Convergence Analysis of the Energy and Helicity Preserving Scheme for Axisymmetric Flows

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Abstract

We give an error estimate for the Energy and Helicity Preserving Scheme (EHPS) in second order finite difference setting on axisymmetric incompressible flows with swirling velocity. With careful and detailed truncation error analysis near the geometric singularity and far field decay estimate for the stream function, we have achieved optimal error bound using weighted energy estimate. A key ingredient in our *a priori* estimate is the permutation identity associated with the Jacobians, which is also a unique feature that distinguishes EHPS from standard finite difference schemes.

1 Introduction

Axisymmetric flow is an important subject in fluid dynamics and has become standard textbook materials (e.g. [2]) as a starting point of theoretical study for complicated flow patterns. Although the number of independent spatial variables is reduced by symmetry, some of the essential feature and complexity of generic 3D flows remains. For example, when the swirling velocity is nonzero, there is a vorticity stretching term present. This is widely believed to account for possible singularity formation for Navier-Stokes and Euler flows. For general smooth initial data, it is well known that the solution remains smooth for short time in Euler [13] and Navier-Stokes flows [14]. A fundamental regularity result concerning the solution of the Navier-Stokes equation is given in the pioneering work of Caffarelli, Kohn and Nirenberg [4]: The one dimensional Hausdorff measure of the singular set is zero. As a consequence, the only possible singularity for axisymmetric Navier-Stokes flows would be on the axis of rotation. This result has motivated subsequent research activities concerning the regularity of axisymmetric solutions of the Navier-Stokes equation. Some regularity and partial regularity results for axisymmetric Euler and Navier-Stokes flows can be found in, for example, [6] and the references therein. To date, the regularity of the Navier-Stokes and Euler flows, whether axisymmetric or not, remains a challenging open problem. For a comprehensive review on the regularity of the Navier-Stokes equation, see [15] and the references therein.

Due to the subtle regularity issue, the numerical simulation of axisymmetric flows is also a challenging subject for computational fluid dynamicists. The earliest attempt of numerical search for potential singularities of axisymmetric flows dates back to the 90's [9, 10]. In a recent work [17], the authors have developed a class of Energy and Helicity Preserving Schemes (EHPS) for incompressible Navier-Stokes and MHD equations. There the authors extended the vorticity-stream formulation of axisymmetric flows given in [9] and proposed a generalized vorticity-stream formulation for 3D Navier-Stokes and MHD flows with coordinate symmetry. In the case of axisymmetric flows, the main difference between EHPS and the formulation in [9] is the expression and numerical discretization of the nonlinear terms. It is shown in [17] that all the nonlinear terms in the Navier-Stokes and MHD equation, including convection, vorticity stretching, geometric source, Lorentz force and electro-motive force, can be written as Jacobians. Associated with the Jacobians is a set of permutation identities which leads naturally to the conservation laws for first and second moments. The main feature of the EHPS schemes is the numerical realization of these conservation laws. In addition to preserving physically relevant quantities, the discrete form of conservation laws provides numerical advantages as well. In particular, the conservation of energy automatically enforces nonlinear stability of EHPS.

A potential difficulty associated with axisymmetric flows is the appearance of $\frac{1}{r}$ factor which becomes infinite at the axis of rotation, therefore sensitive to inconsistent or low order numerical treatment near this 'pole singularity'. In [17], the authors proposed a second order finite difference scheme and handled the pole singularity by shifting the grids half grid length away from the origin. Remarkably, the permutation identities and therefore the energy and helicity identities remain valid in this case. There are alternative numerical treatments proposed in literatures (e.g. [10]) to handle this coordinate singularity. However, rigorous justifications for various pole conditions are yet to be established.

The purpose of this paper is to give a rigorous error estimate of EHPS for axisymmetric flows. To focus on the pole singularity and avoid complication caused by physical boundary conditions, we consider here only the whole space problem with the swirling components of velocity and vorticity decaying fast enough at infinity. The error analysis of numerical methods for NSE with nonslip physical boundary condition has been well studied. We refer the works of Hou and Wetten [11], Liu and Wang [24] to interested readers. Our proof is based on a weighted energy estimate together with a careful and detailed pointwise local truncation error analysis. A major ingredient in our energy estimate is the permutation identities associated with the Jacobians (4.17). These identities are key to the energy and helicity preserving property of EHPS for general symmetric flows. Here the same identities enable us to obtain *a priori* estimate even in the presence of the pole singularity, see section 5 for details. To our knowledge, this is the first rigorous convergence proof for finite difference schemes devised for axisymmetric flows.

In our pointwise local truncation error estimate, a fundamental issue is the identification of smooth flows in the vicinity of the pole. Using a symmetry argument, we show that if the swirling component is even in r (or more precisely, is the restriction of an even function on $r > 0$), the vector field is in fact singular. See Example 1 in section 2 for details. This is an easily overlooked mistake that even appeared in some research papers targeted at numerical search of potential formation of finite time singularities. In addition to the regularity issue at the axis of symmetry, a refined decay estimate for the stream function also plays an important role in our analysis. In general, the stream function only decays as $O((x^2+r^2)^{-1})$ at infinity. Accordingly, we have selected an appropriate combination of weight functions that constitute an r-homogeneous norm. As a result, the slow decay of the stream function is compensated by the fast decay of velocity and vorticity. Overall, we obtain a second order (optimal) error estimate on axisymmetric flows.

The rest of this paper is organized as follows: In section 2, we introduce a proper notion of smoothness for the swirling component of an axisymmetric divergence free vector field. We show that the r-derivatives of even order must vanish on $r = 0^+$. The same holds true for vector fields in Sobolev spaces. In section 3, we show that the generalized vorticity-stream formulation is equivalent to the original Navier-Stokes equation in primitive variable. There is no extra regularity requirement upon switching to the vorticity formulation. In section 4, we recall the energy and helicity preserving property for EHPS and use it to prove our main theorem by energy estimate in section 5. The technical details of the pointwise estimate for the local truncation error is given in the Appendix.

2 Classical and Sobolev Spaces for Axisymmetric Solenoidal Vector Fields

In this section, we establish basic regularity results for axisymmetric vector fields. We will show that the swirling component of a smooth axisymmetric vector field has vanishing even order derivatives in the radial direction at the axis of rotation. This is done in Lemma 1 by a symmetry argument.

Throughout this paper, we will be using the cylindrical coordinate system

$$x = x, \quad y = r\cos\theta, \quad z = r\sin\theta. \tag{2.1}$$

where the x-axis is the axis of rotation. A vector field $\boldsymbol{u}$ is said to be axisymmetric if $\partial_\theta u_x = \partial_\theta u_r = \partial_\theta u_\theta = 0$. Here and throughout this paper, the subscripts of u are used to denote components rather than partial derivatives.

The three basic differential operators in cylindrical coordinate system are given by

$$\nabla u = (\partial_x u)\boldsymbol{e}_x + (\partial_r u)\boldsymbol{e}_r + (\frac{1}{r}\partial_\theta u)\boldsymbol{e}_\theta \tag{2.2}$$

$$\nabla \cdot \boldsymbol{u} = \frac{1}{r}\left(\partial_x(ru_x) + \partial_r(ru_r) + \partial_\theta u_\theta\right) \tag{2.3}$$

$$\nabla \times \boldsymbol{u} = \frac{1}{r}\begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_r & r\boldsymbol{e}_\theta \\ \partial_x & \partial_r & \partial_\theta \\ u_x & u_r & ru_\theta \end{vmatrix} \tag{2.4}$$

Here $\boldsymbol{e}_x$, $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ are the unit vectors in the x, r and θ directions respectively.

Denote by $\mathcal{C}_s^k$ the axisymmetric divergence free subspace of C^k vector fields:

Definition 1 *:*

$$\mathcal{C}_s^k = \{\boldsymbol{u} \in C^k(R^3, R^3), \quad \partial_\theta u_x = \partial_\theta u_r = \partial_\theta u_\theta = 0, \quad \nabla \cdot \boldsymbol{u} = 0\} \tag{2.5}$$

We have the following representation and regularity result for $\mathcal{C}_s^k$:

Lemma 1 *(a) For any $\boldsymbol{u} \in \mathcal{C}_s^k$, there exists a unique (u, ψ) such that*

$$\boldsymbol{u} = u\boldsymbol{e}_\theta + \nabla \times (\psi \boldsymbol{e}_\theta) = \frac{\partial_r(r\psi)}{r}\boldsymbol{e}_x - \partial_x\psi\boldsymbol{e}_r + u\boldsymbol{e}_\theta, \qquad r > 0, \tag{2.6}$$

with

$$u(x,r) \in C^k(R \times \overline{R^+}), \quad \partial_r^{2\ell} u(x, 0^+) = 0 \textit{ for } 0 \le 2\ell \le k, \tag{2.7}$$

and

$$\psi(x,r) \in C^{k+1}(R \times \overline{R^+}), \quad \partial_r^{2\ell}\psi(x,0^+) = 0 \text{ for } 0 \le 2\ell \le k+1. \tag{2.8}$$

(b) *If* (u,ψ) *satisfies (2.7), (2.8) and* $\boldsymbol{u}$ *is given by (2.6) for* $r>0$*, then* $\boldsymbol{u} \in \mathcal{C}_s^k$ *with a removable singularity at* $r=0$*.*

Proof:

Part (a): Since $\boldsymbol{u}$ is axisymmetric, we can write $\boldsymbol{u} = u_x(x,r)\boldsymbol{e}_x + u_r(x,r)\boldsymbol{e}_r + u_\theta(x,r)\boldsymbol{e}_\theta$ for $r>0$. Note that

$$\boldsymbol{e}_z(x,y,z)|_{z=0} = \begin{cases} \boldsymbol{e}_\theta(x,y,z)|_{z=0} & \text{if} \quad y>0 \\ -\boldsymbol{e}_\theta(x,y,z)|_{z=0} & \text{if} \quad y<0 \end{cases} \tag{2.9}$$

Here $\boldsymbol{e}_z$ is the unit vector in the z direction. With slight abuse of notation, we denote the components of $\boldsymbol{u}$ in Cartesian and cylindrical coordinates by

$$\begin{aligned} u_x(x,y,z) &= u_x(x,r) \\ u_y(x,y,z) &= u_r(x,r)\cos\theta - u_\theta(x,r)\sin\theta \\ u_z(x,y,z) &= u_r(x,r)\sin\theta + u_\theta(x,r)\cos\theta \end{aligned} \tag{2.10}$$

where (r,θ) is given by (2.1). It follows that

$$u_z(x,y,z)|_{z=0} = \begin{cases} u_z(x,r,\theta)_{r=y,\theta=0} = u_\theta(x,|y|) & \text{if} \quad y>0 \\ -u_z(x,r,\theta)_{r=-y,\theta=\pi} = -u_\theta(x,|y|) & \text{if} \quad y<0 \end{cases} \tag{2.11}$$

therefore for $y>0$,

$$u_z(x,y,z)|_{z=0} = u_\theta(x,|y|) = u_\theta(x,|-y|) = -u_z(x,-y,z)|_{z=0}. \tag{2.12}$$

Since $u_z \in C^k(R^3)$ and $u_z(x,y,0)$ is odd in y from (2.12), it follows that u_θ has a C^k extension up to $r=0$

$$u(x,r) := \begin{cases} u_\theta(x,r) & \text{if} \quad r>0 \\ 0 & \text{if} \quad r=0 \end{cases} = u_z(x,y,0)|_{y=r}, \quad r \ge 0$$

and all the even r-derivatives vanish at $r=0^+$:

$$\lim_{r\to 0^+} \partial_r^{2\ell} u(x,r) = \lim_{y \to 0^+} \partial_y^{2\ell} u_z(x,y,0) = \partial_y^{2\ell} u_z(x,0,0) = 0, \quad 0 \le 2\ell \le k$$

Hence (2.7) follows.

Next we derive the representation (2.6). Since $\boldsymbol{u}$ is divergence free, (2.3) gives

$$\partial_x(r u_x) + \partial_r(r u_r) = 0,$$

we know from standard argument that there exists a potential $\phi(x,r)$ such that

$$\partial_x \phi = -r u_r, \quad \partial_r \phi = r u_x \tag{2.13}$$

Restricting (2.10) to $\theta = 0$, or equivalently to $z = 0$, $y > 0$, we have

$$u_x(x,r) = u_x(x,y,z)|_{y=r,z=0}, \quad u_r(x,r) = u_y(x,y,z)|_{y=r,z=0}, \quad u_\theta(x,r) = u_z(x,y,z)|_{y=r,z=0} \tag{2.14}$$

From (2.13) and (2.14), it is clear that $\phi(x,r) \in C^{k+1}(R \times R^+)$. Since $\partial_x \phi(x,0^+) = 0$, we may, without loss of generality, assume that $\phi(x,0^+) = 0$. This also determines ψ uniquely. Next we define

$$\psi(x,r) = \frac{\phi(x,r)}{r}, \qquad r > 0. \tag{2.15}$$

It is easy to see that $\psi(x,r) \in C^{k+1}(R \times R^+)$, $\psi(x,0^+) = \partial_r \phi(x,0^+) = 0$ and (2.6) follows for $r > 0$.

It remains to show that $\lim_{r \to 0^+} \partial_r^j \psi(x,r)$ is finite for $1 \le j \le k+1$. To this end, we first establish the following identity by induction:

Claim:

$$(j+1)\partial_r^j \psi(x,0^+) = j \partial_r^{j-1} u_x(x,0^+), \qquad 1 \le j \le k+1. \tag{2.16}$$

Proof of Claim: From (2.13) it follows that

$$\partial_r^j (r\psi) = \partial_r^{j-1}(r u_x), \qquad r > 0. \tag{2.17}$$

When $j = 1$, (2.17) gives

$$\partial_r \psi + \frac{\psi}{r} = u_x$$

and we conclude from l'Hospital's rule that

$$2\partial_r \psi(x,0^+) = u_x(x,0^+),$$

thus (2.16) is verified for $j = 1$.

For $j > 1$, (2.17) gives

$$\partial_r^j \psi - \partial_r^{j-1} u_x + \left(\frac{j \partial_r^{j-1} \psi - (j-1)\partial_r^{j-2} u_x}{r} \right) = 0. \tag{2.18}$$

Now suppose (2.16) is validated for $j = \ell - 1$,

$$\ell \partial_r^{\ell-1} \psi(x, 0^+) - (\ell - 1)\partial_r^{\ell-2} u_x(x, 0^+) = 0, \tag{2.19}$$

from (2.18)-(2.19) and l'Hospital's rule, we can easily derive

$$\partial_r^{\ell} \psi(x, 0^+) - \partial_r^{\ell-1} u_x(x, 0^+) = -\left(\ell \partial_r^{\ell} \psi(x, 0^+) - (\ell - 1)\partial_r^{\ell-1} u_x(x, 0^+)\right)$$

thus (2.16) is validated for $j = \ell$ and inductively for all $1 \leq j \leq k + 1$. This completes the proof of the claim. □

We conclude from (2.14) and (2.16) that $\psi(x, r) \in C^{k+1}(R \times \overline{R^+})$. Moreover, following the same argument outlined in (2.9-2.12), it is easy to show that $u_x(x, y, 0)$ is even in y. Thus (2.8) follows from (2.16). This completes the proof of part (a).

Part (b): Conversely, we now show the regularity of $\boldsymbol{u} = u\boldsymbol{e}_\theta + \nabla \times (\psi \boldsymbol{e}_\theta)$ when (u, ψ) satisfies (2.7) and (2.8). Since $\boldsymbol{u}$ is axisymmetric, it suffices to check the derivatives of $\boldsymbol{u}$ on a cross section, say $\theta = 0$, or $z = 0$, $y \geq 0$.

It is clear from (2.6) and (2.14) that $u_x(x, y, 0)$, $u_y(x, y, 0)$ and $u_z(x, y, 0)$ have continuous x derivatives up to order k on $y \geq 0$. It remains to estimate the y-, z- and mixed derivatives.

From

$$\partial_y = \cos\theta \partial_r - \frac{\sin\theta}{r}\partial_\theta \tag{2.20}$$

$$\partial_z = \sin\theta \partial_r + \frac{\cos\theta}{r}\partial_\theta \tag{2.21}$$

we can derive the following

Proposition 1 *(i)*

$$\partial_y^j F(x, r, \theta) = \partial_r^j F(x, r, \theta) + \sin\theta\, G(x, r, \theta) \tag{2.22}$$

where G consists of the derivatives of F.

(ii)

$$\partial_z^{2m}(f(x, r)\cos\theta) = y \sum_{\ell=0}^{m} a_{\ell,m} z^{2\ell} \left(\frac{1}{r}\partial_r\right)^{\ell+m} \left(\frac{f}{r}\right) \tag{2.23}$$

$$\partial_z^{2m+1}(f(x, r)\cos\theta) = y \sum_{\ell=0}^{m} b_{\ell,m} z^{2\ell+1} \left(\frac{1}{r}\partial_r\right)^{\ell+m+1} \left(\frac{f}{r}\right) \tag{2.24}$$

$$\partial_z^{2m}(g(x,r)\sin\theta) = \sum_{\ell=0}^{m} c_{\ell,m} z^{2\ell+1}\left(\frac{1}{r}\partial_r\right)^{\ell+m}\left(\frac{g}{r}\right) \tag{2.25}$$

$$\partial_z^{2m-1}(g(x,r)\sin\theta) = \sum_{\ell=0}^{m} d_{\ell,m} z^{2\ell}\left(\frac{1}{r}\partial_r\right)^{\ell+m-1}\left(\frac{g}{r}\right) \tag{2.26}$$

for some constants $a_{\ell,m}$, $b_{\ell,m}$, $c_{\ell,m}$ *and* $d_{\ell,m}$.

Proof: From (2.20) and the following identity

$$(\cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta)(F+\sin\theta G_1) = (\cos\theta\partial_r F) + \sin\theta(\cos\theta\partial_r G_1 - \frac{1}{r}\partial_\theta F - \frac{\sin\theta}{r}\partial_\theta G_1), \tag{2.27}$$

it is easy to derive (2.22). This proves part (i).

For part (ii), equations (2.23-2.26) result from substituting $\cos\theta = \frac{y}{r}$, $\sin\theta = \frac{z}{r}$ followed by straight forward calculations. We omit the details. □

Now we proceed to show that all the mixed derivatives of orders up to k are also continuous on $y \geq 0$. For simplicity of presentation, we consider mixed derivatives performed in the following order $\partial_y^j\partial_z^q\partial_x^i$. We start with $\partial_y^j\partial_z^q\partial_x^i u_x$ and analyze for q even and odd separately.

When $q = 2m+1$, we derive from (2.21) and (2.22) that

$$\begin{aligned}
& \partial_y^j\partial_z^{2m+1}\partial_x^i u_x(x,y,0) \\
= \; & \partial_y^j\partial_z^{2m}(\sin\theta\,\partial_r\partial_x^i u_x(x,r))|_{\theta=0,r=y} \\
= \; & \partial_y^j\left(\textstyle\sum_{\ell=0}^{m} c_{\ell,m} z^{2\ell+1}(\frac{1}{r}\partial_r)^{\ell+m}(\frac{\partial_r\partial_x^i u_x(x,r)}{r})\right)|_{z=0,r=y} \\
= \; & 0
\end{aligned} \tag{2.28}$$

Next, when $q = 2m$, it follows from (2.21), (2.22), (2.26), (2.13) and (2.15) that

$$\begin{aligned}
& \partial_y^j\partial_z^{2m}\partial_x^i u_x(x,y,0) \\
= \; & \partial_y^j\partial_z^{2m-1}(\sin\theta\,\partial_r\partial_x^i u_x(x,r))|_{\theta=0,r=y} \\
= \; & (\partial_r^j\partial_z^{2m-1}(\sin\theta\,\partial_r\partial_x^i u_x) + \sin\theta\, G)\,|_{\theta=0,r=y} \\
= \; & \partial_r^j \textstyle\sum_{\ell=0}^{m} d_{\ell,m}(r\sin\theta)^{2\ell}\left(\frac{1}{r}\partial_r\right)^{\ell+m-1}(\frac{\partial_r\partial_x^i u_x(x,r)}{r})|_{\theta=0,r=y} \\
= \; & d_{0,m}\partial_r^j\left(\frac{1}{r}\partial_r\right)^m \partial_x^i u_x(x,r)|_{r=y} \\
= \; & d_{0,m}\partial_r^j\left(\frac{1}{r}\partial_r\right)^{m+1}(r\partial_x^i\psi(x,r))\,|_{r=y}.
\end{aligned} \tag{2.29}$$

From Lemma 1 and Taylor's Theorem, we have

$$\psi(x,r) = a_1(x)r + a_3(x)r^3 + \cdots + a_{2m-1}(x)r^{2m-1} + R_{2m+1}(\psi),$$

where

$$a_n(x) = \frac{1}{n!}\partial_r^n \psi(x,0^+),$$

$$R_{2m+1}(\psi) = \int_0^r \partial_r^{2m+1}\psi(x,s)\frac{(r-s)^{2m}}{(2m)!}ds,$$

and

$$\partial_r^p R_{2m+1}(\psi)(x,0^+) = \begin{cases} 0, & 0 \le p \le 2m \\ \partial_r^p \psi(x,0^+), & 2m+1 \le p \le 2m+1+j \end{cases}. \tag{2.30}$$

It follows from direct calculation that

$$(\frac{1}{r}\partial_r)^{m+1}\left(r\partial_x^i\psi(x,r)\right) = (\frac{1}{r}\partial_r)^{m+1}\left(r\partial_x^i R_{2m+1}(\psi)\right) = \sum_{\ell=0}^{m+1} C_{\ell,m}\frac{\partial_r^\ell\partial_x^i R_{2m+1}(\psi)}{r^{2m+1-\ell}}$$

and

$$\partial_r^j\left(\frac{1}{r}\partial_r\right)^{m+1}(r\partial_x^i\psi(x,r)) = \sum_{\ell=0}^{m+1} C_{\ell,m}\partial_r^j\left(\frac{\partial_r^\ell\partial_x^i R_{2m+1}(\psi)}{r^{2m+1-\ell}}\right) = \sum_{\ell=0}^{m+1}\sum_{p=0}^{j} C_{\ell,m}C'_{p,j}\frac{\partial_r^{p+\ell}\partial_x^i R_{2m+1}(\psi)}{r^{2m+1-\ell+j-p}} \tag{2.31}$$

for some constants $C_{\ell,m}$ and $C'_{p,j}$.

From (2.30), (2.31) and l'Hospital's rule we conclude that

$$\partial_r^j\left(\frac{1}{r}\partial_r\right)^{m+1}(r\partial_x^i\psi)(x,0^+) = \left(\sum_{\ell=0}^{m+1}\sum_{p=0}^{j}\frac{C_{\ell,m}C'_{p,j}}{(2m-1-\ell+j-p)!}\right)\partial_r^{2m+1+j}\partial_x^i\psi(x,0^+). \tag{2.32}$$

Since $\psi \in C^{k+1}(R\times\overline{R^+})$, it follows from (2.29), (2.32) and (2.28) that $\partial_y^j\partial_z^q\partial_x^i u_x(x,y,0)$ is continuous and bounded up to $y = 0^+$ for $j+q+i \le k$.

Next we consider the mixed derivatives of u_y and u_z. In view of (2.10), it suffices to calculate $\partial_y^j\partial_z^q\partial_x^i(f(x,r)\cos\theta + g(x,r)\sin\theta)|_{\theta=0,r=y}$ where f and g are either $\pm\partial_x\psi$ or $\pm u$.

When $q = 2m$, it follows from (2.23) and (2.25) that

$$\begin{aligned} &\partial_y^j\partial_z^{2m}\partial_x^i(f(x,r)\cos\theta + g(x,r)\sin\theta)|_{\theta=0,r=y} \\ =\ &\partial_y^j\partial_z^{2m}(\partial_x^i f(x,r)\cos\theta + \partial_x^i g(x,r)\sin\theta)|_{\theta=0,r=y} \\ =\ &a_{0,m}\partial_r^j\left(r(\tfrac{1}{r}\partial_r)^m(\tfrac{\partial_x^i f}{r})\right)|_{r=y} + c_{0,m}\partial_r^j\left((\tfrac{1}{r}\partial_r)^m(\tfrac{\partial_x^i g}{r})\right)|_{r=y}\end{aligned}$$

From (2.7-2.8), both $-\partial_x \psi(x,r)$ and $u(x,r)$ have local expansions of the form

$$b_1(x)r + b_3(x)r^3 + \cdots + b_{2m-1}(x)r^{2m-1} + R_{2m+1}.$$

Following the same argument above, we can show that both $\partial_y^j \partial_z^{2m} \partial_x^i u_y$ and $\partial_y^j \partial_z^{2m} \partial_x^i u_z$ are continuous and bounded up to $y = 0^+$ for $j + 2m + i \leq k$. The calculations for $\partial_y^j \partial_z^{2m+1} \partial_x^i u_y$ and $\partial_y^j \partial_z^{2m+1} \partial_x^i u_z$ are similar. This completes the proof of Part (b). □

In view of Lemma 1, we now introduce the following function spaces:

Definition 2

$$C_s^k \left(R \times \overline{R^+}\right) = \{f(x,r) \in C^k \left(R \times \overline{R^+}\right), \quad \partial_r^{2j} f(x, 0^+) = 0, \, 0 \leq 2j \leq k\}$$

We can recast Lemma 1 as

Lemma 1'

$$\mathcal{C}_s^k = \{u\boldsymbol{e}_\theta + \nabla \times (\psi \boldsymbol{e}_\theta) \,|\, u \in C_s^k(R \times \overline{R^+}), \, \psi \in C_s^{k+1}(R \times \overline{R^+})\} \tag{2.33}$$

In fact, we will show a counterpart for (2.33) in standard Sobolev spaces: A weak solenoidal axisymmetric vector field admits the representation (2.6) with $u(x,r)$ and $\psi(x,r)$ in certain weighted L^2 and H^k spaces. Moreover, both u and ψ, together with certain even order derivatives have vanishing traces on $r = 0^+$.

We proceed with the following identity for general solenoidal vector fields:

Lemma 2 *If* $\boldsymbol{u} \in C^k(R^3, R^3) \cap H^k(R^3, R^3)$ *and* $\nabla \cdot \boldsymbol{u} = 0$, *then*

$$\|\boldsymbol{u}\|^2_{H^k(R^3,R^3)} = \sum_{\ell=0}^{k} \|(\nabla \times)^\ell \boldsymbol{u}\|^2_{L^2(R^3,R^3)} \tag{2.34}$$

Proof: We prove (2.34) for ℓ even and odd separately.

Since $\nabla \cdot \boldsymbol{u} = 0$, it follows that $\nabla \times \nabla \times \boldsymbol{u} = -\nabla^2 \boldsymbol{u}$. Thus if ℓ is even, we can write

$$\|(\nabla \times)^{2m} \boldsymbol{u}\|_{L^2(R^3,R^3)} = \|(\nabla^2)^m \boldsymbol{u}\|_{L^2(R^3,R^3)} \tag{2.35}$$

When $m = 1$ and $u \in C^k(R^3)$, we can integrate by part to get

$$\int_{R^3} |\nabla^2 u|^2 = \int_{R^3} (\sum_{i_1=1}^{3} \partial_{i_1}^2 u)^2 = \int_{R^3} \sum_{i_1,i_2=1}^{3} \partial_{i_1}^2 u \partial_{i_2}^2 u = \int_{R^3} \sum_{i_1,i_2=1}^{3} (\partial_{i_1} \partial_{i_2} u)^2$$

Similarly, when $m = 2$,

$$\int_{R^3} |(\nabla^2)^2 u|^2 = \int_{R^3} \left((\sum_{i=1}^{3} \partial_i^2)^2 u \right)^2 = \int_{R^3} \sum_{i_1,i_2,i_3,i_4=1}^{3} (\partial_{i_1}^2 \partial_{i_2}^2 u)(\partial_{i_3}^2 \partial_{i_4}^2 u)$$

$$= \sum_{i_1,i_2,i_3,i_4=1}^{3} \int_{R^3} (\partial_{i_1}\partial_{i_2}\partial_{i_3}\partial_{i_4} u)^2.$$

It is therefore easy to see that

$$\int_{R^3} |(\nabla^2)^m u|^2 = \sum_{i_1,\cdots,i_{2m}=1}^{3} \int_{R^3} (\partial_{i_1} \cdots \partial_{i_{2m}} u)^2$$

and consequently for $\boldsymbol{u} \in C^k(R^3, R^3)$, $2m \leq k$,

$$\|(\nabla^2)^m \boldsymbol{u}\|^2_{L^2(R^3,R^3)} = \sum_{i_1,\cdots,i_{2m}=1}^{3} \|(\partial_{i_1} \cdots \partial_{i_{2m}})\boldsymbol{u}\|^2_{L^2(R^3,R^3)}. \tag{2.36}$$

On the other hand, if ℓ is odd, we first write

$$(\nabla\times)^{2m+1}\boldsymbol{u} = \nabla \times (-(\nabla^2))^m \boldsymbol{u} = (-1)^m \nabla \times (\nabla^2)^m \boldsymbol{u}$$

then apply the identity

$$\|\nabla \boldsymbol{v}\|^2_{L^2(R^3,R^3)} = \|\nabla \times \boldsymbol{v}\|^2_{L^2(R^3,R^3)} + \|\nabla \cdot \boldsymbol{v}\|^2_{L^2(R^3)}$$

to get

$$\|(\nabla\times)^{2m+1}\boldsymbol{u}\|_{L^2(R^3,R^3)} = \|\nabla \times (\nabla^2)^m \boldsymbol{u}\|_{L^2(R^3,R^3)} = \|(\nabla^2)^m \nabla \boldsymbol{u}\|_{L^2(R^3,R^3)} \tag{2.37}$$

and from (2.36),

$$\|(\nabla^2)^m \nabla \boldsymbol{u}\|^2_{L^2(R^3,R^3)} = \sum_{i,j=1}^{3} \|(\nabla^2)^m \partial_i u_j\|^2_{L^2(R^3)} = \sum_{i,j=1}^{3} \sum_{i_1,\cdots,i_{2m}=1}^{3} \int_{R^3} (\partial_{i_1} \cdots \partial_{i_{2m}} \partial_i u_j)^2$$

$$= \sum_{i_1,\cdots,i_{2m+1}=1}^{3} \|(\partial_{i_1} \cdots \partial_{i_{2m+1}})\boldsymbol{u}\|^2_{L^2(R^3,R^3)}. \tag{2.38}$$

It follows from (2.35, (2.36), (2.37) and (2.38) that

$$\|\boldsymbol{u}\|^2_{H^k(R^3,R^3)} = \sum_{\ell=0}^{k} \sum_{i_1,\cdots,i_\ell=1}^{3} \|\partial_{i_1}\cdots\partial_{i_\ell}\boldsymbol{u}\|^2_{L^2(R^3,R^3)} = \sum_{\ell=0}^{k} \|(\nabla\times)^\ell \boldsymbol{u}\|^2_{L^2(R^3,R^3)}$$

This completes the proof of Lemma 2. □

In Lemma 3 and Lemma 4 below, we will derive an equivalent representation of the Sobolev norms for axisymmetric solenoidal vector fields.

Lemma 3 *Let* $\boldsymbol{u} \in \mathcal{C}_s^k$ *be represented by* $\boldsymbol{u} = u\boldsymbol{e}_\theta + \nabla\times(\psi\boldsymbol{e}_\theta)$ *with* $u \in C_s^k\left(R\times\overline{R^+}\right)$ *and* $\psi \in C_s^{k+1}\left(R\times\overline{R^+}\right)$*. Then* $(\nabla\times)^\ell\boldsymbol{u} \in \mathcal{C}_s^{k-\ell}$ *and*

$$(\nabla\times)^{2m}\boldsymbol{u} = (\mathcal{L}^m u)\boldsymbol{e}_\theta + \nabla\times((\mathcal{L}^m\psi)\boldsymbol{e}_\theta), \quad \textit{if} \quad 2m \le k,$$

$$(\nabla\times)^{2m+1}\boldsymbol{u} = (\mathcal{L}^{m+1}\psi)\boldsymbol{e}_\theta + \nabla\times((\mathcal{L}^m u)\boldsymbol{e}_\theta), \quad \textit{if} \quad 2m+1 \le k,$$

where

$$\mathcal{L} := -\nabla^2 + \frac{1}{r^2} = -(\partial_r^2 + \frac{1}{r}\partial_r + \partial_x^2) + \frac{1}{r^2}.$$

Moreover

$$\mathcal{L}^m u \in C_s^{k-2m}(R\times\overline{R^+}), \quad \textit{if} \quad 2m \le k,$$

$$\mathcal{L}^{m+1}\psi \in C_s^{k-1-2m}(R\times\overline{R^+}), \quad \textit{if} \quad 2m+1 \le k.$$

Proof: For any $\phi \in C_s^i\left(R\times\overline{R^+}\right)$, we have $\phi\,\boldsymbol{e}_\theta \in \mathcal{C}_s^i$ from Lemma 1 (b). With straight forward calculation using (2.4), it is easy to verify that for $i \ge 2$,

$$\nabla\times\nabla\times(\phi\,\boldsymbol{e}_\theta) = (\mathcal{L}\phi)\boldsymbol{e}_\theta. \tag{2.39}$$

On the other hand, it is clear that

$$\nabla\times\nabla\times(\phi\,\boldsymbol{e}_\theta) \in \mathcal{C}_s^{i-2},$$

and therefore from Lemma 1 (a),

$$\mathcal{L}\phi \in C_s^{i-2}\left(R\times\overline{R^+}\right). \tag{2.40}$$

The Lemma then follows from (2.39) and (2.40). □

Next we proceed to define the weighted Sobolev space for axisymmetric solenoidal vector fields. For $a, b \in C^0\left(R \times \overline{R^+}\right)$, we define the weighted L^2 inner product and norm

$$\langle a, b\rangle = \int_{-\infty}^{\infty}\int_0^{\infty} a(x,r)b(x,r)\, r dx dr, \qquad \|a\|_0^2 = \langle a, a\rangle, \tag{2.41}$$

and for $a, b \in C_s^1\left(R \times \overline{R^+}\right)$, we define the weighted H^1 inner product and norm

$$[a, b] = \langle \partial_x a, \partial_x b\rangle + \langle \partial_r a, \partial_r b\rangle + \langle \frac{a}{r}, \frac{b}{r}\rangle, \qquad \|a\|_1^2 = [a, a]. \tag{2.42}$$

When $a \in C_s^1\left(R \times \overline{R^+}\right)$ and $b \in C_s^1\left(R \times \overline{R^+}\right) \cap C^2\left(R \times R^+\right)$, we also have the following identity from integration by part:

$$\langle a, \mathcal{L}b\rangle = [a, b].$$

If $\boldsymbol{u} = u\boldsymbol{e}_\theta + \nabla \times (\psi \boldsymbol{e}_\theta)$, with $u \in C^0\left(R \times \overline{R^+}\right)$ and $\psi \in C_s^1\left(R \times \overline{R^+}\right)$, it is easy to see that

$$\|\boldsymbol{u}\|_{L^2(R^3,R^3)}^2 = \|u\|_0^2 + \|\psi\|_1^2 \tag{2.43}$$

Higher order Sobolev norms can be defined similarly in terms of u and ψ:

Definition 3 *For $\boldsymbol{u} = u\boldsymbol{e}_\theta + \nabla \times (\psi \boldsymbol{e}_\theta) \in \mathcal{C}_s^k$,*

$$\begin{aligned}\|\boldsymbol{u}\|_{\mathcal{H}_s^k}^2 &:= (\|u\|_0^2 + \|\psi\|_1^2) + (\|\mathcal{L}\psi\|_0^2 + \|u\|_1^2) + (\|\mathcal{L}u\|_0^2 + \|\mathcal{L}\psi\|_1^2) + \cdots \\ &= \sum_{2m\le k}(\|\mathcal{L}^m u\|_0^2 + \|\mathcal{L}^m\psi\|_1^2) + \sum_{2m+1\le k}(\|\mathcal{L}^{m+1}\psi\|_0^2 + \|\mathcal{L}^m u\|_1^2)\end{aligned}$$

In view of Lemma 1, Lemma 2, Lemma 3 and (2.43), we have proved the following

Lemma 4 *If $\boldsymbol{u} \in \mathcal{C}_s^k$, then*

$$\|\boldsymbol{u}\|_{H^k(R^3,R^3)} = \|\boldsymbol{u}\|_{\mathcal{H}_s^k}$$

Denote by C_c the space of compactly supported functions. We can now define the Sobolev spaces for axisymmetric solenoidal vector fields following standard procedure:

Definition 4

$$L_s^2(R \times \overline{R^+}) := \textit{Completion of } C_s^0(R \times \overline{R^+}) \cap C_c(R \times \overline{R^+}) \textit{ with respect to } \|\cdot\|_0$$

$$H_s^1(R \times \overline{R^+}) := \textit{Completion of } C_s^1(R \times \overline{R^+}) \cap C_c(R \times \overline{R^+}) \textit{ with respect to } \|\cdot\|_1$$

$$\mathcal{H}_s^k := \textit{Completion of } \mathcal{C}_s^k \cap C_c(R^3, R^3) \textit{ with respect to } \|\cdot\|_{\mathcal{H}_s^k}$$

Accordingly, we have the following characterization for $\mathcal{H}_s^k$:

Lemma 5 *If $\boldsymbol{u} \in \mathcal{H}_s^k$, then u admits a unique representation*

$$\boldsymbol{u} = u\boldsymbol{e}_\theta + \nabla \times (\psi \boldsymbol{e}_\theta) \tag{2.44}$$

with $u \in L_s^2(R \times \overline{R^+})$ and $\psi \in H_s^1(R \times \overline{R^+})$. Moreover,

$$\mathcal{L}^m u \in L_s^2\left(R \times \overline{R^+}\right), \quad \mathcal{L}^m \psi \in H_s^1\left(R \times \overline{R^+}\right), \quad \text{if } 2m \le k, \tag{2.45}$$

$$\mathcal{L}^{m+1}\psi \in L_s^2\left(R \times \overline{R^+}\right), \quad \mathcal{L}^m u \in H_s^1\left(R \times \overline{R^+}\right), \quad \text{if } 2m+1 \le k, \tag{2.46}$$

and

$$\|\boldsymbol{u}\|_{H^k(R^3,R^3)} = \|\boldsymbol{u}\|_{\mathcal{H}_s^k}. \tag{2.47}$$

Here the equality in (2.44) and the differential operators $\nabla\times$, $\mathcal{L}^m$ are realized in the sense of distribution.

The proof of Lemma 5 is based on standard density argument. We omit the details.

Finally, the counterpart of (2.7) and (2.8) for $\boldsymbol{u} \in \mathcal{H}_s^k$ is given by (2.45), (2.46) and the following trace Lemma:

Lemma 6 *If $v \in H_s^1\left(R \times \overline{R^+}\right)$, then the trace of v on $r = 0$ vanishes.*

Proof: For any $v \in C^1\left(R \times \overline{R^+}\right) \cap \in C_c\left(R \times \overline{R^+}\right)$, we have

$$\int_R |v(x,0)|^2\, dx = -2\int\int_{R\times R^+} v\partial_r v\, dx\, dr \le \int\int_{R\times R^+} \left(\frac{v^2}{r^2} + (\partial_r v)^2\right) r\, dx\, dr \le \|v\|_1^2$$

Since $v(x,0) = 0$ for $v \in C_s^1\left(R \times \overline{R^+}\right)$, the Lemma follows from standard density argument. □

Example 1:

Take $\boldsymbol{u} = u\boldsymbol{e}_\theta$ with $u = e^{-x^2} r^2 e^{-r}$. Note that $u = O(r^2)$ near the axis. Similar functions can be found in literatures as initial data in numerical search for finite time singularities. Although $u \in C^\infty(R \times \overline{R^+})$ and $\boldsymbol{u}$ may appear to be a smooth vector field, it is easy to verify that $\mathcal{L}u(x, 0^+) \ne 0$. Thus from Lemma 1, Lemma 5 and Lemma 6, $\boldsymbol{u}$ is neither in $C^2(R^3, R^3)$ nor in $H^3(R^3, R^3)$.

3 Generalized Vorticity-Stream Formulation for Axisymmetric Flows

3.1 Axisymmetric Formulation of Navier-Stokes Equations

In this section, we show that, under suitable regularity assumptions, any axisymmetric solution of the Navier-Stokes equation

$$
\begin{aligned}
&\partial_t \boldsymbol{u} + (\nabla \times \boldsymbol{u}) \times \boldsymbol{u} + \nabla p = -\nu \nabla \times \nabla \times \boldsymbol{u} \\
&\nabla \cdot \boldsymbol{u} = 0
\end{aligned}
\tag{3.1}
$$

is also a solution to the following axisymmetric formulation of Navier-Stokes solution derived formally in [17]

$$
\begin{aligned}
&u_t + \tfrac{1}{r^2} J\left(ru, r\psi\right) = \nu(\nabla^2 - \tfrac{1}{r^2})u\,, \\
&\omega_t + J\left(\tfrac{\omega}{r}, r\psi\right) = \nu(\nabla^2 - \tfrac{1}{r^2})\omega + J\left(\tfrac{u}{r}, ru\right)\,, \\
&\omega = -(\nabla^2 - \tfrac{1}{r^2})\psi\,,
\end{aligned}
\tag{3.2}
$$

with $\boldsymbol{u} = u\boldsymbol{e}_\theta + \nabla \times (\psi \boldsymbol{e}_\theta)$ and vice versa.

The vorticity formulation for axisymmetric flows (3.2) has appeared in [9] with an alternative expression for the nonlinear terms. In [17], the authors have generalized the vorticity formulation to general symmetric flows with the nonlinear terms recast in Jacobians as in (3.2). Accompanied with the Jacobians is a set of permutation identities which played a key role in the design of energy and helicity preserving scheme and in the convergence proof of the scheme. We will explain the details in section 5.

The axisymmetric Navier-Stokes equation (3.2) can be formally derived from (3.1). A smooth solution of (3.1) also gives rise to a smooth solution of (3.2). Moreover, from Lemma 1 (a), the resulted solution satisfies the pole condition (2.7,2.8) automatically. In other words, if $\boldsymbol{u} \in C^1(0,T;\mathcal{C}_s^k)$ is a solution to (3.1), then the swirling components are in the class

$$
\begin{aligned}
\psi(t;x,r) &\in C^1\left(0,T;C_s^{k+1}(R \times \overline{R^+})\right) \\
u(t;x,r) &\in C^1\left(0,T;C_s^{k}(R \times \overline{R^+})\right) \\
\omega(t;x,r) &\in C^1\left(0,T;C_s^{k-1}(R \times \overline{R^+})\right)
\end{aligned}
\tag{3.3}
$$

However, it is not clear whether smooth solutions of (3.2) also give rise to smooth solutions of (3.1).

For example, in the case of Euler equation, an exact stationary solution to (3.2) with $\nu = 0$ is given by

$$u(t, x, r) = r^2 e^{-r}, \qquad \omega = \psi \equiv 0 \tag{3.4}$$

It is clear that (3.4) is in $C^\infty(R \times \overline{R^+})$, yet the corresponding $\boldsymbol{u} = u\boldsymbol{e}_\theta$ is only in $C^1(R^3, R^3)$ for any t.

In the case of Navier-Stokes equation ($\nu > 0$), (3.2) is an elliptic-parabolic system on a semi-bounded region ($r > 0$). From standard PDE theory, we need to assign boundary values for (ψ, u, ω). The zeroth order part of the pole condition (2.7,2.8) would suffice:

$$\psi(x, 0) = u(x, 0) = \omega(x, 0) = 0. \tag{3.5}$$

It is therefore a natural question to ask if a smooth solution of (3.2, 3.5) in the class

$$\begin{array}{l} \psi(t; x, r) \in C^1\left(0, T; C^{k+1}(R \times \overline{R^+})\right) \\ u(t; x, r) \in C^1\left(0, T; C^{k}(R \times \overline{R^+})\right) \\ \omega(t; x, r) \in C^1\left(0, T; C^{k-1}(R \times \overline{R^+})\right) \end{array} \tag{3.6}$$

will give rise to a smooth solution of (3.2). In other words, is the pole condition (2.7,2.8) automatically satisfied if only the zeroth order part (3.5) is imposed?

The answer to this question is affirmative. We will show in Lemma 9 that (3.3) and (3.6) are indeed equivalent for solutions of (3.2, 3.5). The proof is based on local Taylor expansion. We decompose the proof into several Lemmas.

Lemma 7 *If $v \in C^{2j+2}(R \times \overline{R^+})$ and $\partial_r^{2\ell} v(x, 0^+) = 0$ for $0 \le \ell \le j$, then*

$$\lim_{r \to 0^+} \partial_r^{2j+1}\left(\frac{v(x, r)}{r}\right) = \lim_{r \to 0^+} \frac{1}{2j+2} \partial_r^{2j+2} v(x, r)$$

Proof: Since $v \in C^{2j+2}(R \times \overline{R^+})$, we have

$$v(x, r) = a_1(x) r + a_3(x) r^3 + \cdots + a_{2j+1}(x) r^{2j+1} + R_{2j+2}(v) \tag{3.7}$$

from Taylor's Theorem. Here

$$a_n(x) = \frac{1}{n!} \partial_r^n v(x, 0^+),$$

$$R_{2j+2}(v) = \int_0^r \partial_r^{2j+2} v(x, s) \frac{(r - s)^{2j+1}}{(2j+1)!} ds$$

and

$$\partial_r^i R_{2j+2}(v)(x,0^+) = 0, \quad 0 \le i \le 2j+1, \qquad \partial_r^{2j+2} R_{2j+2}(v)(x,0^+) = \partial_r^{2j+2} v(x,0^+). \tag{3.8}$$

From (3.7), it follows that

$$\partial_r^{2j+1}(\frac{v(x,r)}{r}) = \partial_r^{2j+1}(\frac{R_{2j+2}(v)}{r}) = \sum_{i=0}^{2j+1} C_{2j+1}^i (-1)^i i! \frac{\partial_r^{2j+1-i} R_{2j+2}(v)}{r^{1+i}} \tag{3.9}$$

From (3.8), (3.9) and l'Hospital's rule, we can easily derive

$$\lim_{r\to 0^+} \partial_r^{2j+1}(\frac{v(x,r)}{r}) = \left(\sum_{i=0}^{2j+1} C_{2j+1}^i (-1)^i \frac{1}{i+1} \right) \partial_r^{2j+2} v(x,0^+) = \frac{1}{2j+2} \partial_r^{2j+2} v(x,0^+).$$

This completes the proof of Lemma 7. □

Lemma 8 *If $2j \le k-2$ and*

$$\begin{array}{l} \psi \in C^{k+1}(R\times \overline{R^+}) \cap C_s^{2j}(R\times \overline{R^+}) \\ u \in C^{k}(R\times \overline{R^+}) \cap C_s^{2j}(R\times \overline{R^+}) \\ \omega \in C^{k-1}(R\times \overline{R^+}) \cap C_s^{2j}(R\times \overline{R^+}) \end{array} , \tag{3.10}$$

then the Jacobians $\frac{1}{r^2}J(ru,r\psi)$, $J(\frac{\omega}{r},r\psi)$ and $J(\frac{u}{r},ru)$ are in $C_s^{2j}(R\times \overline{R^+})$.

Proof: From (3.10), it is obvious that $\frac{1}{r^2}J(ru,r\psi) \in C^{k-1}(R\times R^+)$, $J(\frac{\omega}{r},r\psi) \in C^{k-2}(R\times R^+)$ and $J(\frac{u}{r},ru) \in C^{k-1}(R\times R^+)$. Therefore all three Jacobians are in $C^{2j}(R\times R^+)$. It remains to evaluate the r-derivatives of the Jacobians at $(x,0^+)$.

From (3.10), we have the following expansions:

$$\begin{array}{l} \psi = a_1(x)r + \cdots + a_{2j-1}(x)r^{2j-1} + R_{2j+1}(\psi) \\ u = b_1(x)r + \cdots + b_{2j-1}(x)r^{2j-1} + R_{2j+1}(u) \\ \omega = c_1(x)r + \cdots + c_{2j-1}(x)r^{2j-1} + R_{2j+1}(\omega) \end{array}$$

where

$$(a_n(x), b_n(x), c_n(x)) = \frac{1}{n!}\partial_r^n(\psi,u,\omega)(x,0^+)$$

and

$$R_{2j+1}(v) = \int_0^r \partial_r^{2j+1} v(x,s)\frac{(r-s)^{2j}}{(2j)!}ds.$$

Before we continue, we first introduce the following notations for brevity:

Definition 5 *For $0 \le q \le 2j+1$, we define*

1. $f(x,r) = \widehat{O}(r^q)$ *if* $f(x,r) = C(x)r^q$.

2. $g(x,r) = \widetilde{O}(r^q)$ *if* $g(x,r) \in C^{2j+1}(R \times R^+)$ *and*

$$\partial_r^p g(x,0^+) = 0, \quad 0 \le p \le q-1, \quad |\partial_r^q g(x,0^+)| < \infty.$$

Using these notations, we can write

$$R_{2j+1}(\psi,u,\omega) = \widetilde{O}(r^{2j+1})$$

and

$$(\psi,u,\omega) = \widehat{O}(r) + \widehat{O}(r^3) + \cdots + \widehat{O}(r^{2j-1}) + \widetilde{O}(r^{2j+1}). \tag{3.11}$$

It is easy to verify directly that

$$\partial_r^p \widetilde{O}(r^q) = \widetilde{O}(r^{q-p}), \quad p \le q \le 2j+1,$$

and

$$\widetilde{O}(r^q)/r^p = \widetilde{O}(r^{q-p}), \quad p \le q \le 2j+1.$$

From (3.11), we have

$$\frac{1}{r}\partial_x(r\psi),\ \frac{1}{r}\partial_x(ru) = \widehat{O}(r) + \widehat{O}(r^3) + \cdots + \widehat{O}(r^{2j-1}) + \partial_x\widetilde{O}(r^{2j+1})$$

and

$$\frac{1}{r}\partial_r(r\psi),\ \frac{1}{r}\partial_r(ru) = \widehat{O}(1) + \widehat{O}(r^2) + \cdots + \widehat{O}(r^{2j-2}) + \widetilde{O}(r^{2j}).$$

Therefore

$$\begin{aligned} \tfrac{1}{r^2}J(r\psi,ru) &= \widehat{O}(r) + \widehat{O}(r^3) + \cdots + \widehat{O}(r^{2j-1}) \\ &+\widetilde{O}(r^{2j})\left(\widehat{O}(r) + \widehat{O}(r^3) + \cdots + \widehat{O}(r^{2j-1})\right) \\ &+\partial_x\widetilde{O}(r^{2j+1})\left(\widehat{O}(1) + \widehat{O}(r^2) + \cdots + \widehat{O}(r^{2j-2})\right) \\ &+\widehat{O}(r^{2j+3}) + \widehat{O}(r^{2j+5}) + \cdots + \widehat{O}(r^{4j-3}) \end{aligned} \tag{3.12}$$

It is easy to see from (3.12) that

$$\partial_r^{2i}(\frac{1}{r^2}J(ru,r\psi))(x,0^+) = 0, \quad 0 \le i \le j$$

and $\frac{1}{r^2}J\left(ru,r\psi\right) \in C_s^{2j}(R \times \overline{R^+})$. The same argument applies to $J\left(\frac{\omega}{r},r\psi\right)$ and $J\left(\frac{u}{r},ru\right)$. This completes the proof of Lemma 8. □

Lemma 9 *If (ψ, u, ω) is a solution to (3.2, 3.5) in the class (3.6) with $k \geq 3$. Then*

$$\begin{array}{l} \psi \in C_s^{k+1}(R \times \overline{R^+}) \\ u \in C_s^k(R \times \overline{R^+}) \\ \omega \in C_s^{k-1}(R \times \overline{R^+}) \end{array} \tag{3.13}$$

for $0 \leq t \leq T$.

Proof: Let j^* be the largest integer such that $2j^* \leq k-1$. We first show that on $0 \leq t \leq T$,

$$\begin{array}{c} \partial_r^{2\ell}\psi(t,x,0^+) = 0 \\ \partial_r^{2\ell}u(t,x,0^+) = 0 \\ \partial_r^{2\ell}\omega(t,x,0^+) = 0. \end{array} \tag{3.14}$$

for $0 \leq \ell \leq j^*$.

This is done by induction on ℓ. When $\ell = 0$, (3.14) is given by the boundary condition (3.5). Suppose that (3.14) is verified for $\ell = j$ with $j+1 \leq j^*$. We apply $\partial_r^{2j-2}|_{(x,0^+)}$ on both sides of (3.2) and conclude that, in view of Lemma 8,

$$\begin{array}{c} \nu\partial_r^{2j}(\nabla^2 - \frac{1}{r^2})u(x,0^+) = 0, \\ \nu\partial_r^{2j}(\nabla^2 - \frac{1}{r^2})\omega(x,0^+) = 0, \\ \partial_r^{2j}(\nabla^2 - \frac{1}{r^2})\psi(x,0^+) = 0. \end{array}$$

Since

$$(\nabla^2 - \frac{1}{r^2})v = (\partial_x^2 v + \partial_r^2 v + \partial_r(\frac{v}{r})),$$

and

$$\partial_r^{2j}(\nabla^2 - \frac{1}{r^2})v = (\partial_r^{2j+2}v + \partial_r^{2j}\partial_x^2 v + \partial_r^{2j+1}(\frac{v}{r})),$$

it follows from Lemma 7 that $\partial_r^{2j+2}\psi(x,0^+) = \partial_r^{2j+2}u(x,0^+) = \partial_r^{2j+2}\omega(x,0^+) = 0$ thus (3.14) is verified for $\ell = j+1$.

We can continue the induction until (3.14) is verified for $\ell = j^*$ to get

$$\begin{array}{l} \psi \in C^{k+1}(R \times \overline{R^+}) \cap C_s^{2j^*}(R \times \overline{R^+}) \\ u \in C^k(R \times \overline{R^+}) \cap C_s^{2j^*}(R \times \overline{R^+}) \\ \omega \in C^{k-1}(R \times \overline{R^+}) \cap C_s^{2j^*}(R \times \overline{R^+}) \end{array} \tag{3.15}$$

To complete the proof, we proceed with k odd and even separately.

If k is odd, say $k = 2m+1$, then $j^* = m$ and (3.15) can be written as

$$\psi \in C^{2m+2}(R \times \overline{R^+}) \cap C_s^{2m}(R \times \overline{R^+}), \quad u \in C_s^{2m+1}(R \times \overline{R^+}), \quad \omega \in C_s^{2m}(R \times \overline{R^+}). \tag{3.16}$$

Since

$$\partial_r^{2m}(\nabla^2 - \frac{1}{r^2})\psi = (\partial_r^{2m+2}\psi + \partial_r^{2m}\partial_x^2\psi + \partial_r^{2m+1}(\frac{\psi}{r})) = \partial_r^{2m}\omega, \tag{3.17}$$

we conclude from Lemma 7, (3.16) and (3.17) that $\partial_r^{2m+2}\psi(x,0) = 0$, therefore $\psi \in C_s^{2m+2}(R\times \overline{R^+})$.

Similarly, if $k = 2n$, then $j^* = n-1$ and we have from (3.15)

$$\psi \in C^{2n+1}(R\times\overline{R^+})\cap C_s^{2n-2}(R\times\overline{R^+}), \quad u \in C^{2n}(R\times\overline{R^+})\cap C_s^{2n-2}(R\times\overline{R^+}), \quad \omega \in C_s^{2n-1}(R\times\overline{R^+}).$$

Since $2n-2 = k-2$, the assumption in Lemma 8 is satisfied. Therefore we can continue the induction for u to get $\partial_r^{2n}u(x,0^+) = 0$, thus $u \in C_s^{2n}(R\times\overline{R^+})$.

Finally,

$$\partial_r^{2n-2}(\nabla^2 - \frac{1}{r^2})\psi = (\partial_r^{2n}\psi + \partial_r^{2n-2}\partial_x^2\psi + \partial_r^{2n-1}(\frac{\psi}{r})) = \partial_r^{2n-2}\omega$$

and we conclude that $\partial_r^{2n}\psi(x,0^+) = 0$ and $\psi \in C^{2n+1}(R\times\overline{R^+})\cap C_s^{2n}(R\times\overline{R^+}) = C_s^{2n+1}(R\times \overline{R^+})$. This completes the proof of Lemma 9. □

The equivalence of (3.1) and (3.2) in terms of regularity of classical solutions is given by

Theorem 1 *(I) Suppose* $(\boldsymbol{u}, p)$ *be an axisymmetric solution to NSE (3.1) with* $\boldsymbol{u} \in C^1\left(0,T;\mathcal{C}_s^k\right)$, $p \in C^0\left(0,T;C^{k-1}(R^3)\right)$ *and* $k \geq 3$. *Then there is a solution* (ψ, u, ω) *to (3.2) in the class*

$$\begin{array}{l} \psi(t;x,r) \in C^1\left(0,T;C_s^{k+1}(R\times\overline{R^+})\right) \\ u(t;x,r) \in C^1\left(0,T;C_s^{k}(R\times\overline{R^+})\right) \\ \omega(t;x,r) \in C^1\left(0,T;C_s^{k-1}(R\times\overline{R^+})\right) \end{array}$$

and $\boldsymbol{u} = u\boldsymbol{e}_\theta + \nabla\times(\psi\boldsymbol{e}_\theta)$.

(II) Let (ψ, u, ω) *be a solution to (3.2,3.5) in the class*

$$\begin{array}{l} \psi(t;x,r) \in C^1\left(0,T;C^{k+1}(R\times\overline{R^+})\right) \\ u(t;x,r) \in C^1\left(0,T;C^{k}(R\times\overline{R^+})\right) \\ \omega(t;x,r) \in C^1\left(0,T;C^{k-1}(R\times\overline{R^+})\right) \end{array}$$

with $k \geq 3$. *Then*

$$\boldsymbol{u} = u\boldsymbol{e}_\theta + \nabla\times(\psi\boldsymbol{e}_\theta) \in C^1(0,T;\mathcal{C}_s^k)$$

and there is a axisymmetric scalar function $p \in C^0(0,T;C^{k-1}(R^3))$ *such that* $(\boldsymbol{u}, p)$ *is a solution to NSE (3.1).*

Proof:

Part (I): Since $\boldsymbol{u} \in C^1\left(0,T;\mathcal{C}_s^k\right)$ is a solution to (3.1) with $k \geq 3$, it follows that

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u} = \omega \boldsymbol{e}_\theta + \nabla \times (u\boldsymbol{e}_\theta) \in C^1\left(0,T;\mathcal{C}_s^{k-1}\right)$$

is also an axisymmetric solution to the Navier-Stokes equation in vorticity form:

$$\partial_t \boldsymbol{\omega} + \nabla \times (\boldsymbol{\omega} \times \boldsymbol{u}) = -\nu \nabla \times \nabla \times \boldsymbol{\omega} \tag{3.18}$$

Next, we express each term of (3.18) in the cylindrical coordinate as

$$\partial_t \boldsymbol{\omega} = \partial_t \omega \boldsymbol{e}_\theta + \nabla \times (\partial_t u \boldsymbol{e}_\theta), \tag{3.19}$$

$$-\nabla \times \nabla \times \boldsymbol{\omega} = \left((\nabla^2 - \frac{1}{r^2})\omega\right)\boldsymbol{e}_\theta + \nabla \times \left((\nabla^2 - \frac{1}{r^2})u\boldsymbol{e}_\theta\right), \tag{3.20}$$

and

$$\nabla \times (\boldsymbol{\omega} \times \boldsymbol{u}) = \left(J(\frac{\omega}{r}, r\psi) - J(\frac{u}{r}, ru)\right)\boldsymbol{e}_\theta + \nabla \times \left(\frac{1}{r^2}J\left(ru, r\psi\right)\boldsymbol{e}_\theta\right). \tag{3.21}$$

From (3.19-3.21), we can rewrite (3.18) as

$$a\boldsymbol{e}_\theta + \nabla \times (b\boldsymbol{e}_\theta) = \boldsymbol{0}, \tag{3.22}$$

where

$$a = \omega_t + J\left(\frac{\omega}{r}, r\psi\right) - J\left(\frac{u}{r}, ru\right) - \nu(\nabla^2 - \frac{1}{r^2})\omega,$$

and

$$b = u_t + \frac{1}{r^2}J\left(ru, r\psi\right) - \nu(\nabla^2 - \frac{1}{r^2})u.$$

From (3.22), it follows that $a(x,r) = 0$ and $rb(x,r)$ is a constant. Since $b(x,0^+) = 0$ from Lemma 8 and Lemma 9, we conclude that $b(x,r) \equiv 0$ as well. This completes the proof of part (I).

Part (II): From Lemma 9, we know that (ψ, u, ω) satisfies (3.13). Therefore Lemma 1 applies and we have

$$\boldsymbol{u} = u\boldsymbol{e}_\theta + \nabla \times (\psi \boldsymbol{e}_\theta) \in C^1(0,T;\mathcal{C}_s^k)$$

Next we define $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$. From (3.19-3.21), we see that $\boldsymbol{\omega}$ satisfies the Navier-Stokes equation in vorticity formulation (3.18). That is

$$\nabla \times (\partial_t \boldsymbol{u} + \boldsymbol{\omega} \times \boldsymbol{u} + \nu \nabla \times \boldsymbol{\omega}) = \boldsymbol{0}.$$

Thus there exists a function $p : (0, T) \to C^{k-1}(R^3)$ such that

$$\partial_t \boldsymbol{u} + \boldsymbol{\omega} \times \boldsymbol{u} + \nu \nabla \times \boldsymbol{\omega} = -\nabla p \tag{3.23}$$

In other words, $(\boldsymbol{u}, p)$ satisfies the NSE (3.1). Since $\boldsymbol{u} \in C^1(0, T; \mathcal{C}_s^k)$, it follows from (3.23) that $\nabla p \in C^0(0, T; \mathcal{C}_s^{k-2})$. In addition, we can further assign $p(t)$ on a reference point (x_0, r_0) so that $p \in C^0(0, T; C^{k-1}(R^3))$.

By construction, the left hand side of (3.23) is axisymmetric and therefore so is ∇p. In particular

$$\partial_\theta (\nabla p \cdot \boldsymbol{e}_\theta) = \partial_\theta \left(\frac{1}{r} \partial_\theta p \right) = 0.$$

Therefore

$$p = a(x, r)\theta + b(x, r)$$

Since p is continuous and single-valued, we conclude that $a = 0$. In other words, p is axisymmetric. This completes the proof of theorem. □

3.2 Regularity Assumption on Solutions of NSE

The focus of this paper is the convergence rate of EHPS in the presence of the pole singularity. To separate difficulties and avoid complications introduced by physical boundaries, we only consider the whole space problem with exact solution decaying fast at infinity.

To be more specific, we consider the initial data $u(x, 0)$ and $\omega(x, 0)$ to be smooth with compact supports. Since (3.2) is a transport diffusion equation for u and ω with initially finite speed of propagation, we expect u and ω to be essentially compactly supported, at least for short time. For *linear* transport diffusion equations with initial data smooth and compactly supported, the solution together with its derivatives will decay faster than polynomials at infinity for $t > 0$. Some rigorous results concerning the spatial decay rate for the solutions of axisymmetric flows can be found in [6] and the reference therein. In particular, it is shown in [6] that both u and ω decay algebraically at infinity as long as this is the case initially. Here we make a stronger, yet plausible assumption along this direction. The precise form of our assumption is formulated in terms of weighted norms and is less stringent than the analogy we draw from linear transport diffusion equations, see Assumption 1 below.

To quantify our assumption, we first introduce a family of r-homogeneous composite norms and corresponding function spaces which turn out to be natural for our pointwise energy estimate:

Definition 6

$$\|a\|_{\ell,\alpha,\beta} = \sum_{\ell_1+\ell_2=\ell} \| (1+r)^{\alpha}(1+|x|)^{\beta}|\partial_x^{\ell_1}\partial_r^{\ell_2}(\frac{u}{r})| \|_{L^{\infty}(R\times\overline{R^+})}$$

$$|||a|||_{k,\alpha,\beta} = \sum_{0\le\ell\le k} \|a\|_{k-\ell,\alpha-\ell,\beta}$$

$$C_s^{k,\alpha,\beta} = \{a(x,r)\in C_s^k\left(R\times\overline{R^+}\right), |||a|||_{k,\alpha,\beta}<\infty\}$$

In section 5, we will show that EHPS is second order accurate provided the solution satisfies

$$\begin{cases} (\partial_t\psi,\psi,\omega)\in C^0\left(0,T;C_s^{4,\alpha+\frac{7}{2},\beta}\cap C_s^{4,2\alpha+2,2\beta}\right) \\ u\in C^0\left(0,T;C_s^{4,2\alpha+2,2\beta}\cap C_s^{4,2\alpha+2,2\beta}\cap C_s^{1,2,0}\right)\end{cases}, \quad \alpha>\frac{1}{2},\ \beta>\frac{1}{4}. \tag{3.24}$$

In view of (3.24), we formulate our regularity assumption as

Assumption 1

$$(\partial_t\psi,\psi,\omega)\in C^0\left(0,T;C_s^{4,\gamma,\delta}\right),\ u\in C^0\left(0,T;C_s^{4,5,\delta}\right),\quad \gamma>4,\ \delta>\frac{1}{2}. \tag{3.25}$$

Although we expect u,ω and their derivatives to decay faster than any polynomial at infinity, the same expectation is not practical for ψ. As will be shown below, generically ψ only decays like $O((x^2+r^2)^{-1})$ at infinity. This is related to the decay rate of the fundamental solution of Poisson equation in 3D and the vanishing zeroth moments of y, z components of vorticity.

To see this, we start with the integral expression for ψ. From the vorticity-stream relation

$$\nabla\times\nabla\times\boldsymbol{\psi}=\boldsymbol{\omega}$$

and the identification

$$\psi(x,r)=\psi_y(x,y,0)|_{y=r},\qquad \omega(x,r)=\omega_y(x,y,0)|_{y=r},$$

we can derive the following integral formula for ψ [20]:

$$\psi(x,r)=\int_0^{\infty}\int_{-\infty}^{\infty}\omega(x',r')K(x-x',r,r')dx'dr' \tag{3.26}$$

where

$$K(x-x',r,r') = r'\frac{1}{4\pi}\int_0^{2\pi}\frac{\cos\theta}{\sqrt{(x-x')^2+(r-r'\cos\theta)^2+(r'\cos\theta)^2}}d\theta = r'^2\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{r\cos^2\theta}{\rho_+\rho_-(\rho_++\rho_-)}d\theta \tag{3.27}$$

and

$$\rho_\pm^2 = (x-x')^2+(r\pm r'\cos\theta)^2+(r'\cos\theta)^2$$

As a consequence, we have the following far field estimate for K:

Lemma 10

$$|\partial_x^\ell\partial_r^m K(x-x',r,r')| \le C_{\ell,m}(x',r')\left(\sqrt{x^2+r^2}\right)^{-2-\ell-m} \quad as \quad x^2+r^2\to\infty$$

Proof: We will derive a far field estimate for the integrand in (3.27). We first consider a typical term

$$\lim_{x^2+r^2\to\infty}|\partial_x^\ell\partial_r^m\rho|$$

with

$$\rho^2 = (x-x_0)^2+(r-r_0)^2+c_0^2$$

where x_0, r_0 and c_0 are some constants.

With the change of variables

$$r-r_0 = \sigma\cos\lambda$$
$$x-x_0 = \sigma\sin\lambda$$

we can rewrite the x and r derivatives by

$$\partial_r\rho = \partial_r\sqrt{\sigma^2+c_0^2} = (\partial_r\sigma)\partial_\sigma\sqrt{\sigma^2+c_0^2}+(\partial_r\lambda)\partial_\lambda\sqrt{\sigma^2+c_0^2} = \cos\lambda\cdot\frac{\sigma}{\rho}$$
$$\partial_x\rho = \partial_x\sqrt{\sigma^2+c_0^2} = (\partial_x\sigma)\partial_\sigma\sqrt{\sigma^2+c_0^2}+(\partial_x\lambda)\partial_\lambda\sqrt{\sigma^2+c_0^2} = \sin\lambda\cdot\frac{\sigma}{\rho}$$

Therefore by induction,

$$\partial_x^\ell\partial_r^m\rho = P^{\ell,m}(\cos\lambda,\sin\lambda)Q^{\ell,m}(\sigma,\rho)$$

where $P^{\ell,m}(\cos\lambda,\sin\lambda)$ is a polynomial of degree $\ell+m$ in its arguments and $Q^{\ell,m}(\sigma,\rho)$ a rational function of σ and ρ of degree $1-\ell-m$. By degree of a rational function we mean the degree of the numerator subtracting the degree of the denominator.

Since $\sigma = O(\sqrt{x^2+r^2})$ and $\rho = O(\sqrt{x^2+r^2})$, we conclude that

$$|\partial_x^\ell\partial_r^m\rho| = O(\sqrt{x^2+r^2}^{1-\ell-m}).$$

We can now apply the argument above and Leibniz's rule to get

$$\partial_x^\ell \partial_r^m \frac{r}{\rho_+\rho_-(\rho_+ + \rho_-)} = \sum_j^{J_{\ell,m}} \tilde{P}_j^{\ell,m}(\cos\lambda_+, \sin\lambda_+, \cos\lambda_-, \sin\lambda_-)\tilde{Q}_j^{\ell,m}(\sigma_+, \rho_+, \sigma_-, \rho_-, r)$$

where $J_{\ell,m}$ is a finite integer and $\sigma_\pm$ and $\rho_\pm$ are defined by

$$\begin{aligned} r \pm r' \cos\theta &= \sigma_\pm \cos\lambda_\pm \\ x - x_0 &= \sigma_\pm \sin\lambda_\pm \end{aligned}$$

and $\tilde{P}_j^{\ell,m}$, $\tilde{Q}_j^{\ell,m}$ are polynomials and rational functions of degrees $\ell + m$, $-2 - \ell - m$ in their arguments respectively. The Lemma follows by integrating θ over $(0, \frac{\pi}{2})$ in (3.27). $\square$

We close this section by noting that ψ suffers from slow decay rate at infinity as a consequence of (3.26) and Lemma 10. More precisely, $\psi(x, r) \sim O((x^2 + r^2)^{-1})$ in general. This may seem to raise the question whether Assumption 1 is realizable at all. Indeed, a more refined calculation using Lemma 10 shows that the range of γ and δ in (3.25) is not void, provided ω decays fast enough at infinity:

Proposition 2 *If $\gamma + \delta < k + 2$ and $\partial_t\omega, \omega \in C_s^{k,\gamma',\delta'}$ for sufficiently large γ' and δ', then $\partial_t\psi, \psi \in C_s^{k,\gamma,\delta}$.*

4 Energy and Helicity Preserving Scheme

In this section, we outline the derivation of the discrete energy and helicity identities for EHPS. A key ingredient in the derivation is the reformulation of nonlinear terms into Jacobians. The details can be found in [17].

We introduce the standard notation:

$$D_x\phi(x,r) = \frac{\phi(x + \frac{\Delta x}{2}, r) - \phi(x - \frac{\Delta x}{2}, r)}{\Delta x}, \quad D_r\phi(x,r) = \frac{\phi(x, r + \frac{\Delta r}{2}) - \phi(x, r - \frac{\Delta r}{2})}{\Delta r},$$

$$\tilde{D}_x\phi(x,r) = \frac{\phi(x + \Delta x, r) - \phi(x - \Delta x, r)}{2\Delta x}, \quad \tilde{D}_r\phi(x,r) = \frac{\phi(x, r + \Delta r) - \phi(x, r - \Delta r)}{2\Delta r}.$$

and

$$\tilde{\nabla}_h = (\tilde{D}_x, \tilde{D}_r), \quad \tilde{\nabla}_h^\perp = (-\tilde{D}_r, \tilde{D}_x).$$

The finite difference approximation of ∇^2 and the Jacobians are given by

$$\nabla_h^2\psi = D_x\left(D_x\psi\right) + \frac{1}{r}\left(D_r(rD_r\psi)\right)$$

and

$$J_h(f,g) = \frac{1}{3}\left\{\tilde{\nabla}_h f\cdot\tilde{\nabla}_h^{\perp} g + \tilde{\nabla}_h\cdot(f\tilde{\nabla}_h^{\perp} g) + \tilde{\nabla}_h^{\perp}\cdot(g\tilde{\nabla}_h f)\right\} \tag{4.1}$$

Altogether, the finite difference version of EHPS is:

$$\begin{aligned}
&\partial_t u_h + \tfrac{1}{r^2} J_h\left(ru_h, r\psi_h\right) = \nu(\nabla_h^2 - \tfrac{1}{r^2})u_h \\
&\partial_t \omega_h + J_h\left(\tfrac{\omega_h}{r}, r\psi_h\right) = \nu(\nabla_h^2 - \tfrac{1}{r^2})\omega_h + J_h\left(\tfrac{u_h}{r}, ru_h\right) \\
&\omega_h = (-\nabla_h^2 + \tfrac{1}{r^2})\psi_h
\end{aligned} \tag{4.2}$$

To derive the discrete energy and helicity identity, we first introduce the following discrete analogue of weighted inner products

$$\langle a,b\rangle_h = \sum_{i=-\infty}^{\infty}\sum_{j=1}^{\infty}(rab)_{i,j}\,\Delta x\Delta r \tag{4.3}$$

$$[a,b]_h = \left(\sum_{i=-\infty}^{\infty}\sum_{j=1}^{\infty}(r(D_x a)(D_x b))_{i-\frac{1}{2},j} + \sum_{i=-\infty}^{\infty}\sum_{j=1}^{\infty}{}'\,(r(D_r a)(D_r b))_{i,j-\frac{1}{2}}\right)\Delta x\Delta r + \langle\frac{a}{r},\frac{b}{r}\rangle_h \tag{4.4}$$

and the corresponding norms

$$\|a\|_{0,h}^2 = \langle a,a\rangle_h, \qquad \|a\|_{1,h}^2 = [a,a]_h = \|\nabla_h a\|_{0,h}^2 + \|\frac{a}{r}\|_{0,h}^2 \tag{4.5}$$

where

$$\sum_{j=1}^{\infty}{}' = \frac{1}{2}\sum_{j=1} + \sum_{j=2}^{\infty} \tag{4.6}$$

and the grids have been shifted ([18]) to avoid placing the grid point on the axis of rotation:

$$x_i = i\Delta x, \quad i = 0,\pm1,\pm2,\cdots, \quad r_j = (j-\frac{1}{2})\Delta r, \quad j = 1,2,\cdots \tag{4.7}$$

The evaluation of $\tilde{D}_r$ and ∇_h^2 terms in (4.2) at $j=1$ involves the dependent variables u_h,ψ_h,ω_h and the stretching factor $h_3 = |\nabla\theta|^{-1} = r$ at the ghost points $j=0$. In view of Lemma 1, we impose the following reflection boundary condition across the axis of rotation:

$$u_h(i,0) = -u_h(i,1), \quad \psi_h(i,0) = -\psi_h(i,1), \quad \omega_h(i,0) = -\omega_h(i,1). \tag{4.8}$$

Furthermore, we take even extension for the coordinate stretching factor $h_3 = |\nabla\theta|^{-1} = r$ which appears in the evaluation of the Jacobians at $j=1$:

$$h_3(i,0) = h_3(i,1). \tag{4.9}$$

We will show in the remaining sections that the extensions (4.8) and (4.9) indeed give rise to a discrete version of energy and helicity identity and optimal local truncation error. As a consequence, fully second order accuracy of EHPS is justified for axisymmetric flows.

Remark 1 *At first glance, the extension (4.9) may seem to contradict (4.7) on the ghost points $j = 0$. A less ambiguous restatement of (4.9) is to incorporate it into (4.2) as*

$$\begin{aligned} &\partial_t u_h + \tfrac{1}{r^2} J_h\left(|r|u_h, |r|\psi_h\right) = \nu(\nabla_h^2 - \tfrac{1}{r^2})u_h \\ &\partial_t \omega_h + J_h\left(\tfrac{\omega_h}{|r|}, |r|\psi_h\right) = \nu(\nabla_h^2 - \tfrac{1}{r^2})\omega_h + J_h\left(\tfrac{u_h}{|r|}, |r|u_h\right) \quad \text{on } (x_i, r_j), \quad j \geq 1 \qquad (4.10) \\ &\omega_h = (-\nabla_h^2 + \tfrac{1}{r^2})\psi_h \end{aligned}$$

The following identities are essential to the discrete energy and helicity identity and the error estimate:

Lemma 11 *Suppose (a, b, c) satisfies the reflection boundary condition*

$$a(i,0) = -a(i,1), \quad b(i,0) = -b(i,1), \quad c(i,0) = -c(i,1)$$

and define

$$T_h(a,b,c) := \frac{1}{3}\sum_{i=-\infty}^{\infty}\sum_{j=1}^{\infty}\left(c\tilde{\nabla}_h a \cdot \tilde{\nabla}_h^{\perp} b + a\tilde{\nabla}_h b \cdot \tilde{\nabla}_h^{\perp} c + b\tilde{\nabla}_h c \cdot \tilde{\nabla}_h^{\perp} a\right)_{i,j}. \qquad (4.11)$$

Then

$$\sum_{i=-\infty}^{\infty}\sum_{j=1}^{\infty} c_{i,j} J_h(a,b)_{i,j} = T_h(a,b,c), \qquad (4.12)$$

and

$$\langle a, (-\nabla_h^2 + \frac{1}{r^2})b\rangle_h = [a,b]_h. \qquad (4.13)$$

Proof: We first derive (4.12). In view of (4.1) and (4.11), it suffices to show that

$$\sum_j \sum_i c\tilde{\nabla}_h \cdot (a\tilde{\nabla}_h^{\perp} b) = -\sum_{i,j} a\tilde{\nabla}_h c \cdot \tilde{\nabla}_h^{\perp} b \qquad (4.14)$$

$$\sum_i \sum_j c\tilde{\nabla}_h^{\perp} \cdot (b\tilde{\nabla}_h a) = -\sum_{i,j} b\tilde{\nabla}_h^{\perp} c \cdot \tilde{\nabla}_h a \qquad (4.15)$$

or, since there is no boundary terms in the x direction, simply

$$\sum_{i=-\infty}^{\infty}\sum_{j=1}^{\infty}(f\tilde{D}_r g)_{i,j} = -\sum_{i=-\infty}^{\infty}\sum_{j=1}^{\infty}(g\tilde{D}_r f)_{i,j} \tag{4.16}$$

with $f = c$ and $g = b\tilde{D}_x a - a\tilde{D}_x b$.

It is straight forward to verify that

$$\sum_{i=-\infty}^{\infty}\sum_{j=1}^{\infty}(f\tilde{D}_r g)_{i,j} = -\sum_{i=-\infty}^{\infty}\sum_{j=1}^{\infty}(g\tilde{D}_r f)_{i,j} - \sum_{i=-\infty}^{\infty}(f_{i,0}g_{1,0} + g_{i,0}f_{i,1}).$$

In the derivation of the discrete energy and helicity identities (see (4.18-4.20) below), a typical triplet (a, b, c) is given by, say, $a = r\psi_h$, $b = ru_h$ and $c = \frac{u_h}{r}$. From the reflection boundary condition (4.8) and (4.9), we have

$$f_{i,0} = -f_{i,1}, \quad g_{i,0} = g_{i,1}.$$

This gives (4.16), and therefore (4.14), (4.15) and (4.12).

Next we derive (4.13). From the identity

$$\sum_{j=1}^{\infty} f_j(g_{j+\frac{1}{2}} - g_{j-\frac{1}{2}}) = -\sum_{j=1}^{\infty}{}'(f_j - f_{j-1})g_{j-\frac{1}{2}} - \frac{1}{2}(f_1 + f_0)g_{\frac{1}{2}}$$

and $r_{\frac{1}{2}} = 0$, it is easy show that

$$\sum_{i=-\infty}^{\infty}\sum_{j=1}^{\infty} a_{i,j}D_r(rD_r b)_{i,j} = -\sum_{i=-\infty}^{\infty}\sum_{j=1}^{\infty}{}'(D_r a)_{i,j-\frac{1}{2}}r_{j-\frac{1}{2}}(D_r b)_{i,j-\frac{1}{2}}\,.$$

Therefore (4.13) follows. □

From (4.11), we can easily derive the permutation identities

$$T_h(a, b, c) = T_h(b, c, a) = T_h(c, a, b), \qquad T_h(a, b, c) = -T_h(b, a, c)\,. \tag{4.17}$$

Moreover, from (4.12, 4.13), we can easily derive

$$\begin{aligned}
\langle \upsilon, \partial_t u_h\rangle_h + T_h(ru_h, r\psi_h, \tfrac{\upsilon}{r}) &= \nu\langle \upsilon, (\nabla_h^2 - \tfrac{1}{r^2})u_h\rangle_h \\
[\varphi, \partial_t\psi_h]_h + T_h(\tfrac{\omega_h}{r}, r\psi_h, r\varphi) &= \nu\langle \varphi, (\nabla_h^2 - \tfrac{1}{r^2})\omega_h\rangle_h + T_h(\tfrac{u_h}{r}, ru_h, r\varphi) \\
\langle \xi, \omega_h\rangle_h &= [\xi, \psi_h]_h
\end{aligned} \tag{4.18}$$

for all υ, φ and ξ satisfying

$$\upsilon(i,0) = -\upsilon(i,1), \quad \varphi(i,0) = -\varphi(i,1), \quad \xi(i,0) = -\xi(i,1).$$

As a direct consequence of the permutation identity (4.17), we take $(\upsilon, \varphi) = (u_h, \psi_h)$ in (4.18) and recover the discrete energy identity

$$\frac{d}{dt}\frac{1}{2}(\langle u_h, u_h\rangle_h + [\psi_h, \psi_h]_h) + \nu([u_h, u_h]_h + \langle \omega_h, \omega_h\rangle_h) = 0 \tag{4.19}$$

Similarly, the discrete helicity identity

$$\frac{d}{dt}\langle u_h, \omega_h\rangle_h + \nu([u_h, \omega_h]_h - \langle \omega_h, (\nabla_h^2 - \frac{1}{r^2})u_h\rangle_h) = 0 \tag{4.20}$$

follows by taking $(\upsilon, \varphi) = (\omega_h, u_h)$ in (4.18).

Remark 2 *In the presence of physical boundaries, the no-slip boundary condition gives*

$$\boldsymbol{u}\cdot\boldsymbol{n} = \partial_\tau(r\psi) = 0, \quad \boldsymbol{u}\cdot\boldsymbol{\tau} = \partial_n(r\psi) = 0, \quad \boldsymbol{u}\cdot\boldsymbol{e}_3 = u = 0 \tag{4.21}$$

where $\boldsymbol{\tau} = \boldsymbol{n}\times\boldsymbol{e}_3$ and $\boldsymbol{e}_3$ is the unit vector in θ direction. When the cross section Ω is simply connected, (4.21) reads:

$$u = 0, \quad \psi = 0, \quad \partial_n(r\psi) = 0 \quad on \quad \partial\Omega\,. \tag{4.22}$$

It can be shown that the energy and helicity identities (4.19, 4.20) remains valid the presence of physical boundary conditions [17]. The numerical realization of the no-slip condition (4.22) introduced in [17] is second order accurate and seems to be new even for usual 2D flows. The convergence proof for this new boundary condition will be reported elsewhere.

5 Energy Estimate and the Main Theorem

In this section, we proceed with the main Theorem of error estimate. We denote by ψ_h, u_h, ω_h the numerical solution satisfying

$$\begin{aligned}
&\partial_t u_h + \tfrac{1}{r^2}J_h(u_h, r\psi_h) = \nu(\nabla_h^2 - \tfrac{1}{r^2})u_h \\
&\partial_t \omega_h + J_h\left(\tfrac{\omega_h}{r}, r\psi_h\right) = \nu(\nabla_h^2 - \tfrac{1}{r^2})\omega_h + J_h\left(\tfrac{u_h}{r}, ru_h\right) \\
&\omega_h = (-\nabla_h^2 + \tfrac{1}{r^2})\psi_h
\end{aligned} \tag{5.1}$$

and ψ, u, ω the exact solution to (3.2),

$$\begin{aligned} &\partial_t u + \tfrac{1}{r^2} J_h(ru, r\psi) = \nu(\nabla_h^2 - \tfrac{1}{r^2})u + \mathcal{E}_1 \\ &\partial_t \omega + J_h\left(\tfrac{\omega}{r}, r\psi\right) = \nu(\nabla_h^2 - \tfrac{1}{r^2})\omega + J_h\left(\tfrac{u}{r}, ru\right) + \mathcal{E}_2 \\ &\omega = (-\nabla_h^2 + \tfrac{1}{r^2})\psi + \mathcal{E}_3 \end{aligned} \tag{5.2}$$

where the local truncation errors $\mathcal{E}_j$ can be derived by subtracting (3.2) from (5.2):

$$\begin{aligned} &\mathcal{E}_1 = \tfrac{1}{r^2}(J_h - J)(ru, r\psi) - \nu(\nabla_h^2 - \nabla^2)u \\ &\mathcal{E}_2 = (J_h - J)\left(\tfrac{\omega}{r}, r\psi\right) - \nu(\nabla_h^2 - \nabla^2)\omega - (J_h - J)\left(\tfrac{u}{r}, ru\right) \\ &\mathcal{E}_3 = (\nabla_h^2 - \nabla^2)\psi \end{aligned} \tag{5.3}$$

From (5.1) and (5.2), we see that

$$\partial_t(u - u_h) + \frac{1}{r^2}\left(J_h(ru, r\psi) - J_h(ru_h, r\psi_h)\right) = \nu(\nabla_h^2 - \frac{1}{r^2})(u - u_h) + \mathcal{E}_1 \tag{5.4}$$

$$\begin{aligned} &\partial_t(\omega - \omega_h) + \left(J_h\left(\tfrac{\omega}{r}, r\psi\right) - J_h\left(\tfrac{\omega_h}{r}, r\psi_h\right)\right) \\ = \;&\nu(\nabla_h^2 - \tfrac{1}{r^2})(\omega - \omega_h) + \left(J_h\left(\tfrac{u}{r}, ru\right) - J_h\left(\tfrac{u_h}{r}, ru_h\right)\right) + \mathcal{E}_2 \end{aligned} \tag{5.5}$$

$$(\omega - \omega_h) = (-\nabla_h^2 + \frac{1}{r^2})(\psi - \psi_h) + \mathcal{E}_3 \tag{5.6}$$

For nonlinear problems, it is quite unusual that such an equality in conservative form can be derived for finite difference schemes. In our case, the reflecting boundary condition (4.8) and (4.9) play an important role in the derivation of the following equality:

Proposition 3

$$\begin{aligned} &\tfrac{1}{2}\partial_t(\|u - u_h\|_{0,h}^2 + \|\psi - \psi_h\|_{1,h}^2) + \nu(\|u - u_h\|_{1,h}^2 + \|\omega - \omega_h\|_{0,h}^2) \\ = \;&\langle u - u_h, \mathcal{E}_1\rangle_h + \langle \psi - \psi_h, \mathcal{E}_2 - \partial_t\mathcal{E}_3\rangle_h - T_h\left(\tfrac{u-u_h}{r}, r(u - u_h), r\psi\right) \\ &-T_h\left(r(\psi - \psi_h), \tfrac{(\omega-\omega_h)}{r}, r\psi\right) + T_h\left(r(\psi - \psi_h), \tfrac{u}{r}, r(u - u_h)\right) \end{aligned} \tag{5.7}$$

Proof:

We take the weighted inner product of $u - u_h$ with (5.4) to get

$$\begin{aligned} &\tfrac{1}{2}\partial_t\|u - u_h\|_{0,h}^2 + \langle u - u_h, \tfrac{1}{r^2}\left(J_h(ru, r\psi) - J_h(ru_h, r\psi_h)\right)\rangle_h \\ = \;&\nu\langle u - u_h, (\nabla_h^2 - \tfrac{1}{r^2})(u - u_h)\rangle_h + \langle u - u_h, \mathcal{E}_1\rangle_h\,. \end{aligned} \tag{5.8}$$

The second term on the left hand side of (5.8) can be rewritten as

$$\begin{aligned} & \langle u-u_h, \tfrac{1}{r^2}\left(J_h(ru,r\psi)-J_h(ru_h,r\psi_h)\right)\rangle_h \\ = \; & T_h\left(\tfrac{u-u_h}{r},ru,r\psi\right)-T_h\left(\tfrac{u-u_h}{r},ru_h,r\psi_h\right) \\ = \; & -T_h\left(\tfrac{u-u_h}{r},r(u-u_h),r(\psi-\psi_h)\right)+T_h\left(\tfrac{u-u_h}{r},r(u-u_h),r\psi\right)+T_h\left(\tfrac{u-u_h}{r},ru,r(\psi-\psi_h)\right). \end{aligned} \tag{5.9}$$

In addition, from (4.13)

$$\nu\langle u-u_h,(\nabla_h^2-\frac{1}{r^2})(u-u_h)\rangle_h=-\nu[u-u_h,u-u_h]_h=-\nu\|u-u_h\|_{1,h}^2\,.$$

Thus

$$\begin{aligned} & \tfrac{1}{2}\partial_t\|u-u_h\|_{0,h}^2-T_h\left(\tfrac{u-u_h}{r},r(u-u_h),r(\psi-\psi_h)\right)+\nu\|u-u_h\|_{1,h}^2 \\ = \; & \langle u-u_h,\mathcal{E}_1\rangle_h-T_h\left(\tfrac{u-u_h}{r},r(u-u_h),r\psi\right)-T_h\left(\tfrac{u-u_h}{r},ru,r(\psi-\psi_h)\right). \end{aligned} \tag{5.10}$$

Similarly, we take the weighted inner product of $\psi-\psi_h$ with (5.5) and proceed as (5.9-5.10) to get

$$\begin{aligned} & \tfrac{1}{2}\partial_t\|\psi-\psi_h\|_{1,h}^2+T_h\left(r(\psi-\psi_h),\tfrac{(\omega-\omega_h)}{r},r\psi\right)+\nu\|\omega-\omega_h\|_{0,h}^2 \\ = \; & -T_h\left(r(\psi-\psi_h),\tfrac{(u-u_h)}{r},r(u-u_h)\right)+T_h\left(r(\psi-\psi_h),\tfrac{u}{r},r(u-u_h)\right) \\ & +T_h\left(r(\psi-\psi_h),\tfrac{(u-u_h)}{r},ru\right)+\langle\psi-\psi_h,\mathcal{E}_2-\partial_t\mathcal{E}_3\rangle_h\,. \end{aligned} \tag{5.11}$$

Next, we apply (4.13) twice to get

$$\nu\langle(\psi-\psi_h),(\nabla_h^2-\frac{1}{r^2})(\omega-\omega_h)\rangle_h=\nu\langle(\nabla_h^2-\frac{1}{r^2})(\psi-\psi_h),\omega-\omega_h\rangle_h=-\nu\|\omega-\omega_h\|_{0,h}^2$$

and (5.7) follows. This completes the proof of the Proposition. □

We proceed to the estimate of the right hand side of (5.7). We start with the following elementary identities:

Proposition 4 *Define*

$$(\widetilde{A}_x f)_{i,j}=\frac{1}{2}(f_{i+1,j}+f_{i-1,j}),\qquad (\widetilde{A}_r f)_{i,j}=\frac{1}{2}(f_{i,j+1}+f_{i,j-1}).$$

The following estimates hold for $j\geq 1$:

$$|\widetilde{D}_r(ra)|\leq C|\widetilde{A}_r a|+Cr|\widetilde{D}_r a| \tag{5.12}$$

$$|\widetilde{D}_r(\frac{a}{r})| \le C\frac{|\widetilde{A}_r a|}{r^2} + C\frac{|\widetilde{D}_r a|}{r} \tag{5.13}$$

$$|\widetilde{A}_r(ra)| \le Cr\widetilde{A}_r|a| \tag{5.14}$$

$$|\Delta r\widetilde{D}_r a| \le \widetilde{A}_r|a|, \quad |\Delta x\widetilde{D}_x a| \le \widetilde{A}_x|a| \tag{5.15}$$

Remark 3 *As in Remark 1, the stretching factor r in the arguments of left hand side of (5.12-5.14) satisfy the even extension (4.9). A more precise statement for, say, (5.12) is given by*

$$|\widetilde{D}_r(|r|a)|_{i,j} \le C|\widetilde{A}_r a|_{i,j} + Cr_j|\widetilde{D}_r a|_{i,j}, \qquad j \ge 1.$$

For simplicity of presentation, we will adopt the expression as in (5.12-5.14) through rest of the paper.

Proof of Proposition 4:

It is easy to verify that

$$\widetilde{D}_r(fg) = (\widetilde{A}_r f)(\widetilde{D}_r g) + (\widetilde{A}_r g)(\widetilde{D}_r f), \quad \widetilde{D}_x(fg) = (\widetilde{A}_x f)(\widetilde{D}_x g) + (\widetilde{A}_x g)(\widetilde{D}_x f)$$

A straight forward calculation shows that

$$(\tilde{A}_r|r|)_j \le Cr_j, \quad |\,\tilde{D}_r|r|\,|_j \le C$$

and

$$\tilde{A}_r(\frac{1}{|r|})_j \le C\frac{1}{r_j}, \quad |\tilde{D}_r(\frac{1}{|r|})|_j \le C\frac{1}{r_j^2}$$

for $j \ge 1$. The estimates (5.12-5.14) then follows. The proof for (5.15) is also straight forward. □

We now proceed to estimate the trilinear forms on the right hand side of (5.7)

Lemma 12 *For a, b and $c \in C_s^1(R \times \overline{R^+})$, we have*

$$|T_h\left(ra, rb, \frac{c}{r}\right)| \le C\|a\|_{1,h}\|b\|_{1,h}|\!|\!|c|\!|\!|_{1,2,0} \tag{5.16}$$

and

$$|T_h\left(\frac{a}{r}, rb, rc\right)| \le C\|a\|_{0,h}\|b\|_{1,h}|\!|\!|c|\!|\!|_{2,2,0}. \tag{5.17}$$

Proof: We begin with the proof of (5.16). First we expand the left hand side as

$$\begin{aligned} T_h\left(ra, rb, \tfrac{c}{r}\right) = &\ \tfrac{1}{3}\left(\langle \tfrac{c}{r^2}, \tilde\nabla_h(ra)\cdot\tilde\nabla_h^{\perp}(rb)\rangle_h + \langle a, \tilde\nabla_h(rb)\cdot\tilde\nabla_h^{\perp}(\tfrac{c}{r})\rangle_h + \langle b, \tilde\nabla_h(\tfrac{c}{r})\cdot\tilde\nabla_h^{\perp}(ra)\rangle_h\right) \\ = &\ \tfrac{1}{3}(I_1+I_2+I_3) \end{aligned}$$

and estimate the I_j's term by term. We have

$$|I_1| = |\langle \frac{c}{r^2}, \tilde\nabla_h(ra)\cdot\tilde\nabla_h^{\perp}(rb)\rangle_h| = |\langle c, \frac{1}{r}\tilde D_r(ra)\tilde D_x(b) - \tilde D_x(a)\frac{1}{r}\tilde D_r(rb)\rangle_h|,$$

therefore

$$|I_1| \le C\langle |c|, (|\frac{\tilde A_r(a)}{r}| + |\tilde D_r(a)|)|\tilde D_x(b)| + (|\frac{\tilde A_r(b)}{r}| + |\tilde D_r(b)|)|\tilde D_x(a)|\rangle_h \le C\|a\|_{1,h}\|b\|_{1,h}\|c\|_{0,1,0}$$

follows from (5.12), Hölder inequality and the estimate $|c| = |r\frac{c}{r}| \le \|c\|_{0,1,0}$.

Next,

$$\begin{aligned} |I_2| \le &\ C\langle |a|, |\tfrac{\tilde A_r(b)}{r}| + |\tilde D_r(b)||\tilde D_x(c)|\rangle_h + C\langle |a|, |\tilde D_r(c)||\tilde D_x(b)|\rangle_h + C\langle \tfrac{|a|}{r}, |A_r(c)||\tilde D_x(b)|\rangle_h \\ = &\ C\langle \tfrac{|a|}{r}, |\tfrac{\tilde A_r(b)}{r}| + |\tilde D_r(b)||r\tilde D_x(c)|\rangle_h + C\langle \tfrac{|a|}{r}, |r\tilde D_r(c)||\tilde D_x(b)|\rangle_h + C\langle \tfrac{|a|}{r}, |A_r(c)||\tilde D_x(b)|\rangle_h \\ \le &\ C\|a\|_{1,h}\|b\|_{1,h}(\|c\|_{0,1,0} + \|c\|_{1,2,0}) \le C\|a\|_{1,h}\|b\|_{1,h}|\!|\!|c|\!|\!|_{1,2,0} \end{aligned}$$

The estimate for I_3 is similar.

Next we proceed with (5.17). Since

$$|T_h\left(\frac{a}{r}, rb, rc\right)| = |\langle a, \frac{1}{r^2}J_h(rb,rc)\rangle_h| \le \|a\|_{0,h}\|\frac{1}{r^2}J_h(rb,rc)\|_{0,h},$$

we first give a pointwise estimate of the integrand $J_h(rb, rc)$:

$$\begin{aligned} 3J_h(rb,rc) = &\ \widetilde D_r(rb)\widetilde D_x(rc) - \widetilde D_x(rb)\widetilde D_r(rc) + \widetilde D_r\left(rb\widetilde D_x(rc)\right) - \widetilde D_x\left(rb\widetilde D_r(rc)\right) \\ &+ \widetilde D_x\left(rc\widetilde D_r(rb)\right) - \widetilde D_r\left(rc\widetilde D_x(rb)\right) \\ = &\ \widetilde D_r(rb)(I+\widetilde A_r)\widetilde D_x(rc) - \widetilde D_x(rb)(I+\widetilde A_x)\widetilde D_r(rc) + (\widetilde A_r - \widetilde A_x)(rb)\widetilde D_r\widetilde D_x(rc) \\ &+ (\tilde A_x - \tilde A_r)(rc)\tilde D_x\tilde D_r(rb) + \widetilde D_x(rc)\widetilde A_x\widetilde D_r(rb) - \widetilde D_r(rc)\widetilde A_r\widetilde D_x(rb) \\ = &\ \widetilde D_r(rb)(I+\widetilde A_r)\widetilde D_x(rc) - \widetilde D_x(rb)(I+\widetilde A_x)\widetilde D_r(rc) + (\widetilde A_r - \widetilde A_x)(rb)\widetilde D_r\widetilde D_x(rc) \\ &+ \tfrac{1}{2}\Delta x^2\widetilde D_r\widetilde D_x(rb)D_x^2(rc) - \tfrac{1}{2}\Delta r^2\widetilde D_r\widetilde D_x(rb)D_r^2(rc) \\ &+ \widetilde D_x(rc)\widetilde A_x\widetilde D_r(rb) - \widetilde D_r(rc)\widetilde A_r\widetilde D_x(rb) \end{aligned} \tag{5.18}$$

Here I is the identity operator and we have used the identities

$$\tilde{A}_x = \frac{1}{2}\Delta x^2 D_x^2 + I, \quad \tilde{A}_r = \frac{1}{2}\Delta r^2 D_r^2 + I$$

in the second equality of (5.18).

From (5.12), we have

$$|\widetilde{D}_r(rb)(I+\widetilde{A}_r)\widetilde{D}_x(rc)| \le Cr^2(|\widetilde{D}_r b| + \frac{|\widetilde{A}_r b|}{r})\|\partial_x c\|_{L^\infty} \tag{5.19}$$

$$|\widetilde{D}_x(rb)(I+\widetilde{A}_x)\widetilde{D}_r(rc)| \le Cr^2|\widetilde{D}_x b||\partial_r c| + \frac{|c|}{r}\|_{L^\infty} \le Cr^2|\widetilde{D}_x b|(\|c\|_{0,0,0} + \|c\|_{1,1,0}) \tag{5.20}$$

From (5.14) and (5.15), we can similarly derive the remaining terms in (5.18):

$$|(\widetilde{A}_r - \widetilde{A}_x)(rb)\widetilde{D}_r\widetilde{D}_x(rc)| \le Cr^2\frac{(\widetilde{A}_r+\widetilde{A}_x)|b|}{r}\|\partial_x\partial_r(rc)\|_{L^\infty} \le Cr^2\frac{(\widetilde{A}_r+\widetilde{A}_x)|b|}{r}(\|c\|_{1,1,0} + \|c\|_{2,2,0}), \tag{5.21}$$

$$|\frac{1}{2}\Delta x^2\widetilde{D}_r\widetilde{D}_x(rb)D_x^2(rc)| \le C\frac{(\Delta x)^2}{\Delta r}|\widetilde{A}_r(r\widetilde{D}_x(b))D_x^2(rc)| \le Cr^2\frac{\Delta r}{r}\widetilde{A}_r|\widetilde{D}_x b|\|c\|_{2,2,0} \tag{5.22}$$

$$|\frac{1}{2}\Delta r^2\widetilde{D}_r\widetilde{D}_x(rb)D_r^2(rc)| \le C\Delta r|\widetilde{A}_r\widetilde{D}_x(rb)|\|\partial_r^2(rc)\|_{L^\infty} \le Cr^2\frac{\Delta r}{r}\widetilde{A}_r|\widetilde{D}_x b|\|\!|c|\!\|_{2,2,0} \tag{5.23}$$

$$|\widetilde{A}_x\widetilde{D}_r(rb)\widetilde{D}_x(rc)| \le Cr^2|\widetilde{A}_x(\frac{1}{r}\widetilde{D}_r(rb))|\|\partial_x c\|_{L^\infty} \le Cr^2\widetilde{A}_x|\widetilde{D}_r b|\|c\|_{1,1,0} \tag{5.24}$$

and

$$|\widetilde{A}_r\widetilde{D}_x(rb)\widetilde{D}_r(rc)| \le Cr^2\widetilde{A}_r|\widetilde{D}_x b|\|\partial_r c\|_{L^\infty} \le Cr^2\widetilde{A}_r|\widetilde{D}_x b|\|\!|c|\!\|_{1,1,0} \tag{5.25}$$

From (5.19-5.25), we can estimate the weighted L^2 norm of $\frac{1}{r^2}J_h(rb, rc)$ by

$$\|\frac{1}{r^2}J_h(rb,rc)\|_{0,h} \le C\|(|\widetilde{D}_x b| + |\widetilde{D}_r b| + \frac{|b|}{r})\|_{0,h}\|\!|c|\!\|_{2,2,0} \le C\|b\|_{1,h}\|\!|c|\!\|_{2,2,0}$$

and (5.17) follows. □

From Proposition 3, we can derive

$$\begin{aligned}
&\tfrac{1}{2}\partial_t(\|u-u_h\|_{0,h}^2 + \|\psi-\psi_h\|_{1,h}^2) + \nu(\|u-u_h\|_{1,h}^2 + \|\omega-\omega_h\|_{0,h}^2) \\
\le\ & |\langle u-u_h, \mathcal{E}_1\rangle_h| + |\langle \psi-\psi_h, \mathcal{E}_2 - \partial_t\mathcal{E}_3\rangle_h| + C\|u-u_h\|_{0,h}\|u-u_h\|_{1,h}\|\!|\psi|\!\|_{2,2,0} \\
&+C\|\omega-\omega_h\|_{0,h}\|\psi-\psi_h\|_{1,h}\|\!|\psi|\!\|_{2,2,0} + C\|\psi-\psi_h\|_{1,h}\|u-u_h\|_{1,h}\|\!|u|\!\|_{1,2,0}.
\end{aligned} \tag{5.26}$$

Since

$$\|\frac{a}{r}\|_{0,h} \le \|a\|_{1,h},$$

we can further estimate the first two terms on the right hand side of (5.26) by

$$|\langle u-u_h,\mathcal{E}_1\rangle_h| = |\langle \frac{u-u_h}{r}, r\mathcal{E}_1\rangle_h| \le \frac{\nu}{4}\|u-u_h\|_{1,h}^2 + \frac{1}{\nu}\|r\mathcal{E}_1\|_{0,h}^2$$

and

$$|\langle \psi-\psi_h, \mathcal{E}_2-\partial_t\mathcal{E}_3\rangle_h| \le \|\psi-\psi_h\|_{1,h}^2 + \|r(\mathcal{E}_2-\partial_t\mathcal{E}_3)\|_{0,h}^2$$

We now conclude from Hölder's inequality to get

$$\begin{aligned} & \tfrac{1}{2}\partial_t(\|u-u_h\|_{0,h}^2 + \|\psi-\psi_h\|_{1,h}^2) + \tfrac{\nu}{4}(\|u-u_h\|_{1,h}^2 + \|\omega-\omega_h\|_{0,h}^2) \\ \le\ & \|\psi-\psi_h\|_{1,h}^2 + \tfrac{C}{\nu}\|r\mathcal{E}_1\|_{0,h}^2 + \|r\mathcal{E}_2\|_{0,h}^2 + \|r\partial_t\mathcal{E}_3\|_{0,h}^2 + \tfrac{C}{\nu}\|u-u_h\|_{0,h}^2 \|\!|\psi|\!\|_{2,2,0}^2 \qquad (5.27) \\ & +\tfrac{C}{\nu}\|\psi-\psi_h\|_{1,h}^2 \|\!|\psi|\!\|_{2,2,0}^2 + \tfrac{C}{\nu}\|\psi-\psi_h\|_{1,h}^2 \|\!|u|\!\|_{1,2,0}^2 \end{aligned}$$

With (5.27), it remains to estimate the local truncation errors $\|r\mathcal{E}_1\|_{0,h}$, $\|r\mathcal{E}_2\|_{0,h}$ and $\|r\partial_t\mathcal{E}_3\|_{0,h}$. We summarize the results in Lemma 13 below. The proof will be given in the Appendix.

Lemma 13 *Let (ψ, u, ω) be a solution of the axisymmetric Navier Stokes equation (3.2) with*

$$(\partial_t\psi, \psi, u, \omega) \in C^0(0,T; C_s^4)$$

and $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$ defined by (5.2). Then we have the following pointwise local truncation error estimate for $\alpha, \beta \in R$:

$$r|\mathcal{E}_1| \le C\frac{\Delta x^2+\Delta r^2}{(1+r)^{2\alpha}(1+|x|)^{2\beta}}\left(\|\!|\psi|\!\|_{4,\alpha+\frac{7}{2},\beta}\|\!|u|\!\|_{4,\alpha+\frac{7}{2},\beta} + \|\!|u|\!\|_{4,2\alpha+2,2\beta}\right) \qquad (5.28)$$

$$r|\mathcal{E}_2| \le C\frac{\Delta x^2+\Delta r^2}{(1+r)^{2\alpha}(1+|x|)^{2\beta}}\left(\|\!|\psi|\!\|_{4,\alpha+\frac{7}{2},\beta}\|\!|\omega|\!\|_{4,\alpha+\frac{7}{2},\beta} + \|\!|u|\!\|_{4,\alpha+\frac{7}{2},\beta}^2 + \|\!|\omega|\!\|_{4,2\alpha+2,2\beta}\right) \qquad (5.29)$$

and

$$r|\partial_t\mathcal{E}_3| \le C\frac{\Delta x^2+\Delta r^2}{(1+r)^{2\alpha}(1+|x|)^{2\beta}}\|\!|\partial_t\psi|\!\|_{4,2\alpha+2,2\beta} \qquad (5.30)$$

Finally, we have the error estimate:

Theorem 2 *Let (ψ, u, ω) be a solution of the axisymmetric Navier Stokes equation (3.2) satisfying*

$$(\partial_t\psi, \psi, \omega) \in C^0\left(0,T; C_s^{4,\gamma,\delta}\right),\ u \in C^0\left(0,T; C_s^{4,5,\delta}\right), \qquad \gamma > 4,\ \delta > \frac{1}{2}. \qquad (5.31)$$

Then

$$\sup_{[0,T]}\left(\|u-u_h\|_{0,h}^2+\|\psi-\psi_h\|_{1,h}^2\right)+\int_0^T(\|u-u_h\|_{1,h}^2+\|\omega-\omega_h\|_{0,h}^2)dt\le C(\Delta x^2+\Delta r^2)^2 \quad (5.32)$$

where $C = C(\psi, u, \nu, T)$.

Proof: From Lemma 13, we have

$$\begin{aligned} &\|r\mathcal{E}_1\|_{0,h}^2+\|r\mathcal{E}_2\|_{0,h}^2+\|r\partial_t\mathcal{E}_3\|_{0,h}^2 \\ \le\ & C(\Delta x^4+\Delta r^4)\left(\sum_{i=-\infty}^{\infty}\sum_{j=1}^{\infty}\frac{r_j\Delta r\Delta x}{(1+r_j)^{4\alpha}(1+|x_i|)^{4\beta}}\right)\left(\|\!|(\psi,u,\omega)|\!\|_{4,\alpha+\frac{7}{2},\beta}^4+\|\!|(u,\omega,\partial_t\psi)|\!\|_{4,2\alpha+2,2\beta}^2\right). \end{aligned}$$

Since $\displaystyle\sum_{i=-\infty}^{\infty}\sum_{j=1}^{\infty}\frac{r_j\Delta r\Delta x}{(1+r_j)^{4\alpha}(1+|x_i|)^{4\beta}}$ is convergent if $\alpha>\frac{1}{2}$, $\beta>\frac{1}{4}$, it follows that

$$\|r\mathcal{E}_1\|_{0,h}^2+\|r\mathcal{E}_2\|_{0,h}^2+\|r\partial_t\mathcal{E}_3\|_{0,h}^2\le C(\Delta x^4+\Delta r^4)\left(\|\!|(\psi,u,\omega)|\!\|_{4,\gamma,\delta}^4+\|\!|(u,\omega,\partial_t\psi)|\!\|_{4,\gamma,\delta}^2\right) \quad (5.33)$$

provided $\gamma > 4$, $\delta > \frac{1}{2}$.

We conclude that, under assumption (5.31), we have $\psi\in C_s^{2,2,0}$, $u\in C_s^{1,2,0}$ and

$$\begin{aligned} &\tfrac{1}{2}\partial_t(\|u-u_h\|_{0,h}^2+\|\psi-\psi_h\|_{1,h}^2)+\tfrac{\nu}{4}(\|u-u_h\|_{1,h}^2+\|\omega-\omega_h\|_{0,h}^2) \\ \le\ & C\|u-u_h\|_{0,h}^2+C\|\psi-\psi_h\|_{1,h}^2+C(\Delta x^4+\Delta r^4) \end{aligned}$$

follows from (5.27) and (5.33). The error estimate (5.32) then follows from Gronwall's inequality. □

6 Appendix: Local Truncation Error Analysis – Proof of Lemma 13

In this section, we proceed with the local truncation error estimate. All the assertions in Lemmas 14-17 are pointwise estimates on the grid points (x_i, r_j), $j \ge 1$. For brevity, we omit the indices (i, j) whenever it is obvious.

We start with the estimates of the diffusion terms in (5.3).

Lemma 14 *If* $a\in C_s^4(R\times\overline{R^+})$ *and* $\alpha,\beta\in R$, *we have*

$$r|(\nabla_h^2-\nabla^2)a|\le\left(\Delta x^2+\Delta r^2\right)\frac{1}{(1+r)^\alpha(1+|x|)^\beta}\|\!|a|\!\|_{4,\alpha+2,\beta}$$

Proof:

Since $a \in C^4_s(R \times \overline{R^+})$, the odd extension of a given by

$$\tilde{a}(x,r) = \begin{cases} a(x,r), & \text{if } r \geq 0 \\ -a(x,-r), & \text{if } r < 0 \end{cases}$$

is in $C^4(R^2)$. It follows that

$$\nabla_h^2 a = \left(D_x^2 + D_r^2 + \frac{\widetilde{D}_r}{r}\right) a = \nabla^2 a + \frac{1}{12}\Delta x^2 \partial_x^4 a|_{(\xi,r)} + \Delta r^2 \left(\frac{1}{12}\partial_r^4 a|_{(x,\eta_1)} + \frac{1}{6}\frac{1}{r}(\partial_r^3 a)_{(x,\eta_2)}\right)$$

is valid for all $j \geq 1$ with $\xi \in (x - \Delta x, x + \Delta x)$ and $\eta_1, \eta_2 \in (r - \Delta r, r + \Delta r)$.

Thus

$$\begin{aligned}
& r|(\nabla_h^2 - \nabla^2)a| \\
\leq \; & C\left(\Delta x^2 + \Delta r^2\right)\left(r|\partial_x^4(r\tfrac{a}{r})|_{(\xi,r)}| + r|\partial_r^4(r\tfrac{a}{r})|_{(x,\eta_1)}| + |\partial_r^3(r\tfrac{a}{r})|_{(x,\eta_2)}|\right) \\
\leq \; & C\left(\Delta x^2 + \Delta r^2\right)\left(\tfrac{r(\|a\|_{4,\alpha+2,\beta}+\|a\|_{3,\alpha+1,\beta})}{(1+r)^{\alpha+1}(1+|\xi|)^\beta} + \tfrac{r(\|a\|_{4,\alpha+2,\beta}+\|a\|_{3,\alpha+1,\beta})}{(1+\eta_1)^{\alpha+1}(1+|x|)^\beta} + \tfrac{\|a\|_{3,\alpha+1,\beta}+\|a\|_{2,\alpha,\beta}}{(1+\eta_2)^{\alpha}(1+|x|)^\beta}\right) \\
\leq \; & C\left(\Delta x^2 + \Delta r^2\right)\tfrac{1}{(1+r)^\alpha(1+|x|)^\beta}|||a|||_{4,\alpha+2,\beta}
\end{aligned}$$

□

Next we proceed with the estimates for the Jacobians, starting with their typical factors:

Lemma 15 *For $a \in C^4_s(R \times \overline{R^+})$, $\alpha, \beta \in R$, we have*

$$\widetilde{D}_x(\frac{a}{r}) = \partial_x(\frac{a}{r}) + O(1)\Delta x^2 \frac{1}{(1+r)^\alpha(1+|x|)^\beta}|||a|||_{3,\alpha,\beta} \tag{6.1}$$

$$\widetilde{D}_x(ra) = \partial_x(ra) + O(1)r^2\Delta x^2 \frac{1}{(1+r)^\alpha(1+|x|)^\beta}|||a|||_{3,\alpha,\beta} \tag{6.2}$$

$$\widetilde{D}_r(\frac{a}{r}) = \partial_r(\frac{a}{r}) + O(1)\frac{\Delta r^2}{r^3} \frac{1}{(1+r)^\alpha(1+|x|)^\beta}|||a|||_{3,\alpha+3,\beta} \tag{6.3}$$

$$\widetilde{D}_r(ra) = \partial_r(ra) + O(1)\frac{\Delta r^2}{r} \frac{1}{(1+r)^\alpha(1+|x|)^\beta}|||a|||_{3,\alpha+3,\beta} \tag{6.4}$$

Proof:

We begin with (6.1) and (6.2).

Since

$$(\widetilde{D}_x - \partial_x)f = \frac{\Delta x^2}{6}\partial_x^3 f_{|(\xi,r)}, \quad \xi \in (x - \Delta x, x + \Delta x),$$

it follows that

$$\left|(\widetilde{D}_x-\partial_x)(\frac{a}{r})\right| = \frac{\Delta x^2}{6}\left|\partial_x^3(\frac{a}{r})\right|_{|(\xi,r)} \le C\Delta x^2\frac{1}{(1+r)^\alpha(1+|x|)^\beta}\|a\|_{3,\alpha,\beta}$$

and

$$|(\widetilde{D}_x-\partial_x)(ra)| = \frac{\Delta x^2}{6}\left|\partial_x^3(r^2\frac{a}{r})_{|(\xi,r)}\right| \le Cr^2\Delta x^2\frac{1}{(1+r)^\alpha(1+|x|)^\beta}\|a\|_{3,\alpha,\beta}$$

For (6.3) and (6.4), the estimate is a little more complicated due to our reflection boundary condition (4.8) and (4.9). We estimate for $j>1$ and $j=1$ separately.

When $j>1$, we have

$$(\widetilde{D}_r-\partial_r)f = \frac{1}{6}\Delta r^2\partial_r^3 f|_{(x,\eta)}, \quad \eta\in(r-\Delta r, r+\Delta r).$$

Therefore we have

$$\left|(\widetilde{D}_r-\partial_r)(\frac{a}{r})\right| = \frac{\Delta r^2}{6}\left|\partial_r^3(\frac{a}{r})\right|_{(x,\eta)} \le C\frac{\Delta r^2}{r^3}\frac{1}{(1+r)^\alpha(1+|x|)^\beta}\|a\|_{3,\alpha+3,\beta}$$

and

$$\begin{aligned}|(\widetilde{D}_r-\partial_r)(ra)| &\le C\Delta r^2\left|\partial_r^3(r^2\tfrac{a}{r})\right|_{(x,\eta)}\\ &\le C\tfrac{\Delta r^2}{r}\tfrac{1}{(1+r)^\alpha(1+|x|)^\beta}(\|a\|_{3,\alpha+3,\beta}+\|a\|_{2,\alpha+2,\beta}+\|a\|_{1,\alpha+1,\beta})\end{aligned}$$

When $j=1$, we have

$$\left|\partial_r(\frac{a}{r})\right|_{j=1} = C\frac{\Delta r^2}{r_1^3}r_1\left|\partial_r(\frac{a}{r})\right|_{j=1} \le C\frac{\Delta r^2}{r_1^3}\frac{1}{(1+r_1)^\alpha(1+|x|)^\beta}\|a\|_{1,\alpha+1,\beta}$$

In addition, since $r_1=\frac{\Delta r}{2}$, we apply (4.9) to get

$$\left|\widetilde{D}_r(\frac{a}{r})\right|_{j=1} = \left|\frac{\frac{a_2}{r_2}+\frac{a_1}{r_1}}{2\Delta r}\right| = \left|C\frac{\Delta r^2}{r_1^3}\left(\frac{a_2}{r_2}+\frac{a_1}{r_1}\right)\right| \le C\frac{\Delta r^2}{r_1^3}\frac{1}{(1+r_1)^\alpha(1+|x|)^\beta}\|a\|_{0,\alpha,\beta}$$

and (6.3) follows.

(6.4) can be proved similarly,

$$\widetilde{D}_r(ra)_{j=1} = \frac{\frac{3}{2}\Delta r a_2+\frac{1}{2}\Delta r a_1}{2\Delta r} = \frac{3}{4}a_2+\frac{1}{4}a_1,$$

$$|a_1| \le C\frac{\Delta r^2}{r_1}\left|\frac{a_1}{r_1}\right| \le C\frac{\Delta r^2}{r_1}\frac{1}{(1+r_1)^\alpha(1+|x|)^\beta}\|a\|_{0,\alpha,\beta},$$

and

$$|a_2| \le C\frac{\Delta r^2}{r_1}\left|\frac{a_2}{r_2}\right| \le C\frac{\Delta r^2}{r_1}\frac{1}{(1+r_1)^\alpha(1+|x|)^\beta}\|a\|_{0,\alpha,\beta}.$$

Therefore

$$\left|\widetilde{D}_r(ra)\right|_{j=1} \le C\frac{\Delta r^2}{r_1}\frac{1}{(1+r_1)^\alpha(1+|x|)^\beta}\|a\|_{0,\alpha,\beta}$$

In addition,

$$|\partial_r(ra)|_{j=1} \le \left(r^2\left|\partial_r(\frac{a}{r})\right| + 2r\left|\frac{a}{r}\right|\right)_{j=1} \le C\frac{\Delta r^2}{r_1}\frac{1}{(1+r_1)^\alpha(1+|x|)^\beta}(\|a\|_{1,\alpha+1,\beta}+\|a\|_{0,\alpha,\beta}),$$

and (6.4) follows. □

We now continue with the pointwise estimates for the Jacobi terms $\frac{1}{r}|J_h(ra,rb)-J(ra,rb)|$ and $r|J_h\left(\frac{a}{r},rb\right)-J\left(\frac{a}{r},rb\right)|$. Since

$$\begin{aligned}\tfrac{3}{r^2}J_h(ra,rb) = \;& \widetilde{D}_x(\tfrac{a}{r})\widetilde{D}_r(rb)-\widetilde{D}_r(ra)\widetilde{D}_x(\tfrac{b}{r})+\widetilde{D}_x\left(\tfrac{a}{r}\widetilde{D}_r(rb)-\tfrac{b}{r}\widetilde{D}_r(ra)\right)\\ &+\tfrac{1}{r^2}\widetilde{D}_r\left(r^2b\widetilde{D}_xa-r^2a\widetilde{D}_xb\right),\end{aligned} \tag{6.5}$$

$$\begin{aligned}3J_h\left(\tfrac{a}{r},rb\right) = \;& \widetilde{D}_x(\tfrac{a}{r})\widetilde{D}_r(rb)-\widetilde{D}_r(\tfrac{a}{r})\widetilde{D}_x(rb)+\widetilde{D}_x\left(\tfrac{a}{r}\widetilde{D}_r(rb)-rb\widetilde{D}_r(\tfrac{a}{r})\right)\\ &+\widetilde{D}_r\left(b\widetilde{D}_xa-a\widetilde{D}_xb\right),\end{aligned} \tag{6.6}$$

it suffices to estimate the terms in (6.5) and (6.6) individually. We summarize them as the following

Lemma 16 *If $a,b\in C_s^4(R\times\overline{R^+})$ and $\alpha_1,\alpha_2,\beta_1,\beta_2\in R$, then*

$$\begin{aligned}& r|\widetilde{D}_r(\tfrac{a}{r})\widetilde{D}_x(rb)-\partial_r(\tfrac{a}{r})\partial_x(rb)|+\tfrac{1}{r}|\widetilde{D}_r(rb)\widetilde{D}_x(ra)-\partial_r(rb)\partial_x(ra)|\\ &\le C(\Delta x^2+\Delta r^2)\tfrac{1}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}}|||a|||_{3,\alpha_1+\frac{5}{2},\beta_1}|||b|||_{3,\alpha_2+\frac{5}{2},\beta_2}\end{aligned} \tag{6.7}$$

$$\begin{aligned}& r|\widetilde{D}_x(\tfrac{a}{r}\widetilde{D}_r(rb))-\partial_x(\tfrac{a}{r}\partial_r(rb))|+r|\widetilde{D}_x(ra\widetilde{D}_r(\tfrac{b}{r}))-\partial_x(ra\partial_r(\tfrac{b}{r}))|\\ &\le C(\Delta x^2+\Delta r^2)\tfrac{1}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}}|||a|||_{3,\alpha_1+\frac{5}{2},\beta_1}|||b|||_{4,\alpha_2+\frac{7}{2},\beta_2}\end{aligned} \tag{6.8}$$

$$\begin{aligned}& r|\widetilde{D}_r(a\widetilde{D}_xb)-\partial_r(a\partial_xb)|+\tfrac{1}{r}|\widetilde{D}_r(r^2a\widetilde{D}_xb)-\partial_r(r^2a\partial_xb)|\\ &\le C(\Delta x^2+\Delta r^2)\tfrac{1}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}}|||a|||_{3,\alpha_1+\frac{5}{2},\beta_1}|||b|||_{4,\alpha_2+\frac{7}{2},\beta_2}\end{aligned} \tag{6.9}$$

Proof:

Since (6.2-6.3) are valid for any α, $\beta\in R$, we have

$$\widetilde{D}_x(\frac{a}{r}) = \partial_x(\frac{a}{r})+O(1)\Delta x^2\frac{1}{(1+r)^{\alpha_1+\lambda}(1+|x|)^{\beta_1}}|||a|||_{3,\alpha_1+\lambda,\beta_1} \tag{6.10}$$

$$\widetilde{D}_x(ra) = \partial_x(ra)+O(1)r^2\Delta x^2\frac{1}{(1+r)^{\alpha_1+\lambda}(1+|x|)^{\beta_1}}|||a|||_{3,\alpha_1+\lambda,\beta_2} \tag{6.11}$$

$$\widetilde{D}_r(\frac{a}{r}) = \partial_r(\frac{a}{r}) + O(1)\frac{\Delta r^2}{r^3}\frac{1}{(1+r)^{\alpha_1+\lambda}(1+|x|)^{\beta_1}}|||a|||_{3,\alpha_1+\lambda+3,\beta_1} \tag{6.12}$$

$$\widetilde{D}_r(ra) = \partial_r(ra) + O(1)\frac{\Delta r^2}{r}\frac{1}{(1+r)^{\alpha_1+\lambda}(1+|x|)^{\beta_1}}|||a|||_{3,\alpha_1+\lambda+3,\beta_1} \tag{6.13}$$

and

$$\widetilde{D}_x(\frac{b}{r}) = \partial_x(\frac{b}{r}) + O(1)\Delta x^2\frac{1}{(1+r)^{\alpha_2+\mu}(1+|x|)^{\beta_2}}|||b|||_{3,\alpha_2+\mu,\beta_2} \tag{6.14}$$

$$\widetilde{D}_x(rb) = \partial_x(rb) + O(1)r^2\Delta x^2\frac{1}{(1+r)^{\alpha_2+\mu}(1+|x|)^{\beta_2}}|||b|||_{3,\alpha_2+\mu,\beta_2} \tag{6.15}$$

$$\widetilde{D}_r(\frac{b}{r}) = \partial_r(\frac{b}{r}) + O(1)\frac{\Delta r^2}{r^3}\frac{1}{(1+r)^{\alpha_2+\mu}(1+|x|)^{\beta_2}}|||b|||_{3,\alpha_2+\mu+3,\beta_2} \tag{6.16}$$

$$\widetilde{D}_r(rb) = \partial_r(rb) + O(1)\frac{\Delta r^2}{r}\frac{1}{(1+r)^{\alpha_2+\mu}(1+|x|)^{\beta_2}}|||b|||_{3,\alpha_2+\mu+3,\beta_2} \tag{6.17}$$

for any λ, $\mu \in R$. We apply (6.12), (6.15) with $\lambda = -\frac{1}{2}$, $\mu = \frac{5}{2}$ to get

$$\begin{aligned}
& r|\widetilde{D}_r(\tfrac{a}{r})\widetilde{D}_x(rb) - \partial_r(\tfrac{a}{r})\partial_x(rb)| \\
= \;& r|\widetilde{D}_r(\tfrac{a}{r})\widetilde{D}_x(rb) - \partial_r(\tfrac{a}{r})\widetilde{D}_x(rb) + \partial_r(\tfrac{a}{r})\widetilde{D}_x(rb) - \partial_r(\tfrac{a}{r})\partial_x(rb)| \\
= \;& O(|\widetilde{D}_x(rb)|)\tfrac{\Delta r^2}{r^2}\tfrac{1}{(1+r)^{\alpha_1-\frac{1}{2}}(1+|x|)^{\beta_1}}|||a|||_{3,\alpha_1+\frac{5}{2},\beta_1} + O(|\partial_r(\tfrac{a}{r})|)r^3\Delta x^2\tfrac{1}{(1+r)^{\alpha_2+\frac{5}{2}}(1+|x|)^{\beta_2}}|||b|||_{3,\alpha_2+\frac{5}{2},\beta_2}
\end{aligned}$$

Moreover, since

$$r^3|\partial_r(\frac{a}{r})| \le \frac{1}{(1+r)^{\alpha_1-\frac{5}{2}}(1+|x|)^{\beta_1}}\|a\|_{1,\alpha_1+\frac{1}{2},\beta_1}$$

and

$$|\widetilde{D}_x(rb)| = |\partial_x(rb)(\xi,r)| \le r^2\frac{1}{(1+r)^{\alpha_2+\frac{1}{2}}(1+|x|)^{\beta_2}}\|b\|_{1,\alpha_2+\frac{1}{2},\beta_2}$$

it follows that

$$\begin{aligned}
& r|\widetilde{D}_r(\tfrac{a}{r})\widetilde{D}_x(rb) - \partial_r(\tfrac{a}{r})\partial_x(rb)| \\
\le \;& C(\Delta x^2 + \Delta r^2)\tfrac{1}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}}(|||a|||_{3,\alpha_1+\frac{5}{2},\beta_1}\|b\|_{1,\alpha_2+\frac{1}{2},\beta_2} + \|a\|_{1,\alpha_1+\frac{1}{2},\beta_1}|||b|||_{3,\alpha_2+\frac{5}{2},\beta_2}) \\
\le \;& C(\Delta x^2 + \Delta r^2)\tfrac{1}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}}|||a|||_{3,\alpha_1+\frac{5}{2},\beta_1}|||b|||_{3,\alpha_2+\frac{5}{2},\beta_2}
\end{aligned}$$

Similarly, from (6.17) and (6.10), we have

$$\begin{aligned}
&r|\widetilde{D}_x(\tfrac{a}{r})\widetilde{D}_r(rb)-\partial_x(\tfrac{a}{r})\partial_r(rb)|\\
=\ &r|\widetilde{D}_x(\tfrac{a}{r})\widetilde{D}_r(rb)-\widetilde{D}_x(\tfrac{a}{r})\partial_r(rb)+\widetilde{D}_x(\tfrac{a}{r})\partial_r(rb)-\partial_x(\tfrac{a}{r})\partial_r(rb)|\\
=\ &O(|\widetilde{D}_x(\tfrac{a}{r})|)\Delta r^2\tfrac{1}{(1+r)^{\alpha_2-\frac{1}{2}}(1+|x|)^{\beta_2}}\|b\|_{3,\alpha_2+\frac{5}{2},\beta_2}+O(|\partial_r(rb)|)r\Delta x^2\tfrac{1}{(1+r)^{\alpha_1+\frac{5}{2}}(1+|x|)^{\beta_1}}\|a\|_{3,\alpha_1+\frac{5}{2},\beta_1}\\
\le\ &C\Delta r^2\tfrac{1}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}}\|a\|_{1,\alpha_1+\frac{1}{2},\beta_1}\|b\|_{3,\alpha_2+\frac{5}{2},\beta_2}+C\Delta x^2\tfrac{1}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}}\|a\|_{3,\alpha_1+\frac{5}{2},\beta_1}\|b\|_{1,\alpha_2+\frac{1}{2},\beta_2}\\
\le\ &C(\Delta x^2+\Delta r^2)\tfrac{1}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}}\|a\|_{3,\alpha_1+\frac{5}{2},\beta_1}\|b\|_{3,\alpha_2+\frac{5}{2},\beta_2}
\end{aligned}$$

This gives (6.7).

For (6.8), we have

$$\begin{aligned}
&\widetilde{D}_x(f\widetilde{D}_rg)-\partial_x(f\partial_rg)\\
=\ &\widetilde{D}_x(f(\widetilde{D}_r-\partial_r)g)+(\widetilde{D}_x-\partial_x)(f\partial_rg)\\
=\ &\partial_x(f(\widetilde{D}_r-\partial_r)g)_{|(\xi,r)}+\tfrac{1}{6}\Delta x^2\partial_x^3(f\partial_rg)_{|(x,\eta)}\\
=\ &(\partial_xf)((\widetilde{D}_r-\partial_r)g)_{|(\xi,r)}+f((\widetilde{D}_r-\partial_r)\partial_xg)_{|(\xi,r)}+\tfrac{1}{6}\Delta x^2\partial_x^3(f\partial_rg)_{|(x,\eta)}.
\end{aligned}\tag{6.18}$$

We proceed with individual terms in (6.18) with $f=\frac{a}{r}$ and $g=rb$. From (6.15) with $\mu=-\frac{1}{2}$, we have

$$\begin{aligned}
r\left|(\partial_x\tfrac{a}{r})(\widetilde{D}_r-\partial_r)(rb)\right| &\le C|\partial_x(\tfrac{a}{r})|\Delta r^2\tfrac{1}{(1+r)^{\alpha_2-\frac{1}{2}}(1+|x|)^{\beta_2}}\|b\|_{3,\alpha_2+\frac{5}{2},\beta_2}\\
&\le C\Delta r^2\tfrac{1}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}}\|a\|_{1,\alpha_1+\frac{1}{2},\beta_1}\|b\|_{3,\alpha_2+\frac{5}{2},\beta_2}
\end{aligned}$$

Similarly, from (6.17)

$$\begin{aligned}
&r\left|\tfrac{a}{r}(\widetilde{D}_r-\partial_r)\partial_x(rb)\right|\\
\le\ &C\Delta r^2|\tfrac{a}{r}|\tfrac{1}{(1+r)^{\alpha_2+\frac{1}{2}}(1+|x|)^{\beta_2}}\|\partial_xb\|_{3,\alpha_2+\frac{7}{2},\beta_2}\\
\le\ &C\Delta r^2\tfrac{1}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}}\|a\|_{0,\alpha_1-\frac{1}{2},\beta_1}\|b\|_{4,\alpha_2+\frac{7}{2},\beta_2}
\end{aligned}$$

$$r\left|\Delta x^2\partial_x^3(\tfrac{a}{r}\partial_r(rb))_{|(x,\eta)}\right|$$
$$\leq \; C\Delta x^2\left|r\partial_x^3\left(\tfrac{a}{r}\right)\partial_r(rb)+r\left(\tfrac{a}{r}\right)\partial_x^3\partial_r(rb)\right|$$
$$\leq \; C\Delta x^2\tfrac{1}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}}\left(\|a\|_{3,\alpha_1+\frac{5}{2},\beta_1}\|b\|_{1,\alpha_2+\frac{1}{2},\beta_2}+\|a\|_{0,\alpha_1-\frac{1}{2},\beta_1}\|b\|_{3,\alpha_2+\frac{7}{2},\beta_2}\right)$$
$$\leq \; C\Delta x^2\tfrac{1}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}}|||a|||_{3,\alpha_1+\frac{5}{2},\beta_1}|||b|||_{4,\alpha_2+\frac{7}{2},\beta_2}$$

Therefore,

$$r|\widetilde{D}_x(\frac{a}{r}\widetilde{D}_r(rb))-\partial_x(\frac{a}{r}\partial_r(rb))|\leq C(\Delta x^2+\Delta r^2)\frac{1}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}}|||a|||_{3,\alpha_1+\frac{5}{2},\beta_1}|||b|||_{4,\alpha_2+\frac{7}{2},\beta_2}$$

Similarly, we have

$$r|\widetilde{D}_x(ra\widetilde{D}_r(\frac{b}{r}))-\partial_x(ra\partial_r(\frac{b}{r}))|\leq C(\Delta x^2+\Delta r^2)\frac{1}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}}|||a|||_{3,\alpha_1+\frac{5}{2},\beta_1}|||b|||_{4,\alpha_2+\frac{7}{2},\beta_2}$$

This gives (6.8).

We continue with (6.9). For the first term, we write

$$\widetilde{D}_r(a\widetilde{D}_xb)-\partial_r(a\partial_xb)=\widetilde{D}_r(a(\widetilde{D}_x-\partial_x)b)+(\widetilde{D}_r-\partial_r)(a\partial_xb)$$

Since $a,b\in C_s^4(R\times\overline{R^+})$, by extending a,b to an odd function across $r=0$, we see that the extended $a\widetilde{D}_xb$ is in $C^4(R^2)$, therefore

$$\begin{aligned}\widetilde{D}_r(a(\widetilde{D}_x-\partial_x)b)\; &=\partial_r(a(\widetilde{D}_x-\partial_x)b)|_{(x,\eta)}\\ &=\left(\partial_ra(\widetilde{D}_x-\partial_x)b+a(\widetilde{D}_x-\partial_x)(\partial_rb)\right)|_{(x,\eta)}\\ &=\tfrac{\Delta x^2}{6}\left(\partial_ra\partial_x^3b|_{(\xi_1,\eta)}+a\partial_x^3\partial_rb|_{(\xi_2,\eta)}\right)\end{aligned}$$

and

$$r|\widetilde{D}_r(a(\widetilde{D}_x-\partial_x)b)|\leq C\Delta x^2\frac{1}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}}|||a|||_{1,\alpha_1+\frac{1}{2},\beta_1}|||b|||_{4,\alpha_2+\frac{7}{2},\beta_2}.$$

Similarly, the extended $a\partial_xb$ is in $C^3(R^2)$, we have

$$\begin{aligned}r|(\widetilde{D}_r-\partial_r)(a\partial_xb)|\; &=r\tfrac{\Delta r^2}{6}\partial_r^3(a\partial_xb)|_{(x,\eta)}\\ &\leq C\Delta r^2\tfrac{1}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}}|||a|||_{3,\alpha_1+\frac{5}{2},\beta_1}|||b|||_{4,\alpha_2+\frac{7}{2},\beta_2}\end{aligned}$$

The second term in (6.9) can be treated similarly,

$$\frac{1}{r}\widetilde{D}_r(r^2a\widetilde{D}_xb) - \frac{1}{r}\partial_r(r^2a\partial_xb) = \frac{1}{r}\widetilde{D}_r(r^2a(\widetilde{D}_x-\partial_x)b) + \frac{1}{r}(\widetilde{D}_r-\partial_r)(r^2a\partial_xb) \tag{6.19}$$

Again, the extensions of $r^2a(\widetilde{D}_x-\partial_x)b$ and $r^2a\partial_xb$ are in $C^3(R^2)$ and $C^4(R^2)$ respectively, we can directly estimate these two terms by

$$\begin{aligned}
\tfrac{1}{r}\widetilde{D}_r(r^2a(\widetilde{D}_x-\partial_x)b) &= \tfrac{1}{r}\partial_r(r^2a(\widetilde{D}_x-\partial_x)b)|_{(x,\eta)} \\
&= \tfrac{1}{r}\left(\Big(\partial_r(r^2a)(\widetilde{D}_x-\partial_x)b\Big)_{(x,\eta)} + \Big(r^2a(\widetilde{D}_x-\partial_x)(\partial_rb)\Big)_{(x,\eta)}\right) \\
&= \tfrac{\Delta x^2}{6}\Big(((r\partial_ra+2a)\partial_x^3b)_{(\xi_1,\eta)} + (ra\partial_x^3(\partial_rb))_{(\xi_2,\eta)}\Big)
\end{aligned} \tag{6.20}$$

and

$$\frac{1}{r}(\widetilde{D}_r-\partial_r)(r^2a\partial_xb) = \frac{\Delta r^2}{r}\partial_r^3(r^2a\partial_xb)_{(x,\eta)} = \frac{\Delta r^2}{r}\partial_r^3\left(r^4\frac{a}{r}\partial_x(\frac{b}{r})\right)_{(x,\eta)} \tag{6.21}$$

From (6.19), (6.20) and (6.21), we easily have

$$\begin{aligned}
&|\tfrac{1}{r}\widetilde{D}_r(r^2a(\widetilde{D}_x-\partial_x)b)| \\
\le\ & C\Delta x^2\tfrac{1}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}}\Big(|||a|||_{1,\alpha_1+\frac{1}{2},\beta_1}|||b|||_{3,\alpha_2+\frac{5}{2},\beta_2} + \|a\|_{0,\alpha_1-\frac{1}{2},\beta_1}|||b|||_{4,\alpha_2+\frac{7}{2},\beta_2}\Big) \\
\le\ & C\Delta x^2\tfrac{1}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}}|||a|||_{1,\alpha_1+\frac{1}{2},\beta_1}|||b|||_{4,\alpha_2+\frac{7}{2},\beta_2}
\end{aligned} \tag{6.22}$$

and

$$\frac{1}{r}\left|(\widetilde{D}_r-\partial_r)(r^2a\partial_xb)\right| \le C\Delta r^2\frac{1}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}}|||a|||_{3,\alpha_1+\frac{5}{2},\beta_1}|||b|||_{4,\alpha_2+\frac{7}{2},\beta_2} \tag{6.23}$$

From (6.22) and (6.23), we conclude that

$$\begin{aligned}
&\left|\tfrac{1}{r}\widetilde{D}_r(r^2a\widetilde{D}_xb) - \tfrac{1}{r}\partial_r(r^2a\partial_xb)\right| \\
\le\ & C(\Delta x^2+\Delta r^2)\tfrac{1}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}}|||a|||_{3,\alpha_1+\frac{5}{2},\beta_1}|||b|||_{4,\alpha_2+\frac{7}{2},\beta_2}
\end{aligned}$$

This gives (6.9) and completes the proof of Lemma 16. □

As a direct consequence of Lemma 16, we have the pointwise estimate for the Jacobians

Lemma 17 *If $a, b \in C_s^4(R \times \overline{R^+})$, then*

$$\frac{1}{r}|J_h(ra, rb) - J(ra, rb)| \leq C(\Delta x^2 + \Delta r^2)\frac{1}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}}\|a\|_{4,\alpha_1+\frac{7}{2},\beta_1}\|b\|_{4,\alpha_2+\frac{7}{2},\beta_2}$$

$$r|J_h\left(\frac{a}{r}, rb\right) - J\left(\frac{a}{r}, rb\right)| \leq C(\Delta x^2 + \Delta r^2)\frac{1}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}}\|a\|_{4,\alpha_1+\frac{7}{2},\beta_1}\|b\|_{4,\alpha_2+\frac{7}{2},\beta_2}$$

for any $\alpha_1, \alpha_2, \beta_1, \beta_2 \in R$.

From (5.3), Lemma 14 and Lemma 17, we can easily derive (5.28-5.30). This completes the proof of Lemma 13. □

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