## ODD MINIMUM CUT SETS AND b-MATCHINGS REVISITED

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ABSTRACT. The famous Padberg–Rao separation algorithm for b-matching polyhedra can be implemented to run in  $\mathcal{O}(|V|^2|E|\log(|V|^2/|E|))$  time in the uncapacitated case, and in  $\mathcal{O}(|V||E|^2\log(|V|^2/|E|))$  time in the capacitated case. We give a new and simple algorithm for the capacitated case which can be implemented to run in  $\mathcal{O}(|V|^2|E|\log(|V|^2/|E|))$  time.

Key Words: matching, polyhedra, separation.

#### 1. INTRODUCTION

Let G = (V, E) be an undirected graph, let  $b \in \mathbb{Z}_+^V$  be a vector of vertex capacities and let  $u \in \mathbb{Z}_+^E$  be a vector of edge capacities. A *u*-capacitated *b*-matching is a family of edges, possibly containing multiple copies, such that:

- for each  $i \in V$ , there are at most  $b_i$  edges in the family incident on i;
- at most  $u_e$  copies of edge e are used.

If we define for each edge e the integer variable  $x_e$ , representing the number of times e appears in the matching, then the incidence vectors of u-capacitated b-matchings are the solutions to:

$$\sum_{e \in \delta(i)} x_e \le b_i, \quad \text{for all } i \in V \tag{1}$$

$$0 \le x_e \le u_e$$
, for all  $e \in E$  (2)

$$x_e \in \mathbb{Z}, \quad \text{for all } e \in E.$$
 (3)

Here, as usual,  $\delta(i)$  represents the set of vertices incident on *i*.

The convex hull in  $\mathbb{R}^E$  of solutions to (1) - (3) is called the *u*-capacitated *b*matching polytope. Edmonds and Pulleyblank (see [Edm65] and [Pul73]) gave a complete linear description of this polytope. It is described by the *degree inequalities* (1), the *bounds* (2) and the following *blossom inequalities*:

$$\sum_{e \in E(W)} x_e + \sum_{f \in F} x_f \le \left\lfloor \frac{b(W) + \sum_{f \in F} u_f}{2} \right\rfloor,$$
  
for all  $W \subset V, F \subset \delta(W)$  with  $b(W) + \sum_{f \in F} u_f$  odd. (4)

Here, E(W) (respectively,  $\delta(W)$ ) represents the set of edges with both end-vertices (respectively, exactly one end-vertex) in W, b(W) denotes  $\sum_{i \in W} b_i$ .

An important special case is where the upper bounds  $u_e$  are not present (or, equivalently,  $u_{ij} \ge \max\{b_i, b_j\}$  for all  $\{i, j\} \in E$ ). The associated (uncapacitated) *b*-matching polytope is described by the degree inequalities, the non-negativity

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41	gorithm	1	Minimum	T-cut	[PR82]	
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# Input: Graph G, set $T \subset V$ , and weights $c \in \mathbb{Q}_{+}^{E}$ . Output: A minimum T-cut. 1: Compute a cut-tree for the graph G with weights c and terminal vertex set T. 2: For each of the n-1 edges of the cut-tree do 3: Let $\delta(U)$ denote the cut induced by the cut-tree edge. 4: Check the cut: Compute the parity $|T \cap U| \mod 2$ and the weight c(U) of the cut. 5: If adequate, store U. 6: End for 7: Output the best T-cut U.

inequalities  $x_e \ge 0$  for all  $e \in E$ , and the simplified blossom inequalities

$$\sum_{e \in E(W)} x_e \le \left\lfloor \frac{b(W)}{2} \right\rfloor, \quad \text{for all } W \subset V \text{ with } b(W) \text{ odd.}$$
(5)

In their seminal paper, [PR82] devised a combinatorial, polynomial-time separation algorithm for b-matching polytopes. A separation algorithm is a procedure which, given a rational vector  $x^* \in \mathbb{Q}^E$  lying outside of the polytope, finds a linear inequality which is valid for the polytope yet violated by  $x^*$ . Clearly, testing if a degree inequality or bound is violated can be performed in linear time, so the main contribution of [PR82] is to identify violated blossom inequalities.

For uncapacitated b-matching, Padberg & Rao reduce the separation problem to the computation of a minimum *T*-cut, for which they give a generic algorithm, see Algorithm 1. We will give the definition of the minimum *T*-cut problem in the next section. Abbreviating n := |V| and m := |E|, this algorithm involves the solution of up to n-1 maximum flow problems on a graph with n+1 vertices and n+medges. Using the well-known pre-flow push algorithm [GT88] to solve the max-flow problems, this leads to an overall running time of  $O(n^2 m \log n^2/m)$ .

The Padberg-Rao separation algorithm for *capacitated b-matching*, however, is substantially more time-consuming. It involves the computation of a minimum Tcut on a special graph, the so-called *split graph*, which has up to n + m + 1 vertices and up to 2m + n edges. Up to n + m - 1 maximum flow problems may be required to be computed. Using the pre-flow push algorithm, this leads to a worst-case running time of  $O(m^3 \log n)$ . In 1987, [GH87] observed that the above-mentioned max-flow problems can in fact be carried out on graphs with only O(n) vertices and O(m) edges. Although the idea behind this is simple, it reduces the overall running time for the capacitated case to  $O(nm^2 \log n^2/m)$ .

In this paper, we propose a new separation algorithm for the capacitated case whose running time is the same as that for the uncapacitated case. As well as being faster than the Padberg-Rao and Grötschel-Holland approaches, the new algorithm is much simpler and easier to implement. It also has a surprisingly simple proof of correctness.

Our results also apply to the case of *perfect capacitated b-matchings*.

Algorithm 2 Blossom minimization					
Input:					
Graph G, set $T \subset V$ , and weights $c, c' \in \mathbb{Q}_+^E$ .					
Output:					
A minimum blossom.					
1: Compute a cut-tree for G with weights $\min(c, c')$ and terminal vertex set V.					
2: For each of the $n-1$ edges of the cut-tree do					
3: Let $\delta(U)$ denote the cut induced by the cut-tree edge.					
4: Check the cut:					
Compute $\beta(U)$ as in (8).					
5: If adequate, store $U$ along with the arg-min $F$ .					
6: End for					
7: Output the best blossom $(U, F)$ .					

As well as being of interest in the context of matching, the algorithm has an important application to the *Traveling Salesman Problem* (TSP). The special blossom inequalities obtained when  $b_i = 2$  for all i and  $u_e = 1$  for all e are valid for the TSP, and facet-inducing under mild conditions, see [GP79a], [GP79b]. Thus we obtain a faster exact separation algorithm for the TSP as a by-product. In fact, the algorithm is applicable to a general class of cutting planes for integer programs, called  $\{0, 1/2\}$ -Chvátal-Gomory cuts, see [CF96].

Parts of the contents of this paper appeared in the proceedings of the Xth IPCO conference [LRT04]. However, the proof of correctness of the algorithm is now substantially facilitated.

## 2. Algorithms for minimum T-cut and blossom minimization

Given a graph G = (V, E), an even-cardinality set  $T \subset V$  and non-negative rational edge-capacities  $c \in \mathbb{Q}_+^E$ , the *minimum* T-cut problem asks for an odd cut  $(U, \mathsf{C}U)$  (where  $\mathsf{C}U$  is the complement of U in the vertex set) such that the set  $U \subset V$  is T-odd, i.e.,  $|T \cap U|$  is an odd number, and which minimizes, subject to this condition, the submodular function

$$U \mapsto c(U) := \sum_{e \in \delta(U)} c_e.$$

In 1982, Padberg & Rao gave the first polynomial-time combinatorial algorithm for computing a minimum T-cut, see Algorithm 1. The key ingredient is the computation of a Gomory-Hu cut-tree [GH61] in step 1. Given a graph G = (V, E), a set  $X \subset V$ , and non-negative rational vector of edge-capacities  $c \in \mathbb{Q}_+^E$ , a *cut-tree with terminal vertex set* X for G and c consists of a mapping  $\pi \colon V \to X$  with  $\pi(x) = x$ for all  $x \in X$ , and an adjacency relation  $\sim$  on the set X. (We adopt the convention that the edges of G will be denoted by xy, and the edges of the cut-tree by  $x \sim y$ .) The adjacency relation shall make the set of terminal vertices into a tree. An additional condition is required to hold. Deleting an edge  $x \sim y$  of the cut-tree partitions the set X into two sets  $X_x$  and  $X_y$ , and thus defines a cut  $(U, \overline{U})$  in G by letting  $U := \pi^{-1}(X_x)$  and  $\overline{U} := \pi^{-1}(X_y)$ . We call this the cut *induced* by the edge  $x \sim y$  of the cut-tree. Now, the condition which is required is the following:

for  $x, y \in X$  with  $x \sim y$ , the cut induced by this edge of the cut-tree shall be a minimum (s, t)-cut in G with respect to the capacities c. (6)

With the algorithm given by Gomory & Hu, a cut-tree can be computed in time  $O(|X|nm\log n^2/m)$ .

In Algorithm 1, the time for "checking the cut" in step 4 is negligible (the values c(U) even come for free with the Gomory-Hu algorithm), and hence the Padberg-Rao method for computing a minimum *T*-cut runs in time  $O(|T|nm \log n^2/m)$ , as mentioned in the introduction.

Now we come to the blossom separation algorithm of Padberg & Rao [PR82]. Reformulating and generalizing, we say that a *blossom* is a pair (U, F) consisting of a set of vertices  $U \subset V$  and a set of edges  $F \subset \delta(U)$  with the property that  $|T \cap U| + |F|$  is an odd number. Then, if two non-negative rational weight vectors  $c, c' \in \mathbb{Q}_+^E$  are given for the edges of G, the blossom separation problem is equivalent to the problem of producing a blossom whose *value* 

$$\beta(U,F) := \sum_{e \in \delta(U) \backslash F} c_e + \sum_{f \in F} c'_f$$

is strictly less than, one, if it exists. For the sake of completeness, we describe how this equivalence is established. Padberg & Rao [PR82] introduced, for each  $u \in V$ , the term  $s_u := b_u - \sum_{e \in \delta(i)} x_e$ , which is the *slack* of the corresponding degree inequality computed with respect to a given vector x. Then they showed that the blossom inequality (4) can be re-written in the form:

$$\sum_{u \in W} s_u + \sum_{e \in \delta(i)} x_e + \sum_{e \in F} (u_e - x_e) \ge 1.$$

$$\tag{7}$$

To decide if, for a given x, sets W and F exist which violate (7), we define, in a canonical and straight forward manner, a graph  $G^*$ , capacities c and c' and a set T of vertices of  $G^*$ , in such a way that a blossom with value strictly less than one gives rise to a violated inequality (7) and vice-versa. Let  $G^*$  be constructed by adding a new vertex v to G = (V, E) and connecting it with an edge vu to every  $u \in V$ . Then for each  $e \in E$ , we let

$$(c_e, c'_e) := \begin{cases} (x_e, u_e - x_e) & \text{if } u_e \text{ is odd} \\ (\min(x_e, u_e - x_e), \infty) & \text{if } u_e \text{ is even} \end{cases}$$

For the edges vu of  $G^*$ , we let  $c_{uv} := s_u$  and  $c'_{uv} := \infty$ . Finally, we define T as the set of all vertices u for which the value  $b_u$  is odd, and we let  $v \in T$  iff  $\sum_u b_u$  is odd. Now it is easy to see that for each blossom (U, F) in  $G^*$  with  $v \notin U$ , the inequality (7) with  $W := U \cap V$  is violated by  $1 - \beta(U, F)$ . Note that  $\beta(U, F) = \beta(\mathcal{C}U, F)$ .

As mentioned above, the blossom separation Algorithm of Padberg & Rao [PR82] is considerably more complex than the minimum T-cut algorithm. It requires to construct a special graph  $\hat{G}$  with m + n vertices and 2m edges, on which then a minimum T-cut is computed.

We now give an algorithm for what we call the *blossom minimization problem*: given G, T and c, c' as above, find a blossom (U, F) which minimizes  $\beta(U, F)$ . The blossom minimization algorithm is displayed as Algorithm 2. For fixed  $U \subset V$ , it has been observed by Padberg & Rinaldi [PR90] that

$$\beta(U) := \min\left\{\beta(U, F) \mid F \subset \delta(U), |T \cap U| + |F| \text{ odd}\right\}$$
(8)

can be computed in time  $O(|\delta(U)|)$  by first tentatively taking  $F := \{e \in \delta(U) \mid c'_e < c_e\}$ . Now if  $|T \cap U| + |F|$  is odd, we have found a minimizing F. Otherwise, find  $f \in \delta(U)$  minimizing  $|c_f - c'_f|$  over  $f \in \delta(U)$ , because then the symmetric difference of F and  $\{f\}$  minimizes  $\beta(U, \cdot)$ .

This implies that the loop 2–6 in Algorithm 2 runs in time  $O(n^2)$  and that the running time of Algorithm 2 is dominated by the computation of the cut-tree in step 1, which amounts to  $O(n^2 m \log n^2/m)$ .

The similarity between the Padberg-Rao minimum T-cut Algorithm 1 and our blossom minimization Algorithm 2 is striking. Moreover, in the next section, we give a short and elegant proof of correctness of Algorithm 2, which is similar to a proof of correctness of Algorithm 1 given by Rizzi [Riz02]. At this point, we might note that  $\beta(\cdot)$ , unlike  $c(\cdot)$ , is not in general submodular.

### 3. A simple proof of the correctness of Algorithm 2

Let a cut-tree for G with terminal vertex set X be given, where  $X \supset T$ . We say that an edge  $x \sim y$  of the cut-tree is T-odd, if the sets of the bipartition of X defined by  $x \sim y$  are T-odd. Thus, the set of T-odd edges of the cut-tree form what is called a T-join, and an edge in the cut-tree induces a T-cut in G if and only if the edge is T-odd. The next theorem is the keystone of the correctness of Algorithm 1. For the sake of clarity, we repeat the proof of [Riz02].

**Theorem 3.1** ([PR82]). One of the T-odd edges of the cut-tree induces a minimum T-cut in G.

*Proof.* Let U be a minimum T-cut. Now U is a T-odd set, hence there exists an odd number of T-odd cut-tree edges leaving  $T \cap U$ . Let  $x \sim y$  be one of them, and let S be the minimum (x, y)-cut it induces by (6). Since U is an (x, y)-cut, we have  $c(S) \leq c(U)$ , and since  $x \sim y$  is an T-odd edge, S defines a minimum T-cut.  $\Box$ 

Now we come to the proof of correctness of Algorithm 2.

**Theorem 3.2.** One of the edges of of the cut-tree computed in Algorithm 2 induces the a set U which minimizes  $\beta(\cdot)$ .

*Proof.* Let U be a set which minimizes  $\beta(\cdot)$ . Further, define the set T' as the symmetric difference of T with all sets  $\{u, v\}$  for all  $e = uv \in E$   $c'_e < c_e$ .

Case 1: U is T'-odd. The proof of Theorem 3.1 shows that there exists a T'-odd edge of the cut-tree which induces a minimizer of  $\beta(\cdot)$ .

Case 2: U is not T'-odd. Let  $f = x'y' \in \delta(U)$  have the minimal value of  $|c_f - c'_f|$ among all edges in  $\delta(U)$ . On the path from x' to y' in the cut-tree, at least one edge  $x \sim y$  has one end in U and the other not in U. Let S be the minimum (x, y)-cut defined by this edge. Abbreviating  $w := \min(c, c')$ , we then have

$$\beta(U) = w(U) + |c_f - c'_f| \ge w(S) + |c_f - c'_f| \ge \beta(S).$$

The first inequality holds since U is an (x, y)-cut. As for the second, if S is T'-odd, then S minimizes  $\beta$  since  $w(S) \leq w(S) + |c_f - c'_f| \leq \beta(U)$ ; but if  $|T' \cap S|$  is even, then  $(S, \{f\})$  is a blossom whence  $w(S) + |c_f - c'_f| = \beta(S, \{f\}) \geq \beta(S)$ .

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