Partially Punctual Metric Temporal Logic is Decidable

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Metric Temporal Logic $\mathsf{MTL}[\mathsf{U}_I, \mathsf{S}_I]$ is one of the most studied real time logics. It exhibits considerable diversity in expressiveness and decidability properties based on the permitted set of modalities and the nature of time interval constraints I. Henzinger et al., in their seminal paper showed that the non-punctual fragment of MTL called MITL is decidable. In this paper, we sharpen this decidability result by showing that the partially punctual fragment of MTL (denoted PMTL) is decidable over strictly monotonic finite point wise time. In this fragment, we allow either punctual future modalities, or punctual past modalities, but never both together. We give two satisfiability preserving reductions from PMTL to the decidable logic $\mathsf{MTL}[\mathsf{U}_I]$. The first reduction uses simple projections, while the second reduction uses a novel technique of temporal projections with oversampling. We study the trade-off between the two reductions: while the second reduction allows the introduction of extra action points in the underlying model, the equisatisfiable $\mathsf{MTL}[\mathsf{U}_I]$ formula obtained is exponentially succinct than the one obtained via the first reduction, where no oversampling of the underlying model is needed. We also show that PMTL is strictly more expressive than the fragments $\mathsf{MTL}[\mathsf{U}_I, \mathsf{S}]$ and $\mathsf{MTL}[\mathsf{U}, \mathsf{S}_I]$.

I. Introduction

Metric Temporal Logic MTL is a well established logic useful for specifying quantitative properties of real time systems. The main modalities of MTL are U_I (read "until I") and S_I (read "since I"), where I is a time interval with end points in $\mathbb{N}$. These formulae are interpreted over timed behaviours or timed words. A formula $a\,\mathsf{U}_{[2,3]}b$ holds at a position i of a timed word iff there is a position j strictly in the future of i where b holds, and at all intermediate positions between i and j, a holds good; moreover, the difference in the time stamps of i and j must lie in the interval [2,3]. Similarly, $a\,\mathsf{S}_{[2,3]}b$ holds good at a point i iff there is a position j strictly in the past of i where b holds, and at all intermediate positions between i and j a holds; further, the difference in the time stamps between i and j lie in the interval [2,3]. The intervals I can be bounded of the form $\langle l, u\rangle$, or unbounded of the form $\langle l, \infty)$, with $l, u \in \mathbb{N}$, and $\langle$ represents left closed or left open, while $\rangle$ represents right closed or right open intervals. The unary modalities $\Diamond_I$ (read "fut I") and $\diamondminus_I$ (read "past I") are special cases of until and since: $\Diamond_I a = true\,\mathsf{U}_I a$ and $\diamondminus_I a = true\,\mathsf{S}_I a$.

The satisfiability question for various fragments of MTL has evoked lot of interest and work over the past years. In their seminal paper, Alur and Henzinger showed that the satisfiability of $\mathsf{MTL}[\mathsf{U}_I, \mathsf{S}_I]$ is undecidable, while the satisfiability of the "non-punctual" fragment MITL of $\mathsf{MTL}[\mathsf{U}_I, \mathsf{S}_I]$ is decidable. As the name suggests, the non-punctual fragment disallows punctual intervals I: these are intervals of the form $[t, t]$. The satisfiability of the future only fragment of MTL, viz., $\mathsf{MTL}[\mathsf{U}_I]$ was open for a long time, till Ouaknine and Worrell [12] showed its decidability via a reduction to 1-clock alternating timed automata. Even though the logic $\mathsf{MTL}[\mathsf{U}_I, \mathsf{S}]$ is more expressive than $\mathsf{MTL}[\mathsf{U}_I]$, it was shown to be decidable [3] by an equisatisfiable reduction to $\mathsf{MTL}[\mathsf{U}_I]$. The decidability of the unary fragment $\mathsf{MTL}[\Diamond_I, \diamondminus_I]$ has remained open for a long time, it was recently shown undecidable [7]. The only fragment whose decidability is unknown is thus, the "partially punctual fragment" of MTL, where we allow punctualities only in the future or in the past modalities, but never in both. The main result of this paper is the decidability of the partially punctual fragment of MTL for finite strictly monotonic timed words; our results can be adapted to work for weakly monotonic finite words.

II. Metric Temporal Logic

Let Σ be a finite set of propositions. A finite timed word over Σ is a tuple $\rho = (\sigma, \tau)$ where σ and τ are sequences $\sigma_1\sigma_2\ldots\sigma_n$ and $\tau_1\tau_2\ldots\tau_n$ respectively, with $\sigma_i \in 2^{\Sigma} - \emptyset$, and $t_i \in \mathbb{R}_{\geq 0}$ for $1 \leq i \leq n$. Let $dom(\rho)$ be the set of positions $\{1, 2, \ldots, n\}$ in the timed word. Let $\Sigma = \{a, b\}$. An example of a timed word is $(\{a, b\}, 0.3)(\{b\}, 0.7)(\{a\}, 1.1)$. ρ is strictly monotonic iff $t_i < t_{i+1}$ for all $i, i+1 \in dom(\rho)$. Otherwise, it is weakly monotonic. Given Σ, the formulae of MTL are built from Σ using boolean connectives and time constrained versions of the modalities U and S as follows:
$\varphi ::= a(\in \Sigma)\ |true\ |\varphi \wedge \varphi \mid \neg\varphi \mid \varphi\,\mathsf{U}_I\varphi \mid \varphi\,\mathsf{S}_I\varphi$
where I is an open, half-open or closed interval with end points in $\mathbb{N} \cup \{\infty\}$.

Formulae of MTL are interpreted over timed words over a chosen set of propositions. Let φ be an MTL formula. If φ is interpreted over timed words over Δ, then we say that φ is interpreted over Δ. Note that this is different from saying φ is built from a set of propositions Σ: this just means that the propositions in φ are taken from Σ.

Given a finite timed word ρ, and an MTL formula φ, in the pointwise semantics, the temporal connectives of φ quantify over a finite set of positions in ρ. For an alphabet Σ, a timed word $\rho = (\sigma, \tau)$, a position $i \in dom(\rho)$, and an MTL formula φ, the satisfaction of φ at a position i of ρ is denoted $(\rho, i) \models \varphi$, and is defined as follows:

$\rho, i \models a \leftrightarrow a \in \sigma_i$
$\rho, i \models \neg\varphi \leftrightarrow \rho, i \not\models \varphi$
$\rho, i \models \varphi_1 \wedge \varphi_2 \leftrightarrow \rho, i \models \varphi_1$ and $\rho, i \models \varphi_2$
$\rho, i \models \varphi_1 \mathsf{U}_I \varphi_2 \leftrightarrow \exists j > i,\ \rho, j \models \varphi_2, t_j - t_i \in I,$ and $\rho, k \models \varphi_1\ \forall\ i < k < j$
$\rho, i \models \varphi_1 \mathsf{S}_I \varphi_2 \leftrightarrow \exists\ j < i,\ \rho, j \models \varphi_2,\ t_i - t_j \in I,$ and $\rho, k \models \varphi_1\ \forall\ j < k < i$

ρ satisfies φ denoted $\rho \models \varphi$ iff $\rho, 1 \models \varphi$. Let $L(\varphi) = \{\rho \mid \rho, 1 \models \varphi\}$. The set of all timed words over Σ is denoted $T\Sigma^*$.

A non-punctual interval has the form $\langle a, b\rangle$ with $a \neq b$. We denote by $\mathsf{MTL}[\mathsf{U}_I, \mathsf{S}_{np}]$ the class of MTL formulae with non-punctual past modalities. Similarly, $\mathsf{MTL}[\mathsf{U}_{np}, \mathsf{S}_I]$ is the class of MTL formulae with non-punctual future modalities. The class of partially punctual MTL formulae, PMTL consists of all formulae with non-punctual future or non-punctual past. $\mathsf{PMTL} = \mathsf{MITL} \cup \mathsf{MTL}[\mathsf{U}_{np}, \mathsf{S}_I] \cup \mathsf{MTL}[\mathsf{U}_I, \mathsf{S}_{np}]$.

Additional temporal connectives are defined in the standard way: we have the constrained future and past eventuality operators $\Diamond_I a \equiv true\, \mathsf{U}_I a$ and $\Diamond\!\!\!\!-_I a \equiv true\, \mathsf{S}_I a$, and their duals $\Box_I a \equiv \neg\Diamond_I \neg a$, $\boxminus_I a \equiv \neg\Diamond\!\!\!\!-_I \neg a$. Weak versions of operators are defined as : $\Diamond^{\mathsf{w}} a = a \vee \Diamond a, \Box^{\mathsf{w}} a = a \wedge \Box a$, $a\, \mathsf{U}^{\mathsf{w}} b = b \vee [a \wedge (a\, \mathsf{U} b)]$.

III. Temporal Projections

In this section, we discuss the notion of "temporal projections" that are central to this paper. We discuss two kinds of temporal projections: simple projections, and oversampling projections.

A. Simple Extensions and Projections

(Σ, X)-*simple extensions*: Let Σ, X be finite sets of propositions such that $\Sigma \cap X = \emptyset$. A (Σ, X)-simple extension is a timed word ρ over $X \cup \Sigma$ such that at any point $i \in dom(\rho)$, $\sigma_i \cap \Sigma \neq \emptyset$. For $\Sigma = \{a, b\}, X = \{c, d\}$, $(\{a\}, 0.2)(\{b, c, d\}, 0.3)(\{b, d\}, 1.1)$ is a (Σ, X)-simple extension. However, $(\{a\}, 0.2)(\{c, d\}, 0.3)(\{b, d\}, 1.1)$ is not a (Σ, X)-simple extension for the same choice of Σ, X, since for the position $i = 2$, $\{c, d\} \cap \Sigma = \emptyset$.

Simple Projections: Consider a (Σ, X)-simple extension ρ. We define the *simple projection* of ρ with respect to X, denoted $\rho \setminus X$ as the word obtained by erasing the symbols of X from each σ_i. Note that $dom(\rho) = dom(\rho \setminus X)$. For example, if $\Sigma = \{a, c\}$, $X = \{b\}$, and $\rho = (\{a, b, c\}, 0.2)(\{b, c\}, 1)(\{c\}, 1.3)$, then $\rho \setminus X = (\{a, c\}, 0.2)(\{c\}, 1)(\{c\}, 1.3)$. $\rho \setminus X$ is thus, a timed word over Σ. If the underlying word ρ is *not* a (Σ, X)-simple extension, then the simple projection of ρ with respect to X is *undefined*.

Equisatisfiability modulo Simple Projections: Given MTL formulae ψ and ϕ, we say that ϕ is equisatisfiable to ψ *modulo simple projections* iff there exist disjoint sets Σ, X such that

1) ϕ is interpreted over Σ, and ψ is interpreted over $\Sigma \cup X$,
2) For any timed word ρ over $\Sigma \cup X$, $(\rho \models \psi) \rightarrow$ ρ is a (Σ, X)-simple extension and $\rho \setminus X \models \phi$,
3) For any timed word ρ over Σ such that $\rho \models \phi$, $\exists$ a (Σ, X)-simple extension ρ' such that $\rho' \models \psi$, and $\rho' \setminus X = \rho$.

We denote by $\phi = \exists X.\psi$, the fact that ϕ is equisatisfiable to ψ modulo simple projections.

Extended Normal Form(ENF): Given a formula φ built from $\Sigma' \supseteq \Sigma$, the extended normal form of φ with respect to Σ denoted $ENF_\Sigma(\varphi)$ is the formula $\varphi \wedge \Box(\bigvee \Sigma)$.

Lemma 1 (Boolean Closure Lemma). *Let φ_1, φ_2 be formulae built from Σ. Let ψ_1, ψ_2 be formulae built from $\Sigma \cup X_1$ and $\Sigma \cup X_2$ respectively. Let $\Sigma_i = \Sigma \cup X_i$ for $i = 1, 2$, and let $X_1 \cap X_2 = \emptyset$. Then, $(\varphi_1 = \exists X_1.\psi_1$ and $\varphi_2 = \exists X_2.\psi_2) \rightarrow \varphi_1 \wedge \varphi_2 = \exists(X_1 \cup X_2).(\psi_1 \wedge \psi_2)$.*

Proof. The proof can be found in Appendix A. ☐

B. Flattening

Let $\varphi \in \mathsf{MTL}[\mathsf{U}_I, \mathsf{S}_I]$ built from Σ. Given any sub-formula ψ_i of φ, and a fresh symbol $b_i \notin \Sigma$, $T_i = \Box^{\mathsf{w}}(\psi_i \leftrightarrow b_i)$ is called a *temporal definition* and b_i is called a *witness*. Let $\psi = \varphi[b_i/\psi_i]$ be the formula obtained by replacing all occurrences of ψ_i in φ, with the witness b_i. Flattening is done recursively until we have replaced all future/past modalities of interest with witness variables, obtaining $\varphi_{flat} = \psi \wedge T$, where T is the conjunction of all temporal definitions. Let W be the set of all witness propositions. For example, consider the formula $\varphi = a\, \mathsf{U}_{[0,3]}(c\, \mathsf{S}(\Diamond\!\!\!\!-_{[0,1]} d))$. Replacing the $\mathsf{S}, \Diamond\!\!\!\!-$ modalities with witness propositions w_1 and w_2 we get $\psi = a\, \mathsf{U}_{[0,3]} w_1$, along with the temporal definitions $T_1 = \Box^{\mathsf{w}}(w_1 \leftrightarrow (c\, \mathsf{S} w_2))$ and $T_2 = \Box^{\mathsf{w}}(w_2 \leftrightarrow \Diamond\!\!\!\!-_{[0,1]} d)$. Hence, $\varphi_{flat} = \psi \wedge T_1 \wedge T_2$ is obtained by flattening the $\mathsf{S}, \Diamond\!\!\!\!-$ modalities from φ. Here $W = \{w_1, w_2\}$. Note that φ_{flat} is a formula built from $\Sigma \cup W$.

Given a timed word ρ over Σ, flattening marks precisely positions in ρ satisfying ψ_i with witnesses b_i. This marked word ρ' over $\Sigma \cup W$ satisfies φ_{flat} iff $\rho \models \varphi$. Hence, we have $\varphi = \exists W.ENF_\Sigma(\varphi_{flat})$. $ENF_\Sigma(\varphi_{flat})$ ensures that any timed word ρ' over $\Sigma \cup W$ that satisfies φ_{flat} is indeed a (Σ, W)-simple extension. $L(ENF_\Sigma(\varphi_{flat}))$ is the set of all those (Σ, W)-simple extensions ρ' satisfying φ_{flat} such that $\rho' \setminus W = L(\varphi)$.

C. Oversampled Behaviours and Projections

(Σ, X)-*oversampled behaviours*: Let Σ, X be finite sets of propositions such that $\Sigma \cap X = \emptyset$. A (Σ, X)-oversampled behaviour is a timed word $\rho' = (\sigma', \tau')$ over $X \cup \Sigma$, such that $\sigma'_1 \cap \Sigma \neq \emptyset$ and $\sigma'_{|dom(\rho')|} \cap \Sigma \neq \emptyset$. For $\Sigma = \{a, b\}, X = \{c, d\}$, $(\{a\}, 0.2)(\{c, d\}, 0.3)(\{a, b\}, , 0.7)(\{b, d\}, 1.1)$ is a (Σ, X) oversampled behaviour, while $(\{a\}, 0.2)(\{c, d\}, 0.3)(\{c\}, 1.1)$ is not. If ρ is a (Σ, X)-oversampled behaviour, then points i where $\bigvee \Sigma$ is not true are called *non-action points*. Hence, in any (Σ, X)-oversampled behaviour, the first as well as the last points are action points.

Oversampled Projections: Given a (Σ, X)-oversampled behaviour $\rho' = (\sigma', \tau')$, we define the *oversampled projection* of ρ' with respect to Σ, denoted $\rho' \downarrow X$ as the timed word obtained by deleting points i for which $\sigma'_i \cap \Sigma = \emptyset$, and then erasing the symbols of X from the remaining points j ($\sigma'_j \cap \Sigma \neq \emptyset$). The result of oversampling, $\rho{=}\rho' \downarrow X$ is a timed word over Σ. If $\rho = \rho' \downarrow X$, there exists a strictly increasing function $f : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, m\}$ such that $n = |dom(\rho)|$, $m = |dom(\rho')|$, and

- $f(1) = 1$, $\sigma_1 = \sigma'_1 \cap \Sigma$, $\tau_1 = \tau'_1$, and
- $f(n) = m$, $\sigma_n = \sigma'_m \cap \Sigma$, $\tau_n = \tau'_m$, and
- For $1 \leq i \leq n-1$, $f(i) = j$ and $f(i+1) = k$ iff
  - $\sigma_i = \sigma'_j \cap \Sigma$, and $\tau_i = \tau'_j$,
  - $\sigma_{i+1} = \sigma'_k \cap \Sigma$, and $\tau_{i+1} = \tau'_k$,
  - For all $j < l < k$, $\sigma'_l \cap \Sigma = \emptyset$.

For $\rho' = (\{a\}, 0.2)(\{a, c\}, 0.7)(\{c\}, 0.9)(\{b, d\}, 1.1)$, a (Σ, X)-oversampled behaviour for $\Sigma = \{a, b\}, X = \{c, d\}$, we have $\rho' \downarrow X = (\{a\}, 0.2)(\{a\}, 0.7)(\{b\}, 1.1)$. We have $f : \{1, 2, 3\} \rightarrow \{1, 2, 3, 4\}$ with $f(1) = 1, f(2) = 2$, and $f(3) = 4$.

Equisatisfiability modulo Oversampled Projections: Given MTL formulae ψ and ϕ, we say that ϕ is equisatisfiable to ψ *modulo oversampled projections* iff there exist disjoint sets X, Σ such that

1) ϕ is interpreted over Σ, and ψ over $\Sigma \cup X$,
2) For any (Σ, X)-oversampled behaviour ρ', $\rho' \models \psi \rightarrow \rho' \downarrow X \models \phi$
3) For any timed word ρ over Σ such that $\rho \models \phi$, there exists a (Σ, X)-oversampled behaviour ρ' such that $\rho' \models \psi$, and $\rho' \downarrow X = \rho$.

We denote by $\phi = \exists \downarrow X.\psi$ the fact that ϕ is equisatisfiable to ψ modulo oversampled projections. The above conditions establish the existence of *some* (Σ, X)-oversampled behaviour ρ' corresponding to ρ that satisfies ψ, when ρ satisfies φ. If condition 3 above holds for all possible (Σ, X)-oversampled behaviours, i.e,

- if for any timed word ρ over Σ such that $\rho \models \varphi$, all (Σ, X)-oversampled behaviours ρ' for which $\rho' \downarrow X = \rho$ satisfy ψ,

then we say that φ and ψ are *equivalent modulo oversampled projections* and denote it by $\varphi = \forall \downarrow .\psi$

Oversampled Normal Form (ONF): Let ψ be a formula built from $\Sigma \cup X$. Let act denote $\bigvee \Sigma$. The *oversampled normal form* with respect to Σ of ψ denoted $ONF_\Sigma(\psi)$ is obtained by replacing recursively

- all subformulae of the form $a \in \Sigma$ by $a \wedge act$,
- all subformulae of the form $\phi_i \mathsf{U}_I \phi_j$ with $(act \rightarrow ONF_\Sigma(\phi_i)) \mathsf{U}_I (ONF_\Sigma(\phi_j) \wedge act)$,
- all subformulae of the form $\phi_i \mathsf{S}_I \phi_j$ with $(act \rightarrow ONF_\Sigma(\phi_i)) \mathsf{S}_I (ONF_\Sigma(\phi_j) \wedge act)$.
- all subformulae of the form $\Box_I \phi$ with $\Box_I(act \rightarrow ONF_\Sigma(\phi))$, and all subformulae of the form $\Diamond_I \phi$ with $\Diamond_I(\phi \wedge act)$.

and conjuncting the resultant formulae with $act \wedge (\Box\bot \rightarrow act)$.

Let $\psi = \varphi_1 \mathsf{U}_I (\varphi_2 \wedge \Box\varphi_3)$, and $\zeta_i{=}ONF_\Sigma(\varphi_i)$ for $i{=}1, 2, 3$. Then $ONF_\Sigma(\psi){=}(act{\rightarrow}\zeta_1) \mathsf{U}_I (act \wedge [\zeta_2 \wedge \Box(act{\rightarrow}\zeta_3)]) \wedge act \wedge (\Box\bot{\rightarrow}act)$ where act denotes $\bigvee \Sigma$. Proofs of Lemmas 2, 3 and 4 can be found in Appendices B, C and D.

Lemma 2 (Oversampling Closure Lemma)**.** *Let φ be a formula built from Σ. Then $\varphi = \forall \downarrow .ONF_\Sigma(\varphi)$.*

Lemma 3. *Let φ be a formula built from Σ and let $\zeta = ONF_\Sigma(\varphi)$. Then, $\zeta = \forall \downarrow \zeta$.*

Lemma 4. *Consider formulae φ_1, φ_2 built from Σ. Let ψ_1, ψ_2 be formulae built from $\Sigma \cup X_1$ and $\Sigma \cup X_2$ respectively. Let $X = X_1 \cup X_2$, $\Sigma_i = \Sigma \cup X_i$ for $i = 1, 2$, and $X_1 \cap X_2 = \emptyset$. Let $\zeta_1 = ONF_{\Sigma_1}(\psi_1)$ and $\zeta_2 = ONF_{\Sigma_2}(\psi_2)$. Then, $\varphi_1 = \exists \downarrow X_1.\zeta_1$ and $\varphi_2 = \exists \downarrow X_2.\zeta_2 \rightarrow$ $\varphi_1 \wedge \varphi_2 = \exists \downarrow X.(\zeta_1 \wedge \zeta_2)$.*

Lemma 5. *Let $\varphi \in \mathsf{MTL}[\mathsf{U}_I, \mathsf{S}_I]$ be built from Σ, and W be the set of witness variables obtained while flattening φ. Then $\varphi = \exists \downarrow W.ONF_\Sigma(\varphi_{flat})$.*

IV. Decidability of $\mathsf{MTL}[\mathsf{U}_I, \mathsf{S}_{np}]$

In this section, we show that the class $\mathsf{MTL}[\mathsf{U}_I, \mathsf{S}_{np}]$ is decidable, by giving a satisfiability preserving reduction to $\mathsf{MTL}[\mathsf{U}_I]$. Given a timed word ρ, and a non-singular past modality of the form $\psi = \diamondminus_{\langle l,u\rangle}\varphi$, Lemma 6 establishes a relationship between time stamps of the points in ρ where ψ holds and the time stamps of points where φ holds in ρ with respect to l, u.

Lemma 6. *Given a timed word $\rho = (\sigma, \tau)$ and a point $i \in dom(\rho)$. Let $first_\alpha$ and $last_\alpha$ denote respectively the first and last occurrences of $\alpha \in \Sigma$ in ρ. $\rho, i \models \neg(\diamondminus_{\langle l,u\rangle}\alpha)$ iff*

(a) *$\tau_i \sim_1 \tau_{first_\alpha} + l$, where $\sim_1$ is $<$ when $\langle$ is $[$, and $\sim_1$ is $\leq$ when $\langle$ is $($, or*
(b) *$\tau_i \sim_2 \tau_{last_\alpha} + u$, where $\sim_2$ is $>$ when $\rangle$ is $]$, and $\sim_2$ is $\geq$ when $\rangle$ is $)$,or*
(c) *$\tau_i \in \langle \tau_j + u, \tau_k + l\rangle$ for all points $j, k (j < k)$ where α holds consecutively (that is there does not exist any point z, $j < z < k$ where α holds). Note that in this case $\tau_j + u \leq \tau_k + l$.*

Proof. We prove the lemma for intervals of the form $[l, u)$. The proof can be extended for other type of intervals also. Assume that $\rho, i \models \diamondminus_{[l,u)}\alpha$. We then show that $\neg(\tau_i < \tau_{first_\alpha} + l)$ and $\neg(\tau_i \geq \tau_{last_\alpha} + u)$ and $\neg(\tau_i \in [\tau_j + u, \tau_k + l))$ for consecutive points j, k where α holds.

1) Let $\tau_i < \tau_{first_\alpha} + l$. $\rho, i \models \diamondminus_{[l,u)}\alpha$ implies that there is a point i' such that $\tau_{i'} \in (\tau_i - u, \tau_i - l]$, such that $\rho, i' \models \alpha$. Then, $\tau_{i'} \leq \tau_i - l < \tau_{first_\alpha}$, contradicting that $first_\alpha$ is the first point where α holds.
2) Let $\tau_i \geq \tau_{last_\alpha} + u$. Again, $\rho, i \models \diamondminus_{[l,u)}\alpha$ implies that there is a point i' such that $\tau_{i'} \in (\tau_i - u, \tau_i - l]$ such that $\rho, i' \models \alpha$. We then have $\tau_{i'} > \tau_i - u \geq \tau_{last_\alpha}$, contradicting that $last_\alpha$ is the last point where α holds.
3) Assume that there exist consecutive points $j < k$ where α holds. Also, let $\tau_i \in [\tau_j + u, \tau_k + l)$. $\rho, i \models \diamondminus_{[l,u)}\alpha$ implies that there exists a point i' such that $\tau_{i'} \in (\tau_i - u, \tau_i - l]$ and $\rho, i' \models \alpha$. Also, $\tau_i - u \in [\tau_j, \tau_k + (l - u))$ and $\tau_i - l \in [\tau_j + (u - l), \tau_k)$. This gives $\tau_j < \tau_{i'} < \tau_k$

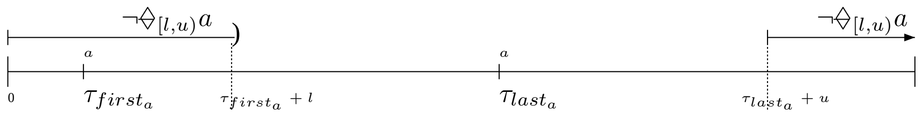

Fig. 1. Cases (a) and (b) of Lemma 6 : $\neg\diamondminus_{[l,u)} a$ holds in $[0, \tau_{first_a} + l)$ and $[\tau_{last_a} + u, \infty)$

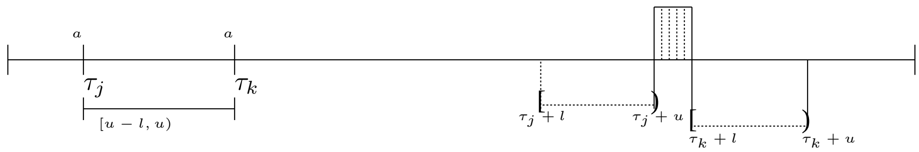

Fig. 2. Case (c) Lemma 6: $\neg\diamondminus_{[l,u)} a$ holds in shaded region

contradicting the assumption that j, k are consecutive points where α holds.

The converse can be found in Appendix E. Figure 1 illustrates regions for cases (a) and (b), while Figure 2 illustrates the region for case (c). In the rest of the paper, we refer to regions in case(a) as Region I, regions in case(b) as Region II and regions in case (c) as Region III. □

In the rest of this section, we show the decidability of $\mathsf{MTL}[\,\mathsf{U}_I, \diamondminus_{np}]$ by reducing any formula $\varphi \in \mathsf{MTL}[\,\mathsf{U}_I, \diamondminus_{np}]$ to a formula $\psi \in \mathsf{MTL}[\,\mathsf{U}_I]$. We have two techniques for this proof: one using oversampling projections, and the other, using simple projections.

A. Elimination of Past with Oversampled Projections

In this section, given a formula φ in $\mathsf{MTL}[\,\mathsf{U}_I, \diamondminus_{np}]$ built from Σ, we synthesize a formula $\psi \in \mathsf{MTL}[\,\mathsf{U}_I]$ built from $\Sigma \cup X$ equisatisfiable to φ *modulo oversampled projections*, whose size is *linear* in $|\varphi|$. Starting with a timed word ρ over Σ, we synthesize an (Σ, X)-oversampled behaviour ρ' such that $\rho \models \varphi$ iff $\rho' \models \psi$.

1) Start with a formula $\varphi \in \mathsf{MTL}[\,\mathsf{U}_I, \diamondminus_{np}]$ built from Σ, and a timed word ρ over Σ,
2) Flatten φ obtaining φ_{flat}. Let W be the witness propositions used. φ_{flat} is a formula built from $\Sigma \cup W$, with $\Sigma \cap W = \emptyset$.
3) Let $T = \bigwedge_{i=1}^{k} T_i$ be the conjunction of all temporal definitions in φ_{flat}. Each T_i has the form $\Box^{\mathsf{w}}(b \leftrightarrow \diamondminus_{\langle l,u\rangle} a)$, with $l, u \in \mathbb{R}_{\geq 0} \cup \{\infty\}$, and $\bigwedge_{i=1}^{k} T_i$ is built from $\Sigma \cup W$. $\varphi_{flat} = \psi \wedge T$, with $\psi \in \mathsf{MTL}[\,\mathsf{U}_I]$. We know from Lemma 5 that $\varphi = \exists \downarrow W.ONF_\Sigma(\varphi_{flat})$.
4) For $i = 1, 2, \ldots, k$, let $\Sigma_i = \Sigma \cup W \cup X_i$, where X_i are a set of fresh propositions, such that $X_i \cap X_j = \emptyset$ for $i \neq j$. Synthesize a formula $\zeta_i = ONF_{\Sigma_i}(\varphi'_i) \in \mathsf{MTL}[\mathsf{U}_I]$ over Σ_i such that $ONF_\Sigma(T_i) = \exists \downarrow X_i.\zeta_i$.
5) Using Lemma 4, $\bigwedge_{i=1}^{k} \zeta_i \in \mathsf{MTL}[\,\mathsf{U}_I]$ is such that $ONF_\Sigma(\bigwedge_{i=1}^{k} T_i) = \exists \downarrow X. \bigwedge_{i=1}^{k} \zeta_i$, for $X = \bigcup_{i=1}^{k} X_i$.

Lemma 7 and Lemma 8 show how to synthesize an equisatisfiable formula in $\mathsf{MTL}[\,\mathsf{U}_I]$ corresponding to $ONF_\Sigma(T_i)$. Lemma 7 shows step 4 for intervals of the form $[l, \infty)$, while Lemma 8 shows step 4 for bounded intervals of the form $[l, u)$. The results of these lemmas can be extended to work for any interval $\langle l, u\rangle$. If all the past modalities involved have unbounded intervals, then we get an equivalent formula, as shown by Lemma 7.

Lemma 7. *Consider a temporal definition $T = \Box^{\mathsf{w}}[b \leftrightarrow \diamondminus_{[l,\infty)} a]$ built from $\Sigma \cup W$. Then we can synthesize a formula $\psi \in \mathsf{MTL}[\,\mathsf{U}_I]$ built from $\Sigma \cup W$ equivalent to $ONF_\Sigma(T)$.*

Proof. It can be shown that $[\Box^{\mathsf{w}}\alpha \vee \{\alpha\ \mathsf{U}^{\mathsf{w}}[(a \wedge act) \wedge \Box^{\mathsf{w}}_{[0,l)}(act \rightarrow \neg b)]\}]\Box^{\mathsf{w}}[(a \wedge act) \rightarrow \Box_{[l,\infty)}(act \rightarrow b)]$[1] is equivalent to $ONF_\Sigma(T)$, for $\alpha = (act \rightarrow (\neg a \wedge \neg b))$. Details in Appendix F. □

Lemma 8. *Consider a temporal definition $T = \Box^{\mathsf{w}}[b \leftrightarrow \diamondminus_{[l,u)} a]$, built from $\Sigma \cup W$. Then we can synthesize a formula $\psi \in \mathsf{MTL}[\,\mathsf{U}_I]$ built from $\Sigma \cup W \cup X$ linear in the size of $ONF_\Sigma(T)$, such that $ONF_\Sigma(T) = \exists \downarrow X.\psi$.*

Proof. We start with $ONF_\Sigma(T)$ and a (Σ, W) oversampled behaviour ρ'. Let $dom(\rho') = \{1, 2, \ldots, n\}$. If there exists a point $i \in dom(\rho')$ marked $act \wedge a$, then we want to ensure that all points j in $dom(\rho')$ marked act such that $\tau'_j \in [\tau'_i + l, \tau'_i + u)$ are marked b. This is enforced by the following formula:

- $\mathsf{MARK}_b : \Box^{\mathsf{w}}[(a \wedge act) \rightarrow \Box_{[l,u)}(act \rightarrow b)]$

MARK_b enforces the direction $act \rightarrow (\diamondminus_{[l,u)}(a \wedge act) \rightarrow b)$ of $ONF_\Sigma(T)$. Marking points of ρ' with $\neg b$ is considerably more involved. We use Lemma 6 to characterize the points where $\neg\diamondminus_{[l,u)} a$ holds, and use this to ensure that such points are marked $\neg b$. Recall that by Lemma 6, such points can be classified into three regions.

Region I consists of all those points to the left of $\tau_{first_a} + l$. In any model, these points are described by the formula $\mathsf{MARK}_{first} = \Box^{\mathsf{w}}(\neg a \wedge \neg b) \vee (\neg a \wedge \neg b)\ \mathsf{U}^{\mathsf{w}}(a \wedge \Box^{\mathsf{w}}_{[0,l)} \neg b)$[2], which says that there are no b's in $[0, \tau_{first_a} + l)$. Region II consists of all points in $[\tau_{last_a} + u, \infty)$. In any model, these points are captured by the formula $\mathsf{MARK}_{last} = \Box^{\mathsf{w}}(\Box\neg a \rightarrow \Box_{[u,\infty)} \neg b)$, which says that there are no b's in $[\tau_{last_a} + u, \infty)$.

Let us now discuss how to mark points lying in region III with $\neg b$. Recall that these are the points in $[\tau_j + u, \tau_k + l)$ for any two consecutive points j, k such that $a \in \sigma_j, \sigma_k$, but $a \notin \sigma_h, j < h < k$. Consider j, k as two consecutive points where a holds. If $\tau_k - \tau_j \leq u - l$, then clearly, there are no

[1] when $l = 0$, $\alpha\ \mathsf{U}^{\mathsf{w}}[a \wedge act \wedge \neg b]$

[2] when $l = 0$, $\Box^{\mathsf{w}}[(\neg a \wedge \neg b) \vee [(\neg a \wedge \neg b)\ \mathsf{U}^{\mathsf{w}}(a \wedge \neg b)]]$

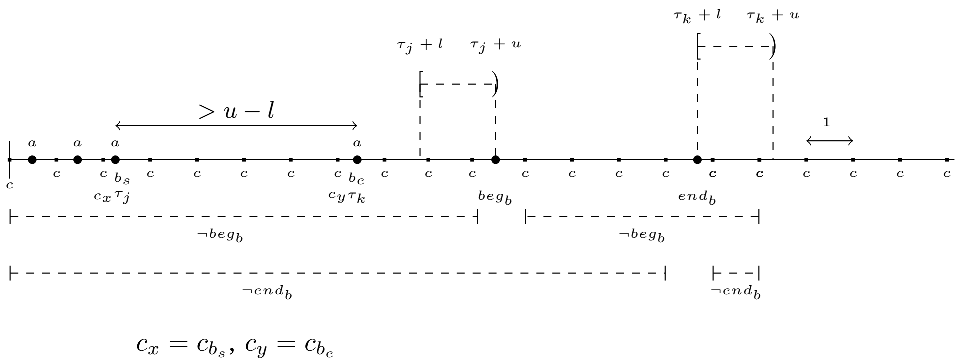

Fig. 3. Marking $[\tau_j + u, \tau_k + l)$ with $\neg b$

points in $[\tau_j + u, \tau_k + l)$ to be marked $\neg b$. Assume now that $\tau_k - \tau_j > u - l$. We need to mark exactly the points falling in $[\tau_j + u, \tau_k + l)$ with $\neg b$. It is quite possible that, we dont have the points g, h in $dom(\rho')$ such that $\tau_g = \tau_j + u$ and $\tau_h = \tau_k + l$. Here, we use the idea of oversampled projections, to obtain a behaviour ρ'' from ρ', by adding extra points to $dom(\rho')$. Corresponding to every pair j, k of consecutive a points, such that $\tau_k - \tau_j > u - l$, we add points x, y to $dom(\rho')$, such that $\tau_x = \tau_j + u$ and $\tau_y = \tau_k + l$. We mark these new points with fresh propositions beg_b and end_b respectively. We then say that between beg_b and end_b, no b can occur. To pindown the points x, y correctly, we mark the points j, k respectively with fresh propositions b_s and b_s.

To summarize the marking scheme, given a (Σ, W)-oversampled behaviour ρ' satisfying $ONF_\Sigma(T)$, where $T = \Box^{\mathsf{w}}[b \leftrightarrow \diamondminus_{[l,u)} a]$, we construct a $(\Sigma \cup W, X)$-oversampled behaviour ρ'' from ρ', such that

- ρ'' is obtained by introducing extra points to $dom(\rho')$. These extra points are related to consecutive a points $j, k \in dom(\rho')$, when $\tau_k - \tau_j > u - l$. For such $j, k \in dom(\rho')$, we add points x, y to $dom(\rho'')$ such that $\tau_x = \tau_j + u$ and $\tau_y = \tau_k + l$. The fresh propositions used so far, consists of symbols $\{b_s, b_e, beg_b, end_b\} \subseteq X$.
- Symbols b_s and b_e represent the "start" and "end" positions j, k. Thus, b_s holds at a point where $a \wedge act$ is true, and where the next consecutive occurrence of a is $> u - l$ distance apart. Similarly, b_e holds at a point where $a \wedge act$ is true, and where the previous occurrence of a is $> u - l$ distance apart. Once we mark τ_j with b_s and τ_k with b_e, the points at $\tau_j + u$ and $\tau_k + l$ are marked beg_b and end_b respectively. Once we have the points beg_b and end_b marked, we assert that between any consecutive pair of beg_b and end_b, all points of ρ' are marked $\neg b$.
- We need to make sure that the beg_b and end_b occurring in ρ'' are legitimate with respect to b_s and b_e: That is, there must be no "free occurrence" of beg_b and end_b. Any occurrence of beg_b and end_b should witness b_s and b_e at exactly u and l distance in the past respectively. This can be done adding extra points at all integer timestamps and restricting the free occurrences of beg_b, end_b in every unit interval.

Now we write formulae in $\mathsf{MTL}[\,\mathsf{U}_I]$ that implement the above, which will hold good on the $(\Sigma \cup W, X)$-oversampled behaviour ρ'' from ρ'.

- Mark b_s and b_e at points j and k: The conjunction of the following two formulae is denoted $\mathsf{MARK}_{j,k}$.
  $\Box^{\mathsf{w}}(b_s \leftrightarrow (a \wedge act \wedge (act \rightarrow \neg a)\ \mathsf{U}_{(u-l,\infty)}(a \wedge act)))$,
  $\Box^{\mathsf{w}}(b_e \leftrightarrow (a \wedge act \wedge (act \rightarrow \neg a)\ \mathsf{S}(b_s \wedge act)))$[3]
- Mark beg_b and end_b appropriately at $\tau_j + u$ and $\tau_k + l$ respectively. The conjunction of the following two formulae is denoted $\mathsf{MARK}_{beg,end}$.
  $\Box^{\mathsf{w}}(b_s \leftrightarrow (\Diamond^{\mathsf{w}}_{[0,u)}\Box\bot \vee [\Box_{(u,u+1)}\neg beg_b \wedge \Diamond_{[u,u+1)} beg_b \wedge \Box_{(u-1,u)}\neg beg_b]))$,
  $\Box^{\mathsf{w}}(b_e \leftrightarrow (\Diamond^{\mathsf{w}}_{[0,l)}\Box\bot \vee [\Box_{(l-1,l)}\neg end_b \wedge \Diamond_{(l-1,l]} end_b \wedge \Box_{(l,l+1)}\neg end_b]))$[4]
- Note that the above formula only asserts where beg_b and end_b should occur. We must assert that all other remaining points beg_b and end_b do not occur. This is done as follows:
  - First mark all integer timestamps with a fresh proposition c. The following formula is denoted MARK_c.
    $c \wedge \Box^{\mathsf{w}}(c \rightarrow [\Diamond_{(0,1)}\Box\bot \vee (\Box_{(0,1)}\neg c \wedge \Diamond_{(0,1]} c)]$
  - We identify the points between b_s and b_e by uniquely marking the closest integral point before b_s with c_{b_s} and and the closest integral point before b_e with c_{b_e}. Recall that b_s and b_e were marked at τ_j and τ_k; thus, c_{b_s} and c_{b_e} get marked respectively at points $\lfloor \tau_j \rfloor$ and $\lfloor \tau_k \rfloor$. We then assert that beg_b can occur at a point t only if there is a c_{b_s} in $(t - u - 1, t - u]$. Thus, given that c_{b_s} is marked at $\lfloor \tau_j \rfloor$, beg_b is marked only in $[\lfloor \tau_j \rfloor + u, \lfloor \tau_j \rfloor + u + 1)$. However, by formula $\mathsf{MARK}_{beg,end}$, we disallow beg_b in $(\tau_j + u, \tau_j + u + 1)$ and $(\tau_j + u - 1, \tau_j + u)$. Thus, we obtain a unique marking for beg_b. In a similar manner, we obtain a unique marking for end_b, given b_e. The conjunction of the following formulae denoted MARK_{c_b} marks

[3] S can be removed from $\mathsf{MTL}[\,\mathsf{U}_I, \mathsf{S}]$ obtaining equisatisfiable formula in $\mathsf{MTL}[\,\mathsf{U}_I]$ modulo simple projections [3], details in Appendix L

[4] when $l = 0$, $\Box^{\mathsf{w}}([b_e \leftrightarrow end_b] \wedge [b_e \rightarrow \neg b])$

c_{b_s} and c_{b_e}, and controls the marking of beg_b and end_b correctly:
$\Box^{\mathsf{w}}[c_{b_s} \leftrightarrow (c \wedge \Diamond^{\mathsf{w}}_{[0,1)} b_s)] \wedge \Box^{\mathsf{w}}[c_{b_e} \leftrightarrow (c \wedge \Diamond^{\mathsf{w}}_{[0,1)} b_e)]$
$\Box^{\mathsf{w}}[c \wedge \neg c_{b_s} \rightarrow \Box^{\mathsf{w}}_{[u,u+1)} \neg beg_b]$
$\Box^{\mathsf{w}}[c \wedge \neg c_{b_e} \rightarrow \Box^{\mathsf{w}}_{[l,l+1)} \neg end_b]$
Note that these formula do not restrict the behavior of beg_b and end_b in the prefix $[0, u]$. At these timepoints beg_b and end_b should not occur. Here we assert that $\Box^{\mathsf{w}}_{[0,u)}(\neg beg_b \wedge \neg end_b)$

- Now that we have precisely placed beg_b and end_b, we can assert at all points of ρ' between beg_b and end_b, $\neg b$ holds. This formula is denoted $\mathsf{MARK}_{\neg b}$.
$\Box^{\mathsf{w}}\{beg_b \rightarrow (\neg end_b \wedge (act \rightarrow \neg b))\, \mathsf{U}^{\mathsf{w}} end_b\}$

Figure 3 illustrates marking of $\neg beg_b$.

Let $\mathsf{MARK} = \mathsf{MARK}_b \wedge \mathsf{MARK}_{first} \wedge \mathsf{MARK}_c \wedge \mathsf{MARK}_{last} \wedge \mathsf{MARK}_{j,k} \wedge \mathsf{MARK}_{beg,end} \wedge \mathsf{MARK}_{\neg b} \wedge \mathsf{MARK}_{c_b}$.[5] Let $\Sigma_i = \Sigma \cup W \cup X$, for $X = \{b_e, b_s, beg_b, end_b, c, c_{b_s}, c_{b_e}\}$. Then, ρ'' is a $(\Sigma \cup W, X)$-oversampled behaviour such that $\rho'' \models ONF_{\Sigma_i}(\mathsf{MARK})$ iff $\rho' \models ONF_{\Sigma}(T)$. That is, $ONF_{\Sigma}(T) = \exists \downarrow X.ONF_{\Sigma_i}(\mathsf{MARK})$. A detailed proof of correctness can be seen in Appendix G. ☐

Theorem 1. *For every $\varphi \in \mathsf{MTL}[\,\mathsf{U}_I,\, \mathsf{S}_{np}]$ over Σ, we can construct ψ_{fut} in $\mathsf{MTL}[\,\mathsf{U}_I]$ over $\Sigma' \supseteq \Sigma$ such that $\varphi = \exists \downarrow X.\psi_{fut}$, $X = \Sigma' - \Sigma$.*

Proof. Follows from the fact that S_{np} can be expressed using S and $\diamondminus_{np}$[6] [3] and elimination of S [3], [8]. ☐

By symmetry, using reflection [8], the satisfiability of $\mathsf{MTL}[\,\mathsf{U}_{np},\, \mathsf{S}_I]$ can be reduced to the satisfiability of $\mathsf{MTL}[\,\mathsf{U}_I,\, \mathsf{S}_{np}]$. Hence, the satisfiability of $\mathsf{MTL}[\,\mathsf{U}_{np},\, \mathsf{S}_I]$ is also decidable.

B. Elimination of Past with Simple Projections

This section is devoted to showing that given any $\varphi \in \mathsf{MTL}[\mathsf{U}_I, \diamondminus_{np}]$ built from Σ, we can synthesize $\varphi' \in \mathsf{MTL}[\mathsf{U}_I]$ built from Σ' such that $\varphi = \exists X.\varphi'$, where $X = \Sigma' - \Sigma$. The main steps are similar to the case of oversampling projections. Here are the steps:

1) Start with a formula $\varphi \in \mathsf{MTL}[\,\mathsf{U}_I, \diamondminus_{np}]$ built from Σ, and a timed word ρ over Σ. Flatten φ obtaining $\varphi_{flat} = \psi \wedge \bigwedge_{i=1}^{k} T_i$. Each T_i is a temporal definition of the form $\Box^{\mathsf{w}}(b_i \leftrightarrow \diamondminus_{\langle l,u \rangle} a_i)$, and $\psi \in \mathsf{MTL}[\,\mathsf{U}_I]$. Let w_i be the fresh witness variable introduced in the temporal definition T_i. Let $W = \{w_1, \ldots, w_n\}$ be the set of all the witness variables.
2) As discussed in section III-B, $\varphi = \exists W.ENF_{\Sigma}(\varphi_{flat})$.
3) We now synthesize *modulo simple projections*, formulae in $\mathsf{MTL}[\,\mathsf{U}_I]$ equisatisfiable with $ENF_{\Sigma}(T_i)$ for $i = 1, 2, \ldots, k$, modulo simple projections.
4) Start with $ENF_{\Sigma}(T_1)$, a formula built from $\Sigma \cup W$. Let $\Sigma_1 = \Sigma \cup W$. We synthesize a formula $\varphi_1 \in \mathsf{MTL}[\,\mathsf{U}_I]$ built from $\Delta_1 = \Sigma \cup W \cup X_1$ such that $ENF_{\Sigma}(T_1) = \exists X_1 \varphi_1$.
5) Repeat step 5 for $ENF_{\Sigma}(T_i)$ for all $2 \leq i \leq k$, obtaining formulae $\varphi_i \in \mathsf{MTL}[\,\mathsf{U}_I]$ built from some $\Delta_i \supseteq \Sigma_1$ such that $ENF_{\Sigma}(T_i) = \exists X_i.\varphi_i$ in each case. The choice of Δ_i is such that $(\Delta_i - \Sigma_1) \cap (\Delta_j - \Sigma_1) = \emptyset$ for $i \neq j$.
6) Using Lemma 1, we obtain $ENF_{\Sigma}(\varphi_{flat}) = ENF_{\Sigma}(\psi \wedge \bigwedge_{i=1}^{k} T_i) = \exists X.[\psi \wedge \bigwedge_{i=1}^{k} \varphi_i]$, where $X = \bigcup_{i=1}^{k} X_i$. Then we get $\varphi = \exists W.ENF_{\Sigma}(\varphi_{flat}) = \exists W.[\exists X.(\psi \wedge \bigwedge_{i=1}^{k} \varphi_i)]$.
7) Steps 1-7 show that $\psi \wedge \bigwedge_{i=1}^{k} \varphi_i \in \mathsf{MTL}[\,\mathsf{U}_I]$ is equisatisfiable to φ modulo simple projections.

Lemma 9 explains how to eliminate temporal definitions of the form $\Box^{\mathsf{w}}[b \leftrightarrow \diamondminus_{\langle l,\infty)}(a)]$, while Lemma 10 explains how to eliminate temporal definitions of the form $\Box^{\mathsf{w}}[b \leftrightarrow \diamondminus_{\langle l,u \rangle}(a)]$. If all the past modalities involved have unbounded intervals, then we get an equivalent formula, as shown by Lemma 9.

Lemma 9. *Consider the temporal definition $T = \Box^{\mathsf{w}}[b \leftrightarrow \diamondminus_{[l,\infty)}(a)]$ built from $\Sigma \cup W$. Then we can synthesize a formula $\psi \in \mathsf{MTL}[\,\mathsf{U}_I]$ built from $\Sigma \cup W$ equivalent to T.*

Proof. It can be shown that $[\Box^{\mathsf{w}}(\neg a) \vee \Box^{\mathsf{w}}[a \rightarrow \Box_{[l,\infty)} b]] \wedge [\Box^{\mathsf{w}}(\neg a \wedge \neg b) \vee (\neg a \wedge \neg b)\, \mathsf{U}^{\mathsf{w}}(a \wedge \Box^{\mathsf{w}}_{[0,l)} \neg b)]$[7] is equivalent to T. Details can be found in Appendix I. ☐

Lemma 10. *Consider the temporal definition $T = \Box^{\mathsf{w}}[b \leftrightarrow \diamondminus_{\langle l,u \rangle}(a)]$ built from $\Sigma \cup W$. We can synthesize a formula $\psi \in \mathsf{MTL}[\mathsf{U}_I]$ built from $\Sigma \cup W \cup X$ such that $ENF_{\Sigma}(T) = \exists X.\psi$.*

Proof. We prove the lemma for temporal definitions of the form $T = \Box^{\mathsf{w}}[b \leftrightarrow \diamondminus_{[l,u)}(a)]$. The proof can be extended to all kinds of intervals $\langle l, u \rangle$.

Note that T is the conjunction of $C_1 = \Box^{\mathsf{w}}[b \leftarrow \diamondminus_{[l,u)} a]$ and $C_2 = \Box^{\mathsf{w}}[b \rightarrow \diamondminus_{[l,u)} a]$. Consider a timed word ρ over $\Sigma \cup W$. ρ satisfies C_1 iff, for all points $j \in dom(\rho)$, if there exists a point $i \in dom(\rho)$, with $\tau_i \in (\tau_j - u, \tau_j - l]$ and $a \in \sigma_i$, then $b \in \sigma_j$. Clearly, such models ρ are such that whenever $a \in \sigma_i$, then $b \in \sigma_j$ for all $j \in dom(\rho)$ such that $\tau_j \in [\tau_i + l, \tau_i + u)$. Let $\mathsf{MARK}_b = \Box^{\mathsf{w}}[a \rightarrow \Box_{[l,u)} b]$. Clearly, $\rho \models \mathsf{MARK}_b$ iff $\rho \models C_1$.

For a word ρ to satisfy T, the above conditions are not enough, since they only characterize points in the model where b hold. The formula $\mathsf{MARK}_b \in \mathsf{MTL}[\mathsf{U}_I]$ does not characterize points where b should not hold. Models satisfying MARK_b can allow a point where b as well as $\neg\diamondminus_{[l,u)} a$ holds. Our next goal is therefore, to find a formula $\mathsf{MARK}_{\neg b} \in \mathsf{MTL}[\,\mathsf{U}_I]$ which is equisatisfiable to C_2. Then $\mathsf{MARK}_b \wedge \mathsf{MARK}_{\neg b}$ is the formula in $\mathsf{MTL}[\,\mathsf{U}_I]$ that is equi-satisfiable to T.

We use Lemma 6 to characterize the points where $\neg\diamondminus_{[l,u)} a$ holds, and use this to ensure that such points are marked $\neg b$. Recall that by Lemma 6, such points can be classified into three regions. Points lying in Regions I,II are handled by the formulae $\mathsf{MARK}_{first}, \mathsf{MARK}_{last}$ given in Lemma 8. So far,

[5] when $l = 0$, conjunct $\Box^{\mathsf{w}}([a \wedge \Box_{[0,u)} \neg a \wedge \Diamond_{[0,u]} a] \rightarrow \neg b)$ to MARK

[6] For instance, we can write $a\,\mathsf{S}_{[l,r)} b$ as $\diamondminus_{[l,r)} b \wedge (a\,\mathsf{S} b) \wedge \boxminus_{[0,l)}(a \wedge a\,\mathsf{S} b)$, for $r = l+1, \infty$

[7] when $l = 0$, $[[\Box^{\mathsf{w}}(\neg a) \vee \Box^{\mathsf{w}}[a \rightarrow \Box_{[l,\infty)} b]] \wedge [\Box^{\mathsf{w}}(\neg a \wedge \neg b) \vee (\neg a \wedge \neg b)\, \mathsf{U}^{\mathsf{w}}(a \wedge \neg b)]$

we have the conjunction $\mathsf{MARK}_{first} \wedge \mathsf{MARK}_{last} \wedge \mathsf{MARK}_b$ of formulae in $\mathsf{MTL}[\,\mathsf{U}_I]$.

Region III consists of all points in $[\tau_j + u, \tau_k + l)$ for any pair of consecutive "a" points j, k ($a \in \sigma_j, \sigma_k$ and for all $j < h < k$, $a \notin \sigma_h$). The difficulty in marking points in $[\tau_j + u, \tau_k + l)$ with $\neg b$ is :

1) Points $p_1, p_2 \in dom(\rho)$ with $\tau_{p_1} = \tau_j + u$ and $\tau_{p_2} = \tau_k + l$ may not be present in $dom(\rho)$;
2) The length of the region $[\tau_j + u, \tau_k + l)$ may not be an integer. If it were, we can pin down points in $[\tau_j + u, \tau_k + l)$ by anchoring at points j, k since l, u are integers.

Unless we can pin down these points or mark this region uniquely, we may end up marking lesser points than necessary with $\neg b$ or may mark a point already marked b with $\neg b$, giving rise to inconsistencies. The rest of the proof is devoted to showing how we can indeed pin down the set of points between $\tau_j + u$ and $\tau_k + l$.

Since we may not have the points $\tau_j + u$ and $\tau_k + l$, we try to get points as close as possible to $\tau_j + u$ and $\tau_k + l$, by considering an over approximation of the interval $[\tau_j + u, \tau_k + l)$. The idea is to express $[\tau_j + u, \tau_k + l)$ as the intersection of two intervals $I^1_{j,k}$ and $I^2_{j,k}$, both having integer length, and such that it is possible to pin down $I^1_{j,k}$ and $I^2_{j,k}$. For this, we consider the intervals $I^1_{j,k} = [\tau_k + l - d, \tau_k + l)$ and $I^2_{j,k} = [\tau_j + u, \tau_j + u + d)$ where $d = \lceil \tau_k - \tau_j \rceil + (l - u)$. Note that d is the closest integer that is larger than the actual duration of the interval $[\tau_j + u, \tau_k + l)$. Also, $\tau_k + l - d \le \tau_j + u$ and $\tau_k + l \le \tau_j + u + d$. Hence, $[\tau_j + u, \tau_k + l) \subseteq I^1_{j,k} \cap I^2_{j,k}$. We now pin down points in the intersection $I^1_{j,k} \cap I^2_{j,k}$ and mark them $\neg b$. Towards getting the points in the intersection, we allow marking points $i \in dom(\rho)$ with fresh witness propositions, obtaining from ρ, a simple extension ρ'.

In the following, we explain the choice of these propositions, the marking scheme to obtain ρ', and formulae in $\mathsf{MTL}[\mathsf{U}_I, \mathsf{S}]$[8] which enforce these markings.

Case 1: If $\tau_k - \tau_j \le u - l$ for consecutive points j, k with $a \in \sigma_j, \sigma_k$. Then $[\tau_j + u, \tau_k + l)$ is the empty interval and $d = \lceil \tau_k - \tau_j \rceil + (l - u) \le 0$ and hence no action need to be taken. Figure 4 illustrates this case.

Case 2: If $\tau_k - \tau_j \in (u - l, u]$. Then the interval $[\tau_j + u, \tau_k + l)$ is non-empty, and $1 \le d = \lceil \tau_k - \tau_j \rceil + (l - u) \le l$.

1) We introduce two propositions a_0, a_1 that marks all positions $i \in dom(\rho)$ such that $a \in \sigma_i$ with a unique element from $\{a_0, a_1\}$. The position $first_a$ is marked a_0; if consecutive a's are at a distance $> u - l$, then they are marked by exactly one of a_i and a_{1-i} respectively, for $i \in \{0, 1\}$ such that they alternate; if consecutive a's are at a distance $\le u - l$, they are both marked with exactly the same a_i, $i \in \{0, 1\}$. A consecutive a_i, a_{1-i} pair "flags" attention : they play a role, in marking some interval with $\neg b$. The conjunction of the following formulae, denoted MARK_a implements these:
   a) $\Box^{\mathsf{w}}((a_0 \vee a_1) \leftrightarrow a) \wedge \Box^{\mathsf{w}}(\neg a_0 \vee \neg a_1)$
   b) $\neg a\, \mathsf{U}^{\mathsf{w}}(a \wedge a_0)$
   c) $\bigwedge_{i \in \{0,1\}} \Box^{\mathsf{w}}[F_1 \wedge F_2]$ where
      $F_1 : (a_i \wedge \Box_{[0,u-l]} \neg a) \to \Box \neg a \vee (\neg a\, \mathsf{U}(a \wedge a_{1-i}))$[9],
      $F_2 : (a_i \wedge \Diamond_{[0,u-l]} a) \to \neg a\, \mathsf{U}(a \wedge a_i)$.
2) To easily identify the intervals $I^1_{j,k}$ and $I^2_{j,k}$, we mark the points $j, k \in dom(\rho)$ with propositions beg_{db} and end_{db}. The d in suffix is $d = \lceil \tau_k - \tau_j \rceil + (l - u)$, the b in suffix is the witness proposition for $\Diamond\!\!\!\!-_{[l,u)} a$, while beg, end signify the beginning and end of respective consecutive a positions. To correctly get the d, we need to check the closest unit interval corresponding to $\tau_k - \tau_j$: for instance, if $\tau_k - \tau_j = (u - l) + 0.4$, then we know $\tau_k - \tau_j \in (u - l, u - l + 1]$. In this case, $\lceil \tau_k - \tau_j \rceil = u - l + 1$, and hence, $d = 1$. We need to do this for all the $l - 1$ possibilities : $\tau_k - \tau_j \in (t, t + 1]$, where $t \in \{u - l, \ldots, u - 1\}$. In each case, the symbols marking the respective consecutive a's will be $beg_{t+1+l-u\ b}$ and $end_{t+1+l-u\ b}$, where $t + 1 = \lceil \tau_k - \tau_j \rceil$.
   To summarize, we introduce propositions $\{beg_{db}, end_{db} \mid 1 \le d \le l\}$ to mark two consecutive a's that are at a distance in $(u - l, u]$. The d in the suffix is the closest integer $\ge$ the duration of the interval $[\tau_j + u, \tau_k + l)$. This is used in the next step to mark correctly the intervals $I^1_{j,k}$ and $I^2_{j,k}$, both of which have duration d : Identifying points j, k with beg_{db} and end_{db}, $I^1_{j,k}$ is the interval $[\tau_{end_{db}} + l - d, \tau_{end_{db}} + l)$ while $I^2_{j,k}$ is the interval $[\tau_{beg_{db}} + u, \tau_{beg_{db}} + d + u)$. Note that a unique value of d will only satisfy formula 2(a) below: that value is $d = \lceil \tau_k - \tau_j \rceil + (l - u) = t + 1 + l - u$.
   The following formulae implement this idea by ensuring that beg_{db} and end_{db} indeed correspond to consecutive points j, k with $a \in \sigma_j, \sigma_k$. For $t \in \{u - l, \ldots, u - 1\}$, and $d \in \{1, \ldots, l\}$,
   a) $\Box^{\mathsf{w}}(beg_{t+1+l-u\ b} \leftrightarrow (a \wedge (\neg a\, \mathsf{U}_{(t,t+1]} a)))$.
   b) $\Box^{\mathsf{w}}(end_{db} \leftrightarrow (a \wedge (\neg a\, \mathsf{S}\ beg_{db})))$.

   Let $\mathsf{MARK}_{beg,end,d}$ be the conjunction of the above formulae.
3) The propositions beg_{db} and end_{db} now help us in identifying the relevant points in the intersection of $I^1_{j,k}$ and $I^2_{j,k}$ as follows: Recall that points j, k marked with beg_{db}, end_{db} are also marked with one of a_0, a_1 such that $\{beg_{db}, a_i\} \subseteq \sigma_j$ iff $\{end_{db}, a_{1-i}\} \subseteq \sigma_k$. We now identify the points in $I^1_{j,k} = [\tau_{end_{db}} + l - d, \tau_{end_{db}} + l)$ by marking them with a proposition y_{cb} iff $a_{1-c} \in \sigma_k$. Likewise, all the points in $I^2_{j,k} = [\tau_{beg_{db}} + u, \tau_{beg_{db}} + d + u)$ are marked with a proposition x_{cb} iff $a_c \in \sigma_j$. It can be observed now that points in $I^1_{j,k} \cap I^2_{j,k}$ will be marked with both x_{cb}, y_{cb}. Such points are marked $\neg b$. Figure 5 illustrates this. This is implemented by the conjunction of the following formulae, denoted $\mathsf{MARK}_{x,y,c}$:
   a) $\bigwedge_{c \in \{0,1\}} \Box^{\mathsf{w}}((beg_{db} \wedge a_c) \to \Box_{[u,u+d)} x_{cb})$
   b) $\bigwedge_{c \in \{0,1\}} \Box^{\mathsf{w}}((end_{db} \wedge a_c) \to \Box^{\mathsf{w}}_{[l-d,l)} y_{1-c\ b}))$

[8] S can be removed from $\mathsf{MTL}[\,\mathsf{U}_I,\ \mathsf{S}]$ obtaining equisatisfiable formula in $\mathsf{MTL}[\,\mathsf{U}_I]$ modulo simple projections [3], details in Appendix L

[9] Note that points j, k with consecutive a's, such that $\tau_k - \tau_j > u$ also are marked by a_i, a_{1-i}

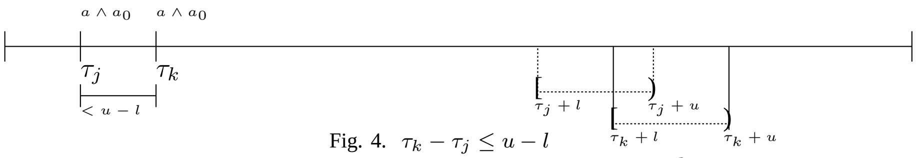

Fig. 4. $\tau_k - \tau_j \leq u - l$

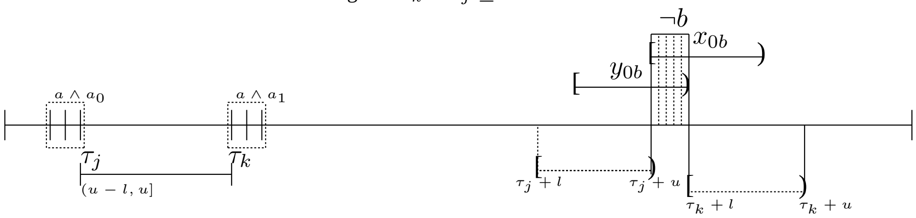

Fig. 5. $\tau_k - \tau_j \in (u - l, u]$. The shaded region indicates $x_{0b} \wedge y_{0b}$. This region is marked $\neg b$

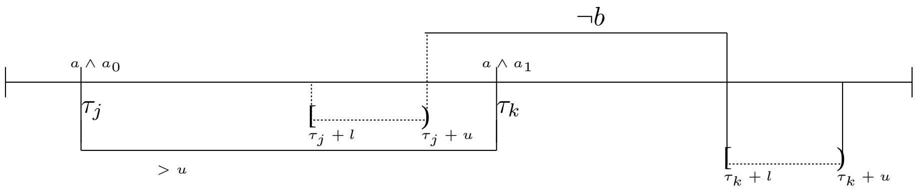

Fig. 6. $\tau_k - \tau_j > u$

4) Let $\mathsf{MARK}_{\neg b,c}$ denote $\Box^{\mathsf{w}}((x_{cb} \wedge y_{cb}) \rightarrow \neg b)$, $c \in \{0, 1\}$.

Case 2 Summary: We mark consecutive points j, k having a that are apart by a distance in $(u - l, u]$ with a_c, a_{1-c}, $c \in \{0, 1\}$, and with beg_{db}, end_{db} respectively, where d is the closest integer that is $\geq \lceil \tau_k - \tau_j \rceil + l - u$. The bit $c \in \{0, 1\}$ and the value d help in marking all points in $[\tau_k + l - d, \tau_k + l)$ with y_{cb} and all points in $[\tau_j + u, \tau_j + u + d)$ with x_{cb}. Points marked both x_{cb}, y_{cb} are then marked $\neg b$.

Case 3: $\tau_k - \tau_j \in (u, \infty)$. Then again, $[\tau_j + u, \tau_k + l)$ is non-empty.[10] Then $d = \lceil \tau_k - \tau_j \rceil + (l - u) > l$. Figure 6 illustrates this case.

1) We introduce propositions $\{b^1_\infty, b^2_\infty\}$ to mark consecutive a's that are more than u distance apart. We assert that $\neg b$ holds in the $[0, l)$ future of b^2_∞; also $\neg b$ holds at all points that are at a distance $\geq u$ from b^1_∞ and that lie before b^2_∞. We first mark such consecutive points j, k with propositions b^1_∞ and b^2_∞. Let $\mathsf{MARK}_{succ,\infty}$ be the conjunction of the following formulae:
   a) $\Box^{\mathsf{w}}(b^1_\infty \leftrightarrow (a \wedge \neg a\, \mathsf{U}_{(u,\infty)} a))$
   b) $\Box^{\mathsf{w}}(b^2_\infty \leftrightarrow (a \wedge \neg a\, \mathsf{S} b^1_\infty))$
2) Next we assert that points in $(\tau_j + u, \tau_k]$ and in $[\tau_k, \tau_k + l)$ be marked $\neg b$. This is implemented by the conjunction of the following formulae, denoted $\mathsf{MARK}_{\neg b,\infty}$:
   a) $\Box^{\mathsf{w}}((b^1_\infty \wedge \Diamond^{\mathsf{w}}_{[0,u)} b) \rightarrow (\Diamond^{\mathsf{w}}_{[0,u)}(b \wedge \neg b\, \mathsf{U} b^2_\infty))$
   b) $\Box^{\mathsf{w}}((b^1_\infty \wedge \Box^{\mathsf{w}}_{[0,u)} \neg b) \rightarrow (\neg b \wedge \neg b\, \mathsf{U} b^2_\infty))$
   c) $\Box^{\mathsf{w}}(b^2_\infty \rightarrow \Box^{\mathsf{w}}_{[0,l)} \neg b)$

Purpose of Extra Propositions: The extra propositions introduced are $X = \{a_0, a_1, x_{0b}, x_{1b}, y_{0b}, y_{1b}, b^1_\infty, b^2_\infty\} \cup \{beg_{db}, end_{db} \mid 1 \leq d \leq l\}$.

[10] If $l = 0$, case 2 gives an empty interval. Case 3 deals with $> u$ distance. For a's which are u apart, we add the formula $(a \wedge \Box_{[0,u)} \neg a \wedge \Diamond_{[0,u]} a) \rightarrow \Diamond_{[0,u]} \neg b$

1) First of all, a_0, a_1 are chosen to enable marking points in $I^1_{j,k}, I^2_{j,k}$ with x_{0b}, y_{0b} or x_{1b}, y_{1b}, depending on whether the point j was marked a_0 or a_1. Consider three consecutive points j, k, h where a holds, with $\tau_k - \tau_j, \tau_h - \tau_k \in [u - l, u]$. Clearly, we are looking at points in $I^1_{j,k}, I^2_{j,k}$ and $I^1_{k,h}, I^2_{k,h}$. If we just had x_b, y_b to mark these intervals, then we get points in $I^1_{j,k}, I^1_{k,h}$ marked with y_b, and points in $I^2_{j,k}, I^2_{k,h}$ marked x_b. There is a possibility as illustrated by the example below, that points marked x_b in $I^2_{j,k}$ intersect with points marked y_b in $I^1_{h,k}$. By our technique of marking points with both x_b, y_b as $\neg b$, this could give rise to inconsistency. For example, consider $[l, u) = [6, 7)$, $\tau_j = 3.1, \tau_k = 4.8, \tau_h = 5.9$. Clearly, $\tau_k - \tau_j, \tau_h - \tau_k \in (1, 7]$. For $d_1 = \lceil \tau_k - \tau_j \rceil + (l - u) = 1$, the over approximations of the interval $[\tau_j + u, \tau_k + l) = [10.1, 10.8)$ are $[\tau_k + l - d_1, \tau_k + l) = [9.8, 10.8) = I^1_{j,k}$, and $[\tau_j + u, \tau_j + u + d_1) = [10.1, 11.1) = I^2_{j,k}$. By construction, points in $[9.8, 10.8) = I^1_{j,k}$ are marked y_b, points in $[10.1, 11.1) = I^2_{j,k}$ are marked x_b. Clearly, points in $[10.1, 10.8)$ have both x_b, y_b marked. Again, the over approximations for the interval $[\tau_k + u, \tau_h + l) = [11.8, 11.9)$ are $I^1_{k,h} = [\tau_h + l - d_2, \tau_h + l) = [10.9, 11.9)$ and $I^2_{k,h} = [\tau_k + u, \tau_k + u + d_2) = [11.8, 12.8)$ for $d_2 = \lceil \tau_h - \tau_k \rceil + (l - u) = 1$. As per the marking scheme, we would mark $[10.9, 11.9)$ with y_b and $[11.8, 12.8)$ with x_b. While this gives us points in $[11.8, 11.9)$ marked with both x_b, y_b, this also gives us points in $[10.9, 11.1)$ marked with both x_b, y_b. We would then mark $\neg b$ for all points in $[10.9, 11.1)$, giving rise to inconsistency, as $[10.9, 11.1)$ is marked b by MARK_b. However, had we marked $[9.8, 10.8) = I^1_{j,k}$ with y_{0b}, $[10.1, 11.1) = I^2_{j,k}$ with x_{0b}, $[10.9, 11.9) = I^1_{k,h}$ with y_{1b} and $[11.8, 12.8) = I^2_{k,h}$ with x_{1b}, the erroneous interval

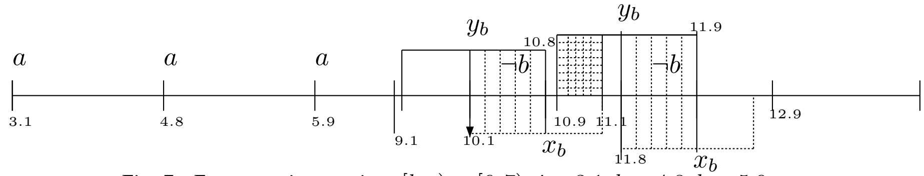

Fig. 7. Erroneous intersection: $[l, u) = [6, 7), j = 3.1, k = 4.8, h = 5.9$.

$[10.9, 11.1)$ is marked with x_{1b} and y_{0b}. Thus, using two bits 0,1, we can rule out marking points having $x_{cb}, y_{1-c\ b}$ with $\neg b$. The situation of erroneous marking is illustrated in Figure 7.

2) Note that it suffices to have only two bits 0,1 and hence propositions $x_{0b}, y_{0b}, x_{1b}, y_{1b}$. We do not need x_{2b}, y_{2b}. Consider any two pairs of points j, k and h, m such that $j < k < h < m$, and j and k, h and m and k and h are all consecutive with respect to a. i.e, there are no points between j, k or k, h or h, m that are marked a. Let $\tau_k - \tau_j, \tau_m - \tau_h > u - l$. Assume further that $a_0 \in \sigma_j$ as per our marking scheme. There are 2 cases :
Case 1: $\tau_h - \tau_k \leq u - l$. In this case k, h will be marked as a_1 and j, m will be marked as a_0. Note that the regions $I^1_{k,h}$ and $I^2_{k,h}$ are empty. No erroneous intersection can happen : $I^2_{j,k}$ is marked x_{0b} while $I^1_{h,m}$ is marked y_{1b}.
Case 2: $\tau_h - \tau_k > u - l$. In this case j, h will be marked as a_0 and k, m will be marked as a_1. Let $d_1 = \lceil \tau_k - \tau_j \rceil + (l - u)$, $d_2 = \lceil \tau_h - \tau_k \rceil + (l - u)$, and $d_3 = \lceil \tau_m - \tau_h \rceil + (l - u)$.
   - Intervals $I^1_{h,m} = [\tau_m + l - d_3, \tau_m + l)$ (marked y_{0b}) and $I^2_{j,k} = [\tau_j + u, \tau_j + u + d_1)$ (marked x_{0b}) are disjoint: we have $\tau_j + u + d_1 < \tau_k + u < \tau_h + l < \tau_m + l - d_3$.
   - Intervals $I^2_{h,m} = [\tau_h + u, \tau_h + u + d_3)$ (marked x_{0b}) and $I^1_{j,k} = [\tau_k + l - d_1, \tau_k + l)$ (marked y_{0b}) are disjoint: $\tau_h + u \geq \tau_k + (u - l) + l \geq \tau_k + l$.

   This shows that for consecutive pairs of a points j, k and h, m where $\tau_k - \tau_j, \tau_h - \tau_k, \tau_m - \tau_h > u - l$, intervals $I^1_{h,m}$ and $I^2_{j,k}$ (respectively $I^2_{h,m}$ and $I^1_{j,k}$) which are marked x_{ib}, y_{ib} will never intersect.

3) The formulae $\mathsf{MARK}_{x,y,c}$ only say where x_{cb}, y_{cb} are marked; they do not disallow occurrences of $x_{1-c\ b}, y_{1-c\ b}$ at those points. We claim that the free occurrences of $x_{1-c\ b}, y_{1-c\ b}$ does not create problems. Note that points marked b by MARK_b and points marked $\neg b$ by $\mathsf{MARK}_{\neg b,c}$, $c \in \{0, 1\}$ are disjoint and span $dom(\rho)$. Let p, q be consecutive points marked a. For every point p with $a \in \sigma_p$, $[\tau_p + l, \tau_p + u)$ is marked b by MARK_b, and $[\tau_p + u, \tau_q + l)$ is marked $\neg b$ by $\mathsf{MARK}_{\neg b,c}$. In case $p = last_a$, then $[\tau_{last_a} + u, \infty)$ is marked $\neg b$ by MARK_{last}. Thus, inducting on the a's in ρ, the union of the points marked b by MARK_b (call it B) and points marked $\neg b$ by $\mathsf{MARK}_{\neg b,c}$ (call it $\bar{B}$) is $dom(\rho)$.
   Thus, there are 2 possibilties for the free occurrence of $x_{1-c\ b}, y_{1-c\ b}$:
   - $x_{1-c\ b}, y_{1-c\ b}$ occur freely in $\bar{B}$. The freely occurring $x_{1-c\ b}, y_{1-c\ b}$ results in marking of $\neg b$ by formula $\mathsf{MARK}_{\neg b,\ 1-c}$; this does not generate inconsistencies, since they are already marked $\neg b$ by $\mathsf{MARK}_{\neg b,c}$.
   - $x_{1-c\ b}, y_{1-c\ b}$ occur freely in B. The freely occurring $x_{1-c\ b}, y_{1-c\ b}$ results in marking of $\neg b$ by formula $\mathsf{MARK}_{\neg b,\ 1-c}$; but these points are already marked b by MARK_b. Thus, at any point p in B, $\rho, p \nvDash x_{1-c\ b} \wedge y_{1-c\ b}$, for $c \in \{0, 1\}$. Thus, at all points of B, the appearance of $x_{1-c\ b}$ and $y_{1-c\ b}$ (if that is the case), is mutually exclusive.

   Thus, free markings of x_{cb}, y_{cb} if at all, they appear, do not come in the way of correctly marking points with $b, \neg b$.

The formula MARK in $\mathsf{MTL}[\mathsf{U}_I, \mathsf{S}]$ obtained as a conjunction of $\mathsf{MARK}_b, \mathsf{MARK}_{first}, \mathsf{MARK}_{last}, \mathsf{MARK}_a, \mathsf{MARK}_{x,y,c}$, $\mathsf{MARK}_{beg,end,d}, \mathsf{MARK}_{succ,\infty}, \mathsf{MARK}_{\neg b,c}$ and $\mathsf{MARK}_{\neg b,\infty}$. is such that $\rho \models ENF_\Sigma(T)$ iff $\rho' \models \mathsf{MARK}$, where $\rho' \setminus X = \rho$. A proof of correctness can be found in Appendix J. Using the technique in [3], we can eliminate the S modality from MARK obtaining an equisatisfiable formula ψ in $\mathsf{MTL}[\mathsf{U}_I]$. ☐

Note that our reduction does not introduce any new punctual modality. Hence, we also have the equivalence modulo simple projection of $\mathsf{MITL}[\mathsf{U}_I, \mathsf{S}_I]$ and $\mathsf{MITL}[\mathsf{U}_I]$.

Theorem 2. *For every $\varphi \in \mathsf{MTL}[\mathsf{U}_I, \mathsf{S}_{np}]$ over Σ, we can construct ψ_{fut} in $\mathsf{MTL}[\mathsf{U}_I]$ over $\Delta = \Sigma \cup \Sigma'$ such that $\varphi = \exists(\Sigma' - \Sigma).\psi_{fut}$.*

Proof. Follows from the fact that S_{np} can be expressed using S and $\diamondminus_{np}$ [3] and elimination of S modulo simple projections [3], [8]. ☐

C. Simple Versus Oversampling Projections: Formulae Size

Consider a formula $\varphi \in \mathsf{MTL}[\mathsf{U}_I, \diamondminus_{np}]$. Assume that the number of past modalities in φ is n, of which there are n_b bounded past modalities and n_u unbounded past modalitties. i.e, $n = n_b + n_u$. Flattening φ results in a linear increase in the size of φ. Converting φ_{flat} to ENF gives a constant size increase. Elimination of unbounded past (Lemma 9) also results in a constant increase in size. During elimination of bounded past modalities $\diamondminus_{[l,u)}$ (Lemma 10), we add $l - 1$ new formulae resulting in $\mathcal{O}(l)$ extra modalities. Thus, the number of extra modalities introduced after elimination of all the n_b temporal definitions corresponding to bounded past modalities is $\leq nl_{max}$, where l_{max} is the maximal lower bound of all bounded past modalities in φ. Hence, the formula obtained by simple projections, ψ has in the worst case, an exponential increase in size over φ. In the case of oversampled projections, it can be seen that both bounded as well as unbounded past modalities contribute to a linear increase in the size of the resultant formulae. In simple projections (Lemma 10), marking $\neg b$ correctly in $[\tau_j + u, \tau_k + l)$ depended on the distance $\tau_k - \tau_j$, resulting in $l - 1$ formulae; in the case of oversampling projections (Lemma 8), this is handled indirectly by the introduction of extra integral points between j and k. However, the formulae needed to introduce these extra points correctly have a constant size. A more detailed complexity analysis can be found in Appendix K.

D. Expressiveness

We wind up this section with a brief discussion about the expressive powers of logics $\mathsf{MTL}[\mathsf{U}_I, \mathsf{S}_{np}]$ and $\mathsf{MTL}[\mathsf{U}_{np}, \mathsf{S}_I]$. The following lemma highlights that even unary modalities $\Diamond_I, \Diamond\!\!\!-_I$ with singular intervals are more expressive than $\mathsf{U}_{np}, \mathsf{S}_{np}$; likewise, non-singular intervals are more expressive than intervals of the form $[0, \infty)$.

Lemma 11. *(i)* $\mathsf{MTL}[\Diamond_I] \not\subseteq \mathsf{MTL}[\mathsf{U}_{np}, \mathsf{S}_I]$, *(ii)* $\mathsf{MTL}[\Diamond_I, \Diamond\!\!\!-_I] \not\subseteq \mathsf{MTL}[\mathsf{U}_I, \mathsf{S}_{np}]$, *and (iii)* $\mathsf{MTL}[\Diamond_{np}, \Diamond\!\!\!-_{np}] \not\subseteq \mathsf{MTL}[\mathsf{U}_I, \mathsf{S}]$.

Proof. The formula $\Diamond_{(0,1)}\{a \wedge \neg\Diamond_{[1,1]}(a \vee b)\}$ in $\mathsf{MTL}[\Diamond_I]$ has no equivalent formula in $\mathsf{MTL}[\mathsf{U}_{np}, \mathsf{S}_I]$. Similarly, the formula $\Diamond\{b \wedge \neg\Diamond\!\!\!-_{[1,1]}(a \vee b)\}$ in $\mathsf{MTL}[\Diamond_I, \Diamond\!\!\!-_I]$ has no equivalent formula in $\mathsf{MTL}[\mathsf{U}_I, \mathsf{S}_{np}]$. The formula $\Diamond_{(1,2)}[a \wedge \neg\Diamond\!\!\!-_{(1,2)}a] \in \mathsf{MTL}[\Diamond_{np}, \Diamond\!\!\!-_{np}]$ has no equivalent formula in $\mathsf{MTL}[\mathsf{U}_I, \mathsf{S}]$. Details in Appendix M. □

V. Discussion

In this paper, we have proposed two notions of equivalence between MTL formulae (with different sets of propositions), which both preserve satisfiability. The notion $\phi = \exists X.\psi$, denoting *equisatisfiability modulo simple projection* denotes that a timed word satisfying ϕ can be extended to a timed word with additional propositions X which satisfies ψ, and a timed word satisfying ψ can be projected to a timed word satisfying ϕ. In both cases the set of time stamps of the letters remains identical. A more elaborate notion, $\phi = \exists \downarrow X.\psi$, denoting *equisatisfiability modulo oversampling projection*, is similar but the models of ψ may have additional time points. Thus, during temporal projection we allow oversampling of the original behaviour by adding new time points. Both forms of temporal projections are useful. They often allow formulae of a more complex logic to be effectively reduced in equisatisfiable manner to formulae of a much simpler logic. This often provides a convenient technique for proving satisfiability. As a significant use of this technique of temporal projections, in the paper, we have shown the decidability of $\mathsf{MTL}[\mathsf{U}_I, \mathsf{S}_{np}]$ over finite strictly monotonic timed words. This logic is more expressive than the previously known decidable fragments of MTL as well as MITL but less expressive than $\mathsf{MTL}[\mathsf{U}_I, \mathsf{S}_I]$. A symmetric proof would allow showing that $\mathsf{MTL}[\mathsf{U}_{np}, \mathsf{S}_I]$ is also decidable. Our result can also be adapted to weakly monotonic finite timed words (see Appendix H). Thus, we have extended the boundary of known decidable fragments of logic MTL over timed words. We note that the proof techniques used for showing decidability of MTL as well as MITL, do not seem to generalize easily to the logic $\mathsf{MTL}[\mathsf{U}_I, \mathsf{S}_{np}]$ considered here. In proving decidablity of $\mathsf{MTL}[\mathsf{U}_I, \mathsf{S}_{np}]$, we have given two different proofs. In the first proof, we reduced $\mathsf{MTL}[\mathsf{U}_I, \mathsf{S}_{np}]$ to $\mathsf{MTL}[\mathsf{U}_I]$ using the notion of oversampled temporal projections. This encoding is relatively simple and results only in linear blowup in formula size. We also gave an alternative reduction using only simple temporal projections, but the reduction turns out to be considerably more complex, and leads to an exponential blow up in formula size.

The technique of temporal projections has been widely used for continuous time MTL. For example, Hirshfeld and Rabinovich [6] used it to eliminate non-singular future operator $\Diamond_{[0,1)}$ in terms of $\Diamond\!\!\!-_{[0,1)}$, U and S. Subsequently, D'souza *et al* [3] as well as Kini *et al* [8] used the technique to remove past operator S_I from $\mathsf{MTL}[\mathsf{U}_I, \mathsf{S}_I]$. Their reduction does not carry over to logic $\mathsf{MTL}[\mathsf{U}_I, \mathsf{S}_{np}]$ over pointwise time which is expressively weak and allows insertion errors. In this paper, we have extended the technique of temporal projections to pointwise time (timed words). One novel aspect of our formulation is that during temporal projection we allow oversampling of the original behaviour by adding new time points. We have demonstrated that the ability of adding such additional points can considerably simplify the reductions. The expressive power of (the two forms of) temporal projections is an interesting topic of future work.

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APPENDIX

A. *Proof of Lemma 1*

We first define the composition of two simple extensions.
Composition of Two Simple Extensions: Consider Σ, X_1, X_2 such that $\Sigma \cap X_1 = \emptyset$ and $\Sigma \cap X_2 = \emptyset$. Let $\Sigma_1 = \Sigma \cup X_1$, $\Sigma_2 = \Sigma \cup X_2$ and $X = X_1 \cup X_2$.

Let $\zeta'_1 = (\sigma^1, \tau^1)$ be a (Σ, X_1)-simple extension and let $\zeta'_2 = (\sigma^2, \tau^2)$ be a (Σ, X_2)-simple extension, such that $\zeta'_1 \setminus X_1 = \zeta'_2 \setminus X_2$. Then the composition of ζ'_1 and ζ'_2 denoted $\zeta'_1 \oplus \zeta'_2$, is a (Σ, X)-simple extension $\zeta' = (\sigma', \tau')$ such that $\sigma'_i = \sigma^1_i \cup \sigma^2_i$ and $\tau_i = \tau^1_i = \tau^2_i$. Note that $(\zeta'_1 \oplus \zeta'_2) \setminus X = \zeta'_1 \setminus X_1 = \zeta'_2 \setminus X_2$, and $\zeta' \setminus X_2 = \zeta'_1$ and $\zeta' \setminus X_1 = \zeta'_2$. Consider the following example:

- Let $\Sigma = \{a, b\}, X_1 = \{c\}, X_2 = \{d\}$,
- $\zeta'_1 = (\{a\}, 0.3)(\{b, c\}, 0.8)(\{b\}, 1.1)$, and
- $\zeta'_2 = (\{a, d\}, 0.3)(\{b, d\}, 0.8)(\{b\}, 1.1)$. Then
- $\zeta'_1 \setminus X_1 = \zeta'_2 \setminus X_2 = (\{a\}, 0.3)(\{b\}, 0.8)(\{b\}, 1.1)$,
- $\zeta'_1 \oplus \zeta'_2 = (\{a, d\}, 0.3)(\{b, c, d\}, 0.8)(\{b\}, 1.1)$,
- $(\zeta'_1 \oplus \zeta'_2) \setminus X_2 = (\{a\}, 0.3)(\{b, c\}, 0.8)(\{b\}, 1.1) = \zeta'_1$,
- $(\zeta'_1 \oplus \zeta'_2) \setminus X_1 = (\{a, d\}, 0.3)(\{b, d\}, 0.8)(\{b\}, 1.1) = \zeta'_2$.

We use the following easy lemma in the proof:

Lemma 12. *Consider Σ, X_1, X_2 such that $\Sigma \cap X_1 = \emptyset$ and $\Sigma \cap X_2 = \emptyset$. Let $\Sigma_1 = \Sigma \cup X_1$, $\Sigma_2 = \Sigma \cup X_2$ and $X = X_1 \cup X_2$. Then $\zeta' = (\zeta' \setminus X_2) \oplus (\zeta' \setminus X_1)$.*

Proof of Lemma 1:

Proof. Assume $\varphi_1 = \exists X_1.\psi_1$, $\varphi_2 = \exists X_2.\psi_2$.

(a) Then, for $i = 1, 2$, and any timed word ρ_i over Σ, such that $\rho_i \models \varphi_i$, we have a (Σ, X_i)-simple extension ρ'_i such that $\rho'_i \models \psi_i$ and $\rho'_i \setminus X_i = \rho_i$.

(b) For any timed word ρ'_i, $\rho'_i \models \psi_i$ implies ρ'_i is a (Σ, X_i)-simple extension such that $\rho'_i \setminus X_i \models \varphi_i$.

Consider $\varphi_1 \wedge \varphi_2$, a formula built over Σ. Also, $\psi_1 \wedge \psi_2$ is a formula built from $\Sigma \cup X_1 \cup X_2$. Let $X = X_1 \cup X_2$.

1) Let ζ' be a timed word over $\Sigma \cup X$ such that $\zeta' \models \psi_1 \wedge \psi_2$. Then $\zeta' \models \psi_i$ for $i = 1, 2$. Since ψ_i is a formula built from Σ_i, and $X_1 \cap X_2 = \emptyset$, we have
   - $\zeta' \models \psi_1 \to \zeta' \setminus X_2 \models \psi_1$, and
   - $\zeta' \models \psi_2 \to \zeta' \setminus X_1 \models \psi_1$.
   - Call $\zeta' \setminus X_1 = \zeta'_2$ and $\zeta' \setminus X_2 = \zeta'_1$.

   Note that ζ'_1 is a (Σ, X_1)-simple extension and ζ'_2 is a (Σ, X_2)-simple extension. This gives, by (b) above that $\zeta'_1 \setminus X_1 \models \varphi_1$ and $\zeta'_2 \setminus X_2 \models \varphi_2$. By Lemma 12, we have $\zeta'_1 \setminus X_1 = \zeta'_2 \setminus X_2$, call it some timed word ζ over Σ. Then $\zeta \models \varphi_1 \wedge \varphi_2$. Also, $\zeta = (\zeta'_1 \setminus X_1) = (\zeta' \setminus X_2) \setminus X_1 = \zeta' \setminus X$.
2) Now let ζ be a timed word over Σ such that $\zeta \models \varphi_1 \wedge \varphi_2$. We have to show that there is a (Σ, X)-simple extension ζ' such that $\zeta' \models \psi_1 \wedge \psi_2$ such that $\zeta' \setminus X = \zeta$.
   Since $\varphi_1 = \exists X_1.\psi_1$, $\varphi_2 = \exists X_2.\psi_2$, we know that for any word ζ over Σ satisfying $\varphi_1 \wedge \varphi_2$, $\zeta \models \varphi_i$. By (a) above, $\zeta \models \varphi_i$ implies there exists (Σ, X_i)-simple extensions ζ'_i such that $\zeta'_i \models \psi_i$, with $\zeta'_i \setminus X_i = \zeta$. Then the composition $\zeta' = \zeta'_1 \oplus \zeta'_2$, of ζ'_1 and ζ'_2 is well-defined. Clearly, ζ' is a (Σ, X)-simple extension obtained by composing the (Σ, X_1)-simple extension ζ'_1 and the (Σ, X_2)-simple extension ζ'_2 such that $\zeta' \setminus X = \zeta$.
   Since $X_1 \cap X_2 = \emptyset$, and ψ_1 is built from from $\Sigma \cup X_1$ and ψ_2 from $\Sigma \cup X_2$, $\zeta'_1 \oplus \zeta'_2$ will not interfere in the satisfiability of either ψ_1 or ψ_2, in a way different from ζ'_1 and ζ'_2: Assume the contrary. That is, $\zeta' \not\models \psi_1 \wedge \psi_2$. That is, $\zeta' \not\models \psi_1$ or $\zeta' \not\models \psi_2$. Let $\zeta' \not\models \psi_1$. If so, then $\zeta' \setminus X_2 \not\models \psi_1$ since ψ_1 has no symbols from X_2 (by assumption X_1 and X_2 are disjoint). But $\zeta' \setminus X_2 = \zeta'_1$, and we know $\zeta'_1 \models \psi_1$, contradicting $\zeta' \not\models \psi_1$. Hence, $\zeta' \models \psi_1 \wedge \psi_2$.

The following example illustrates what might go wrong when $X_1 \cap X_2 \neq \emptyset$. Consider, $\Sigma = \{a, c, d\}$, $X_1 = \{b, e\}$ and $X_2 = \{b, f\}$. Note that $X_1 \cap X_2 = \{b\}$.

Consider formulae $\psi_1 = b \wedge \Box(b \leftrightarrow \Diamond c) \wedge \Box \bigvee \Sigma$ and $\psi_2 = b \wedge \Box(b \leftrightarrow \Diamond a) \wedge \Box \bigvee \Sigma$. Also, let $\varphi_1 = \Diamond c$ and $\varphi_2 = \Diamond a$. Let ζ be the word $(d, 0.1)(c, 0.3)(d, 0.7)(a, 0.9)$ over Σ. Clearly, $\zeta \models \varphi_1 \wedge \varphi_2$.

Consider $\zeta'_1 = (\{d, e, b\}, 0.1)(\{c\}, 0.3)(\{e, d\}, 0.7)(\{a\}, 0.9)$, a (Σ, X_1)-simple extension and the (Σ, X_2)-simple extension $\zeta'_2 = (\{d, f, b\}, 0.1)(\{c, b\}, 0.3)(\{f, b, d\}, 0.7)(\{a\}, 0.9)$. Then, $\zeta'_1 \models \psi_1$, $\zeta'_2 \models \psi_2$, $\zeta'_1 \setminus X_1 = \zeta'_2 \setminus X_2 = \zeta$. However, $\zeta' = (\{d, b, e, f\}, 0.1)(\{c, b\}, 0.3)(\{d, b, e, f\}, 0.7)(\{a\}, 0.9)$, the composition of ζ'_1 and ζ'_2 is such that $\zeta' \not\models (\psi_1 \wedge \psi_2)$. ☐

B. *Proof of Lemma 2*

Proof. The proof follows by structural induction on φ.

- Let ρ be a timed word over Σ such that $\rho \models \varphi$. We have to show that for all (Σ, X)-oversampled behaviour ρ' such that $\rho' \downarrow X = \rho$ holds, $\rho' \models ONF_\Sigma(\varphi)$.
  Consider a (Σ, X)-oversampled behaviour ρ', such that $\rho' \downarrow X = \rho$. Then, there exists a strictly increasing function $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\}$ such that $n = |dom(\rho)|$, $m = |dom(\rho')|$, and
  - $f(1) = 1$, $\sigma_1 = \sigma'_1 \cap \Sigma$, $\tau_1 = \tau'_1$, and
  - $f(n) = m$, $\sigma_n = \sigma'_m \cap \Sigma$, $\tau_n = \tau'_m$, and
  - For $1 \leq i \leq n-1$, $f(i) = j$ and $f(i+1) = k$ iff
    - $\sigma_i = \sigma'_j \cap \Sigma$, and $\tau_i = \tau'_j$,
    - $\sigma_{i+1} = \sigma'_k \cap \Sigma$, and $\tau_{i+1} = \tau'_k$,
    - For all $j < l < k$, $\sigma'_l \subseteq X$.

  By applying structural induction on depth of φ, we show that $\rho \models \varphi \to \rho' \models ONF_\Sigma(\varphi)$. For depth 0, the base case trivially holds for atomic propositions. For example if $\varphi = a \in \Sigma$, then $ONF_\Sigma(\varphi) = a \wedge act$. Clearly, $\rho, 1 \models a$ iff $\rho', f(1) \models ONF_\Sigma(a)$.
  Assume the result for formulae φ of depth $\leq n-1$. Consider φ as a formula of depth n. Lets discuss the case of formulae of the form $\varphi = \psi_1 \mathsf{U}_I \psi_2$ where ψ_1 and ψ_2 have depth $\leq n-1$.
  If $\rho, i \models \psi_1 \mathsf{U}_I \psi_2$, then there exists $j > i$ where ψ_2 holds, and all points in between i and j satisfy ψ_1. Also, $t_j - t_i \in I$. By the above, there exists a point $f(j) > f(i)$ such that $\sigma'_{f(j)} \models ONF_\Sigma(\psi_2)$ (by induction hypothesis), and $\sigma'_{f(j)} \models act$ (definition of f). Let $\{i_1, \ldots, i_q\}$ be the

set of points between $f(i)$ and $f(j)$. For all $i < l < j$, $f(l) \in \{i_1, \ldots, i_q\}$. Also, $\sigma_{f(l)} \models ONF_\Sigma(\psi_1)$. However, there are points $i_j \in \{i_1, \ldots, i_q\}$ such that $i_j \neq f(l)$ for any $i < l < j$. These points are such that $\sigma'_{i_j} \cap \Sigma = \emptyset$. Now if we look at points between $f(i)$ and $f(j)$, then we have

  – For all points k such that $f(i) < k < f(j)$, we have $\sigma'_k \models ONF_\Sigma(\psi_1)$, or $\sigma'_k \cap \Sigma = \emptyset$.
    i.e, $(\sigma'_k \cap \Sigma \neq \emptyset) \to \sigma'_k \models ONF_\Sigma(\psi_1)$.
  – Recall that if $\sigma'_k \cap \Sigma \neq \emptyset$, then $\sigma'_k \models act$

  The above conditions give us
  $\rho', f(i) \models (act \to ONF_\Sigma(\psi_1)) \, \mathsf{U}_I (act \wedge ONF_\Sigma(\psi_2))$. Also, since ρ' is a (Σ, X)-oversampled behaviour, act holds good at the start and end points. $\rho' \models act$ iff act holds good at the starting point. $\Box\bot$ holds good only at the last point; $\bot$ stands for $false$. Clearly, $\rho \models (\psi_1 \mathsf{U}_I \psi_2)$ implies $\rho' \models (act \to ONF_\Sigma(\psi_1) \, \mathsf{U}_I (act \wedge ONF_\Sigma(\psi_2) \wedge act \wedge (\Box\bot \to act)$. The proof for past modaility is analogous.

- Let ρ' be a (Σ, X)-oversampled behaviour such that $\rho' \models ONF_\Sigma(\varphi)$. We have to show that $\rho' \downarrow X \models \varphi$. In a manner similar to the above, by structural induction of φ, we can show that $\rho' \downarrow X \models \varphi$.

□

C. Proof of Lemma 3

Proof. Follows from Lemma 2 and equivalence of ζ and $ONF_\Sigma(\zeta)$. □

D. Proof of Lemma 4

We first define the composition of two oversampled behaviours.

Composition of Oversampled Behaviours: Let $\rho_1 = (\sigma^1, \tau^1)$ be a (Σ, X_1)-oversampled behaviour and $\rho_2 = (\sigma^2, \tau^2)$ be a (Σ, X_2)-oversampled behaviour such that $\rho_1 \downarrow X_1 = \rho_2 \downarrow X_2$. This condition says that the points in ρ_1 where propositions of Σ hold is exactly same as the points in ρ_2 where propositions of Σ hold; moreover the same propositions of Σ hold at these points of ρ_1 and ρ_2. Let $\Sigma_1 = \Sigma \cup X_1$ and $\Sigma_2 = \Sigma \cup X_2$. We define the composition of ρ_1 and ρ_2 denoted $\rho_1 \boxplus \rho_2$ to be all $(\Sigma, X_1 \cup X_2)$-oversampled behaviours ρ such that $\rho \downarrow X_1 = \rho_2$ and $\rho \downarrow X_2 = \rho_1$. Note that $\rho_1 \boxplus \rho_2$ is guaranteed to exist only when $X_1 \cap X_2 = \emptyset$. The following example illustrates that when $X_1 \cap X_2 \neq \emptyset$, $\rho_1 \boxplus \rho_2$ may not exist.

Consider $\Sigma = \{a, b\}, X_1 = \{c, e\}, X_2 = \{d, e\}$. Let $\rho_1 = (\{a, c\}, 0.1)(\{e\}, 0.3)(\{b, e, c\}, 1)$ be a (Σ, X_1)-oversampled behaviour and $\rho_2 = (\{a\}, 0.1)(\{e\}, 0.3)(\{b, e, d\}, 1)$ be a (Σ, X_2)-oversampled behaviour. Then $\rho_1 \downarrow X_1 = (\{a\}, 0.1)(\{b\}, 1) = \rho_2 \downarrow X_2$. Assume that $\rho \in \rho_1 \boxplus \rho_2$. Then, $\rho \downarrow X_1 = \rho_2$. However, $\rho \downarrow X_1$ will not contain any position i which is marked just with e, since such a position will be eliminated during oversampled projection with respect to X_1. Thus, there can be no such ρ, which after oversampling projections with respect to X_1 will give ρ_2. A similar problem happens when trying to show that $\rho \downarrow X_2 = \rho_1$.

We now give an example to illustrate the composition of two oversampled behaviours. Let $\Sigma = \{a\}, X_1 = \{c\}, X_2 = \{d\}$, $\rho_1 = (\{a\}, 0.1)(\{c\}, 0.5)$ and $\rho_2 = (\{a\}, 0.1)(\{d\}, 0.5)(\{d\}, 0.5)$. $\rho_1 \boxplus \rho_2$ consists of:

- $(\{a\}, 0.1)(\{c\}, 0.5)(\{d\}, 0.5)(\{d\}, 0.5)$
- $(\{a\}, 0.1)(\{d\}, 0.5)(\{d\}, 0.5)(\{c\}, 0.5)$
- $(\{a\}, 0.1)(\{d\}, 0.5)(\{c\}, 0.5)(\{d\}, 0.5)$

Clearly, when the words ρ_1, ρ_2 are weakly monotonic, $\rho_1 \boxplus \rho_2$ can consist of more than one word; however, when ρ_1, ρ_2 are strictly monotonic, $\rho_1 \boxplus \rho_2$ is a unique word. Our proof applies to both weakly monotonic and strictly monotonic words. We use the following easy lemma in the proof:

Lemma 13. *Let $X_1 \cap X_2 = \emptyset$, and $X_1 \cup X_2 = X$. Let ρ be a (Σ, X)-oversampled behaviour, and let $\Sigma_1 = \Sigma \cup X_1$ and $\Sigma_2 = \Sigma \cup X_2$. Then $\rho \in [(\rho \downarrow X_2)] \boxplus [(\rho \downarrow X_1)]$.*

<u>Proof of Lemma 4</u>:

Proof. Given $\varphi_1 = \exists \downarrow X_1.\zeta_1$ and $\varphi_2 = \exists \downarrow X_2.\zeta_2$. We know that by definition,

(a) For any (Σ, X_i)-oversampled behaviour ρ'_i,
    $\rho'_i \models \zeta_i \to (\rho'_i \downarrow X_i) \models \varphi_i$.
(b) For any timed word ρ_i over Σ such that $\rho_i \models \varphi_i$, there exists a (Σ, X_i)-oversampled behaviour ρ'_i such that $\rho'_i \models \zeta_i$ and $\rho'_i \downarrow X_i = \rho_i$.

We now want to show that $\varphi_1 \wedge \varphi_2 = \exists \downarrow X.(\zeta_1 \wedge \zeta_2)$.

1) Let ρ be a timed word over Σ such that $\rho \models \varphi_1 \wedge \varphi_2$. Since $\rho \models \varphi_i$, we have by (b) above, (Σ, X_i)-oversampled behaviours ρ'_i such that $\rho'_i \models \zeta_i$ and $\rho'_i \downarrow X_i = \rho$, for $i = 1, 2$. Hence, $\rho'_1 \boxplus \rho'_2$ is welldefined; let $\rho' \in \rho'_1 \boxplus \rho'_2$.
   Since ζ_i is in the oversampled normal form with respect to Σ_i, by Lemma 3, we have $\zeta_1 = \forall \downarrow .\zeta_1$ and $\zeta_2 = \forall \downarrow .\zeta_2$. We already have $\rho'_i \models \zeta_i$, for $i = 1, 2$. Hence,
   - any (Σ_1, X_2)-oversampled behaviour ρ'' such that $\rho'' \downarrow X_2 = \rho'_1$ will also satisfy ζ_1.
   - any (Σ_2, X_1)-oversampled behaviour ρ''' such that $\rho''' \downarrow X_1 = \rho'_2$ will also satisfy ζ_2.
   - By definition of $\boxplus$, we know that $\rho' \in \rho'_1 \boxplus \rho'_2$ is such that $\rho' \downarrow X_2 = \rho'_1$ and $\rho' \downarrow X_1 = \rho'_2$.
   - Picking $\rho' = \rho'' = \rho'''$, we have $\rho' \models \zeta_1$ and $\rho' \models \zeta_2$.

   Hence $\rho' \in \rho'_1 \boxplus \rho'_2$ satisfies $\zeta_1 \wedge \zeta_2$. Further,
   $\rho' \downarrow X = \{[\rho' \downarrow X_1] \downarrow X_2\} = \{\rho'_2 \downarrow X_2\} = \rho$.
2) Conversely, let ρ' be a (Σ, X)-oversampled behaviour, such that $\rho' \models \zeta_1 \wedge \zeta_2$. Then $\rho' \models \zeta_i$ for $i = 1, 2$. Again, since ζ_i is in the oversampled normal form with respect to Σ_i, by Lemma 3, we have $\zeta_1 = (\forall \downarrow).\zeta_1$ and $\zeta_2 = (\forall \downarrow).\zeta_2$. We already have $\rho' \models \zeta_i$ for $i = 1, 2$. Hence,
   - $\rho' \models \zeta_1 \to \rho' \downarrow X_2 \models \zeta_1$.
   - $\rho' \models \zeta_2 \to \rho' \downarrow X_1 \models \zeta_2$.
   - Let $\rho'_1 = \rho' \downarrow X_2$ and $\rho'_2 = \rho' \downarrow X_1$. Then $\rho'_1 \models \zeta_1$ and $\rho'_2 \models \zeta_2$.

- By (a) above, we have $\rho'_1 \downarrow X_1 \models \varphi_1$ and $\rho'_2 \downarrow X_2 \models \varphi_2$.
- By Lemma 13, $\rho' \in \rho'_1 \boxplus \rho'_2$. Hence, by definition of $\boxplus$, $\rho'_1 \downarrow X_1 = \rho'_2 \downarrow X_2$. Call it ρ, a timed word over Σ. Clearly, $\rho \models \varphi_1 \wedge \varphi_2$ and $\rho = \rho' \downarrow X$.

$\square$

E. Proof of Lemma 6

Proof. We prove the lemma for intervals of the form $[l, u)$. The proof can be extended for other type of intervals also. Assume that $\rho, i \models \diamondminus_{[l,u)}\alpha$. We then show that $\neg(\tau_i < \tau_{first_\alpha} + l)$ and $\neg(\tau_i \geq \tau_{last_\alpha} + u)$ and $\neg(\tau_i \in [\tau_j + u, \tau_k + l))$ for consecutive points j, k where α holds.

1) Let $\tau_i < \tau_{first_\alpha} + l$. $\rho, i \models \diamondminus_{[l,u)}\alpha$ implies that there is a point i' such that $\tau_{i'} \in (\tau_i - u, \tau_i - l]$, such that $\rho, i' \models \alpha$. Then, $\tau_{i'} \leq \tau_i - l < \tau_{first_\alpha}$, contradicting that $first_\alpha$ is the first point where α holds.
2) Let $\tau_i \geq \tau_{last_\alpha} + u$. Again, $\rho, i \models \diamondminus_{[l,u)}\alpha$ implies that there is a point i' such that $\tau_{i'} \in (\tau_i - u, \tau_i - l]$ such that $\rho, i' \models \alpha$. We then have $\tau_{i'} > \tau_i - u \geq \tau_{last_\alpha}$, contradicting that $last_\alpha$ is the last point where α holds.
3) Assume that there exist consecutive points $j < k$ where α holds. Also, let $\tau_i \in [\tau_j + u, \tau_k + l)$. $\rho, i \models \diamondminus_{[l,u)}\alpha$ implies that there exists a point i' such that $\tau_{i'} \in (\tau_i - u, \tau_i - l]$ and $\rho, i' \models \alpha$. Also, $\tau_i - u \in [\tau_j, \tau_k + (l - u))$ and $\tau_i - l \in [\tau_j + (u - l), \tau_k)$. This gives $\tau_j < \tau_{i'} < \tau_k$ contradicting the assumption that j, k are consecutive points where α holds.

Conversely, assume that $\neg(\tau_i < \tau_{first_\alpha} + l)$ and $\neg(\tau_i \geq \tau_{last_\alpha} + u)$ and $\neg(\tau_i \in [\tau_j + u, \tau_k + l))$ for consecutive points j, k where α holds. Then, $\tau_i \in [\tau_{first_\alpha} + l, \tau_{last_\alpha} + u)$. We show that $\rho, i \models \diamondminus_{[l,u)}\alpha$.

1) If $\tau_{first_\alpha} = \tau_{last_\alpha}$, then $\tau_i - u < \tau_{first_\alpha} \leq \tau_i - l$. Clearly, α holds in $(\tau_i - u, \tau_i - l]$, and hence $\rho, i \models \diamondminus_{[l,u)}\alpha$.
2) If $\tau_{first_\alpha} < \tau_{last_\alpha}$, and $\tau_i \in [\tau_{first_\alpha} + l, \tau_{last_\alpha} + u)$. By the condition $\neg(\tau_i \in [\tau_j + u, \tau_k + l))$ for consecutive points j, k where α holds, we have for all consecutive points $j < k$ where α holds, $\tau_i \notin [\tau_j + u, \tau_k + l)$. Combining this with $\tau_i \in [\tau_{first_\alpha} + l, \tau_{last_\alpha} + u)$, we have $\tau_i \in [\tau_k + l, \tau_{last_\alpha} + u)$ or $\tau_i \in [\tau_{first_\alpha} + l, \tau_j + u)$ for $k \leq last_\alpha$ and $j \geq first_\alpha$.
   If $j = first_\alpha$, and if $\tau_i \in [\tau_{first_\alpha} + l, \tau_{first_\alpha} + u)$, and as seen in the first case, $\rho, i \models \diamondminus_{[l,u)}\alpha$. Similar is the case when $k = last_\alpha$. Assume now that $j > first_\alpha$ and $k < last_\alpha$. Considering j' as the immediate point before j where α holds, (there is certainly such a point j', j' could be $first_\alpha$) we have by assumption $\tau_i \notin [\tau_{j'} + u, \tau_j + l)$. This combined with $\tau_i \in [\tau_{first_\alpha} + l, \tau_j + u)$ gives $\tau_i \in [\tau_j + l, \tau_j + u)$. Similarly, considering k' as the immediate next point after k where α holds (there is certainly one such point, k' could be $last_\alpha$) we have by assumption $\tau_i \notin [\tau_k + u, \tau_{k'} + l)$. This combined with $\tau_i \in [\tau_k + l, \tau_{last_\alpha} + u)$ gives $\tau_i \in [\tau_k + l, \tau_k + u)$. Hence, we have $\rho, i \models \diamondminus_{[l,u)}\alpha$.

$\square$

F. Proof of Lemma 7

Proof. Let ρ' be a (Σ, W)-oversampled behaviour. Let $\alpha = (act \rightarrow (\neg a \wedge \neg b))$. Consider the following formulae in $\mathsf{MTL}[\,\mathsf{U}_I]$:

1) $\psi_1 : [\Box^{\mathsf{w}}\alpha \vee \{\alpha\, \mathsf{U}^{\mathsf{w}}[(a \wedge act) \wedge \Box^{\mathsf{w}}_{[0,l)}(act \rightarrow \neg b)]\}]$
2) $\psi_2 : \Box^{\mathsf{w}}[(a \wedge act) \rightarrow \Box_{[l,\infty)}(act \rightarrow b)]$.

Let $\psi = \psi_1 \wedge \psi_2$. We claim that $\rho' \models ONF_\Sigma(T)$ iff $\rho' \models \psi$. Assume $\rho' \models ONF_\Sigma(T)$. Assume the contrary that $\rho' \models \neg\psi_1$. Then, either there is a point marked $act \wedge b$ before the first occurrence of $a \wedge act$, or there is a point marked $act \wedge b$ in the $[0, l)$ future of the first $a \wedge act$. Both of these imply $\neg ONF_\Sigma(T)$ giving contradiction.

Now assume that $\rho' \models \neg\psi_2$. Then some point act in the $[l, \infty)$ future of a certain $a \wedge act$ is marked $\neg b$, which again contradicts $ONF_\Sigma(T)$. Hence $\rho' \models \psi$. The converse can be proved in a similar way. Note that $\psi_1 \wedge \psi_2$ increases a constant number of modalities compared to $ONF_\Sigma(T)$.

$\square$

G. Proof of Correctness of Lemma 8

Proof. We give a proof of correctness on the construction of ρ'' and the formula MARK, showing that $ONF_\Sigma(T) = \exists \downarrow X.ONF_{\Sigma_i}(\mathsf{MARK})$. We start with a (Σ, W)-oversampled behaviour ρ' over $\Sigma \cup W$. We induct on the a's in ρ', and show that a point p of ρ' is marked b iff $\rho', p \models \diamondminus_{[l,u)}a$.

- Given any point q of ρ' marked a, MARK_b marks all points in $[\tau_q + l, \tau_q + u)$ with b.
- Lets look at the first a of ρ'. Recall that the point where a holds for the first time is called $first_a$. The formula MARK_{first} ensures that all points of ρ' that are at a distance $[0, l)$ from $first_a$ are marked $\neg b$. Also, all points in $[0, \tau_{first_a}]$ are also marked $\neg b$. Thus, MARK_{first} accounts for all points in $[0, \tau_{first_a} + l)$, while MARK_b marks all points in $[\tau_{first_a} + l, \tau_{first_a} + u)$ with b.
- Consider a point j in $dom(\rho')$ such that $j > first_a$, $a \in \sigma_j$ and assume that all the a's in $[0, \tau_j]$ have been accounted for: that is, all points in $[0, \tau_j + u)$ of ρ' have been marked with b or $\neg b$ correctly. This is the inductive hypothesis. Now consider the next consecutive a occurring after j, call that point k. If $\tau_k - \tau_j \leq u - l$, then $\tau_k + l \leq \tau_j + u$, and by MARK_b, all points in $[\tau_k + l, \tau_k + u)$ will be marked b. Hence, we are done accounting for $[0, \tau_k + u)$. Hence, assume $\tau_k - \tau_j > u - l$. In this case, $\tau_k + u > \tau_k + l > \tau_j + u$. MARK_b marks all points in $[\tau_k + l, \tau_k + u)$ with b; we need to reason that points in $[\tau_j + u, \tau_k + l)$ will be marked $\neg b$.
  - The formulae $\mathsf{MARK}_{j,k}$, $\mathsf{MARK}_{beg,end}$ mark points j, k respectively with b_s, b_e, and points $\tau_j + u, \tau_k + l$ respectively with beg_b and end_b. Also, $\mathsf{MARK}_{beg,end}$ marks $(\tau_j + u, \tau_j + u + 1)$ as well as $(\tau_j + u - 1, \tau_j + u)$ with $\neg beg_b$. As discussed in Lemma 8, we must assert that all other remaining points beg_b and end_b do not occur. The formula MARK_c first marks all integer points with c. We then identify the points between b_s and b_e by uniquely marking the

closest integral point before b_s with c_{b_s} and and the closest integral point before b_e with c_{b_e}. Recall that b_s and b_e were marked at τ_j and τ_k; thus, c_{b_s} and c_{b_e} get marked respectively at points $\lfloor\tau_j\rfloor$ and $\lfloor\tau_k\rfloor$. We then assert that beg_b can occur at a point t iff there is a c_{b_s} in $(t-u-1, t-u]$. Thus, given that c_{b_s} is marked at $\lfloor\tau_j\rfloor$, beg_b is marked only in $[\lfloor\tau_j\rfloor+u, \lfloor\tau_j\rfloor+u+1)$. However, by formula $\mathsf{MARK}_{beg,end}$, we disallow beg_b in (τ_j+u, τ_j+u+1) and (τ_j+u-1, τ_j+u). Thus, we obtain a unique marking for beg_b. In a similar way, we obtain a unique marking for end_b. Note that the oversampled behaviour ρ'' now has these markings. The formula $\mathsf{MARK}_{\neg b}$ now marks all points of ρ' (or all points marked act in ρ'') between beg_b and end_b with $\neg b$. This takes care of the interval we were interested in: the interval $[\tau_j+u, \tau_k+l)$.
  - Thus, we have now accounted for all points of ρ' in $[0, \tau_k+u)$.
- We are now left with the remaining part $[\tau_k+u, \tau_{|dom(\rho')|}]$. If $k \neq last_a$, we can extend the reasoning above to the next consecutive position after k, which is marked an a. In this way, we can account for all points of ρ' in $[0, \tau_{last_a}+u)$. We just need to reason for $[\tau_{last_a}+u, \tau_{|dom(\rho')|}]$. Consider the point $last_a$. The formula MARK_{last} marks all points of ρ' in the interval $[\tau_{last_a}+u, \tau_{|dom(\rho')|}]$ with $\neg b$.

The above argument shows that all points of ρ' are marked b or $\neg b$ correctly. The $(\Sigma\cup W, X)$-oversampled behaviour ρ'' reflects these markings. When we do an oversampled projection of ρ'' with respect to $X = \{b_e, b_s, beg_b, end_b, c, c_{b_s}, c_{b_e}\}$, we are left with ρ', where at all positions, we have the correct marking with respect to b or $\neg b$. Clearly, a point p of ρ' is marked b iff $\rho', p \models \diamondminus_{[l,u)} a$. Hence, $\rho' \models ONF_\Sigma(T)$ iff $\rho'' \models ONF_{\Sigma\cup W\cup X}(\mathsf{MARK})$.

Conversely, if we start with a $(\Sigma\cup W, X)$-oversampled behaviour ρ'' satisfying $ONF_{\Sigma\cup W\cup X}(\mathsf{MARK})$, then all points p of ρ'' marked act will be marked b iff $\diamondminus_{[l,u)} a$ holds good at p. Then $\rho'' \downarrow X$ will give a word ρ' over $\Sigma\cup W$ that satisfies $ONF_\Sigma(T)$. $\square$

H. Extending Lemma 8 to weakly monotonic timed words

Note that for weakly monotonic words, we need to specify the exact location of beg_b and end_b for a fixed time-stamp. Recall that we mark the time stamp τ_j+u with beg_b, for a pair j, k of consecutive a's at distance $> u-l$.

- Since there are several occurrences of the same time stamp, we want beg_b to the first symbol of the repeating time stamp τ_j+u while dealing with intervals $\langle l, u)$. We then add an extra formula $Fweak_{beg_b} = \Box^{\mathsf{w}}(\boxminus_{[0,0]}\neg beg_b)$ which says that beg_b is not after any symbol α having the same time stamp as beg_b.
- Likewise, while dealing with intervals $\langle l, u]$, beg_b should always be the last symbol at its timestamp τ_j+u. $Lweak_{beg_b} = \Box^{\mathsf{w}}(beg_b \to \Box_{[0,0]}\bot)$ which says that there are no symbols α after beg_b sharing the same time stamp as beg_b.
- In a similar way, the position of end_b depends on the left parantheses of the interval. Recall that we mark end_b at τ_k+l. If the interval is of the form $[l, u\rangle$, then we want end_b to be the first symbol with time stamp τ_k+l. Similarly, if the interval is of the form $(l, u\rangle$, then we want end_b to be the last symbol at time stamp τ_k+l. This can be done similarly as above.

I. Proof of Lemma 9

Proof. The temporal definition T is the conjunction of $C_1 = \Box^{\mathsf{w}}[b \leftarrow \diamondminus_{[l,\infty)} a]$ and $C_2 = \Box^{\mathsf{w}}[b \to \diamondminus_{[l,\infty)} a]$. Models ρ satisfying C_1 are those where b holds at all points i such that a holds somewhere from the beginning of ρ till $\tau_i - l$, that is in the prefix $[0, \tau_i - l]$ of ρ. Clearly, either there is no point marked a in the model, in which case $\Box^{\mathsf{w}}\neg a$ holds, or, whenever there is point i marked a, then b holds at all points in $[\tau_i+l, \infty)$. Thus, C_1 is equivalent to $\psi_1 = \Box^{\mathsf{w}}(\neg a) \vee \Box^{\mathsf{w}}[a \to \Box_{[l,\infty)} b]$.

Models satisfying C_2 are those in which points where $\neg\diamondminus_{[l,\infty)} a$ hold must be marked $\neg b$. Clearly, all points in $[0, l)$ must be marked $\neg b$. Also, if i is the point where a holds for the first time, then all points in $[\tau_i, \tau_i+l)$ should be marked $\neg b$. Thus, the formula $\psi_2 = \Box^{\mathsf{w}}(\neg a \wedge \neg b) \vee (\neg a \wedge \neg b)\ \mathsf{U}^{\mathsf{w}}(a \wedge \Box^{\mathsf{w}}_{[0,l)}\neg b)$ is equivalent to C_2. We thus have a formula $\psi_1 \wedge \psi_2 \in \mathsf{MTL}[\,\mathsf{U}_I]$ equivalent to T. $\square$

J. Proof of Correctness for Lemma 10

Proof. The proof of correctness proceeds in similar lines as Lemma 8. We give a proof of correctness on the construction of ρ' and the formula MARK, showing that $ENF_\Sigma(T) = \exists X.\mathsf{MARK}$. We start with a timed word ρ over $\Sigma\cup W$. We induct on the a's in ρ, and show that a point p of ρ is marked b iff $\rho, p \models \diamondminus_{[l,u)} a$.

- Given any point q of ρ marked a, MARK_b marks all points in $[\tau_q+l, \tau_q+u)$ with b.
- Lets look at the first a of ρ. Recall that the point where a holds for the first time is called $first_a$. The formula MARK_{first} ensures that all points of ρ that are at a distance $[0, l)$ from $first_a$ are marked $\neg b$. Also, all points in $[0, \tau_{first_a}]$ are also marked $\neg b$. Thus, MARK_{first} accounts for all points in $[0, \tau_{first_a}+l)$, while MARK_b marks all points in $[\tau_{first_a}+l, \tau_{first_a}+u)$ with b.
- Consider a point j in $dom(\rho)$ such that $j > first_a$, $a \in \sigma_j$ and assume that all the a's in $[0, \tau_j]$ have been accounted for: that is, all points in $[0, \tau_j+u)$ of ρ have been marked with b or $\neg b$ correctly. This is the inductive hypothesis. Now consider the next consecutive a occurring from j, call that point k. If $\tau_k - \tau_j \leq u-l$, then $\tau_k+l \leq \tau_j+u$, and by MARK_b, all points in $[\tau_k+l, \tau_k+u)$ will be marked b. Hence, we are done accounting for $[0, \tau_k+u)$. Hence, assume $\tau_k - \tau_j \in (u-l, u]$. In this case, $\tau_k+u > \tau_k+l > \tau_j+u$. MARK_b marks all points in $[\tau_k+l, \tau_k+u)$ with b; we need to reason that points in $[\tau_j+u, \tau_k+l)$ will be marked $\neg b$.

- We start marking points of ρ with new propositions, obtaining a simple extension ρ' of ρ. We start marking points where a holds good in ρ with propositions in $\{a_0, a_1\}$.
- Assume that point j is marked a_0, while k is marked a_1 by formula MARK_a. Let $d = \lceil \tau_k - \tau_k \rceil + l - u$, the closest integer $\geq$ the duration of the interval $[\tau_j + u, \tau_k + l)$. Formula $\mathsf{MARK}_{beg,end,d}$ marks j with beg_{db} and point k with end_{db}. Identifying point j as beg_{db} and point k with end_{db}, all points in $I^2_{j,k} = [\tau_{end_{db}} + l - d, \tau_{end_{db}} + l)$ are marked x_{0b} and all points in $I^1_{j,k} = [\tau_{beg_{db}} + u, \tau_{beg_{db}} + u + d)$ are marked y_{0b}. The points in $I^1_{j,k} \cap I^2_{j,k}$ are marked $\neg b$ by $\mathsf{MARK}_{\neg b,0}$.
- Since $[\tau_j + u, \tau_k + l) \subseteq I^1_{j,k} \cap I^2_{j,k}$, we have clearly marked all points in $[\tau_j + u, \tau_k + l)$ with $\neg b$. Also, points in $[\tau_j + u, \tau_k + l)$ are not handled by formula MARK_b, since these points are not in the $[l, u)$-future of any point marked a. Thus, points handled by $\mathsf{MARK}_{\neg b,0}$ and MARK_b are disjoint.
- Recall the discussion in Lemma 10 regarding free occurrences of x_{1b}, y_{1b} : as noted earlier, if $\{x_{1b}, y_{1b}\} \subseteq \sigma_p$ for any $p \in I^1_{j,k} \cap I^2_{j,k}$, there is no problem, since these points are anyway marked $\neg b$; if $\{x_{1b}, y_{1b}\} \subseteq \sigma_p$, for $p \notin I^1_{j,k} \cap I^2_{j,k}$, then either they lie in some $I^1_{h,m} \cap I^2_{h,m}$ corresponding points h, m such that $\tau_m - \tau_h \in (u - l, u]$, or p is a point handled by MARK_b. In the former case, there is no problem, while in the latter case, we get an inconsistent simple extension ρ' from ρ. Since we work only on consistent simple extensions, we rule out simple extensions where of the latter form.
- Thus, to summarize, we have accounted for all points $[0, \tau_k + u)$, being marked by one of $b, \neg b$ in consistent simple extensions.

- We are now left with the remaining part $[\tau_k + u, \tau_{|dom(\rho)|}]$. If $k \neq last_a$, we can extend the reasoning above to the next consecutive position after k, which is marked an a. In this way, we can account for all points of ρ in $[0, \tau_{last_a} + u)$. We just need to reason for $[\tau_{last_a} + u, \tau_{|dom(\rho)|}]$. Consider the point $last_a$. The formula MARK_{last} marks all points of ρ in the interval $[\tau_{last_a} + u, \tau_{|dom(\rho)|}]$ with $\neg b$.

The above argument shows that all points of ρ are marked b or $\neg b$ correctly. The $(\Sigma \cup W, X)$-simple extension ρ' reflects these markings. When we do a simple projection of ρ' with respect to X, we are left with ρ, the timed word over $\Sigma \cup W$ satisfying $ENF_\Sigma(T)$. On this ρ, at all positions, we have the correct marking with respect to b or $\neg b$. Clearly, a point p of ρ is marked b iff $\rho, p \models \diamondminus_{[l,u)} a$. Hence, $\rho \models ENF_\Sigma(T)$ iff $\rho' \models \mathsf{MARK}$.

Conversely, if we start with a timed word ρ' over $\Sigma \cup W \cup X$ satisfying MARK, then any point p of ρ' will be marked b iff $\diamondminus_{[l,u)} a$ holds good at p. Then $\rho' \setminus X$ will give a word ρ over $\Sigma \cup W$ that satisfies $ENF_\Sigma(T)$ iff ρ' is a $(\Sigma \cup W, X)$-simple extension.

□

K. Simple Versus Oversampling Projections: Formulae Size

Consider a formula $\varphi \in \mathsf{MTL}[\mathsf{U}_I, \diamondminus_{np}]$. First we discuss the case of eliminating $\diamondminus_{np}$ by simple projections. Assume that the number of past modalities in φ is n, of which there are n_b bounded past modalities and n_u unbounded past modalities. i.e, $n = n_b + n_u$.

1) The first step is flattening, resulting in φ_{flat}. This only increases the size of the formula linearly in n. Converting φ_{flat} to ENF again increases the size by a constant number; thus, $ENF_\Sigma(\varphi_{flat})$ has a size increase of $\mathcal{O}(n)$ with respect to φ.
2) Let us first look at the n_u unbounded past modalities. By Lemma 9, the elimination of each temporal definition involving an unbounded past modality results in adding 2 formulae $\in \mathsf{MTL}[\mathsf{U}_I]$, and hence, in 3 extra modalities. Thus, after elimination of all the n_u temporal definitions, we get a formula whose size is increased by $\mathcal{O}(n)$.
3) Now let us look at the elimination of the temporal definitions corresponding to the n_b bounded past modalities.
4) Lemma 10 deals with this. Look at formula 2(a) (in Case 2) introduced by Lemma 10. This results in $l - 1$ new formulae, and hence results in $\mathcal{O}(l)$ extra modalities. Thus, the number of extra modalities introduced after elimination of all the n_b temporal definitions corresponding to bounded past modalities is $\leq nl_{max}$, where l_{max} is the maximal lower bound of all bounded past modalities in φ. Assuming constants are encoded in binary, $\mathcal{O}(n_b l_{max})$ is pseudo polynomial; hence, the formula obtained by simple projections, ψ_1 has in the worst case, an exponential increase in size over φ. Just to illustrate, $l_{max} = 10^{10}$ will really blow up!
5) Note that Lemma 10 can further be optimized by changing the formula 2(a), 2(b), 3(a) and 3(b) in Case 2. Recall that formula 2(a) is $\Box^{\mathsf{w}}(x_{t+1+l-u\ b} \leftrightarrow (a \wedge (\neg a\ \mathsf{U}_{(t,t+1]} a)))$, with $t \in \{u - l, \ldots, u - 1\}$, 2(b) is $\Box^{\mathsf{w}}(y_{db} \leftrightarrow (a \wedge (\neg a\ \mathsf{S} x_{db})))$, while formula 3(a) is $\bigwedge_{c \in \{0,1\}} \Box^{\mathsf{w}}((x_{db} \wedge a_c) \rightarrow \Box_{[u,u+d)} x_c)$ and formula 3(b) is $\bigwedge_{c \in \{0,1\}} \Box^{\mathsf{w}}((y_{db} \wedge a_c) \rightarrow \Box^{\mathsf{w}}_{[l-d,l)} y_{1-c}))$, where $d \in \{1, \ldots, l\}$. The "bounding" interval between two consecutive a's was considered as a unit interval here : we were considering the interval lengths to lie in $(u - l, u - l + 1], (u - l + 1, u - l + 2]$ and so on till $(u - 1, u]$. This resulted in $l - 1$ formulae. Had we chosen intervals of size 2 instead of 1, we would have considered the intervals as $(u - l, u - l + 2], (u - l + 2, u - l + 4], \ldots, (u - 2, u]$, resulting in $\frac{l}{2}$ formulae. In general, we could have chosen as "period" any μ that gives rise to $\frac{l}{\mu}$ formulae. Clearly, since Case 2 in Lemma 10 considers $\tau_k - \tau_j \in (u - l, u]$, the maximum period we can consider is $u - l$ i.e, $1 \leq \mu \leq u - l$. When $\mu = u - l$, we get $\frac{l}{u-l}$ formulae. In this case, replacing 2(a),2(b),3(a),3(b), we get

- 2(a) by $\Box^{\mathsf{w}}(x_{t+1+l-u\ b} \leftrightarrow (a \wedge (\neg a\, \mathsf{U}_{(t,t+u-l]} a)))$ for $t \in \{u-l, 2(u-l), \ldots, \frac{l(u-l)}{u-l}\}$,
- 2(b) by $\Box^{\mathsf{w}}(y_{\kappa b} \leftrightarrow (a \wedge (\neg a\, \mathsf{S} x_{\kappa b})))$
- 3(a) by $\bigwedge_{c\in\{0,1\}} \Box^{\mathsf{w}}((x_{\kappa b} \wedge a_c) \rightarrow \Box_{[u,u+\kappa(u-l))} x_c)$
- 3(b) by $\bigwedge_{c\in\{0,1\}} \Box^{\mathsf{w}}((y_{\kappa b} \wedge a_c) \rightarrow \Box^{\mathsf{w}}_{[l-\kappa(u-l),l)} y_{1-c}))$ where $\kappa \in \{1, \ldots, \frac{l}{u-l}\}$.

In this case, we get an increase of $\mathcal{O}(\frac{nl}{u-l})$ over the size of φ, as opposed to an increase of $\mathcal{O}(nl_{max})$. Asymptotically, this is not a big saving, so we can stick to $\mu = 1$.

Now we discuss the case of oversampled projections. Lemma 7 discussed the case of unbounded past modalities and Lemma 8 the case of bounded past modalities. In both cases, it can be seen that the resultant formulae had an increase of size by a constant number, while eliminating each temporal definition. Thus, the total increase of size in the resultant formula $\psi_2 \in \mathsf{MTL}[\,\mathsf{U}_I]$ is only $\mathcal{O}(n)$.

L. *Eliminating* S *from* $\mathsf{MTL}[\,\mathsf{U}_I,\, \mathsf{S}]$

Given a formula $\varphi \in \mathsf{MTL}[\,\mathsf{U}_I,\, \mathsf{S}_I]$ over Σ, we first flatten the formula to obtain formula φ_{flat} over $\Sigma \cup W$. In this section, we elaborate [8], [3] on removing the temporal definitions of the form $[r \leftrightarrow (c\, \mathsf{S} f)]$ from φ_{flat}, using future operators. We use the short form $\mathsf{O}\varphi$ to denote $false\, \mathsf{U}\varphi$.

$[r \leftrightarrow (c\, \mathsf{S} f)]$ will be replaced by a conjunction ν_r of the following future formulae:

- $\varphi_1 : \Box^{\mathsf{w}}(f \rightarrow \mathsf{O} r)$
- $\varphi_2 : \neg r$
- $\varphi_3 : \Box^{\mathsf{w}}[(r \wedge c) \rightarrow \mathsf{O} r]$
- $\varphi_4 : \Box^{\mathsf{w}}[r \wedge (\neg c \wedge \neg f) \rightarrow \mathsf{O}\neg r]$
- $\varphi_5 : \Box^{\mathsf{w}}[(\neg r \wedge \neg f) \rightarrow \mathsf{O}\neg r]$

For example, consider the formula $\varphi = (a \wedge (b \wedge (c\, \mathsf{U}_{(1,2)}[(d\, \mathsf{S} e) \wedge f])))$ built from $\Sigma = \{a, b, c, d, e, f\}$.

The flattened version $\varphi_{flat} = (a \wedge b \wedge w_2) \wedge T_1 \wedge T_2$, where $T_1 = \Box^{\mathsf{w}}[(d\, \mathsf{S} e) \leftrightarrow w_1]$ and $T_2 = \Box^{\mathsf{w}}[w_2 \leftrightarrow c\, \mathsf{U}_{(1,2)}[w_1 \wedge f]]$. φ_{flat} is built from $\Sigma \cup W$, where $W = \{w_1, w_2\}$.

Replace T_1 with ν_{w_1} to obtain the formula $\psi = (a \wedge b \wedge w_2) \wedge \nu_{w_1} \wedge T_2 \in \mathsf{MTL}[\,\mathsf{U}_I]$. ψ is also built from $\Sigma \cup W$ and is equivalent to φ_{flat}. It can be seen that $\varphi = \exists W.\varphi_{flat} = \exists W.\psi$.

M. *Proof of Lemma 11*

We prove that the $\mathsf{MTL}[\mathsf{U}_{np}, \mathsf{S}_I], \mathsf{MTL}[\mathsf{U}_I, \mathsf{S}_{np}]$ are strictly less expressive than $\mathsf{MTL}[\,\mathsf{U}_I,\, \mathsf{S}_I]$ using EF Games. We omit the game strategies here and give the candidate formula and pair of words.

(i) $\mathsf{MTL}[\Diamond_I] \nsubseteq \mathsf{MTL}[\,\mathsf{U}_{np},\, \mathsf{S}_I]$

We consider a formula in $MTL^{pw}[\Diamond_I]$, $\varphi = \Diamond_{(0,1)}\{a \wedge \neg\Diamond_{[1,1]}(a \vee b)\}$. For an n-round game, consider the words $w_1 = W_a W_b$ and $w_2 = W_a W_b'$ with

- $W_a = (a,\delta)(a,2\delta)\ldots(a,i\delta-\kappa)\underline{(a,i\delta)}\ldots(a,n\delta)$
- $W_b = (b,1+\delta)(b,1+2\delta)\ldots(b,1+i\delta-\kappa)\underline{(b,1+i\delta)}\ldots(b,1+n\delta)$
- $W_b' = (b,1+\delta)(b,1+2\delta)\ldots(b,1+(i-1)\delta)(b,1+i\delta-\kappa)(b,1+i\delta)(b,1+(i+1)\delta)\ldots(b,1+n\delta)$

$w_1 \models \varphi$, but $w_2 \nvDash \varphi$. The underlined b in W_b shows that there is a b at distance 1 from a; however, this is not the case with W_b'. The key observation for duplicator's win in an $\mathsf{U}_{NS},\, \mathsf{S}_I$ game is that (a) any non-singular future move of spoiler can be mimicked by the duplicator from $W_a W_b$ or $W_a W_b'$ (b) for any singular past move made by spoiler on $W_a W_b$, duplicator has a reply from $W_a W_b'$. The same holds for any singular past move of spoiler made from $W_a W_b'$.

(ii) $\mathsf{MTL}[\Diamond_I, \diamondminus_I] \nsubseteq \mathsf{MTL}[\,\mathsf{U}_I,\, \mathsf{S}_{np}]$

We consider a formula in $MTL^{pw}[\Diamond_I]$, $\phi' = \Diamond\{b \wedge \neg\diamondminus_{[1,1]}(a \vee b)\}$. We show that there is no way to express this formula in $\mathsf{MTL}[\,\mathsf{U}_I,\, \mathsf{S}_{np}]$. This is symmetrical to (i). For an n round game, consider the words $w_1 = W_a W_b$ and $w_2 = W_a' W_b$ with

- $W_a = (a,\delta)(a,2\delta)\ldots(a,(i-1)\delta)(a,i\delta-\kappa)(a,i\delta)\ldots(a,n\delta)$
- $W_a' = (a,\delta)(a,2\delta)\ldots(a,(i-1)\delta)(a,i\delta)\ldots(a,n\delta)$
- $W_b = (b,1+\delta)(b,1+2\delta)\ldots(b,1+(i-1)\delta)\underline{(b,1+i\delta-\kappa)}(b,1+i\delta)\ldots(b,1+n\delta)$

$w_1 \nvDash \varphi', w_2 \models \varphi'$. The underlined b in W_b shows that there is an a at past distance 1 in W_a, but not in W_a'. The key observation for duplicator's win in an n-round $\mathsf{U}_I,\, \mathsf{S}_{NS}$ game is that (a) any non-singular past move by spoiler from W_a, W_b or from W_a', W_b can be answered by duplicator, (b) for any singular future move made by spoiler on W_a, W_b, duplicator has a reply from W_a', W_b. The same holds for any singular future move of spoiler made from W_a', W_b.

(iii) $\mathsf{MTL}[\Diamond_{np}, \diamondminus_{np}] \nsubseteq \mathsf{MTL}[\,\mathsf{U}_I,\, \mathsf{S}]$. We consider the $\mathsf{MTL}[\Diamond_{np}, \diamondminus_{np}]$ formula $\varphi'' = \Diamond_{(1,2)}[a \wedge \neg\diamondminus_{(1,2)} a]$, and show that there is no way to express it using U_I, S. For an n round game, consider the words $w_1 = W_1 W_2$ and $w_2 = W_1 W_2'$ with

- $W_1 = (a, 0.5+\epsilon)\ldots(a, 0.5+n\epsilon)(a, 0.9+\epsilon)\ldots(a, 0.9+n\epsilon)$
- $W_2 = \underline{(a, 1.5)}(a, 1.6+\epsilon)(a, 1.6+2\epsilon)\ldots(a, 1.6+n\epsilon)$
- $W_2' = (a, 1.6+\epsilon)(a, 1.6+2\epsilon)\ldots(a, 1.6+n\epsilon)$

for a very small $\epsilon > 0$. Clearly, $w_1 \models \varphi'', w_2 \nvDash \varphi''$. The underlined a in W_2 shows the a in (1,2) which has no a in $\diamondminus_{(1,2)}$. The key observation for duplicator's win in an n-round U_I, S game is that (a) when spoiler picks any position in W_1, duplicator can play copy cat, (b) when spoiler picks $(a, 1.5)$ in W_2 as part of a future $(0,1)$ move from W_1, duplicator picks $0.9+n\epsilon$ in W_2'. All until, since moves from the configuration $[(a.1.5), (a, 0.9+n\epsilon)]$ are symmetric.