The non-existence of a $[[13, 5, 4]]$-quantum stabilizer code

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Abstract

We solve one of the oldest problems in the theory of quantum stabilizer codes by proving the non-existence of quantum $[[13, 5, 4]]$-codes.

1 Introduction

After the determination of the parameter spectrum of additive quantum codes of distance 3 (see [2]) the oldest open existence problem for quantum stabilizer codes concerns the parameters $[[13, 5, 4]]$. We give a negative answer:

Theorem 1. *There is no* $[[13, 5, 4]]$*-quantum stabilizer code.*

The reduction of the problem of quantum error-correction to codes in symplectic geometry essentially is in [7]. For a geometric approach see also [6]. We use the following definitions:

Definition 1. *Let k be such that $2k$ is a positive integer. An additive quaternary $[n,k]_4$-code $\mathcal{C}$ (length n, dimension k) is a $2k$-dimensional subspace of $\mathbb{F}_2^{2n}$, where the coordinates come in pairs of two. We view the codewords as n-tuples where the coordinate entries are elements of $\mathbb{F}_2^2$.*

A **generator matrix** *of $\mathcal{C}$ is a binary $(2k,2n)$-matrix whose rows form a basis of the binary vector space $\mathcal{C}$.*

In the case of quantum stabilizer codes we view the ambient space $\mathbb{F}_2^{2n}$ as a binary symplectic space, where each of the n parameter sections corresponds to a hyperbolic plane, equivalently a 2-dimensional symplectic space. Each codeword is therefore a vector in the $2n$-dimensional symplectic geometry over $\mathbb{F}_2$.

Definition 2. *A quaternary quantum stabilizer code is an additive quaternary code C which is contained in its dual, where duality is with respect to the symplectic form.*

Describe C by a generator matrix M. Each of the n coordinate sections contains 2 columns which we view as points in binary projective space. The geometric description of the quantum code is in terms of the system of n lines (the codelines) generated by those n pairs of points.

Definition 3. *Let C be a quaternary additive code of length n, with generator matrix M. The* **strength** *of C is the largest number t such that any t codelines are in general position.*

Observe that the strength $t(C)$ is one less than the dual distance.

Definition 4. *An $[[n,m,d]]$-code C where $m>0$ is a quaternary quantum stabilizer code of binary dimension $n-m$ satisfying the following: any codeword of $C^\perp$ having weight at most $d-1$ is in C.*

The code is **pure** *if $C^\perp$ does not contain codewords of weight $\leq d-1$, equivalently if C has strength $t \geq d-1$.*

An $[[n,0,d]]$-code C is a self-dual quaternary quantum stabilizer code of strength $t=d-1$.

The optimal parameters of quantum stabilizer codes of length ≤ 13 are known, with the sole exception of parameters $[[13,5,4]]$ (see the database in [9]). The remainder of the paper is dedicated to a proof of Theorem 1. Assume C is a $[[13,5,4]]$-quantum code. In the next section we show that C is necessarily pure.

2 The purity of the code

Proposition 1. *Let C be a $[[13, 5, 4]]$-quantum code. Then C is pure.*

In general the geometric objects defined by the column pairs of a generator matrix (which we called codelines) may be lines, points or even the empty set (if the corresponding pair of columns has all entries $= 0$). The following basic fact follows from the definition:

Lemma 1. *Whenever some $\leq d-1$ codelines of a quantum code of distance d are not in general position there is a hyperplane containing all the remaining codelines.*

In the remainder of this section we prove Proposition 1. It follows from Proposition 3.1 of [6] that the codeobjects of C are indeed lines and that no line occurs more than once. Quantum code C is therefore described by a set of 13 different lines in $PG(7, 2)$. Observe that the (quaternary) minimum weight of nonzero words in $C^{\perp}$ therefore is ≥ 2. As we are assuming that C is not pure there are three codelines L_1, L_2, L_3 contained in a subspace $PG(4, 2)$.

Lemma 2. *Let L_i, L_j, L_k be three codelines not in general position. Let $v(\{L_i, L_j, L_k\}) \in C$ a nonzero codeword with support in coordinates i, j, k.*

Observe that $v(\{L_i, L_j, L_k\})$ in Lemma 2 has weight 2 or 3.

The 10 remaining codelines are in a hyperplane H. In the sequel we use basic facts concerning additive quaternary codes, see [4]. The nonexistence of a quaternary additive $[10, 6.5, 4]$ and its dual shows that the family of remaining codelines cannot have strength 3. It follows that L_4, L_5, L_6 are in a subspace $PG(4, 2)$. By Lemma 1 there is a hyperplane containing all codelines $\notin \{L_4, L_5, L_6\}$. This shows that the 7 codelines $\notin \{L_1, \ldots, L_6\}$ are contained in a secundum S (a $PG(5, 2)$). The non-existence of a quaternary $[7, 4, 4]$-code and its dual shows that three of the seven remaining lines (L_7, L_8, L_9, say) are not in general position. It follows from Lemma 1 that $L_{10}, \ldots, L_{13}$ are contained in a subspace $PG(4, 2)$.

We start from the information that some four lines which we now call L_1, L_2, L_3, L_4 are in a subspace $PG(4, 2)$. The codewords $v(\{L_1, L_2, L_3\})$ and $v(\{L_2, L_3, L_4\})$ show that there is a secundum S (a $PG(5, 2)$) containing the remaining 9 codelines. The usual argument, based on the non-existence of a quaternary $[9,6,4]$-code, shows that there is a $PG(4,2)$ containing 6 codelines.

Start again and use the knowledge that some six codelines $L_1,\ldots,L_6$ are contained in a subspace $PG(4,2)$. Applying our argument to subsets of three codelines shows that the remaining 7 codelines are contained in a $PG(4,2)$.

Finally we use the fact some seven codelines $L_1,\ldots,L_7$ are contained in a $PG(4,2)$. Apply our argument to the following triples of codelines:

- $\{L_1,L_2,L_3\}$ yielding $v(\{L_1,L_2,L_3\})$ which we can choose to have nonzero entries in coordinates $1,2$ (and possibly 3),
- $\{L_2,L_3,L_4\}$ where we choose notation such that 2 is in the support of $v(\{L_2,L_3,L_4\})$, and
- $\{L_3,L_4,L_5\}$

This yields the contradiction $L_6=L_7$. Proposition 1 has been proved.

3 The structure of the proof

Let C be a $[[13,5,4]]$ quantum code, described by a set of lines $L_1,\ldots,L_{13}$ in the ambient space U (a $PG(7,2)$). We know that the strength is 3. Let $e_1,\ldots,e_8$ be a basis of the underlying vector space V and choose $L_1=\langle e_1,e_2\rangle,\ L_2=\langle e_3,e_4\rangle$. Consider the factor space $V/\langle e_1,e_2,e_3,e_4\rangle$ and the corresponding $PG(3,2)$ which we call Π. We work in U and in the factor space Π. Because of strength 3 each codeline $L_i, i>2$ defines a line in Π.

Definition 5. *Let g be a line of Π (a $PG(3,2)$). Define the* **weight** *$w(g)$ of g as 2 less than the number of codelines contained in the preimage of g. For points P and planes E of Π define*

$$w(P)=\sum_{P\in g}w(g),\ w(E)=\sum_{g\subset E}w(g).$$

The geometric meaning of $w(P)$ and $w(E)$ is as follows: $w(P)+2$ is the number of codelines which meet the preimage of P (a $PG(4,2)$) nontrivially, $w(E)+2$ is the number of codelines contained in the preimage of E (a hyperplane $PG(6,2)$).

Proposition 2. *We have* $\sum_g w(g) = 11$ *where the sum is over all lines* g *of* Π. *For each line* h *of* Π *the number of lines of our multiset which intersect* h *nontrivially is odd.*

Proof. We think of the multiplicities $w(g)$ as defining a multiset, clearly of 11 lines. Let h be a line of Π. Its preimage under the canonical mapping onto Π is a secundum of the ambient space U. The orthogonality condition of Definition 2 translates as follows in geometric terms: for each secundum S of U the number of codelines meeting S nontrivially is odd (see also [6]). Applying this to the preimage of line h yields our claim. □

We refer to the condition of Proposition 2 as the **quantum condition.** Observe that in the quantum condition the sum is over all lines, including h itself: each of the 35 lines of Π gives a condition, and the sum is over all g.

As C is pure sets of strength 3 play an important role.

4 Sets of strength 3

Definition 6. *A set of objects in a projective space has* **strength** 3 *if any subset of three of those objects are in general position. An* (n, m)*-set is a set of strength* 3 *consisting of* n *lines and* m *points.*

Proposition 3. *Assume* H *is a hyperplane in* U *containing precisely* n *codelines. Then* H *meets the union of the codelines in an* $(n, 13 - n)$*-set whose points meet each hyperplane* S *of* H *in a cardinality whose parity is different from* n.

Proof. Each of the 13 codelines either is contained in H or it meets H in a point. This proves the first part. Let S be a hyperplane of H. Then S is a secundum of U and therefore meets an odd number of codelines. As S does meet the n codelines contained in H the second statement follows. □

In order to obtain bounds on $w(P), w(g), w(E)$ consider the corresponding preimage spaces ($PG(4, 2), PG(5, 2), VPG(6, 2)$, respectively) with their (n, m)-sets formed by the intersection with codelines.

Lemma 3. *A* $(2, m)$*-set of strength* 3 *in* $PG(4, 2)$ *has* $m \leq 4$. *All these sets are embedded in a uniquely determined* $(2, 4)$*-set.*

Proof. The lines are without restriction $L_1 = \langle e_1, e_2 \rangle$, $L_2 = \langle e_3, e_4 \rangle$, the points of the $(2,4)$-set of strength 3 can be chosen as

$$e_5,\ e_1 + e_3 + e_5,\ e_2 + e_4 + e_5,\ e_1 + e_2 + e_3 + e_4 + e_5.$$

□

Of particular importance are the hyperoval in $PG(2,4)$ and the $[7, 3.5, 4]_4$-codes.

Lemma 4. *An* $(n,0)$*-set in* $PG(5,2)$ *has* $n \leq 6$*. For each* n *it is uniquely determined. They are all embedded in the uniquely determined* $(6,0)$*-set, which we call the* **binary hyperoval.** *Consider* (n,m)*-sets in* V_6*. If* $n = 6$*, then* $m = 0$*. If* $n = 5$*, then* $m \leq 2$*. If* $n = 4$*, then* $m \leq 4$*.*

Proof. The first 3 lines can be chosen as usual:

$$L_1 = \langle e_1, e_2 \rangle, L_2 = \langle e_3, e_4 \rangle, L_3 = \langle e_5, e_6 \rangle.$$

There are exactly 27 points, the transversal points, each forming a $(3,1)$-set with $\{L_1, L_2, L_3\}$. Then $L_4 = \langle e_1 + e_3 + e_5, e_2 + e_4 + e_6 \rangle$ is the essentially unique fourth line. There remain 6 points each forming a $(4,1)$-set together with $L_1, \dots, L_4$. These are exactly the six points on the remaining lines

$$L_5 = \langle e_1 + (e_3 + e_4) + e_6, e_2 + e_3 + (e_5 + e_6) \rangle, L_6 = \langle e_1 + e_4 + (e_5 + e_6), e_2 + (e_3 + e_4) + e_5 \rangle$$

of the binary hyperoval. The uniqueness statement follows. □

We chose the term **binary hyperoval** as the $(6,0)$-set in $PG(5,2)$ is the binary image of the hyperoval in $PG(2,4)$. It is well known that the hyperoval has the symmetric group S_6 as its group of automorphisms. The automorphism group of the binary hyperoval has order $3 \times 6!$ where the additional factor 3 stems from the multiplicative group of the field.

As for the case of (n,m)-sets in $PG(6,2)$ we use earlier work in relation to additive $[7, 3.5, 4]_4$-codes, see [3, 5].

Proposition 4. *There is no* $(7,0)$*-set in* $PG(5,2)$ *and no* $(8,0)$*-set in* $PG(6,2)$*. There are precisely three non-equivalent* $(7,0)$*-sets in* $PG(6,2)$*. Exactly one of them defines a self-dual code with respect to the Euclidean form (the dot product).*

Proof. A $(7,0)$-set in V_6 would define an additive $[7,4,4]_4$-code. In the same way an $(8,0)$-set in V_7 would lead to an $[8,4.5,4]_4$-code. Those codes do not exist. □

The classification of $(7,0)$-sets in $PG(6,2)$ has been carried out independently several times, most recently in Danielsen-Parker [8] and Han-Kim [10].

Proposition 5. *Consider the three* $(7,0)$*-sets in* $PG(6,2)$*. The number* c *of points that complete them to a* $(7,1)$*-set is* $c=1$, $c=2$ *and* $c=8$*, respectively. The case of* 8 *extension points occurs when the code generated by the* $(7,0)$*-set is self-dual. This* $(7,0)$*-set can be extended to a uniquely determined* $(7,7)$*-set and to a* $(7,6)$*-set which is uniquely determined up to projectivity.*

Proof. This is a computer result. The self-dual code is the one with 8 extension points. Here it is:

$$\left(\begin{array}{c|c|c|c|c|c|c} L_1 & L_2 & L_3 & L_4 & L_5 & L_6 & L_7 \\ 00 & 00 & 01 & 00 & 01 & 01 & 01 \\ 01 & 00 & 00 & 01 & 00 & 01 & 01 \\ 01 & 01 & 00 & 00 & 01 & 00 & 01 \\ 00 & 00 & 10 & 10 & 10 & 00 & 10 \\ 10 & 00 & 00 & 10 & 10 & 10 & 00 \\ 00 & 10 & 00 & 00 & 10 & 10 & 10 \\ 11 & 11 & 11 & 11 & 11 & 11 & 11 \end{array}\right)$$

The eight extension points are $P_0=(0:0:0:0:0:0:1)$ and the columns of

$$\left(\begin{array}{c|c|c|c|c|c|c} 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right)$$

forming a set E. The $(7,0)$-set has an automorphism group G of order 42 which fixes P_0, preserves the hyperplane H with equation $x_7=0$ and acts transitively on E. Consider the cone with vertex P_0 consisting of the lines from P_0 to the points $L_i \cap H$. The third points on those lines make up E. It follows that the uniquely determined $(7,7)$-set is defined by point set E and the essentially uniquely determined $(7,6)$-set is obtained by omitting one point from E. □

5 The weights in the factor space $PG(3,2)$

Consider the weights $w(g)$ of lines in $\Pi = PG(3,2)$ and the induced weights $w(P), w(E)$ on points and planes.

Lemma 5. *For points, lines, planes of* Π *we have* $w(P) \leq 4, w(g) \leq 3, w(E) \leq 4$.

Proof. The statement on points follows from Lemma 3. Proposition 4 shows that $w(E) \leq 5$. The hyperplane H corresponding to a plane E of weight n yields an $(n+2, 11-n)$-set. Assume $n = 5$. Then there is a $(7,6)$-set in V_7. By Proposition 5 the 7 lines are uniquely determined as only the self-dual cyclic example has more than 2 extension points. There is a uniquely determined $(7,6)$-set in $PG(6,2)$ (see Proposition 71classiprop), but it does not satisfy the quantum condition of Proposition 3. It follows $w(E) \leq 4$. Assume now $w(g) = 4$. The quantum condition shows that it is contained in a plane of weight 5, contradiction. □

We can improve on Lemma 5:

Proposition 6. $w(E) \leq 3$ *for each plane* E *of* Π. *Each hyperplane* H *of* U *contains at most* 5 *codelines. The codelines define a quaternary* $[13,4,8]$*-code.*

Proof. All three statements of the proposition are equivalent. Assume $w(E) = 4$. Assume at first E contains a line g such that $w(g) = 3$. Then in the $PG(5,2)$ corresponding to g we have the lines $L_1, \dots, L_5$ corresponding to an oval in $PG(2,4)$ and $L_6 = \langle e_1+e_4+e_5+e_6, e_7\rangle$ in the hyperplane corresponding to E. Those 6 lines must be completable to a $(6,7)$-system in $PG(6,2)$ which satisfies the quantum condition: each hyperplane of the $PG(6,2)$ must meet the set of 7 extension points in odd cardinality. A computer search shows that this problem has no solution.

Assume next E contains a line g of weight 2. We have the usual lines $L_1, \dots, L_4$ in $PG(5,2)$ and two more lines in the hyperplane which are not in the secundum. By Lemma 4 one of those lines can be chosen as $L =$

$\langle e_1 + e_3 + e_4 + e_6, e_7 \rangle$. It remains to find the one remaining line and the system of 7 points in $PG(6,2)$ completing it to a $(6,7)$-system that satisfies the quantum condition. A computer search shows that there is no solution. At this point we have shown the following:

- Each hyperplane H of U which contains 6 codelines is generated by each 4 of its codelines.

This follows directly from the fact that for each plane E of weight 4 of Π we have $w(g) \leq 1$ for each line $g \subset E$. Observe that we could have started from any pair of codelines instead of L_1, L_2 and considered the hyperplane corresponding to a plane of weight 4 in the factor space.

A computer search showed that there are exactly four families of 6 lines in $PG(6,2)$ satisfying the following:

- Any three of the lines are in general position.

- Any four of the lines generate the ambient space $PG(6,2)$.

Here they are:

$$\left(\begin{array}{c|c|c|c|c|c}
L_1 & L_2 & L_3 & L_4 & L_5 & L_6 \\
10 & 00 & 00 & 10 & 00 & 01 \\
01 & 00 & 00 & 00 & 10 & 10 \\
00 & 10 & 00 & 10 & 10 & 10 \\
00 & 01 & 00 & 00 & 01 & 10 \\
\hline
00 & 00 & 10 & 10 & 01 & 11 \\
00 & 00 & 01 & 00 & 10 & 01 \\
00 & 00 & 00 & 01 & 01 & 01
\end{array}\right)$$

$$\left(\begin{array}{c|c|c|c|c|c}
L_1 & L_2 & L_3 & L_4 & L_5 & L_6 \\
10 & 00 & 00 & 10 & 00 & 11 \\
01 & 00 & 00 & 00 & 10 & 10 \\
00 & 10 & 00 & 10 & 10 & 10 \\
00 & 01 & 00 & 00 & 01 & 11 \\
\hline
00 & 00 & 10 & 10 & 01 & 11 \\
00 & 00 & 01 & 00 & 10 & 11 \\
00 & 00 & 00 & 01 & 01 & 01
\end{array}\right)$$

$$\left(\begin{array}{c|c|c|c|c|c} L_1 & L_2 & L_3 & L_4 & L_5 & L_6 \\ 10 & 00 & 00 & 10 & 00 & 11 \\ 01 & 00 & 00 & 00 & 10 & 10 \\ 00 & 10 & 00 & 10 & 01 & 10 \\ 00 & 01 & 00 & 00 & 10 & 11 \\ \hline 00 & 00 & 10 & 10 & 10 & 11 \\ 00 & 00 & 01 & 00 & 01 & 11 \\ 00 & 00 & 00 & 01 & 01 & 01 \end{array}\right)$$

$$\left(\begin{array}{c|c|c|c|c|c} L_1 & L_2 & L_3 & L_4 & L_5 & L_6 \\ 10 & 00 & 00 & 10 & 10 & 01 \\ 01 & 00 & 00 & 00 & 11 & 10 \\ 00 & 10 & 00 & 10 & 11 & 10 \\ 00 & 01 & 00 & 00 & 01 & 10 \\ \hline 00 & 00 & 10 & 10 & 11 & 11 \\ 00 & 00 & 01 & 00 & 11 & 01 \\ 00 & 00 & 00 & 01 & 01 & 01 \end{array}\right)$$

In each of those cases another computer program shows that the corresponding family F of codelines cannot be completed by a set S of 7 points in $H = PG(6,2)$ which together with the codelines form a $(6,7)$-set of strength 3 and such that the quantum condition is satisfied. □

6 Excluding a special configuration

In this section we show the following:

Proposition 7. *Any five codelines generate either the ambient space U or a hyperplane.*

Assume this is not the case. If some five codelines were in a $PG(4,2)$ then some hyperplane would contain six codelines, contradicting Proposition 6. Assume therefore some five codelines generate a secundum S. In terms of the factor space Π this means there is some line g_0 of weight 3. As $w(E) \leq 3$ for each plane E of Π this implies $w(g) = 0$ for each line $g \neq g_0$ intersecting g_0 nontrivially.

The codelines in S can be chosen as $L_1, \ldots, L_5$ according to Lemma 4. Let now $H \supset S$ be a hyperplane and $\mathcal{M} = \{M_0, \ldots, M_7\}$ the points of intersection with the eight remaining codelines. Then $M_i \notin S$. Without restriction $M_0 = e_7$. Write $M_i = e_7 + w_i$. Then the following conditions must be satisfied:

1. $w_i \notin L_1 \cup \ldots L_5$ for $i = 1, \ldots, 7$.
2. $w_i + w_j \notin L_1 \cup \ldots L_5$ for $i \neq j$.
3. Let W be the $(7, 8)$-matrix with the elements of $\mathcal{M}$ as columns. Then all codewords of the code generated by W have even weights.

Here the last condition represents the quantum condition: each hyperplane of H meets $\mathcal{M}$ in even cardinality.

A computer search showed that up to equivalence there are 12 systems $\mathcal{M}$ satisfying the conditions above.

Here is the structure of the generator matrix that far:

$$
\left(\begin{array}{c|c|c|c|c||c|c|c|c|c|c|c|c}
L_1 & L_2 & L_3 & L_4 & L_5 & L_6 & L_7 & L_8 & L_9 & L_{10} & L_{11} & L_{12} & L_{13} \\
10 & 00 & 00 & 10 & 10 & 00 & 1 & & & & & & \\
01 & 00 & 00 & 01 & 01 & 00 & 0 & & & & & & \\
00 & 10 & 00 & 10 & 11 & 00 & 1 & & & & & & \\
00 & 01 & 00 & 01 & 10 & 00 & 0 & & & & & & \\
\hline
00 & 00 & 10 & 10 & 01 & 00 & 0 & & & & & & \\
00 & 00 & 01 & 01 & 11 & 00 & 0 & & & & & & \\
00 & 00 & 00 & 00 & 00 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 \\
00 & 00 & 00 & 00 & 00 & 01 & 01 & 01 & 01 & 01 & 01 & 01 & 01
\end{array}\right)
$$

For each choice of $\mathcal{M}$ we need to determine the solutions of the problem in $PG(3, 2)$ (the last four rows of the generator matrix). Finally the generator matrix needs to be completed. The computer showed that this completion is impossible.

7 Completing the proof

Let $L_1, \ldots, L_5$ be codelines not generating the ambient space. They generate a hyperplane H. Consider the corresponding $(5, 8)$-set in H. The lines define an additive $[5, 3.5]_4$-code of strength 3. As its dual, a $[5, 1.5, 4]_4$-code, is uniquely determined (corresponding to a set of 5 lines in the Fano plane), the same is true of the code itself. We can therefore choose

$$L_1 = \langle e_1, e_2 \rangle, L_2 = \langle e_3, e_4 \rangle, L_3 = \langle e_5, e_6 \rangle,$$

$$L_4 = \langle e_1 + e_3 + e_5, e_7 \rangle, L_5 = \langle e_1 + e_4 + e_6, e_2 + e_3 + e_7 \rangle.$$

No four of those are on a hyperplane. How many points complete them to a $(5, 1)$-set of strength 3? There are 15 points on the lines, $10 \times 3/2 = 15$ in the intersection of the two spaces generated by two lines and 10×6 further points on spaces generated by two lines. This leaves space for $127 - 90 = 37$ extension points. Within this set of 37 points we have to find a subset $\mathcal{M}$ of eight points which satisfy the conditions

- $\mathcal{M}$ is a cap.
- Secants of $\mathcal{M}$ do not meet any of the lines L_i.
- Let W be the $(7, 8)$-matrix with the elements of $\mathcal{M}$ as columns. Then all codewords of the code generated by W have even weights.

The general form of the generator matrix is

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}	L_{11}	L_{12}	L_{13}
10	00	00	10	10	0							
01	00	00	00	01	0							
00	10	00	10	01	0							
00	01	00	00	10	0							
00	00	10	10	00	0							
00	00	01	00	10	0							
00	00	00	01	01	0							
00	00	00	00	00	01	01	01	01	01	01	01	01

A computer program did the following:

- Determine the solutions $\mathcal{M}$.
- For each solution $\mathcal{M}$ determine the 8 lines in Π completing the projections of the eight points of $\mathcal{M}$ such that the orthogonality condition on the last four rows of the generator matrix are satisfied.

- Complete the generator matrix.

Observe that in the second step the projection to Π may lead to repeated points. This has to be taken into account when adapting the lines in Π to the points of $\mathcal{M}$. The computer search showed that there are no solutions. This completes the proof of Theorem 1.

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