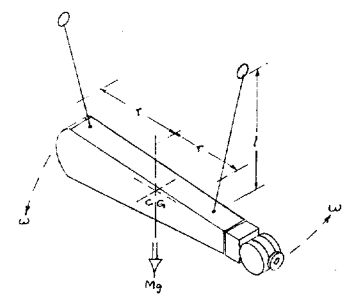

Figure 1. The two wire suspension used for Rotational Inertia Measurement.

of the link, the location of the center of gravity, the distance from the wire attachment points to the axis of rotation, the length of the wires, and the period of rotational oscillation. The inertia about each axis is measured by configuring the link to swing about that axis. Rotational oscillation is started by twisting and releasing the link. If one is careful when releasing the link, it is possible to start fundamental mode oscillation without visibly exciting any of the other modes. The relationship between measured properties and rotational inertia is:

* This method was suggested by Prof. David Powell.

$$I = \frac{Mg * r^2}{\omega^2 * l} \tag{11}$$

where
I is the inertia about the axis of rotation;
Mg is the weight of the link;
r is the distance from each suspension wire to the axis of rotation;
ω is the oscillation frequency in radians per second;
l is the length of the supporting wires.

Measurement of the Motor and Drive Inertia

A parameter identification method was used to learn the total rotational inertia at each joint. This inertia includes the effective motor and drive inertia and the contribution due to the mass of the arm. To make this measurement our control system was configured to command a motor torque proportional to displacement, effecting a torsional spring. By measuring the period of oscillation of the resultant mass-spring system, the total rotational inertia about each joint was determined. By subtracting the arm contributions, determined from direct measurements, from the measured total inertia, the motor and drive inertial contributions were found.

Measurement Tolerance

A tolerance for each direct measurement was established as the measurement was taken. The tolerance values are derived from the precision or smallest graduation of the measuring instrument used, or from the repeatability of the measurement itself. The tolerances are reported where the data are presented. The tolerance values assigned to calculated parameters were determined by RMS combination of the tolerance assigned to each direct measurement contributing to the calculation. The inertia dyadic and center of gravity parameters of link 3 were measured with the wrist attached; the values reported for link 3 alone have been obtained by subtracting the contribution of the wrist from the total of link 3 plus wrist. Tolerance values are reported with the values for link 3 plus wrist, as these are the original measurements.

6. The Measured PUMA 560 Parameters

The mass of links 2 through 6 of the PUMA arm are reported in Table 4; the mass of link 1 in not included because that link was not removed from the base. Separately measured mass and inertia terms are not required for link one because that link rotates only about its own Z axis .

Table 4. Link Masses (kilograms; $\pm 0.01 + 1\%$)

Link	Mass
Link 2	17.40
Link 3	4.80
Link 4*	0.82
Link 5*	0.34
Link 6*	0.09
Link 3 with Complete Wrist	6.04
Detached Wrist	2.24

* Values derived from external dimensions; ±25%.

The positions of the centers of gravity are reported in Table 5. The dimensions r_x, r_y and r_z refer to the x, y and z coordinates of the center of gravity in the coordinate frame attached to the link. The coordinate frames used are assigned by a modified Denavit-Hartenberg method [Craig 85]. In this variant of the Denavit-Hartenberg method, frame i is attached to link i, and axis Z_i lies along the axis of rotation of joint i. The coordinate frame attachments are shown in Figure 2; they are located as follows:

Link 1: Z axis along the axis of rotation, +Z up; +Y1 || +Z2.

Link 2: Z axis along the axis of rotation , +Z away from the base; X-Y plane in the center of the link, with +X toward link 3.

Link 3: Z3 || Z2; X-Y plane is in the center of link 3; +Y is away from the wrist.

Link 4: The origin is at the intersection of the axes of joints 4 5 and 6; +Z4 is along the axis of rotation and directed away from link 2; +Y4 || +Z3 when joint 4 is in the zero position.

Link 5: The origin coincides with that of frame 4; +Z5 is directed away from the base; +Y5 is directed toward link 2 when joint 5 is in the zero position.

Link 6: The origin coincides with that of frame 4; when joints 5 and 6 are in the zero position frame 6 is aligned with frame 4.

Wrist: The dimensions are reported in frame 4.

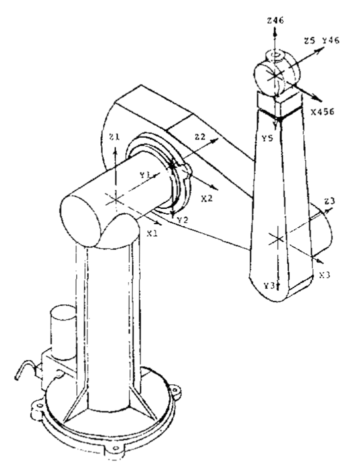

Figure 2. The PUMA 560 in the Zero Position with Attached Coordinate Frames Shown.

The inertia dyadic and effective motor and drive inertia terms are reported in Table 6. For each link, the coordinate frame for the inertia dyadic terms is placed at the center of gravity, parallel to the attached frame used in Table 5. The tolerances assigned to these measurements are shown in parenthesis. No tolerance is associated with the value of I_{zz} for link one because this value was not directly measured; it was computed backwards from the measured total joint inertia. It is not important to distinguish I_{zz1} from the $m_1 \times ry_1{}^2$ term or from the motor and drive inertia at joint one because these contributions are neither configuration dependent nor appear in any term other than a_{11}. The total link 1 inertia measured by the identification method is the sum of I_{zz1} and I_{motor} in table 6.

Table 5. Centers of Gravity. (meters ±0.003)

Link	r_x	r_y	r_z
Link 2	0.068	0.006	-0.016
Link 3	0	-0.070	0.014
Link 3 With Wrist	0	-0.143	0.014
Link 4*	0	0	-0.019
Link 5*	0	0	0
Link 6*	0	0	0.032
Wrist	0	0	-0.064

* Values derived from external dimensions; ±25%.

The effective torsional spring method of inertia measurement was applied at each joint. The motor and drive inertia, I_{motor}, were found by subtracting the inertial contribution due to the arm dynamics, known from direct measurements, from the total inertia measured. The uncertainty in the total inertia measurement is somewhat higher at joint one because of the larger friction at that joint. It was necessary to add positive velocity feedback (damping factor -0.1) to cause joint one to oscillate for several cycles.

Table 6. Diagonal Terms of the Inertia Dyadics and Effective Motor Inertia.

Link	I_{xx}	I_{yy}	I_{zz}	I_{motor}
Link 1	–	–	0.35	1.14(±0.27)
Link 2	0.130 (±3%)	0.524 (±5%)	0.539 (±3%)	4.71 (±0.54)
Link 3	0.066	0.0125	0.086	0.83 (±0.09)
Link 3 With Wrist	0.192 (±4%)	0.0154 (±5%)	0.212 (±4%)	–
Link 4*	1.80×10^{-3}	1.80×10^{-3}	1.30×10^{-3}	0.200 (±0.016)
Link 5*	0.30×10^{-3}	0.30×10^{-3}	0.40×10^{-3}	0.179 (±0.014)
Link 6*	0.15×10^{-3}	0.15×10^{-3}	0.04×10^{-3}	0.193 (±0.015)

* Inertia Diadic terms derived from external dimensions; ±50%.

The gear ratios, maximum motor torque, and break away torque for each joint of the PUMA is reported in Table 7. The maximum motor torque and break away torque values have been taken from data collected during our motor calibration process. The current amplifiers of the Unimate controller are driven by 12 bit D/A converters, so the nominal torque resolution can be obtained by dividing the reported maximum joint torque by 2048.

Table 7. Motor and Drive Parameters

	Joint 1	Joint 2	Joint 3	Joint 4	Joint 5	Joint 6
Gear Ratio	62.61	107.36	53.69	76.01	71.91	76.73
Maximum Torque (N-m)	97.6	186.4	89.4	24.2	20.1	21.3
Break Away Torque (N-m)	6.3	5.5	2.6	1.3	1.0	1.2

7. Conclusion

Explicit dynamic models of complex manipulators are attainable. The PUMA 560 arm is as complex as any 6 dof arm with a spherical wrist, yet a deterministic simplification procedure has produced an explicit model that is more economical than the

RNE algorithm. With the application of a stringent significance criterion, the computational cost of the explicit model is reduced to one fifth that of the recursive alternative. The availability of measured dynamic parameters provides improved accuracy in the calculated forces of motion and simplifies model generation by allowing one to omit zero value parameters. As automatic model generation becomes available and manufacturers become aware of the need for dynamic parameters, we expect to see increasing use of explicit models and measured parameters in the calculation of dynamics for control.

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APPENDIX

The Full Expressions for the Forces of Motion of a Puma 560 Arm

In the following tables the expressions for the elements of the A,B and C matrices and the **g** vector are presented. These expressions are made in terms of constants which have units of inertia or torque, and trigonometric terms that are functions of the joint angles. We have abbreviated the trigonometric functions by writing $S2$ to mean $\sin(q_2)$ and $C5$ to mean cos(q5). When a trigonometric operation is applied to the sum of several angles we write $C23$ to mean $\cos(q_2 + q_3)$ and $S223$ to mean

$\sin(q_2 + q_2 + q_3)$. And when a product of several trigonometric operations on the same joint variable appears we write $CC2$ to mean $\cos(q_2) * \cos(q_2)$ and $CS4$ to mean $\cos(q_4) * \sin(q_4)$. These final abbreviations, $CS4$ etc. , are considered to be factorizations; and the cost of computing these terms is included in the totals reported above.

The position of zero joint angles and coordinate frame attachments to the PUMA arm are shown in Figure 2 above. The modified Denavit-Hartenberg parameters, assigned according to the method presented in [Craig 85], are listed in Table A1.

Table A1. Modified Denavit – Hartenberg Parameters

i	α_{i-1} (degrees)	θ_i	a_{i-1} (meters)	d_i (meters)
1	0	q_1	0	0
2	-90	q_2	0	.2435
3	0	q_3	.4318	-.0934
4	90	q_4	-.0203	.4331
5	-90	q_5	0	0
6	90	q_6	0	0

The equations of the PUMA model constants are presented in Table A2; these constants appear in the dynamic equations of Tables A4 through A7. $I_{zzi}$and m_i refer to the second moment of link i about the z axis of frame i and the mass of link i respectively. The terms a_i and d_i are the Denavit-Harterberg parameters. Terms of the form r_{xi} are the offsets to the center of gravity of link i in the i^{th} coordinate frame. In Table A3 the values of the model constants are listed. The terms I_{mi} are the motor and drive train contribution to inertia at joint i.

The equations for the elements of the kinetic energy matrix, $A(q)$, are presented in Table A4. $A(q)$ is symmetric, so only equations for elements on and above the matrix diagonal are presented.

The equations for the elements of the Coriolis matrix, $B(q)$, are presented in Table A5. The Coriolis terms have been left in the form of a three dimensional array, with a convention for the indices that matches that of the Cristoffel symbols. Element $\beta^{i,kl}$ multiples $\dot{q}_k$ and $\dot{q}_l$ to give a contribution to the torque at joint i. The Coriolis matrix may also be written as a 6 x 15 array, where the 15 columns correspond to the 15 possible combinations of joint velocities. The equations for the elements of the centrifugal matrix, $C(q)$, are presented in Table A6. And the equations for the terms of the gravity vector, $\mathbf{g}(q)$, are presented in Table A7.

A load can be represented in this model by attaching it to the 6^{th} link. In the model the 6^{th} link is assumed to have a center of gravity on the axis of rotation, and to have $I_{xx6} = I_{yy6}$; these restrictions extend to a load represented by changing the 6^{th} link parameters. A more general, though computationally more expensive, method of incorporating a load in the dynamic calculation is presented in [Izaguirre and Paul 1985].

Table A2. Expressions for the Constants Appearing in the Equations of Forces of Motion.

Part I. Inertial Constants

$$I_1 = I_{zz1} + m_1 * r_{y1}^2 + m_2 * d_2^2 + (m_4 + m_5 + m_6) * a_3^2 + m_2 * r_{z2}^2 + (m_3 + m_4 + m_5 + m_6) * (d_2 + d_3)^2 + I_{xx2} + I_{yy3} + 2 * m_2 * d_2 * r_{z2} + m_2 * r_{y2}^2 + m_3 * r_{z3}^2 + 2 * m_3 * (d_2 + d_3) * r_{z3} + I_{zz4} + I_{yy5} + I_{zz6} ;$$

$$I_2 = I_{zz2} + m_2 * (r_{x2}^2 + r_{y2}^2) + (m_3 + m_4 + m_5 + m_6) * a_2^2 ;$$

$$I_3 = -I_{xx2} + I_{yy2} + (m_3 + m_4 + m_5 + m_6) * a_2^2 \; m_2 * r_{x2}^2 - m_2 * r_{y2}^2 ;$$

$$I_4 = m_2 * r_{x2} * (d_2 + r_{z2}) + m_3 * a_2 * r_{z3} + (m_3 + m_4 + m_5 + m_6) * a_2 * (d_2 + d_3) ;$$

$$I_5 = -m_3 * a_2 * r_{y3} + (m_4 + m_5 + m_6) * a_2 * d_4 + m_4 * a_2 * r_{z4} ;$$

$$I_6 = I_{zz3} + m_3 * r_{y3}^2 + m_4 * a_3^2 + m_4 * (d_4 + r_{z4})^2 + I_{yy4} + m_5 * a_3^2 + m_5 * d_4^2 + I_{zz5} + m_6 * a_3^2 + m_6 * d_4^2 + m_6 * r_{z6}^2 + I_{xx6} ;$$

$$I_7 = m_3 * r_{y3}^2 + I_{xx3} - I_{yy3} + m_4 * r_{z4}^2 + 2 * m_4 * d_4 * r_{z4} + (m_4 + m_5 + m_6) * (d_4^2 - a_3^2) + I_{yy4} - I_{zz4} + I_{zz5} - I_{yy5} + m_6 * r_{z6}^2 - I_{zz6} + I_{xx6} ;$$

$$I_8 = -m_4 * (d_2 + d_3) * (d_4 + r_{z4}) - (m_5 + m_6) * (d_2 + d_3) * d_4 \; m_3 * r_{y3} * r_{z3} + m_3 * (d_2 + d_3) * r_{y3} ;$$

$$I_9 = m_2 * r_{y2} * (d_2 + r_{z2}) ;$$

$$I_{10} = 2 * m_4 * a_3 * r_{z4} + 2 * (m_4 + m_5 + m_6) * a_3 * d_4 ;$$

$$I_{11} = -2 * m_2 * r_{x2} * r_{y2} ;$$

$$I_{12} = (m_4 + m_5 + m_6) * a_2 * a_3 ;$$

$$I_{13} = (m_4 + m_5 + m_6) * a_3 * (d_2 + d_3) ;$$

$$I_{14} = I_{zz4} + I_{yy5} + I_{zz6} ;$$

$$I_{15} = m_6 * d_4 * r_{z6} ;$$

$$I_{16} = m_6 * a_2 * r_{z6} ;$$

$$I_{17} = I_{zz5} + I_{xx6} + m_6 * r_{z6}^2 ;$$

$$I_{18} = m_6 * (d_2 + d_3) * r_{z6} ;$$

$$I_{19} = I_{yy4} - I_{xx4} + I_{zz5} - I_{yy5} + m_6 * r_{z6}^2 + I_{xx6} - I_{zz6} ;$$

$$I_{20} = I_{yy5} - I_{xx5} - m_6 * r_{z6}^2 + I_{zz6} - I_{xx6} ;$$

$$I_{21} = I_{xx4} - I_{yy4} + I_{xx5} - I_{zz5} ;$$

$$I_{22} = m_6 * a_3 * r_{z6} ;$$

$$I_{23} = I_{zz6} ;$$

Part II. Gravitional Constants

$$g_1 = -g * ((m_3 + m_4 + m_5 + m_6) * a_2 + m_2 * r_{x2}) ;$$

$$g_2 = g * (m_3 * r_{y3} - (m_4 + m_5 + m_6) * d_4 - m_4 * r_{z4}) ;$$

$$g_3 = g * m_2 * r_{y2} ;$$

$$g_4 = -g * (m_4 + m_5 + m_6) * a_3 ;$$

$$g_5 = -g * m_6 * r_{z6} ;$$

Table A3. Computed Values for the Constants Appearing in the Equations of Forces of Motion.

(Inertial constants have units of kilogram meters-squared)

I_1 =	1.43	± 0.05	I_2 =	1.75	± 0.07
I_3 =	1.38	± 0.05	I_4 =	6.90×10^{-1}	$\pm 0.20\times10^{-1}$
I_5 =	3.72×10^{-1}	$\pm 0.31\times10^{-1}$	I_6 =	3.33×10^{-1}	$\pm 0.16\times10^{-1}$
I_7 =	2.98×10^{-1}	$\pm 0.29\times10^{-1}$	I_8 =	-1.34×10^{-1}	$\pm 0.14\times10^{-1}$
I_9 =	2.38×10^{-2}	$\pm 1.20\times10^{-2}$	I_{10} =	-2.13×10^{-2}	$\pm 0.22\times10^{-2}$
I_{11} =	-1.42×10^{-2}	$\pm 0.70\times10^{-2}$	I_{12} =	-1.10×10^{-2}	$\pm 0.11\times10^{-2}$
I_{13} =	-3.79×10^{-3}	$\pm 0.90\times10^{-3}$	I_{14} =	1.64×10^{-3}	$\pm 0.07\times10^{-3}$
I_{15} =	1.25×10^{-3}	$\pm 0.30\times10^{-3}$	I_{16} =	1.24×10^{-3}	$\pm 0.30\times10^{-3}$
I_{17} =	6.42×10^{-4}	$\pm 3.00\times10^{-4}$	I_{18} =	4.31×10^{-4}	$\pm 1.30\times10^{-4}$
I_{19} =	3.00×10^{-4}	$\pm 14.0\times10^{-4}$	I_{20} =	-2.02×10^{-4}	$\pm 8.00\times10^{-4}$
I_{21} =	-1.00×10^{-4}	$\pm 6.00\times10^{-4}$	I_{22} =	-5.80×10^{-5}	$\pm 1.50\times10^{-5}$
I_{23} =	4.00×10^{-5}	$\pm 2.00\times10^{-5}$			
I_{m1} =	1.14	± 0.27	I_{m2} =	4.71	± 0.54
I_{m3} =	8.27×10^{-1}	$\pm 0.93\times10^{-1}$	I_{m4} =	2.00×10^{-1}	$\pm 0.16\times10^{-1}$
I_{m5} =	1.79×10^{-1}	$\pm 0.14\times10^{-1}$	I_{m6} =	1.93×10^{-1}	$\pm 0.16\times10^{-1}$

(Gravitational constants have units of newton meters)

g_1 =	-37.2	± 00.5	g_2 =	-8.44	± 0.20
g_3 =	1.02	± 0.50	g_4 =	2.49×10^{-1}	$\pm 0.25\times10^{-1}$
g_5 =	-2.82×10^{-2}	$\pm 0.56\times10^{-2}$			

Table A4. The expressions giving the elements of the kinetic energy matrix.
(The Abbreviated Expressions have units of kg-m^2.)

$$\begin{aligned} a_{11} &= I_{m1} + I_1 + I_3 * CC2 + I_7 * SS23 + I_{10} * SC23 + I_{11} * SC2 \\ &\quad + I_{20} * (SS5 * (SS23 * (1 + CC4) - 1) - 2 * SC23 * C4 * SC5) \\ &\quad + I_{21} * SS23 * CC4 + 2 * \{ I_5 * C2 * S23 + I_{12} * C2 * C23 \\ &\quad + I_{15} * (SS23 * C5 + SC23 * C4 * S5) \\ &\quad + I_{16} * C2 * (S23 * C5 + C23 * C4 * S5) \\ &\quad + I_{18} * S4 * S5 + I_{22} * (SC23 * C5 + CC23 * C4 * S5) \} \; ; \\ &\approx 2.57 \; + \; 1.38 * CC2 \; + \; 0.30 * SS23 \; + \; 7.44{\times}10^{-1} * C2 * S23 \; . \end{aligned}$$

$$\begin{aligned} a_{12} &= I_4 * S2 + I_8 * C23 + I_9 * C2 + I_{13} * S23 - I_{15} * C23 * S4 * S5 \\ &\quad + I_{16} * S2 * S4 * S5 + I_{18} * (S23 * C4 * S5 - C23 * C5) \\ &\quad + I_{19} * S23 * SC4 + I_{20} * S4 * (S23 * C4 * CC5 + C23 * SC5) \\ &\quad + I_{22} * S23 * S4 * S5 \; ; \\ &\approx 6.90{\times}10^{-1} * S2 \; - \; 1.34{\times}10^{-1} * C23 \; + \; 2.38{\times}10^{-2} * C2 \; . \end{aligned}$$

$$\begin{aligned} a_{13} &= I_8 * C23 + I_{13} * S23 - I_{15} * C23 * S4 * S5 + I_{19} * S23 * SC4 \\ &\quad + I_{18} * (S23 * C4 * S5 - C23 * C5) + I_{22} * S23 * S4 * S5 \\ &\quad + I_{20} * S4 * (S23 * C4 * CC5 + C23 * SC5) \; ; \\ &\approx -1.34{\times}10^{-1} * C23 \; + \; -3.97{\times}10^{-3} * S23 \; . \end{aligned}$$

$$\begin{aligned} a_{14} &= I_{14} * C23 + I_{15} * S23 * C4 * S5 + I_{16} * C2 * C4 * S5 \\ &\quad + I_{18} * C23 * S4 * S5 - I_{20} * (S23 * C4 * SC5 + C23 * SS5) \\ &\quad + I_{22} * C23 * C4 * S5 \; ; \qquad \approx 0 \; . \end{aligned}$$

$$\begin{aligned} a_{15} &= I_{15} * S23 * S4 * C5 + I_{16} * C2 * S4 * C5 + I_{17} * S23 * S4 \\ &\quad + I_{18} * (S23 * S5 - C23 * C4 * C5) + I_{22} * C23 * S4 * C5 \; ; \\ &\approx 0 \; . \end{aligned}$$

$$a_{16} = I_{23} * (C23 * C5 - S23 * C4 * S5) \; ; \qquad \approx 0 \; .$$

$$\begin{aligned} a_{22} &= I_{m2} + I_2 + I_6 + I_{20} * SS4 * SS5 + I_{21} * SS4 \\ &\quad + 2 * \{ I_5 * S3 + I_{12} * C3 + I_{15} * C5 \\ &\quad + I_{16} * (S3 * C5 + C3 * C4 * S5) + I_{22} * C4 * S5 \} \; ; \\ &\approx 6.79 \; + \; 7.44{\times}10^{-1} * S3 \; . \end{aligned}$$

$$\begin{aligned} a_{23} &= I_5 * S3 + I_6 + I_{12} * C3 + I_{16} * (S3 * C5 + C3 * C4 * S5) \\ &\quad + I_{20} * SS4 * SS5 + I_{21} * SS4 + 2 * \{ I_{15} * C5 + I_{22} * C4 * S5 \} \; ; \\ &\approx .333 \; + \; 3.72{\times}10^{-1} * S3 \; - \; 1.10{\times}10^{-2} * C3 \; . \end{aligned}$$

$$\begin{aligned} a_{24} &= -I_{15} * S4 * S5 - I_{16} * S3 * S4 * S5 + I_{20} * S4 * SC5 \; ; \\ &\approx 0 \; . \end{aligned}$$

$$\begin{aligned} a_{25} &= I_{15} * C4 * C5 + I_{16} * (C3 * S5 + S3 * C4 * C5) \\ &\quad + I_{17} * C4 + I_{22} * S5 \; ; \qquad \approx 0 \; . \end{aligned}$$

$$a_{26} = I_{23} * S4 * S5 \; ; \qquad \approx 0 \; .$$

$$\begin{aligned} a_{33} &= I_{m3} + I_6 + I_{20} * SS4 * SS5 + I_{21} * SS4 \\ &\quad + 2 * \{ I_{15} * C5 + I_{22} * C4 * S5 \} \; ; \qquad \approx 1.16 \; . \end{aligned}$$

$$\begin{aligned} a_{34} &= -I_{15} * S4 * S5 + I_{20} * S4 * SC5 \; ; \\ &\approx -1.25{\times}10^{-3} * S4 * S5 \; . \end{aligned}$$

$$\begin{aligned} a_{35} &= I_{15} * C4 * C5 + I_{17} * C4 + I_{22} * S5 \; ; \\ &\approx 1.25{\times}10^{-3} * C4 * C5 \; . \end{aligned}$$

$$a_{36} = I_{23} * S4 * S5 \; ; \qquad \approx 0 \; .$$

$$a_{44} = I_{m4} + I_{14} - I_{20} * SS5 \; ; \qquad \approx 0.20 \; .$$

$$a_{45} = 0.$$

$$a_{46} = I_{23} * C5 \; ; \qquad \approx 0 \; .$$

$$a_{55} = I_{m5} + I_{17} \; ; \qquad \approx 0.18 \; .$$

$$a_{56} = 0.$$

$$a_{66} = I_{m6} + I_{23} \; ; \qquad \approx 0.19 \; .$$

Table A5. The expressions giving the elements of the Coriolis matrix.
(The Abbreviated Expressions have units of kg-m^2.)

$$\begin{aligned} b_{112} &= 2 * \{ -I_3 * SC2 + I_5 * C223 + I_7 * SC23 - I_{12} * S223 \\ &\quad + I_{15} * (2 * SC23 * C5 + (1 - 2 * SS23) * C4 * S5) \\ &\quad + I_{16} * (C223 * C5 - S223 * C4 * S5) + I_{21} * SC23 * CC4 \\ &\quad + I_{20} * ((1 + CC4) * SC23 * SS5 - (1 - 2 * SS23) * C4 * SC5) \\ &\quad + I_{22} * ((1 - 2 * SS23) * C5 - 2 * SC23 * C4 * S5) \} \\ &\quad + I_{10} * (1 - 2 * SS23) + I_{11} * (1 - 2 * SS2) \; ; \\ &\approx -2.76 * SC2 \; + \; 7.44{\times}10^{-1} * C223 \; + \; 0.60 * SC23 \\ &\quad - \; 2.13{\times}10^{-2} * (1 - 2 * SS23) \; . \end{aligned}$$

$$\begin{aligned} b_{113} &= 2 * \{ I_5 * C2 * C23 + I_7 * SC23 - I_{12} * C2 * S23 \\ &\quad + I_{15} * (2 * SC23 * C5 + (1 - 2 * SS23) * C4 * S5) \\ &\quad + I_{16} * C2 * (C23 * C5 - S23 * C4 * S5) + I_{21} * SC23 * CC4 \\ &\quad + I_{20} * ((1 + CC4) * SC23 * SS5 - (1 - 2 * SS23) * C4 * SC5) \\ &\quad + I_{22} * ((1 - 2 * SS23) * C5 - 2 * SC23 * C4 * S5) \} \\ &\quad + I_{10} * (1 - 2 * SS23) \; ; \\ &\approx 7.44{\times}10^{-1} * C2 * C23 \; + \; 0.60 * SC23 \\ &\quad + \; 2.20{\times}10^{-2} * C2 * S23 \; - \; 2.13{\times}10^{-2} * (1 - 2 * SS23) \; . \end{aligned}$$

$$\begin{aligned} b_{114} &= 2 * \{ -I_{15} * SC23 * S4 * S5 - I_{16} * C2 * C23 * S4 * S5 \\ &\quad + I_{18} * C4 * S5 - I_{20} * (SS23 * SS5 * SC4 - SC23 * S4 * SC5) \\ &\quad - I_{22} * CC23 * S4 * S5 - I_{21} * SS23 * SC4 \} \; ; \\ &\approx -2.50{\times}10^{-3} * SC23 * S4 * S5 \; + \; 8.60{\times}10^{-4} * C4 * S5 \\ &\quad - \; 2.48{\times}10^{-3} * C2 * C23 * S4 * S5 \; . \end{aligned}$$

$$\begin{aligned} b_{115} &= 2 * \{ I_{20} * (SC5 * (CC4 * (1 - CC23) - CC23) \\ &\quad - SC23 * C4 * (1 - 2 * SS5)) - I_{15} * (SS23 * S5 - SC23 * C4 * C5) \\ &\quad - I_{16} * C2 * (S23 * S5 - C23 * C4 * C5) + I_{18} * S4 * C5 \\ &\quad + I_{22} * (CC23 * C4 * C5 - SC23 * S5) \} \; ; \\ &\approx -2.50{\times}10^{-3} * (SS23 * S5 - SC23 * C4 * C5) \\ &\quad - \; 2.48{\times}10^{-3} * C2 * (S23 * S5 - C23 * C4 * C5) \\ &\quad + \; 8.60{\times}10^{-4} * S4 * C5. \end{aligned}$$

$$b_{116} = 0 \; .$$

$$\begin{aligned} b_{123} &= 2 * \{ -I_8 * S23 + I_{13} * C23 + I_{15} * S23 * S4 * S5 \\ &\quad + I_{18} * (C23 * C4 * S5 + S23 * C5) + I_{19} * C23 * SC4 \\ &\quad + I_{20} * S4 * (C23 * C4 * CC5 - S23 * SC5) \\ &\quad + I_{22} * C23 * S4 * S5 \} \; ; \\ &\approx 2.67{\times}10^{-1} * S23 \; - \; 7.58{\times}10^{-3} * C23 \; . \end{aligned}$$

$$\begin{aligned} b_{124} &= -I_{18} * 2 * S23 * S4 * S5 + I_{19} * S23 * (1 - (2 * SS4)) \\ &\quad + I_{20} * S23 * (1 - 2 * SS4 * CC5) - I_{14} * S23 \; ; \qquad \approx 0 \; . \end{aligned}$$

$$\begin{aligned} b_{125} &= I_{17} * C23 * S4 + I_{18} * 2 * (S23 * C4 * C5 + C23 * S5) \\ &\quad + I_{20} * S4 * (C23 * (1 - 2 * SS5) - S23 * C4 * 2 * SC5) \; ; \\ &\approx 0 \; . \end{aligned}$$

$$b_{126} = -I_{23} * (S23 * C5 + C23 * C4 * S5) \; ; \qquad \approx 0 \; .$$

$$b_{134} = b_{124} \; . \qquad b_{135} = b_{125} \; . \qquad b_{136} = b_{126} \; .$$

$$\begin{aligned} b_{145} &= 2 * \{ I_{15} * S23 * C4 * C5 + I_{16} * C2 * C4 * C5 \\ &\quad + I_{18} * C23 * S4 * C5 + I_{22} * C23 * C4 * C5 \} + I_{17} * S23 * C4 \\ &\quad - I_{20} * (S23 * C4 * (1 - 2 * SS5) + 2 * C23 * SC5) \; ; \\ &\approx 0 \; . \end{aligned}$$

$$b_{146} = I_{23} * S23 * S4 * S5 \; ; \qquad \approx 0 \; .$$

$$b_{156} = -I_{23} * (C23 * S5 + S23 * C4 * C5) \; ; \qquad \approx 0 \; .$$

$$b_{212} = 0 \; . \qquad b_{213} = 0 \; .$$

$$\begin{aligned} b_{214} &= I_{14} * S23 + I_{19} * S23 * (1 - (2 * SS4)) \\ &\quad + 2 * \{ -I_{15} * C23 * C4 * S5 + I_{16} * S2 * C4 * S5 \\ &\quad + I_{20} * (S23 * (CC5 * CC4 - 0.5) + C23 * C4 * SC5) \\ &\quad + I_{22} * S23 * C4 * S5 \} \; ; \\ &\approx 1.64{\times}10^{-3} * S23 \; - \; 2.50{\times}10^{-3} * C23 * C4 * S5 \; + \\ &\quad 2.48{\times}10^{-3} * S2 * C4 * S5 \; + \; 0.30{\times}10^{-3} * S23 * (1 - (2 * SS4)) \; . \end{aligned}$$

$$\begin{aligned} b_{215} &= 2 * \{ -I_{15} * C23 * S4 * C5 + I_{22} * S23 * S4 * C5 \\ &\quad + I_{16} * S2 * S4 * C5 \} - I_{17} * C23 * S4 \\ &\quad + I_{20} * (C23 * S4 * (1 - 2 * SS5) - 2 * S23 * SC4 * SC5) \; ; \\ &\approx -2.50{\times}10^{-3} * C23 * S4 * C5 \; + \; 2.48{\times}10^{-3} * S2 * S4 * C5 \\ &\quad - \; 6.42{\times}10^{-4} * C23 * S4 \; . \end{aligned}$$

$$b_{216} = -b_{126} \; .$$

$$\begin{aligned} b_{223} &= 2 * \{ -I_{12} * S3 + I_5 * C3 + I_{16} * (C3 * C5 - S3 * C4 * S5) \} \; ; \\ &\approx 2.20{\times}10^{-2} * S3 \; + \; 7.44{\times}10^{-1} * C3 \; . \end{aligned}$$

$$b_{224} = 2*\{-I_{16}*C3*S4*S5+I_{20}*SC4*SS5 +I_{21}*SC4-I_{22}*S4*S5\}\ ;$$
$$\approx -2.48\times10^{-3}*C3*S4*S5\ .$$

$$b_{225} = 2*\{-I_{15}*S5+I_{16}*(C3*C4*C5-S3*S5) +I_{20}*SS4*SC5+I_{22}*C4*C5\}\ ;$$
$$\approx -2.50\times10^{-3}*S5\ +\ 2.48\times10^{-3}*(C3*C4*C5-S3*S5)\ .$$

$b_{226} = 0\ .$ $\quad b_{234} = b_{224}\ .$

$b_{235} = b_{225}\ .$ $\quad b_{236} = 0\ .$

$$b_{245} = 2*\{-I_{15}*S4*C5-I_{16}*S3*S4*C5\} -I_{17}*S4+I_{20}*S4*(1-2*SS5)\ ; \qquad \approx 0\ .$$

$$b_{246} = I_{23}*C4*S5\ ; \qquad \approx 0\ .$$

$$b_{256} = I_{23}*S4*C5\ ; \qquad \approx 0\ .$$

$b_{312} = 0\ .$ $\quad b_{313} = 0\ .$

$$b_{314} = 2*\{-I_{15}*C23*C4*S5+I_{22}*S23*C4*S5 +I_{20}*(S23*(CC5*CC4-0.5)+C23*C4*SC5)\} +I_{14}*S23+I_{19}*S23*(1-(2*SS4))\ ;$$
$$\approx -2.50\times10^{-3}*C23*C4*S5\ +\ 1.64\times10^{-3}*S23 +\ 0.30\times10^{-3}*S23*(1-2*SS4)\ .$$

$$b_{315} = 2*(-I_{15}*C23*S4*C5+I_{22}*S23*S4*C5) -I_{17}*C23*S4 +I_{20}*S4*(C23*(1-2*SS5)-2*S23*C4*SC5)\ ;$$
$$\approx -2.50\times10^{-3}*C23*S4*C5\ -\ 6.42\times10^{-4}*C23*S4\ .$$

$b_{316} = -b_{136}\ .$ $\quad b_{323} = 0\ .$

$$b_{324} = 2*\{I_{20}*SC4*SS5+I_{21}*SC4-I_{22}*S4*S5\}\ ;$$
$$\approx 0\ .$$

$$b_{325} = 2*\{-I_{15}*S5+I_{20}*SS4*SC5+I_{22}*C4*C5\}\ ;$$
$$\approx -2.50\times10^{-3}*S5\ .$$

$b_{326} = 0\ .$ $\quad b_{334} = b_{324}\ .$

$b_{335} = b_{325}\ .$ $\quad b_{336} = 0\ .$

$$b_{345} = -I_{15}*2*S4*C5-I_{17}*S4+I_{20}*S4*(1-2*SS5)\ ;$$
$$\approx -2.50\times10^{-3}*S4*C5\ .$$

$b_{346} = b_{246}\ .$ $\quad b_{356} = b_{256}\ .$ $\quad b_{412} = -b_{214}\ .$

$b_{413} = -b_{314}\ .$ $\quad b_{414} = 0\ .$

$$b_{415} = -I_{20}*(S23*C4*(1-2*SS5)+2*C23*SC5) -I_{17}*S23*C4\ ;$$
$$\approx -6.42\times10^{-4}*S23*C4\ .$$

$b_{416} = -b_{146}\ .$ $\quad b_{423} = -b_{324}\ .$ $\quad b_{424} = 0\ .$

$$b_{425} = I_{17}*S4+I_{20}*S4*(1-2*SS5)\ ;$$
$$\approx 6.42\times10^{-4}*S4\ .$$

$b_{426} = -b_{246}\ .$ $\quad b_{434} = 0\ .$

$b_{435} = b_{425}\ .$ $\quad b_{436} = -b_{346}\ .$

$$b_{445} = -I_{20}*2*SC5\ ; \qquad \approx 0\ .$$

$$b_{446} = 0\ ;$$

$$b_{456} = -I_{23}*S5\ ; \qquad \approx 0\ .$$

$b_{512} = -b_{215}\ .$ $\quad b_{513} = -b_{315}\ .$ $\quad b_{514} = -b_{415}\ .$

$b_{515} = 0\ .$ $\quad b_{516} = -b_{156}\ .$ $\quad b_{523} = -b_{325}\ .$

$b_{524} = -b_{425}\ .$ $\quad b_{525} = 0\ .$ $\quad b_{526} = -b_{256}\ .$

$b_{534} = b_{524}\ .$ $\quad b_{535} = 0\ .$ $\quad b_{536} = -b_{356}\ .$

$b_{545} = 0\ .$ $\quad b_{546} = -b_{456}\ .$ $\quad b_{556} = 0\ .$

$b_{612} = b_{126}\ .$ $\quad b_{613} = b_{136}\ .$ $\quad b_{614} = b_{146}\ .$

$b_{615} = b_{156}\ .$ $\quad b_{616} = 0\ .$ $\quad b_{623} = 0\ .$

$b_{624} = b_{246}\ .$ $\quad b_{625} = b_{256}\ .$ $\quad b_{626} = 0\ .$

$b_{634} = b_{624}\ .$ $\quad b_{635} = b_{625}\ .$ $\quad b_{636} = 0\ .$

$b_{645} = b_{456}\ .$ $\quad b_{646} = 0\ .$ $\quad b_{656} = 0\ .$

Table A6. The expressions for the terms of the centrifugal matrix. (The Abbreviated Expressions have units of kg-m^2.)

$$c_{11} = 0\ .$$

$$c_{12} = +I4*C2-I8*S23-I9*S2+I13*C23 +I_{15}*S23*S4*S5+I_{16}*C2*S4*S5 +I_{18}*(C23*C4*S5+S23*C5)+I_{19}*C23*SC4 +I_{20}*S4*(C23*C4*cC5-S23*SC5) +I_{22}*C23*S4*S5\ ;$$
$$\approx\ 6.90\times10^{-1}*C2\ +\ 1.34\times10^{-1}*S23\ -\ 2.38\times10^{-2}*S2\ .$$

$$c_{13} = 0.5*b_{123}\ .$$

$$c_{14} = -I_{15}*S23*S4*S5-I_{16}*C2*S4*S5 +I_{18}*C23*C4*S5+I_{20}*S23*S4*SC5 -I_{22}*C23*S4*S5\ ; \qquad \approx 0\ .$$

$$c_{15} = -I_{15}*S23*S4*S5-I_{16}*C2*S4*S5\ ; +I_{18}*(S23*C5+C23*C4*S5)-I_{22}*C23*S4*S5$$
$$\approx\ 0\ .$$

$c_{16} = 0\ .$ $\quad c_{21} = -0.5*b_{112}\ .$

$c_{22} = 0\ .$ $\quad c_{23} = 0.5*b_{223}\ .$

$$c_{24} = -I_{15}*C4*S5-I_{16}*S3*C4*S5+I_{20}*C4*SC5\ ;$$
$$\approx\ 0\ .$$

$$c_{25} = -I_{15}*C4*S5+I_{16}*(C3*C5-S3*C4*S5) +I_{22}*C5\ ; \qquad \approx 0\ .$$

$c_{26} = 0\ .$ $\quad c_{31} = -0.5*b_{113}\ .$

$c_{32} = -c_{23}\ .$ $\quad c_{33} = 0\ .$

$$c_{34} = -I_{15}*C4*S5+I_{20}*C4*SC5\ ;$$
$$\approx\ -1.25\times10^{-3}*C4*S5\ .$$

$$c_{35} = -I_{15}*C4*S5+I_{22}*C5\ ; \qquad \approx c_{34}\ .$$

$c_{36} = 0\ .$ $\quad c_{41} = -0.5*b_{114}\ .$ $\quad c_{42} = -0.5*b_{224}\ .$

$c_{43} = 0.5*b_{423}\ .$ $\quad c_{44} = 0\ .$ $\quad c_{45} = 0\ .$

$c_{46} = 0\ .$ $\quad c_{51} = -0.5*b_{115}\ .$ $\quad c_{52} = -0.5*b_{225}\ .$

$c_{53} = 0.5*b_{523}\ .$ $\quad c_{54} = -0.5*b_{445}\ .$ $\quad c_{55} = 0\ .$

$c_{56} = 0\ .$ $\quad c_{61} = 0\ .$ $\quad c_{62} = 0\ .$

$c_{63} = 0\ .$ $\quad c_{64} = 0\ .$ $\quad c_{65} = 0\ .$

$c_{66} = 0\ .$

Table A7. Gravity Terms.
(The Abbreviated Expressions have units of newton-meters.)

$$g_1 = 0\ .$$

$$g_2 = g1*C2+g2*S23+g3*S2+g4*C23\ ; +g5*(S23*C5+C23*C4*S5)$$
$$\approx -37.2*C2\ -\ 8.4*S23\ +\ 1.02*S2\ .$$

$$g_3 = g2*S23+g4*C23+g5*(S23*C5+C23*C4*S5)\ ;$$
$$\approx -8.4*S23\ +\ 0.25*C23\ .$$

$$g_4 = -g5*S23*S4*S5\ ;$$
$$\approx 2.8\times10^{-2}*S23*S4*S5\ .$$

$$g_5 = g5*(C23*S5+S23*C4*C5)\ ;$$
$$\approx -2.8\times10^{-2}*(C23*S5\ +\ S23*C4*C5)\ .$$

$$g_6 = 0\ .$$