Resilient Delayed Impulsive Control for Consensus of Multiagent Networks Subject to Malicious Agents

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***Abstract*—Impulsive control is widely applied to achieve the consensus of multiagent networks (MANs). It is noticed that malicious agents may have adverse effects on the global behaviors, which, however, are not taken into account in the literature. In this study, a novel delayed impulsive control strategy based on sampled data is proposed to achieve the resilient consensus of MANs subject to malicious agents. It is worth pointing out that the proposed control strategy does not require any information on the number of malicious agents, which is usually required in the existing works on resilient consensus. Under appropriate control gains and sampling period, a necessary and sufficient graphic condition is derived to achieve the resilient consensus of the considered MAN. Finally, the effectiveness of the resilient delayed impulsive control is well demonstrated via simulation studies.**

***Index Terms*—Communication delays, impulsive control, multiagent network (MAN), resilient consensus, trusted agent.**

I. Introduction

DURING the past decades, the studies on multiagent networks (MANs) have raised much attention due to the practical applications in robot navigation and task planning [1], [2]; cyber-physical systems [3], [4]; distributed online optimization [5], [6]; and other aspects. The MAN is a complex dynamic network that can produce macrocooperative behaviors through the interactions between agents and the environment. In the studies on MANs, the consensus is one of the most significant problems. A wide variety of results on the consensus has been obtained by using different control methods, including event-triggered control [7], [8]; sampled-data control [9], [10]; impulsive control [11]–[14]; model-based predictive control [15], [16]; and adaptive control [17], [18]. Among these control methods, the impulsive control is a discontinuous control, which is widely used in the real-world applications [19], such as biological control of pesticide, fuel-optimal control of satellites, and control of robot manipulators. More recently, the impulsive control has been explored in the consensus of MANs in different contexts, such as double-integrator dynamics [20], [21]; communication delays [21]–[25]; switching topologies [21], [26]; asynchronous networks [27]; and event-triggered communications [8]. However, all the impulsive controls mentioned above are implemented under the situation where there are no attacks. The MANs working in a complex environment have many vulnerable points for attacks. For example, multiarmy vehicles on the battlefield face attacks from hidden enemies and unknown explosions. The attacks may make some agents become uncontrolled. Such agents are called malicious agents. For the MANs with malicious agents, the existing impulsive consensus control algorithms may become invalid. Thus, it is essential to design a novel impulsive consensus algorithm for the MANs with malicious agents.

Great progresses [28]–[36] have been made in the study of consensus of MANs with malicious agents, which is called resilient consensus. Resilient consensus requires that an agreement is achieved among the agents behaving normally for any possible malicious agents. Moreover, the value of agreement is confined within a bounded range determined by the initial conditions of the agents behaving normally. One of the earliest works on resilient consensus can go back to [32], which is based on the f-total/f-local assumption (i.e., there are at most f malicious agents in the entire network/each neighborhood). The authors proposed a classical resilient consensus algorithm by filtering out the largest f and smallest f values from neighbors. It was proved that the resilient consensus can be ensured if the communication topology satisfies a newly proposed graphic condition $(f+1, f+1)$-robustness/$(2f+1, 1)$-robustness. From then on, the resilient consensus problem has been explored in different contexts, including double-integrator dynamics [30], communication delays [28], time-varying topologies [34], asynchronous networks [33], event-triggered communications [35], quantization [29], and differential privacy requirements [36]. All of the above-mentioned works on resilient consensus require the same graphic conditions as [32], where the highly connected communication topology is required. In order to

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relax the connectivity requirements, Abbas *et al.* [31] introduced trusted agents for the first-order discrete-time MAN under the f-total/f-local assumption. The trusted agents can resist attacks to avoid becoming the uncontrolled malicious agents by increasing the safety investments. The result in [31] shows that the resilient consensus can be ensured if the communication topology satisfies the condition $(f+1, f+1)$-robustness/$(2f + 1, 1)$-robustness with trusted nodes. One common feature in the above works [28]–[36] on resilient consensus is that an upper bound f on the number of malicious agents is required to be known. However, the upper bound f on the number of the malicious agents may be difficult to obtain, especially in the context of distributed control. Thus, how to design a resilient consensus algorithm without using any information on the number of malicious agents is still unsolved.

In addition, the coordination of MANs is dependent on the digital communication among agents. Thus, the communication delays are usually inevitable. There have been lots of results [21]–[26] on designing the consensus algorithms for MANs based on delayed information. Instead of the whole spectrum of delayed information, only the sampled position data is used to design the resilient consensus algorithm for double-integrator MANs in this article. The advantages of using only sampled position data are to utilize less information and save energy.

Motivated by the above observations, this article aims to utilize the impulsive control strategy with only sampled position data to solve the resilient consensus problem of double-integrator MANs in the presence of communication delays. More challenging, we consider the situation where any information on the number of the malicious agents is unknown. The main contributions are summarized as follows.

1) Based on the impulsive control, we propose a novel algorithm to mitigate the adverse effects of malicious agents on the considered MAN. The advantage of the proposed algorithm is that it does not require any information on the number of the malicious agents. However, the existing resilient consensus algorithms [28]–[36] usually need to know an upper bound on the number of the malicious agents.
2) Under the proposed resilient delayed impulsive control algorithm, a necessary and sufficient graphic condition is obtained to ensure resilient consensus of MANs with communication delays. Compared with the existing results based on graph robustness [28]–[36], the obtained graphic condition is more intuitive and easier to verify.
3) To the best of our knowledge, the existing works on the impulsive consensus do not consider the presence of malicious agents. The resilient consensus problem is solved by using the impulsive control for the first time.

The remainder part of this article is as follows. In Section II, necessary preliminaries on the graph theory are provided and the problem is formulated. The convergence analysis of the resilient delayed impulsive control is presented in Section III. In Section IV, simulation studies are performed to demonstrate the results. Finally, the conclusions are summarized in Section V.

Notation: Let $\mathbb{R}$, $\mathbb{N}$, and $\mathbb{Z}_+$ represent the sets of real numbers, natural numbers, and positive integers, respectively. Symbols $\mathbb{R}^{n\times m}$ and $\mathbb{R}^n$ are the sets of $n \times m$ real matrices and n-dimensional real column vectors, respectively. Let $\mathrm{diag}(a_1, \ldots, a_n)$ denote the diagonal matrix with a_i as its ith diagonal entries. In particular, if $a_i = 1$ for all $i = 1, \ldots, n$, the diagonal matrix is called the identity matrix of dimension n, denoted by I_n. Let $\vec{1}_n$ represent the n-dimensional column vector with all entries being 1. The matrix inequality $A \geq 0$ means that each entry in A is greater than or equal to 0. $|\mathcal{H}|$ represents the number of elements in set $\mathcal{H}$. Let $\mathrm{co}\{x_1, \ldots, x_n\}$ represent the set $\{x | x = \mu_1 x_1 + \cdots + \mu_n x_n, \mu_1 + \cdots + \mu_n = 1, \mu_1, \ldots, \mu_n \geq 0\}$. Symbols $\max\{\cdot\}$ and $\min\{\cdot\}$ are used to denote, respectively, the maximum and minimum values of all entries of the vectors. rand(t)[a,b] is the function that randomly takes value within interval $[a, b]$.

II. Preliminaries and Problem Formulation

A. Graph Theory

In this subsection, some necessary notions and preliminaries on graph theory are introduced.

Let $\mathcal{G}(\mathcal{V}, \mathcal{E})$ be a directed graph (digraph) with node set $\mathcal{V} = \{1, \ldots, n\}$ and edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. The edge $(j, i) \in \mathcal{E}$ means that node i can obtain information from node j, where node j is called a neighbor of node i. The set of all neighbors of node i is denoted by $\mathcal{N}_i$. The adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ associated with digraph $\mathcal{G}$ is defined by $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$, otherwise, $a_{ij} = 0$. Self-loops are not allowed, that is, $a_{ii} = 0$. The Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{n\times n}$ is defined by $l_{ij} = -a_{ij}$ for $j \neq i$, and $l_{ii} = \sum_{j=1}^{n} a_{ij}$.

An in-neighbor of a nonempty set of nodes $\mathcal{V}_\mathcal{S}$ is the node $j \in \mathcal{V} \backslash \mathcal{V}_\mathcal{S}$ for which there exists a directed edge $(j, i) \in \mathcal{E}$ for some node $i \in \mathcal{V}_\mathcal{S}$. In a digraph $\mathcal{G}$, if there exists one node, called root, which has a directed path to all other nodes, then this digraph $\mathcal{G}$ is said to contain a directed spanning tree. A digraph $\mathcal{G}_\mathcal{S}(\mathcal{V}_\mathcal{S}, \mathcal{E}_\mathcal{S})$ is called a subgraph of $\mathcal{G}(\mathcal{V}, \mathcal{E})$ if $\mathcal{V}_\mathcal{S} \subseteq \mathcal{V}$ and $\mathcal{E}_\mathcal{S} \subseteq \mathcal{E}$. Furthermore, if $\mathcal{E}_\mathcal{S} = \{(j, i) : (j, i) \in \mathcal{E}, i, j \in \mathcal{V}_\mathcal{S}\}$, then $\mathcal{G}_\mathcal{S}(\mathcal{V}_\mathcal{S}, \mathcal{E}_\mathcal{S})$ is called the subgraph induced by $\mathcal{V}_\mathcal{S}$.

Next, we will give several lemmas, which play crucial roles in the proof of the obtained theorems in this article.

Lemma 1 [2]: Given a digraph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, the following two conditions are equivalent:

i) $\mathcal{G}(\mathcal{V}, \mathcal{E})$ contains a directed spanning tree;
ii) For any pair of nonempty disjoint subsets $\mathcal{V}_1, \mathcal{V}_2 \subset \mathcal{V}$, either $\mathcal{V}_1$ has an in-neighbor or $\mathcal{V}_2$ has an in-neighbor.

Based on Lemma 1, it is easy to obtain the following lemma.

Lemma 2: If $\mathcal{G}(\mathcal{V}, \mathcal{E})$ does not contain a directed spanning tree, then there exists a pair of nonempty disjoint subsets $\mathcal{V}_1$, $\mathcal{V}_2 \subset \mathcal{V}$ such that for any $i \in \mathcal{V}_1$, $j \in \mathcal{V}_2$, there hold that $\mathcal{N}_i \cap (\mathcal{V} \backslash \mathcal{V}_1) = \emptyset$ and $\mathcal{N}_j \cap (\mathcal{V} \backslash \mathcal{V}_2) = \emptyset$.

Remark 1: Lemma 2 will be used to prove that an interconnected structure among trusted agents is necessary for achieving resilient consensus under the proposed algorithm.

Lemma 3: For any pair of nonempty disjoint subsets $\mathcal{Z}_1, \mathcal{Z}_2 \subset \mathcal{V}$, if a subgraph $\mathcal{G}_\mathcal{S}(\mathcal{V}_\mathcal{S}, \mathcal{E}_\mathcal{S}) \subseteq \mathcal{G}$ satisfies the following two conditions:

i) $\mathcal{G}_{\mathcal{S}}$ contains a directed spanning tree;
ii) For each node $i \in \mathcal{V}$, either $i \in \mathcal{V}_{\mathcal{S}}$ or $\mathcal{N}_i \cap \mathcal{V}_{\mathcal{S}} \neq \emptyset$, then there holds that $\mathcal{X}_{\mathcal{Z}_1}^{\mathcal{V}_{\mathcal{S}}} \cup \mathcal{X}_{\mathcal{Z}_2}^{\mathcal{V}_{\mathcal{S}}} \neq \emptyset$, where $\mathcal{X}_{\mathcal{Z}_i}^{\mathcal{V}_{\mathcal{S}}} = \{j \in \mathcal{Z}_i : \mathcal{N}_j \cap (\mathcal{V}_{\mathcal{S}} \backslash \mathcal{Z}_i) \neq \emptyset\}$, $i = 1, 2$.

Proof: For any pair of nonempty disjoint subsets $\mathcal{Z}_1, \mathcal{Z}_2 \subset \mathcal{V}$, we discuss the following three cases:

1) $\mathcal{Z}_1 \cap \mathcal{V}_{\mathcal{S}} = \emptyset$: According to condition 2), each node in $\mathcal{Z}_1$ has at least a neighbor in $\mathcal{V}_{\mathcal{S}} \backslash \mathcal{Z}_1$. Hence, it is obtained that $\mathcal{X}_{\mathcal{Z}_1}^{\mathcal{V}_{\mathcal{S}}} \neq \emptyset$.
2) $\mathcal{Z}_1 \cap \mathcal{V}_{\mathcal{S}} \neq \emptyset$, $\mathcal{Z}_2 \cap \mathcal{V}_{\mathcal{S}} = \emptyset$: Similar to the above case, it is obtained that $\mathcal{X}_{\mathcal{Z}_2}^{\mathcal{V}_{\mathcal{S}}} \neq \emptyset$.
3) $\mathcal{Z}_1 \cap \mathcal{V}_{\mathcal{S}} \neq \emptyset$, $\mathcal{Z}_2 \cap \mathcal{V}_{\mathcal{S}} \neq \emptyset$: According to condition 1), there exists a node i having a directed path to all other nodes in $\mathcal{V}_{\mathcal{S}}$. If $i \in \mathcal{Z}_1$, then there exists a node in $\mathcal{Z}_2 \cap \mathcal{V}_{\mathcal{S}}$ which has a neighbor in $\mathcal{V}_{\mathcal{S}} \backslash \mathcal{Z}_2$. Hence, there holds that $\mathcal{X}_{\mathcal{Z}_2}^{\mathcal{V}_{\mathcal{S}}} \neq \emptyset$. If $i \notin \mathcal{Z}_1$, then there exists a node in $\mathcal{Z}_1 \cap \mathcal{V}_{\mathcal{S}}$ which has a neighbor in $\mathcal{V}_{\mathcal{S}} \backslash \mathcal{Z}_1$. Hence, there holds that $\mathcal{X}_{\mathcal{Z}_1}^{\mathcal{V}_{\mathcal{S}}} \neq \emptyset$. ■

Remark 2: In Theorem 1, we will prove that if the subgraph induced by trusted agents satisfies the conditions i) and ii) in Lemma 3, the resilient consensus can be guaranteed under the proposed algorithm.

B. Problem Formulation

Consider a MAN with time-varying communication delays, in which there exist the unknown number of the malicious agents. The communications among agents are described by the digraph $\mathcal{G}(\mathcal{V}, \mathcal{E})$. The agents are governed by the following dynamics:

$$\begin{cases} \dot{x}_i(t) = v_i(t), \quad \dot{v}_i(t) = u_i(t), & i \in \mathcal{V}_{\mathcal{N}} \\ x_i(t)\text{:arbitrary}, & i \in \mathcal{V}_{\mathcal{M}} \end{cases} \tag{1}$$

where $x_i(t)$, $v_i(t)$, and $u_i(t) \in \mathbb{R}$ represent the position, velocity, and control input of agent $i \in \mathcal{V}$, respectively. $\mathcal{V} = \mathcal{V}_{\mathcal{N}} \cup \mathcal{V}_{\mathcal{M}}$, $\mathcal{V}_{\mathcal{N}}$ and $\mathcal{V}_{\mathcal{M}}$ denote the sets of normal agents and malicious agents, respectively. A small subset of normal agents is trusted agents, which behave normally even under attacks. The set of trusted agents is denoted by $\mathcal{V}_{\mathcal{T}}$ ($\mathcal{V}_{\mathcal{T}} \subseteq \mathcal{V}_{\mathcal{N}}$). Assume that each normal agent knows which information packets are from trusted agents.

The control input $u_i(t)$ of normal agent $i \in \mathcal{V}_{\mathcal{N}}$ is dependent on the received information from its neighbor set $\mathcal{N}_i$. In order to save energy and utilize less information, each agent measures its own position state $x_i(t)$ and sends it to other agents through the communication links only at the discrete instant $t_k = kT$, where T is the sampling period and $k \in \mathbb{N}$. Because of the presence of the communication delays, each agent i may only receive the delayed position state $x_j(t_k - \tau_{ij}[k])$ from neighbor j at the sampling instant t_k, where $\tau_{ij}[k]$ is the corresponding communication delay. Assume that the communication delays are bounded. Thus, there exists a positive real number τ such that the communication delay $\tau_{ij}[k] \leq \tau$ for $i, j \in \mathcal{V}$, $k \in \mathbb{N}$. Since the information is only measured and sent at the sampling instants, there exists $d_{ij}[k] \in \mathbb{N}$ such that $t_k - \tau_{ij}[k] = t_{k-d_{ij}[k]}$ for any $k \in \mathbb{N}$. In particular, we assume that each agent i can acquire its own information $x_i(t), v_i(t)$ immediately at each sampling instant t_k.

The objective of this article is to design $u_i(t)$ to make MAN (1) achieve the resilient consensus. Inspired by [28]–[36], the definition of the resilient consensus is given as follows.

Definition 1: Resilient consensus of MAN (1) is said to be achieved if for any possible set of malicious agents $\mathcal{V}_{\mathcal{M}} \subseteq \mathcal{V} \backslash \mathcal{V}_{\mathcal{T}}$, any initial positions and initial velocities, the following conditions are satisfied:

i) *Agreement Condition:* For some constant $c \in \mathbb{R}$, it holds that $\lim_{t \to \infty} x_i(t) = c$ for any $i \in \mathcal{V}_{\mathcal{N}}$.
ii) *Safety Condition:* There exists a bounded interval Δ, called safety interval, determined by the initial positions and initial velocities of normal agents such that $x_i(t) \in \Delta$ for any $i \in \mathcal{V}_{\mathcal{N}}$, $t \in [0, \infty)$.

Remark 3: Except for trusted agents, all other agents may become malicious agents under attacks. The control objective is to ensure consensus under any possible set of malicious agents $\mathcal{V}_{\mathcal{M}} \subseteq \mathcal{V} \backslash \mathcal{V}_{\mathcal{T}}$. If there exists a set of malicious agents such that the agreement condition or safety condition is not satisfied, then resilient consensus cannot be said to be achieved.

C. Resilient Delayed Impulsive Control

How to eliminate or mitigate the impacts of malicious agents is a central problem for achieving resilient consensus of MAN (1). A trusted region is proposed to filter the received information. Based on the filtering method, we propose a novel resilient delayed impulsive control, which is given as the following three steps:

Step 1: At each sampling instant $t_k = kT$, each normal agent i obtains the delayed information from its neighbor set $\mathcal{N}_i$. The trusted region of agent i at t_k is defined as $\mathcal{R}_i[k] = [x^i_{\min}(t_k), x^i_{\max}(t_k)]$, where

$$x^i_{\max}(t_k) = \max\{x_j(t_{k-d_{ij}[k]}) : j \in (\mathcal{V}_{\mathcal{T}} \cap \mathcal{N}_i) \cup \{i\}\},$$

$$x^i_{\min}(t_k) = \min\{x_j(t_{k-d_{ij}[k]}) : j \in (\mathcal{V}_{\mathcal{T}} \cap \mathcal{N}_i) \cup \{i\}\}.$$

Step 2: For agent $i \in \mathcal{V}_{\mathcal{N}}$, if the obtained information $x_j(t_{k-d_{ij}[k]}) \in \mathcal{R}_i[k]$, let $a_{ij}[k] = a_{ij}$. Otherwise, let $a_{ij}[k] = 0$. For $j \notin \mathcal{N}_i$, let $a_{ij}[k] = 0$.

Step 3: The control input $u_i(t)$ of agent $i \in \mathcal{V}_{\mathcal{N}}$ is designed as follows:

$$u_i(t) = \sum_{k=0}^{\infty} \left[-p_1 \sum_{j \in \mathcal{N}_i} a_{ij}[k] \big(x_i(t_k) - x_j\big(t_{k-d_{ij}[k]}\big)\big) - p_2 v_i(t_k) \right] \delta(t - t_k) \tag{2}$$

where $\delta(\cdot)$ is the Dirac impulsive function, that is, $\delta(t) = 0$ for $t \neq 0$ and $\int_{-\infty}^{+\infty} \delta(t) dt = 1$, $p_1, p_2 > 0$ are the control gains.

Remark 4: To utilize less information and save energy, only sampled position data with delays from neighbors is used to design $u_i(t)$. The control input $u_i(t)$ only changes the velocity to some constant instantaneously at the sampling instants, the execution time of which is much smaller than the impulsive interval.

Remark 5: The resilient delayed impulsive control does not require any information on the number of the malicious agents. However, the existing works [28]–[36] on resilient consensus require the prior knowledge of an upper bound f on the number of the malicious agents, which may be hard to obtain in some cases, such as the real-time death toll in military wars, and the number of damaged sensors in large-scale sensor networks in cold weather.

III. Convergence Analysis

Since the communication delays are bounded, there exists $d \in \mathbb{Z}_+$ such that $d_{ij}[k] \le d$ for $i, j \in \mathcal{V}$, $k \in \mathbb{N}$. We denote the number of normal agents $|\mathcal{V}_\mathcal{N}|$ in MAN (1) by n_1. Assume that $\mathcal{V}_\mathcal{N} = \{1, \ldots, n_1\}$ and $\mathcal{V}_\mathcal{M} = \{n_1+1, \ldots, n\}$. Let

$$\begin{aligned} x(t) &= [x_1(t), \ldots, x_n(t)]^T \\ x^N(t) &= [x_1(t), \ldots, x_{n_1}(t)]^T \\ v(t) &= [v_1(t), \ldots, v_n(t)]^T \\ v^N(t) &= [v_1(t), \ldots, v_{n_1}(t)]^T \\ z(t_k) &= [x(t_k)^T, x(t_{k-1})^T, \ldots, x(t_{k-d})^T]^T \end{aligned}$$

and

$$z^N(t_k) = [x^N(t_k)^T, x^N(t_{k-1})^T, \ldots, x^N(t_{k-d})^T]^T.$$

From (1) and (2), the evolution of agent $i \in \mathcal{V}_\mathcal{N}$ is written equivalently as

$$\begin{cases} \left.\begin{aligned}\dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= 0,\end{aligned}\right. \quad t \in (t_k, t_{k+1}], \\ \Delta v_i(t_k) = -p_1 \sum_{j=1}^{n} l_{ij}[k] x_j(t_{k-d_{ij}[k]}) - p_2 v_i(t_k), \end{cases} \tag{3}$$

where $\Delta v_i(t_k) = \lim_{t \to t_k^+} v_i(t) - v_i(t_k)$, $l_{ij}[k] = -a_{ij}[k]$ for $j \neq i$, and $l_{ii}[k] = \sum_{j \neq i} a_{ij}[k]$. $v_i(t)$ is left continuous at t_k and invariable at $(t_k, t_{k+1}]$. Thus, we have

$$\begin{cases} x_i(t_{k+1}) = x_i(t_k) + \Bigg[-p_1 \sum_{j=1}^{n} l_{ij}[k] x_j\big(t_{k-d_{ij}[k]}\big) \\ \qquad\qquad + (1-p_2) v_i(t_k)\Bigg] T \\ v_i(t_{k+1}) = -p_1 \sum_{j=1}^{n} l_{ij}[k] x_j\big(t_{k-d_{ij}[k]}\big) \\ \qquad\qquad + (1-p_2) v_i(t_k). \end{cases} \tag{4}$$

Let

$$(A_\alpha[k])_{ij} = \begin{cases} a_{ij}[k], & \text{if } d_{ij}[k] = \alpha \text{ and } j \in \mathcal{N}_i \\ 0, & \text{otherwise.} \end{cases}$$

$$D[k] = \left[\operatorname{diag}\left(\sum_{j=1}^{n} a_{1j}[k], \ldots, \sum_{j=1}^{n} a_{n_1,j}[k]\right) \; 0_{n_1 \times (n-n_1)}\right],$$

$$L[k] = [D[k] - A_0[k], -A_1[k], \ldots, -A_d[k]].$$

Thus, (4) can be written into the matrix form

$$\begin{cases} x^N(t_{k+1}) = \big([I_{n_1} \; 0] - p_1 T L[k]\big) z(t_k) \\ \qquad\qquad + (1-p_2) T v^N(t_k), \\ v^N(t_{k+1}) = -p_1 L[k] z(t_k) + (1-p_2) v^N(t_k). \end{cases} \tag{5}$$

Furthermore, for $k \in \mathbb{Z}_+$, we have

$$\begin{aligned} x^N(t_{k+1}) &= \big[W_1[k] \; W_2[k]\big] \begin{bmatrix} z(t_k) \\ z(t_{k-1}) \end{bmatrix} \\ &= \big[W_3[k] \; W_4[k]\big] \begin{bmatrix} z(t_k) \\ x^N(t_{k-1}) \end{bmatrix} \end{aligned} \tag{6}$$

where

$$\begin{aligned} W_1[k] &= W_3[k] = (2-p_2)\big[I_{n_1} \; 0\big] - p_1 T L[k], \\ W_2[k] &= (p_2 - 1)\big[I_{n_1} \; 0\big], \end{aligned}$$

and

$$W_4[k] = (p_2 - 1) I_{n_1}.$$

Assumption 1: The sampling period T and the control gains p_1 and p_2 satisfy the following inequalities:

$$\begin{cases} p_1 > 0 \\ 1 \le p_2 < 2 \\ 0 < T < (2-p_2)/(p_1 \max\limits_{1 \le i \le n} l_{ii}). \end{cases} \tag{7}$$

The inequalities (7) ensure that $[\, W_3[k] \; W_4[k]\,] \ge 0$, which plays a vital role in achieving resilient consensus.

Lemma 4: Assume that the inequalities (7) are satisfied, then it is established that

$$\begin{aligned} x_i(t_{k+1}) \in \operatorname{co}\{&x_1(t_k), \ldots, x_n(t_k), \\ &x_1(t_{k-1}), \ldots, x_n(t_{k-1}), \\ &\vdots \qquad\qquad \vdots \\ &x_1(t_{k-d}), \ldots, x_n(t_{k-d})\} \end{aligned}$$

for any $i \in \mathcal{V}_\mathcal{N}$ and $k \in \mathbb{Z}_+$.

Proof: From the definition of $L[k]$, the row sums of $L[k]$ are all equal to zero. Therefore, based on (6), the row sums of $[\, W_3[k] \; W_4[k]\,]$ are all equal to one, that is

$$\big[W_3[k] \; W_4[k]\big] \vec{1}_{n_1+n(d+1)} = \vec{1}_{n_1}.$$

Moreover, it holds that $[W_3[k] \; W_4[k]] \ge 0$ under the inequalities (7). Based on the definitions of $z(t_k)$ and $x^N(t_{k-1})$, we can conclude that

$$\begin{aligned} x_i(t_{k+1}) \in \operatorname{co}\{&x_1(t_k), \ldots, x_n(t_k) \\ &x_1(t_{k-1}), \ldots, x_n(t_{k-1}) \\ &\vdots \qquad\qquad \vdots \\ &x_1(t_{k-d}), \ldots, x_n(t_{k-d})\} \end{aligned}$$

for any $i \in \mathcal{V}_\mathcal{N}$ and $k \in \mathbb{Z}_+$. ∎

In the following, the main theoretical results of this article will be presented.

Theorem 1: Assume that the inequalities (7) are satisfied, then the resilient consensus of MAN (1) can be achieved under the resilient delayed impulsive control (2) if and only if the

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subgraph $\mathcal{G}_{\mathcal{T}}(\mathcal{V}_{\mathcal{T}}, \mathcal{E}_{\mathcal{T}})$ induced by trusted agents satisfies the following two conditions:

i) $\mathcal{G}_{\mathcal{T}}$ contains a directed spanning tree;

ii) For each node $i \in \mathcal{V}$, either $i \in \mathcal{V}_{\mathcal{T}}$ or $\mathcal{N}_i \cap \mathcal{V}_{\mathcal{T}} \neq \emptyset$. Moreover, the safety interval is given by

$$\Delta = \big[\min\{z^N(0)\} + (1-p_2)T\max\{0, v^N(0)\} \quad \max\{z^N(0)\} + (1-p_2)T\min\{0, v^N(0)\}\big]. \tag{8}$$

Proof: (Sufficiency) First, we prove the agreement condition. Let

$$\overline{x}(t_k) = \max\{z^N(t_k)\}, \quad \underline{x}(t_k) = \min\{z^N(t_k)\}, \quad k \in \mathbb{N}.$$

For any $i \in \mathcal{V}_{\mathcal{N}}$, if its neighbor $j \in \mathcal{V}_{\mathcal{M}}$ satisfies $x_j(t_{k-d_{ij}[k]}) \notin [\min\{z^N(t_k)\}, \max\{z^N(t_k)\}]$, then there holds that

$$x_j\big(t_{k-d_{ij}[k]}\big) \notin \mathcal{R}_i[k]$$

based on the resilient delayed impulsive control algorithm. Thus, we have $a_{ij}[k] = 0$, which means that $x_j(t_{k-d_{ij}[k]})$ is filtered out. As a result, it is derived from (6) and Lemma 4 that

$$\max\{x^N(t_{k+1})\} \leq \max\big\{z^N(t_k), x^N(t_{k-1})\big\} = \max\{z^N(t_k)\} = \overline{x}(t_k). \tag{9}$$

Note that

$$\begin{aligned} \max\{x^N(t_k)\} &\leq \max\{z^N(t_k)\} = \overline{x}(t_k) \\ \max\{x^N(t_{k-1})\} &\leq \max\{z^N(t_k)\} = \overline{x}(t_k) \\ &\vdots \\ \max\{x^N(t_{k+1-d})\} &\leq \max\{z^N(t_k)\} = \overline{x}(t_k) \end{aligned}$$

combining these with (9) yields

$$\begin{aligned} \overline{x}(t_{k+1}) &= \max\{z^N(t_{k+1})\} \\ &= \max\{x^N(t_{k+1}), x^N(t_k), \dots, x^N(t_{k+1-d})\} \\ &\leq \overline{x}(t_k). \end{aligned}$$

Therefore, $\{\overline{x}(t_k)\}$ is a nonincreasing sequence. Similarly, it can be proved that $\{\underline{x}(t_k)\}$ is a nondecreasing sequence. Then, $\{\overline{x}(t_k)\}$ and $\{\underline{x}(t_k)\}$ are both bounded by $[\underline{x}(0), \overline{x}(0)]$. Based on the monotone convergence theorem, the limits of $\{\overline{x}(t_k)\}$ and $\{\underline{x}(t_k)\}$ exist, denoted, respectively, by a and b. As long as we prove $a = b$, the agreement condition can be guaranteed. It obviously holds that $a \geq b$. We next prove $a = b$ by contradiction.

Assume that $a > b$. The minimum positive entry of $[\,W_3[k]\ W_4[k]\,]$ is denoted by θ for all $k \in \mathbb{N}$. Under Assumption 1, one has $\theta \in (0, 1)$. From $a > b$, there exists $\varepsilon_0 > 0$ such that $b+\varepsilon_0 < a-\varepsilon_0$. Randomly choose ε satisfying

$$0 < \varepsilon < \theta^{(d+1)n_1}\varepsilon_0/(1-\theta^{(d+1)n_1}).$$

Subsequently, we construct a sequence $\{\varepsilon_l\}$ defined by $\varepsilon_{l+1} = \theta\varepsilon_l - (1-\theta)\varepsilon, \quad l = 0, 1, \dots, (d+1)n_1 - 1$. Obviously, $\{\varepsilon_l\}$ is a strictly decreasing sequence. Moreover, there holds that

$$\varepsilon_{(d+1)n_1} = \theta^{(d+1)n_1}\varepsilon_0 - (1-\theta^{(d+1)n_1})\varepsilon > 0.$$

The convergence of $\{\overline{x}(t_k)\}$ and $\{\underline{x}(t_k)\}$ indicates that there exists $K(\varepsilon) \in \mathbb{Z}_+$ such that $\overline{x}(t_k) < a + \varepsilon$ and $\underline{x}(t_k) > b - \varepsilon$ for $k \geq K(\varepsilon)$. Define

$$\mathcal{H}_1[K(\varepsilon)+l] := \big\{j \in \mathcal{V}_{\mathcal{N}}: x_j\big(t_{K(\varepsilon)+l}\big) > a - \varepsilon_l\big\}$$

$$\mathcal{H}_2[K(\varepsilon)+l] := \big\{j \in \mathcal{V}_{\mathcal{N}}: x_j\big(t_{K(\varepsilon)+l}\big) < b + \varepsilon_l\big\}$$

where $l = 0, 1, \dots, (d+1)n_1$. Since there holds that $a - \varepsilon_l > b + \varepsilon_l$, we obtain

$$\mathcal{H}_1[K(\varepsilon)+l] \cap \mathcal{H}_2[K(\varepsilon)+l] = \emptyset. \tag{10}$$

Next, we prove that at least one of the following conditions is satisfied:

$$\text{1)}\quad \mathcal{H}_1[K(\varepsilon)+n_1(d+1)-l] = \emptyset, l = 0, 1, \dots, d+1 \tag{11}$$

$$\text{2)}\quad \mathcal{H}_2[K(\varepsilon)+n_1(d+1)-l] = \emptyset, l = 0, 1, \dots, d+1 \tag{12}$$

Because $\{\overline{x}(t_k)\}$ is a nonincreasing sequence with the limit a, we obtain

$$\begin{aligned} \overline{x}(t_{K(\varepsilon)+d}) &= \max\{z^N(t_{K(\varepsilon)+d})\} \\ &= \max\{x^N(t_{K(\varepsilon)+d}), \dots, x^N(t_{K(\varepsilon)})\} \geq a. \end{aligned}$$

Therefore, there exists $i \in \mathcal{V}_{\mathcal{N}}$ such that at least one of the following conditions is satisfied:

1) $x_i(t_{K(\varepsilon)+d}) \geq a > a - \varepsilon_d$;

2) $x_i(t_{K(\varepsilon)+d-1}) \geq a > a - \varepsilon_{d-1}$;

$\vdots$

$d+1) x_i(t_{K(\varepsilon)}) \geq a > a - \varepsilon_0$.

As a result, we have

$$\bigcup_{l=0}^{d} \mathcal{H}_1[K(\varepsilon)+l] \neq \emptyset. \tag{13}$$

For $\mathcal{H}_1[K(\varepsilon)]$, we can prove that it is nonempty by contradiction. Assume that $\mathcal{H}_1[K(\varepsilon)] = \emptyset$, which means that $x_j(t_{K(\varepsilon)}) \leq a - \varepsilon_0$ for all $j \in \mathcal{V}_{\mathcal{N}}$. Furthermore, it follows from (6) and Lemma 4 that:

$$\begin{aligned} x_j\big(t_{K(\varepsilon)+1}\big) &\leq (1-\theta)\overline{x}\big(t_{K(\varepsilon)}\big) + \theta(a-\varepsilon_0) \\ &\leq (1-\theta)(a+\varepsilon) + \theta(a-\varepsilon_0) \\ &= a - \varepsilon_1 \end{aligned} \tag{14}$$

for all $j \in \mathcal{V}_{\mathcal{N}}$. Therefore, we have $\mathcal{H}_1[K(\varepsilon)+1] = \emptyset$. Similarly, it can be proved that $\mathcal{H}_1[K(\varepsilon)+l] = \emptyset$ for $l = 2, \dots, d$. Then, we obtain

$$\bigcup_{l=0}^{d} \mathcal{H}_1[K(\varepsilon)+l] = \emptyset$$

which is contrary to (13). Therefore, one has $\mathcal{H}_1[K(\varepsilon)] \neq \emptyset$. Similarly, it is established that $\mathcal{H}_2[K(\varepsilon)] \neq \emptyset$. As a result, it follows from (10) that $\mathcal{H}_1[K(\varepsilon)]$ and $\mathcal{H}_2[K(\varepsilon)]$ are a pair of nonempty and disjoint subsets. From Lemma 3, since $\mathcal{G}_{\mathcal{T}}(\mathcal{V}_{\mathcal{T}}, \mathcal{E}_{\mathcal{T}})$ satisfies the conditions i) and ii), we have

$$\mathcal{X}^{\mathcal{V}_{\mathcal{T}}}_{\mathcal{H}_1[K(\varepsilon)]} \cup \mathcal{X}^{\mathcal{V}_{\mathcal{T}}}_{\mathcal{H}_2[K(\varepsilon)]} \neq \emptyset. \tag{15}$$

If $\mathcal{X}^{\mathcal{V}_{\mathcal{T}}}_{\mathcal{H}_1[K(\varepsilon)]} \neq \emptyset$, then for any $h \in \mathcal{X}^{\mathcal{V}_{\mathcal{T}}}_{\mathcal{H}_1[K(\varepsilon)]} \subseteq \mathcal{H}_1[K(\varepsilon)]$, there exists a trusted agent $e \in \mathcal{N}_h \cap (\mathcal{V}_{\mathcal{T}} \backslash \mathcal{H}_1[K(\varepsilon)])$ satisfying $x_e(t_{K(\varepsilon)}) \leq a - \varepsilon_0$. Based on the discussion above, one has $e \notin \mathcal{H}_1[K(\varepsilon) + l]$ for $0 \leq l \leq d$. Therefore, we can obtain the following results:

$$\begin{aligned}
x_e(t_{K(\varepsilon)}) &\leq a - \varepsilon_0 < a - \varepsilon_d,\\
x_e(t_{K(\varepsilon)+1}) &\leq a - \varepsilon_1 < a - \varepsilon_d,\\
&\vdots\\
x_e(t_{K(\varepsilon)+d-1}) &\leq a - \varepsilon_{d-1} < a - \varepsilon_d,\\
x_e(t_{K(\varepsilon)+d}) &\leq a - \varepsilon_d.
\end{aligned}$$

From (6) and Lemma 4, at least one of the above results is used in the update process at $t_{K(\varepsilon)+d+1}$, it is derived that

$$\begin{aligned}
x_h(t_{K(\varepsilon)+d+1}) &\leq (1-\theta)\bar{x}(t_{K(\varepsilon)+d}) + \theta(a - \varepsilon_d)\\
&\leq (1-\theta)(a+\varepsilon) + \theta(a - \varepsilon_d)\\
&= a - \varepsilon_{d+1}.
\end{aligned}$$

Therefore, one has $h \notin \mathcal{H}_1[K(\varepsilon) + d + 1]$. In addition, if a normal agent $j \notin \mathcal{H}_1[K(\varepsilon)]$, then it can be easily obtained that $j \notin \mathcal{H}_1[K(\varepsilon) + d + 1]$. We have $|\mathcal{H}_1[K(\varepsilon) + d + 1]| < |\mathcal{H}_1[K(\varepsilon)]|$. Similarly, we have $|\mathcal{H}_2[K(\varepsilon) + d + 1]| < |\mathcal{H}_2[K(\varepsilon)]|$ if $\mathcal{X}^{\mathcal{V}_{\mathcal{T}}}_{\mathcal{H}_2[K(\varepsilon)]} \neq \emptyset$. As a result, we conclude that

$$\begin{aligned}
&|\mathcal{H}_1[K(\varepsilon) + d + 1]| + |\mathcal{H}_2[K(\varepsilon) + d + 1]|\\
&\quad < |\mathcal{H}_1[K(\varepsilon)]| + |\mathcal{H}_2[K(\varepsilon)]|.
\end{aligned}$$

If at least one of $\mathcal{H}_1[K(\varepsilon) + d + 1]$ and $\mathcal{H}_2[K(\varepsilon) + d + 1]$ is the empty set, correspondingly, we can easily prove that $\mathcal{H}_1[K(\varepsilon) + l] = \emptyset$ or $\mathcal{H}_2[K(\varepsilon) + l] = \emptyset$ for all $l \in \{d + 1, \ldots, n_1(d + 1)\}$. Therefore, at least one of (11) and (12) is satisfied. If both $\mathcal{H}_1[K(\varepsilon) + d + 1]$ and $\mathcal{H}_2[K(\varepsilon) + d + 1]$ are nonempty, we can similarly prove that

$$\begin{aligned}
&|\mathcal{H}_1[K(\varepsilon) + 2(d + 1)]| + |\mathcal{H}_2[K(\varepsilon) + 2(d + 1)]|\\
&\quad < |\mathcal{H}_1[K(\varepsilon) + d + 1]| + |\mathcal{H}_2[K(\varepsilon) + d + 1]|.
\end{aligned}$$

Due to the fact that $|\mathcal{H}_1[K(\varepsilon)]| + |\mathcal{H}_2[K(\varepsilon)]| \leq n_1$, by analogy, at least one of (11) and (12) is satisfied. Assume that (11) is satisfied. That is to say, for any $i \in \mathcal{V}_{\mathcal{N}}$, the following results are established:

$$\begin{aligned}
x_i(t_{K(\varepsilon)+n_1(d+1)}) &\leq a - \varepsilon_{n_1(d+1)} < a\\
x_i(t_{K(\varepsilon)+n_1(d+1)-1}) &\leq a - \varepsilon_{n_1(d+1)-1} < a\\
&\vdots\\
x_i(t_{K(\varepsilon)+n_1(d+1)-d}) &\leq a - \varepsilon_{n_1(d+1)-d} < a.
\end{aligned}$$

Therefore, we have

$$\max\{z^N(t_{K(\varepsilon)+n_1(d+1)})\} = \bar{x}(t_{K(\varepsilon)+n_1(d+1)}) < a.$$

However, $\{\bar{x}(t_k)\}$ is a nonincreasing sequence with the limit a, which means that $\bar{x}(t_{K(\varepsilon)+n_1(d+1)}) \geq a$. It is a contradiction. The assumption $a > b$ does not hold. We have $a = b$. The agreement condition is proved.

Next, we prove the safety condition.

1) *When $t = t_k$:* From $p_2 \geq 1$, it is easy to know that $[\min\{z^N(0)\}, \max\{z^N(0)\}] \subseteq \Delta$. Therefore, we have $\underline{x}(0) \in \Delta$ and $\bar{x}(0) \in \Delta$. From the monotonicity and the definition of $\{\bar{x}(t_k)\}$ and $\{\underline{x}(t_k)\}$, one has $x_i(t_k) \in \Delta$ for any $i \in \mathcal{V}_{\mathcal{N}}$ and $k \in \mathbb{N}$.
2) *When $t \neq t_k$:* Based on the resilient delayed impulsive control (2), $v_i(t)$ is constant at each sampling interval. Therefore, $x_i(t)$ is monotonous at each sampling interval. From the above result that $x_i(t_k) \in \Delta$, we obtain $x_i(t) \in \Delta$ for any $i \in \mathcal{V}_{\mathcal{N}}$ and $t \in (t_k, t_{k+1})$.

In summary, we have $x_i(t) \in \Delta$ for any $i \in \mathcal{V}_{\mathcal{N}}$ and $t \in [0, \infty)$. The safety condition is proved.

(Necessity) We prove it by contradiction. The following two cases should be taken into consideration.

1) Assume that the subgraph $\mathcal{G}_{\mathcal{T}}$ does not satisfy the condition i): From Lemma 2, there exists a pair of nonempty disjoint subsets $\mathcal{V}_{\mathcal{T}_1}, \mathcal{V}_{\mathcal{T}_2} \subset \mathcal{V}_{\mathcal{T}}$ such that for any $i \in \mathcal{V}_{\mathcal{T}_1}$, $j \in \mathcal{V}_{\mathcal{T}_2}$, there hold that $\mathcal{N}_i \cap (\mathcal{V}_{\mathcal{T}} \backslash \mathcal{V}_{\mathcal{T}_1}) = \emptyset$ and $\mathcal{N}_j \cap (\mathcal{V}_{\mathcal{T}} \backslash \mathcal{V}_{\mathcal{T}_2}) = \emptyset$. Let $d_{ij}[k] = 0$ for $i, j \in \mathcal{V}, k \in \mathbb{N}$. We consider the following initial condition:

$$\begin{cases} x_j(0) = c_1,\ v_j(0) = 0,\ j \in \mathcal{V}_{\mathcal{T}_1}\\ x_j(0) = c_2,\ v_j(0) = 0,\ j \in \mathcal{V} \backslash \mathcal{V}_{\mathcal{T}_1} \end{cases}$$

where c_1 and c_2 are the arbitrary real numbers not equal to each other. For any $i \in \mathcal{V}_{\mathcal{T}_1}$, from the resilient delayed impulsive control algorithm, one has $\mathcal{R}_i[0] = \{c_1\}$. Therefore, we have $u_i(t) = 0$ for $t \in [0, t_1)$. Then, it is derived that $v_i(t_1) = v_i(0) = 0$ and $x_i(t_1) = x_i(0) = c_1$ for any $i \in \mathcal{V}_{\mathcal{T}_1}$. Similarly, we have

$$x_i(t_k) = x_i(t_{k-1}) = \ldots = x_i(t_1) = x_i(0) = c_1$$

for $k \in \mathbb{Z}_+$. The same is true for any $j \in \mathcal{V}_{\mathcal{T}_2}$, we obtain $x_j(t_k) = c_2$ for $k \in \mathbb{Z}_+$. Due to the fact that $c_1 \neq c_2$, resilient consensus is not achieved.

2) Assume that the subgraph $\mathcal{G}_{\mathcal{T}}$ does not satisfy the condition ii): According to the condition ii), there exists a nontrusted agent i such that $\mathcal{N}_i \cap \mathcal{V}_{\mathcal{T}} = \emptyset$. Assume that the nontrusted agent $i \in \mathcal{V}_{\mathcal{N}} \backslash \mathcal{V}_{\mathcal{T}}$ and $\mathcal{N}_i \subseteq \mathcal{V}_{\mathcal{M}}$. Let $d_{ij}[k] = 0$ for $i, j \in \mathcal{V}, k \in \mathbb{N}$. Consider the following initial condition:

$$\begin{cases} x_j(0) > r_1,\ v_j(0) = 0,\ j \in \mathcal{N}_i \cup \{i\}\\ x_j(0) < r_1,\ v_j(0) = 0,\ j \notin \mathcal{N}_i \cup \{i\} \end{cases}$$

where r_1 is the arbitrary real number. For the nontrusted agent $i \in \mathcal{V}_{\mathcal{N}} \backslash \mathcal{V}_{\mathcal{T}}$, from the resilient delayed impulsive control algorithm, one has $\mathcal{R}_i[0] = \{x_i(0)\}$. Therefore, we have $u_i(t) = 0$ for $t \in [0, t_1)$. Then, it is derived that $v_i(t_1) = v_i(0) = 0$ and $x_i(t_1) = x_i(0) > r_1$. Similarly, we have

$$x_i(t_k) = x_i(t_{k-1}) = \ldots = x_i(t_1) = x_i(0) > r_1$$

for $k \in \mathbb{Z}_+$. For any $q \in \mathcal{V}_{\mathcal{N}} \backslash \{i\}$, from the fact that

$$x^N(t_1) = \left(\begin{bmatrix} I_{n_1} & 0 \end{bmatrix} - p_1 TL[0]\right)x(0)$$

we have $x_q(t_1) \in \text{co}\{x_1(0), x_2(0), \ldots, x_n(0)\}$. Based on the resilient delayed impulsive control algorithm, the positions of the neighbors of agent q greater than or equal to r_1 will be filtered out. Therefore, one has $x_q(t_1) < r_1$. From (6), we have

$$x^N(t_2) = \left[W_3[1] \; W_4[1] \right] \begin{bmatrix} x(t_1) \\ x^N(0) \end{bmatrix}. \quad (16)$$

Based on the resilient delayed impulsive control algorithm, it is obtained that $x_q(t_2) < r_1$ for $q \in \mathcal{V_N} \backslash \{i\}$. Similarly, it is derived that $x_q(t_k) < r_1$ for $k \in \mathbb{Z}_+$. However, we have $x_i(t_k) > r_1$ for $k \in \mathbb{Z}_+$. Therefore, resilient consensus is not achieved. ■

Corollary 1: Assume that the inequalities (7) are satisfied, then the resilient consensus of MAN (1) without communication delays can be achieved if and only if the subgraph $\mathcal{G_T}(\mathcal{V_T}, \mathcal{E_T})$ induced by trusted agents satisfies the conditions i) and ii) in Theorem 1. Moreover, the safety interval is given by

$$\Delta = \big[\min\{x^N(0)\} + (1 - p_2)T \max\{0, v^N(0)\}, \max\{x^N(0)\} + (1 - p_2)T \min\{0, v^N(0)\}\big].$$

Remark 6: Based on Theorem 1 and Corollary 1, the proposed resilient delayed impulsive control can ensure the resilient consensus no matter what the number of the malicious agents is. However, in the previous results [28]–[36], the communication topologies have to be constructed according to the upper bound f on the number of the malicious agents. In other words, the MANs only can resist a limited number of malicious agents for the given communication topology. Actually, except for trusted agents, all other agents may become malicious agents under attacks. Once the number of malicious agents is greater than the pre-set parameter f, the resilient consensus cannot be guaranteed.

Remark 7: Compared with the traditional graphic conditions based on graph robustness in [28]–[36], the graphic conditions established in Theorem 1 and Corollary 1 are more intuitive and easier to check and verify.

IV. Simulation Studies

In this section, simulation examples are given to confirm the results in this article.

Consider MAN (1), respectively, cooperating on the digraphs $\mathcal{G}_1$ and $\mathcal{G}_2$, which are shown in Fig. 1(a) and (b). The green nodes represent trusted agents. Except for trusted agents, all other agents may become malicious agents under attacks. It is easy to verify that the subgraphs $\mathcal{G}_{\mathcal{T}_1}$ and $\mathcal{G}_{\mathcal{T}_2}$ induced by trusted agents both satisfy the conditions i) and ii) in Theorem 1. Throughout the simulation studies, for the adjacency matrix A, we take $a_{ij} = 1/|\mathcal{N}_i|$ if $j \in \mathcal{N}_i$, otherwise, let $a_{ij} = 0$. The control gains p_1 and p_2 and the sampling period T are taken as 1, 1.5, and 0.25s, respectively. Since the information is only measured and sent at the sampling instant $t_k = kT, k \in \mathbb{N}$, the communication delays are assumed to choose randomly from {0.25s, 0.5s, 0.75s, 1s}.

First, we confirm the sufficiency of Theorem 1. We assume that node 5 in Fig. 1(a) is the malicious agent evolving with

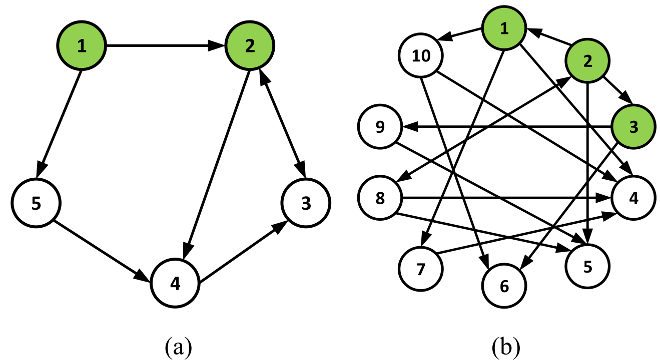
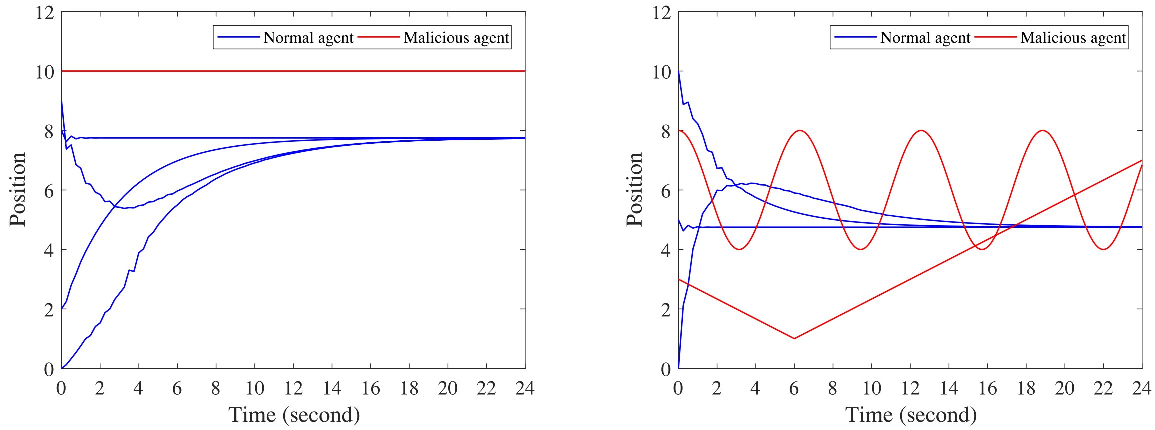


Fig. 1. (a) Digraph $\mathcal{G}_1$, where the subgraph $\mathcal{G}_{\mathcal{T}_1}$ induced by the set of trusted agents $\mathcal{V}_{\mathcal{T}_1} = \{1, 2\}$ satisfies the conditions i) and ii) in Theorem 1. (b) Digraph $\mathcal{G}_2$, where the subgraph $\mathcal{G}_{\mathcal{T}_2}$ induced by the set of trusted agents $\mathcal{V}_{\mathcal{T}_2} = \{1, 2, 3\}$ satisfies the conditions i) and ii) in Theorem 1.

$x_5(t) = 10$. The initial condition of normal agents is assumed to be $[x_1(t), x_2(t), x_3(t), x_4(t); v_1(t), v_2(t), v_3(t), v_4(t)] = [8, 2, 0, 9; 3, 4, 1, 6], t \in [-1, 0]$ s. Based on (8), the safety interval $\Delta = [-0.75, 9]$. Under the proposed resilient delayed impulsive control, the corresponding trajectories are shown in Fig. 2(a). We can see that the agreement condition and safety condition in Definition 1 are both satisfied in this case. Furthermore, we consider the case that nodes 4 and 5 in Fig. 1(a) are the malicious agents evolving with $x_4(t) = 2\cos t + 6$ and $x_5(t) = 1 + |t - 6|/3$. The initial condition of normal agents $[x_1(t), x_2(t), x_3(t); v_1(t), v_2(t), v_3(t)] = [5, 10, 0; 3, 4, 1], t \in [-1, 0]$s. Thus, it is obtained that the safety interval $\Delta = [-0.5, 10]$. From Fig. 2(b), it is obvious that the agreement condition and safety condition are also both satisfied.

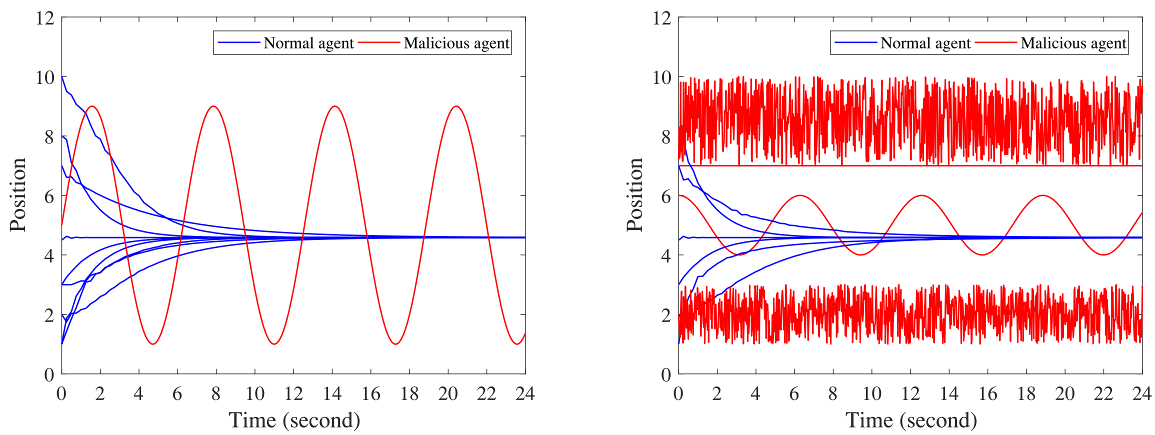
In order to further confirm the correctness of the sufficiency. The digraph $\mathcal{G}_2$ shown in Fig. 1(b) is considered as the communication topology of MAN (1). Node 10 is assumed to be the malicious agent evolving with $x_{10}(t) = 5 + 4\sin t$. The initial condition of normal agents is $[8, 4.5, 3, 1, 7, 2, 10, 1, 3; -6, -1, 1, 2, 1.5, 3, 0, 0, 0]$, $t \in [-1, 0]$s. By calculation, we have $\Delta = [0.625, 10.75]$. The trajectories in Fig. 3(a) illustrate that the agreement condition and safety condition are both satisfied. Next, we assume that nodes 7–10 in Fig. 1(b) are the malicious agents. It is assumed that they, respectively, evolve with $x_7(t) = 7$, $x_8(t) = rand(t)[1, 3]$, $x_9(t) = 5 + \cos t$, and $x_{10}(t) = rand(t)[7, 10]$. The initial condition of normal agents is $[8, 4.5, 3, 1, 7, 2; -6, -1, 1, 2, 1.5, 3]$, $t \in [-1, 0]$s. Thus, we have $\Delta = [0.625, 8.75]$ from (8). Under the resilient delayed impulsive control, the trajectories of MAN (1) are shown in Fig. 3(b). As expected, the agreement condition and safety condition are also both satisfied.

Next, we confirm the necessity of the conditions i) and ii) in Theorem 1 for achieving resilient consensus. For the digraph $\mathcal{G}_2$ in Fig. 1(b), we assume that node 10 is the malicious agent evolving with $x_{10}(t) = 5+4\sin t$. If the directed edge (2, 3) in digraph $\mathcal{G}_2$ is removed, then only condition ii) in Theorem 1 is satisfied. In this case, the trajectories of MAN (1) are shown in Fig. 4(a). Obviously, the agreement condition is not satisfied. Therefore, condition i) in Theorem 1 is necessary for



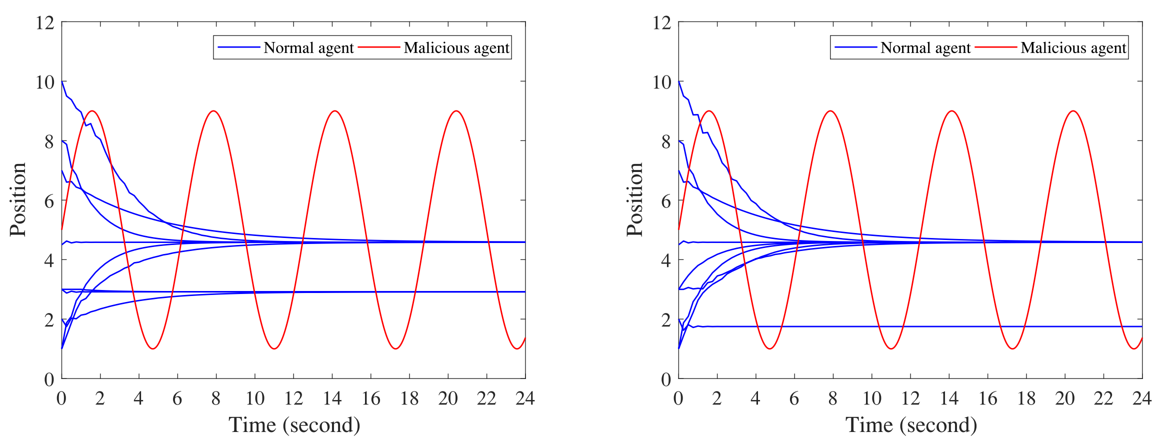
(a) (b)

Fig. 2. (a) Trajectories of MAN (1) cooperating on Fig. 1(a), where node 5 is assumed to be the malicious agent evolving with $x_5(t) = 10$. (b) Trajectories of MAN (1) cooperating on Fig. 1(a), where nodes 4 and 5 are assumed to be the malicious agents evolving with $x_4(t) = 2\cos t + 6$, $x_5(t) = 1 + |t - 6|/3$.



(a) (b)

Fig. 3. (a) Trajectories of MAN (1) cooperating on Fig. 1(b), where node 10 is assumed to be the malicious agent evolving with $x_{10}(t) = 5 + 4\sin t$. (b) Trajectories of MAN (1) cooperating on Fig. 1(b), where nodes 7–10 are assumed to be the malicious agents evolving with $x_7(t) = 7$, $x_8(t) = rand(t)[1, 3]$, $x_9(t) = 5 + \cos t$, and $x_{10}(t) = rand(t)[7, 10]$.



(a) (b)

Fig. 4. (a) Trajectories of MAN (1) cooperating on the digraph, where $\mathcal{G}_{\mathcal{T}}$ only satisfies the condition ii) in Theorem 1. (b) Trajectories of MAN (1) cooperating on the digraph, where $\mathcal{G}_{\mathcal{T}}$ only satisfies the condition i) in Theorem 1.

achieving resilient consensus of MAN (1). If we remove the directed edge $(3, 6)$ in digraph $\mathcal{G}_2$, then only the condition i) in Theorem 1 is satisfied. After applying the resilient delayed impulsive control, we can see that the agreement condition is also not satisfied from Fig. 4(b). Therefore, condition ii) in Theorem 1 is also necessary for achieving resilient consensus.

In summary, the sufficiency and necessity of Theorem 1 are confirmed by the above simulation studies.

V. Conclusion

In this article, we have designed a novel resilient delayed impulsive control to achieve the resilient consensus of MAN (1), in which the number of the unexpected malicious agents is unknown. It has been proved that the resilient consensus can be achieved if and only if certain graphic conditions are satisfied. By introducing trusted agents, we have shown that the resilient consensus can be achieved even in the sparse communication topology. Simulation studies have been presented to validate the resilient delayed impulsive control.

Compared with the traditional resilient consensus algorithms, the proposed resilient delayed impulsive control algorithm does not require any information on the number of the malicious agents. However, the resilient delayed impulsive control also has some cons. For details, all agents are not treated equally. The received information from the trusted agents requires to be labeled, namely, whether an agent is trusted or not needs to be previously known. This deficiency needs to be overcome in the future.

It is of interest to further develop the proposed resilient consensus control algorithm, such as switching topologies, event-triggered communications, and quantization.

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