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SEMISMOOTH NEWTON METHODS FOR TIME-OPTIMAL CONTROL FOR A CLASS OF ODES*

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Abstract. Time-optimal control problems for a class of linear multi-input systems are considered. The problems are regularized and the asymptotic and monotone behavior of the regularization procedure is investigated. For the regularized problems the applicability of semismooth Newton methods is verified. First numerical tests are presented which show that the proposed approach is different from other methods in that it does not rely on a priori information about the switching structure.

Key words. semismooth Newton method, time-optimal control

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1. Introduction.

This paper addresses time-optimal control for a class of linear multi-input controls systems for ordinary differential equations. Due to their practical relevance and inherent structural difficulties, time-optimal control problems have received a considerable amount of attention for decades. Much of the literature up to the late 1960s is covered in [HL]. Many recent results can be found, or are referenced, in [BPW, KLM, MO]. Time-optimal control for infinite dimensional systems is considered in [Fa], for example.

The optimality system associated to time-optimal control problems with pointwise constraints on the controls is complicated due to lack of smoothness of the optimal controls. In fact, the first order optimality system for time-optimal control problems contains a multivalued operation which impedes the use of fast numerical methods. For this reason we introduce a regularization to the time-optimal problem. In section 2 the behavior of the solutions of the regularized problems as the regularization parameter ε tends to zero is investigated. In particular, monotonic structure of the solutions with respect to ε is shown. An optimality system for the regularized problems is derived under a condition which is stronger than controllability and weaker than normality. The optimal controls of the regularized problems are $W^{1,\infty}$ regular and converge to a minimum norm solution of the original problem as the regularization parameter tends to zero.

The optimality system of the regularized problems is still not C^1, and thus second order methods with local quadratic convergence order are not directly applicable. However, sufficient conditions will be obtained in section 3 which imply that semismooth Newton methods [IK2] are well-posed and locally superlinearly convergent.

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Section 4 contains a brief description of numerical results. We compare the chosen regularization to an alternative one, which has stronger regularization properties. Since the optimal controls of the original time-optimal problems are typically not continuous, it appears that our choice of regularization which leads to $W^{1,\infty}$ regularized controls is preferable over other regularization strategies which provide smoother controls. More detailed numerical tests are available in [XK].

Let us note that the approach that we propose for solving time-optimal problems deviates from traditional approaches, which are frequently grouped into direct and indirect methods. Indirect methods based on multiple shooting techniques [Ke] solve the two point boundary value problem describing first order necessary conditions. Equipped with a good initial guess for all unknowns, including the switching function, the shooting method is reported to converge fast and to generate very accurate solutions. The methods that we propose also originate from the first order condition, but it is different from the shooting method in that it does not require accurate information about the switching structure in advance.

Direct methods, on the other hand, consider time-optimal problems as genuine nonlinear programming problems. They are used in several variants, which frequently involve reparametrization of the controls as the unknowns. The new unknowns can be the switching times, as in [MB], or the arc durations, as in [KN].

2. The time-optimal problem and its regularization. Consider the time-optimal control problem for the linear multi-input system,

$$
\text{(P)} \qquad \begin{cases} \min_{\tau\geq 0} \int_0^\tau dt \\ \text{subject to} \\ \qquad \frac{d}{dt}x(t) = Ax(t) + Bu(t),\ |u(t)|_{\ell^\infty} \leq 1,\ x(0) = x_0,\ x(\tau) = x_1, \end{cases}
$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $x_0 \in \mathbb{R}^n$, $x_1 \in \mathbb{R}^n$ are given, $u(t) \in \mathbb{R}^m$, u is measurable, and $|\cdot|_{\ell^\infty}$ denotes the infinity-norm on $\mathbb{R}^m$. The columns of B are denoted by b_i. It is assumed that x_1 can be reached in finite time by an admissible control. Then (P) admits a solution with optimal time denoted by τ^*, and associated state x^* and control u^*.

The first order optimality system for (P) can be expressed in terms of the adjoint p and the Hamiltonian

$$
H(x, u, p_0, p) = p_0 + p^T(Ax + Bu)
$$

as

$$
\begin{cases} \dot{x} = Ax + Bu,\ x(0) = x_0,\ x(\tau) = x_1, \\ -\dot{p} = A^T p, \\ u = \operatorname{argmin}_{|v|_{\ell^\infty}\leq 1} H(x, v, p_0, p) \text{ a.e. in } (0, \tau), \\ p_0 + p(t)^T(Ax(t) + Bu(t)) = 0,\ p_0 \geq 0, \end{cases} \tag{2.1}
$$

where the superscript T denotes transposition; see, e.g., [MS, Chap. V, pp. 109–110]. Further, p is not identically 0, and thus there exists a nontrivial vector $q \in \mathbb{R}^n$ such that

$$
p(t) = \exp\left(A^T(\tau - t)\right) q.
$$

Due to the special structure of H, the optimal control can be expressed as

$$u_i = -\sigma(b_i^T p) = -\sigma\big(b_i^T \exp\left(-A^T(\tau - t)\right) q\big), \tag{2.2}$$

where σ denotes the coordinatewise operation

$$\sigma(s) \in \begin{cases} -1 & \text{if } s < 0, \\ [-1, 1] & \text{if } s = 0, \\ 1 & \text{if } s > 0. \end{cases} \tag{2.3}$$

The last equation in (2.1) holds everywhere rather than almost everywhere (a.e.) on $[0, \tau]$. In fact, p and x are continuous, and $p(t)^T Bu(t) = -\sum_{i=1}^m |p(t)^T b_i|$.

Let us recall the notions of controllability and normality, which will be referred to as follows:

$$\begin{cases} \text{The pair } (A, B) \text{ is called controllable if} \\ \text{rank } \{B, AB, \ldots, A^{n-1}B\} = n. \end{cases} \tag{2.4}$$

$$\begin{cases} \text{The pair } (A, B) \text{ is called normal if } (A, b_i) \\ \text{is controllable for all columns } b_i \text{ of } B. \end{cases} \tag{2.5}$$

Normality of (A, B) implies controllability. Moreover, if (A, B) is normal, then the optimal control u^* to (P) is unique, bang-bang, and piecewise constant; see, e.g., [MS, HL].

The requirement

$$p_0 > 0 \tag{2.6}$$

is referred to as strict transversality. In this case it can be assumed that $p_0 = 1$, which can be achieved by scaling q. If strict transversality holds, then (x^*, u^*, τ^*) is a strict local minimum, in the sense that there exists $\delta > 0$ such that x_1 is not in the attainable set for $t \in (\tau^* - \delta, \tau^*)$; see [HL, p. 89].

With (2.6) holding, we can express the optimality condition as

$$\begin{cases} \dot{x} = Ax + Bu, \; x(0) = x_0, \; x(\tau) = x_1, \\ -\dot{p} = A^T p, \\ u = \operatorname{argmin}_{|v|_{\ell^\infty} \le 1} H(x, v, p) \text{ a.e. in } (0, \tau), \\ 1 + p(\tau)^T (Ax(\tau) + Bu(\tau)) = 0. \end{cases} \tag{2.7}$$

Here we eliminate the variable p_0 from the notation for H since it was fixed to be 1.

Introducing the transformation $\hat{t} = \frac{t}{\tau}$ and setting

$$\hat{x}(\hat{t}) = x(\tau\hat{t}) = x(t), \quad \hat{p}(\hat{t}) = p(\tau\hat{t}) = p(t), \quad \hat{u}(\hat{t}) = u(\tau\hat{t}) = u(t),$$

we obtain the following equivalent system to (2.7), where, for the ease of presentation, we omit the superscript hats

$$\begin{cases} \dot{x} = \tau(Ax + Bu), \; x(0) = x_0, \; x(1) = x_1, \\ -\dot{p} = \tau A^T p, \\ u = \operatorname{argmin}_{|v|_{\ell^\infty} \le 1} H(x, v, p) \text{ a.e. in } (0, \tau), \\ 1 + p(1)^T (Ax(1) + Bu(1)) = 0. \end{cases} \tag{2.8}$$

The nondifferentiable operation involved in characterizing the optimal control,

$$u = -\sigma(B^T p)$$

(cf. (2.2)), prohibits the use of Newton-type methods for solving (2.8) numerically.

Therefore a family of regularized problems given by

$$(\mathrm{P}_\varepsilon) \qquad \begin{cases} \min_{\tau \geq 0} \int_0^\tau (1 + \frac{\varepsilon}{2}\,|u(t)|^2)\,dt \\ \text{subject to} \\ \qquad \frac{d}{dt}x(t) = Ax(t) + Bu(t),\ |u(t)|_{\ell^\infty} \leq 1,\ x(0) = x_0,\ x(\tau) = x_1, \end{cases}$$

with $\varepsilon > 0$ considered. The norm $|\cdot|$ used in the cost-functional denotes the Euclidean norm. It is straightforward to argue for the existence of a solution $(u_\varepsilon, x_\varepsilon, \tau_\varepsilon)$.

Convergence of the solutions $(x_\varepsilon,\, p_\varepsilon, u_\varepsilon, \tau_\varepsilon)$ of (P_ε) to a solution $(x^*,\, p^*,\, u^*,\, \tau^*)$ of (P) is considered next. Note that τ^* is unique.

PROPOSITION 2.1. *For every* $0 < \varepsilon_0 < \varepsilon_1$ *and any solution* $(\tau^*,\, u^*)$ *of* (P)*, we have*

$$\tau^* \leq \tau_{\varepsilon_0} \leq \tau_{\varepsilon_1} \leq \tau^* \left(1 + \frac{\varepsilon_1}{2}\right), \tag{2.9}$$

$$|u_{\varepsilon_1}|_{L^2(0,\,\tau_{\varepsilon_1})} \leq |u_{\varepsilon_0}|_{L^2(0,\,\tau_{\varepsilon_0})} \leq |u^*|_{L^2(0,\,\tau^*)}. \tag{2.10}$$

If u^* *is a bang-bang solution, then*

$$0 \leq |u^*|^2_{L^2(0,\,\tau^*)} - |u_\varepsilon|^2_{L^2(0,\,\tau_\varepsilon)} \leq \ \mathit{meas}\ \{t \in [0,\tau^*] : |u_\varepsilon(t)| < 1\} \tag{2.11}$$

for every $\varepsilon > 0$.

Proof. From the definition of τ^* and τ_ε, we have

$$\tau^* \leq \tau_\varepsilon \quad \text{for every} \ \ \varepsilon > 0$$

and

$$\tau_\varepsilon + \frac{\varepsilon}{2} \int_0^{\tau_\varepsilon} \ |u_\varepsilon|^2\,dt \leq \tau^* + \frac{\varepsilon}{2} \int_0^{\tau^*} \ |u^*|^2\,dt;$$

hence

$$|u_\varepsilon|_{L^2(0,\,\tau_\varepsilon)} \leq |u^*|_{L^2(0,\,\tau^*)} \text{ and } \tau^* \leq \tau_\varepsilon \leq \tau^* \left(1 + \frac{\varepsilon}{2}\right).$$

For $0 < \varepsilon_0 < \varepsilon_1$ we have

$$\int_0^{\tau_{\varepsilon_0}} \left(1 + \frac{\varepsilon_0}{2}\,|u_{\varepsilon_0}|^2\right)\,dt \leq \int_0^{\tau_{\varepsilon_1}} \left(1 + \frac{\varepsilon_0}{2}\,|u_{\varepsilon_1}|^2\right)\,dt,$$

where we used the fact that the pair $(\tau_{\varepsilon_0}, u_{\varepsilon_0})$ is optimal for $(\mathrm{P}_{\varepsilon_0})$. Adding $\frac{1}{2}(\varepsilon_1 - \varepsilon_0) \int_0^{\tau_{\varepsilon_1}} \ |u_{\varepsilon_1}|^2\,dt$ on both sides implies that

$$\begin{aligned} \tau_{\varepsilon_0} + \frac{\varepsilon_1}{2} \int_0^{\tau_{\varepsilon_1}} \ |u_{\varepsilon_1}|^2\,dt + \frac{\varepsilon_0}{2} \left(\int_0^{\tau_{\varepsilon_0}} \ |u_{\varepsilon_0}|^2\,dt - \int_0^{\tau_{\varepsilon_1}} \ |u_{\varepsilon_1}|^2\,dt\right) \\ \leq \int_0^{\tau_{\varepsilon_1}} \left(1 + \frac{\varepsilon_1}{2}\,|u_{\varepsilon_1}|^2\right) dt \leq \tau_{\varepsilon_0} + \frac{\varepsilon_1}{2} \ \int_0^{\tau_{\varepsilon_0}} \ |u_{\varepsilon_0}|^2\,dt. \end{aligned} \tag{2.12}$$

It follows from (2.12) that

$$\varepsilon_1\left(\int_0^{\tau_{\varepsilon_1}} |u_{\varepsilon_1}|^2\,dt - \int_0^{\tau_{\varepsilon_0}} |u_{\varepsilon_0}|^2\,dt\right) \le \varepsilon_0\left(\int_0^{\tau_{\varepsilon_1}} |u_{\varepsilon_1}|^2\,dt - \int_0^{\tau_{\varepsilon_0}} |u_{\varepsilon_0}|^2\,dt\right),$$

and hence

$$|u_{\varepsilon_1}|_{L^2(0,\,\tau_{\varepsilon_1})} \le |u_{\varepsilon_0}|_{L^2(0,\,\tau_{\varepsilon_0})}. \tag{2.13}$$

From the first inequality in (2.12), we obtain

$$\tau_{\varepsilon_0} + \frac{\varepsilon_0}{2}\,|u_{\varepsilon_0}|^2_{L^2(0,\,\tau_{\varepsilon_0})} \le \tau_{\varepsilon_1} + \frac{\varepsilon_0}{2}\,|u_{\varepsilon_1}|^2_{L^2(0,\,\tau_{\varepsilon_1})},$$

and by (2.13),

$$\tau^* \le \tau_{\varepsilon_0} \le \tau_{\varepsilon_1}.$$

These estimates imply (2.9) and (2.10).

If u^* is bang-bang, then

$$\begin{aligned} 0 &\le |u^*|^2_{L^2(0,\tau^*)} - |u_\epsilon|^2_{L^2(0,\tau_\varepsilon)} \\ &\le \int_{\{t\in(0,\tau^*):|u_\varepsilon(t)|<1\}} (1-|u_\varepsilon(t)|^2)\,dt \;\le\; \text{meas}\{t\in(0,\tau^*):|u_\varepsilon(t)|<1\}, \end{aligned}$$

and thus (2.11) holds. ∎

THEOREM 2.2. *For $\varepsilon\to 0^+$ we have $\tau_\varepsilon\to\tau^*$, and every convergent subsequence of solutions $\{(u_\varepsilon,\,x_\varepsilon)\}_{\varepsilon>0}$ to* (P$_\varepsilon$) *converges in $L^2(0,\,\tau_\epsilon;\mathbb{R}^m)\times W^{1,2}(0,\,\tau_\epsilon;\,\mathbb{R}^n)$ to a solution $(u^*,\,x^*)$ of* (P)*, where u^* is a minimum norm solution.*

Here convergence of u_ε to u^* is defined as

$$\int_0^1 |u_\varepsilon(\tau_\varepsilon\,t) - u^*(\tau^*\,t)|^2\,dt \to 0$$

and is defined analogously for $\{x_\varepsilon\}$ and for weak convergence.

Proof. The first claim follows from Proposition 2.1. Since $\{u_\varepsilon(\tau_\varepsilon\cdot)\}_{\varepsilon>0}$ and $\{x_\varepsilon(\tau_\varepsilon\cdot)\}_{\varepsilon>0}$ are bounded in $L^2(0,\,1;\mathbb{R}^m)$ and $W^{1,2}(0,\,1;\,\mathbb{R}^n)$ there exist weak accumulation points $u^*\in L^2(0,\tau^*;\mathbb{R}^m)$ and $x^*\in W^{1,2}(0,\,\tau^*;\,\mathbb{R}^n)$. Subsequently we avoid subsequential indices. Passing to the limit in $\dot x_\varepsilon(\tau_\varepsilon\,\cdot) = \tau_\varepsilon(Ax_\varepsilon(\tau_\varepsilon\,\cdot)+Bu_\varepsilon(\tau_\varepsilon\,\cdot))$ and $x_\varepsilon(0)=x_0,\ x_\varepsilon(\tau_\varepsilon)=x_1$, it follows that x^* is admissible. Due to weak closedness of $\{u\in L^2(0,\,1;\mathbb{R}^m):\,|u(x)|_{\ell^\infty}\le 1 \text{ a.e.}\}$ we have that u^* is admissible as well. Since

$$\lim_{\varepsilon\to 0^+}\tau_\varepsilon + \frac{\varepsilon}{2}\int_0^{\tau_\varepsilon} |u_\varepsilon|^2\,dt = \tau^*,$$

the triple $(\tau^*,\,u^*,\,x^*)$ is optimal for (P). By Proposition 2.1 and weak lower semicontinuity of norms,

$$\limsup_{\varepsilon\to 0}\,|u_\varepsilon|_{L^2(0,\tau_\varepsilon)} \le |u^*|_{L^2(0,\tau^*)} \le \liminf_{\varepsilon\to 0}\,|u_\varepsilon|_{L^2(0,\tau_\varepsilon)}, \tag{2.14}$$

and hence $\lim_{\varepsilon\to 0}\,|u_\varepsilon|_{L^2(0,\tau_\varepsilon;\mathbb{R}^m)} = |u^*|_{L^2(0,\tau^*;\mathbb{R}^m)}$. As a consequence, u_ε and x_ε converge strongly in $L^2(0,\,\tau_\varepsilon)$ (respectively, in $W^{1,2}(0,\,\tau_\varepsilon;\,\mathbb{R}^n)$) to u^* and x^*. Let $\hat u$ denote another optimal control for (P) with $|\hat u|<|u^*|$. Then by (2.10) and (2.14),

$$\limsup_{\varepsilon\to 0}\,|u_\varepsilon|_{L^2(0,\tau_\varepsilon;\mathbb{R}^m)} \le |\hat u|_{L^2(0,\tau^*;\mathbb{R}^m)} < |u^*|_{L^2(0,\tau^*;\mathbb{R}^m)} \le \liminf_{\varepsilon\to 0}\,|u_\varepsilon|_{L^2(0,\tau_\varepsilon;\mathbb{R}^m)},$$

which is a contradiction. Consequently (P) has a minimal norm control, and the claimed strong convergence properties hold. ∎

COROLLARY 2.3. *If* (2.5) *holds, then the solution* u^* *to* (P) *is unique and bang-bang, and* $u_\varepsilon \to u^*$ *in* L^2 *as* $\varepsilon \to 0^+$.

Proof. Equation (2.5) implies that the solution to (P) is unique and bang-bang. The remainder of the corollary follows from Theorem 2.2. ∎

We turn to the optimality condition for (P_ε). Let

$$\sigma_\varepsilon(s) \in \begin{cases} -1 & \text{if } s \le -\varepsilon, \\ \frac{s}{\varepsilon} & \text{if } |s| < \varepsilon, \\ 1 & \text{if } s \ge \varepsilon. \end{cases} \tag{2.15}$$

If σ_ε is applied to a vector, then it acts coordinatewise.

We shall use the following controllability assumption which is stronger than controllability and weaker than normality:

(H1) There exists i^* such that (A, b_{i^*}) is controllable.

THEOREM 2.4. *Assume that* (H1) *holds, and let* $(x_\varepsilon, u_\varepsilon, \tau_\varepsilon)$ *be a solution of* (P_ε). *If there exist* $\eta > 0$ *and an interval* $I_{i^*} \subset (0,1)$ *such that*

$$|\,(\hat{u}_\varepsilon)_{i^*}(t)|_{\ell^\infty} \le 1 - \eta \quad \textit{for a.e. } t \in I_{i^*}, \tag{2.16}$$

then there exists an adjoint state p_ε *such that*

$$\begin{cases} \dot{x}_\varepsilon = Ax_\varepsilon + Bu_\varepsilon,\ x_\varepsilon(0) = x_0,\ x_\varepsilon(\tau_\varepsilon) = x_1, \\ -\dot{p}_\varepsilon = A^T p_\varepsilon, \\ u_\varepsilon = -\sigma_\varepsilon(B^T p_\varepsilon), \\ 1 + \frac{\epsilon}{2}|u_\varepsilon(\tau_\varepsilon)|^2_{\mathbb{R}^m} + p_\varepsilon(\tau_\varepsilon)^T(Ax_\varepsilon(\tau_\varepsilon) + Bu_\varepsilon(\tau_\varepsilon)) = 0. \end{cases} \tag{2.17}$$

Proof. We use a Lagrange multiplier argument for the reparametrized formulation of (P_ε), which is given by

$$\begin{cases} \min_{\tau \ge 0} \int_0^1 (\tau + \frac{\tau\varepsilon}{2}\,|\hat{u}(t)|^2)\,dt, \\ \text{subject to} \\ \frac{d}{dt}\hat{x}(t) = \tau(A\hat{x}(t) + B\hat{u}(t)),\ \hat{u} \in C,\ \hat{x}(0) = x_0,\ \hat{x}(1) = x_1, \end{cases} \tag{2.18}$$

where $C = \{\hat{u} \in L^2(0,1;\mathbb{R}^m) : |\hat{u}|_{\ell^\infty} \le 1\}$. Here $(\hat{u}, \tau)$ are treated as independent variables and $\hat{x}$ as a dependent variable. Further, $\hat{u} \in C$ is considered as an explicit constraint, and a Lagrange multiplier μ_0 is introduced for the constraint $e(\hat{u}, \tau) = \hat{x}(1) - x_1 = 0$. The resulting Lagrangian is

$$\mathcal{L}(\hat{u}, \tau, \mu_0) = \int_0^1 \left(\tau + \frac{\tau\varepsilon}{2}\,|\hat{u}(t)|^2\right)\,dt + \mu_0^T(\hat{x}(1) - x_1),$$

where $\hat{x}(1)$ is defined through the differential equation and the initial condition.

We now argue that $e : C \times \mathbb{R} \subset L^2(0,1;\mathbb{R}^m) \times \mathbb{R} \to \mathbb{R}^n$ satisfies the regular point condition in the sense of Maurer and Zowe [MZ]; see also [IK2]. Thus we have to verify that

$$0 \in \operatorname{int} \{e'(\hat{u}_\varepsilon, \tau_\varepsilon)\big((C - \hat{u}_\varepsilon) \times \mathbb{R}\big)\}, \tag{2.19}$$

where $e'(\hat{u}_\varepsilon, \tau_\varepsilon)$ denotes the linearization of e at $(\hat{u}_\varepsilon(\cdot\,\tau_\varepsilon), \tau_\varepsilon)$.

Considering $e'(\hat{u}_\varepsilon, \tau_\varepsilon)$ in directions $\delta\tau = 0$ and δu satisfying $(\delta u)_i = 0$ for $i \neq i^*$ and $(\delta u)_{i^*} = 0$ in $(0,1) \setminus (\alpha, \alpha + \delta)$, with $(\alpha, \alpha+\delta) := I_{i^*}$, we find

$$\begin{aligned} e'(\hat{u}_\varepsilon, \tau_\varepsilon)(\delta u - \hat{u}_\varepsilon, 0) &= \textstyle\int_\alpha^{\alpha+\delta} e^{\tau A(1-t)}\, \tau\, b_{i^*}\, (\delta u(t) - \hat{u}_\varepsilon(t))_{i^*}\, dt \\ &= \textstyle\int_0^\delta e^{\tau A(\delta - t)}\, \tilde{b}_{i^*}\, (\delta u(t+\alpha) - \hat{u}_\varepsilon(t+\alpha))_{i^*}\, dt, \end{aligned}$$

where $\tilde{b}_{i^*} = \tau e^{\tau A(1-\delta-\alpha)} b_{i^*}$. Then

$$e'(\hat{u}_\varepsilon, \tau_\varepsilon)(\delta u - \hat{u}_\varepsilon, 0) = \int_0^\delta e^{\tau A(\delta - t)} \tilde{b}_{i^*} (\delta\tilde{u}(t) - \tilde{u}_\varepsilon(t))_{i^*}\, dt,$$

where $\delta\tilde{u}_{i^*}(t) = \delta u_{i^*}(t+\alpha), \tilde{u}_{\varepsilon,i^*}(t) = \hat{u}_{\varepsilon,i^*}(t+\alpha)$. Note that by (2.16),

$$\{(\delta\tilde{u} - \tilde{u}_\varepsilon)_{i^*} : [0,\delta] \to \mathbb{R}^1 \;\mid\; |(\delta\tilde{u})_{i^*}| \leq 1\} \supset S := \left\{v : [0,\delta] \to \mathbb{R}^1\,,\ |v| \leq \frac{\eta}{2}\right\}.$$

Observe that controllability of (A, b_{i*}) implies that $(A, \tilde{b}_{i*})$ is controllable as well. Controllability of the single input system $(A, \tilde{b}_{i^*})$ implies that

$$0 \in \operatorname{int} \left\{ \int_0^\delta e^{\tau A(\delta - t)} \tilde{b}_{i^*} v\, dt \mid v \in S \right\}. \tag{2.20}$$

In fact, the set on the right of (2.20) contains 0 and has nonempty interior; see, e.g., [LM, pp. 77, 133]. Moreover, if 0 were a boundary point of this set, then the corresponding control $v = 0$ would be an extremal control, which is impossible; see, e.g., [LM, p. 133]. Now (2.20) implies (2.19).

With the regular point condition satisfied, we can conclude the necessary optimality

$$\begin{aligned} &\mathcal{L}_\tau(\hat{u}_\varepsilon, \tau_\varepsilon, \mu_0) = 0, \\ &\mathcal{L}_u(\hat{u}_\varepsilon, \tau_\varepsilon, \mu_0)(\delta u - \hat{u}_\varepsilon) \geq 0 \text{ for all } \delta u \text{ with } |\delta u|_{\ell^\infty} \leq 1. \end{aligned} \tag{2.21}$$

From the second property in (2.21), we have

$$\int_0^1 (\varepsilon \hat{u}_\varepsilon + B^T e^{\tau A^T(1-t)} \mu_0)(\delta u - \hat{u}_\varepsilon)\, dt \geq 0.$$

Setting

$$p(t) = e^{-\tau A^T t} q \text{ with } q = e^{\tau A^T} \mu_0, \tag{2.22}$$

this implies

$$\int_0^1 (\varepsilon \hat{u}_\varepsilon + B^T \hat{p}_\varepsilon)(\delta u - \hat{u}_\varepsilon)\, dt \geq 0$$

for all δu as in (2.21). The second and third claims in (2.17) follow with $p_\varepsilon(t) = \hat p_\varepsilon(\tau^{-1}\, t)$.

Exploiting the first property in (2.21) implies that

$$\mathcal{L}_\tau(\hat u_\varepsilon, \tau_\varepsilon, \mu_0) = 1 + \frac{\varepsilon}{2}\int_0^1 |\hat u_\varepsilon|^2\, dt$$
$$+\ \mu_0^T\left(Ae^{\tau A}x_0 + \int_0^1 e^{\tau A(1-t)}B\hat u_\varepsilon(t)\, dt + \int_0^1 A(1-t)e^{\tau A(1-t)}\tau B\hat u_\varepsilon(t)\, dt\right)$$
$$= 1 + \frac{\varepsilon}{2}\int_0^1 |\hat u_\varepsilon|^2\, dt + \mu_0^T\bigg(Ae^{(\tau(1-t)+\tau t)A}x_0$$
$$+ \int_0^1 e^{\tau A(1-t)}B\hat u_\varepsilon(t)\, dt + \int_0^1 Ae^{\tau A(1-t)}\int_0^t e^{\tau A(t-s)}\tau B\hat u_\varepsilon(s)\bigg)\, ds\, dt = 0.$$

This implies

$$\mathcal{L}_\tau(\hat u_\varepsilon, \tau_\varepsilon, \mu_0) = 1 + \frac{\varepsilon}{2}\int_0^1 |\hat u_\varepsilon|^2\, dt + \int_0^1 p^T(t)\left(A\hat x_\varepsilon(t) + B\hat u_\varepsilon(t)\right) dt = 0. \tag{2.23}$$

From $u_\varepsilon = -\sigma_\varepsilon(B^Tp_\varepsilon)$ we conclude that $u_\varepsilon \in W^{1,\infty}(0,\tau_\varepsilon;\mathbb{R}^m)$.

We introduce the Hamiltonian for (P_ε) as

$$H_\varepsilon(x,u,p) = 1 + \frac{\varepsilon}{2}|u|^2_{\mathbb{R}^m} + p^T(Ax+Bu).$$

It is constant along the optimal solution. In fact we have a.e. on $(0,1)$

$$\frac{d}{dt}H_\varepsilon(\hat x_\varepsilon, \hat u_\varepsilon, \hat p_\varepsilon) = \varepsilon \hat u_\varepsilon^T\frac{d}{dt}\hat u_\varepsilon + \frac{1}{\tau_\varepsilon}\frac{d}{dt}\hat p_\varepsilon^T\frac{d}{dt}\hat x_\varepsilon + \hat p_\varepsilon^T A\frac{d}{dt}\hat x_\varepsilon + \hat p_\varepsilon^T B\frac{d}{dt}\hat u_\varepsilon$$
$$= (\varepsilon\hat u_\varepsilon + B^T\hat p_\varepsilon)^T\frac{d}{dt}\hat u_\varepsilon = 0.$$

Combined with (2.23), this implies that

$$1 + \frac{\varepsilon}{2}|\hat u_\varepsilon|^2_{\mathbb{R}^m} + \hat p_\varepsilon^T(A\hat x_\varepsilon + B\hat u_\varepsilon) = 0 \text{ on } [0,1].$$

This implies the claim. ☐

The proof revealed extra regularity of u_ε.

COROLLARY 2.5. *Under the assumptions of Theorem* 2.4, *we have that* $u_\varepsilon \in W^{1,\infty}(0,\tau_\varepsilon;\mathbb{R}^m)$.

Remark 2.1. Condition (2.16) requires that the modulus of at least one of the coordinates of $\hat u_\varepsilon$ is not a.e. equal to 1. Once it is known from Corollary 2.5 that $\hat u_\varepsilon$ is continuous, this amounts to requiring that at least one of the coordinates of u^* switches from 1 to -1 or vice versa.

Under the assumptions of Theorem 2.4, the first order necessary optimality condition for (P_ε) after the transformation $t \to \frac{t}{\tau}$ is given by

$$\begin{cases} \dot x = \tau(Ax+Bu),\ x(0) = x_0,\ x(1) = x_1, \\ -\dot p = \tau A^T p, \\ u = -\sigma_\varepsilon(B^Tp), \\ 1 + \frac{\epsilon}{2}|u(1)|^2 + p(1)^T(Ax(1) + Bu(1)) = 0, \end{cases} \tag{2.24}$$

where for convenience of notation the dependence on ε and the superscript hat were dropped.

In the following section we shall investigate semismooth Newton methods for solving (2.24).

We close this section with a simple example which illustrates some of the features of the regularization approach.

Example 2.1. Consider the two-dimensional time-optimal problem for the simple control system

$$\begin{cases} \dot{x}_1 = u_1, \\ \dot{x}_2 = u_2, \end{cases}$$

where A is the zero and B the identity matrix, with initial condition $(1, \frac{1}{2})$ and terminal condition the origin. This system is controllable but not normal. The optimal time is $\tau^* = 1$, and the first coordinate of an optimal control is uniquely determined to be $u_1^* = -1$, with associated state $x_1 = 1 - t$. There are infinitely many choices for optimal solutions u_2^* of bang-bang and non-bang-bang type. The associated constant adjoints are $(p_1, p_2) = (1, 0)$. They satisfy

$$u = -\sigma(p) \in \begin{pmatrix} -1 \\ [-1, 1] \end{pmatrix}.$$

The transversality condition $1 + p^T B u = 0$ is satisfied.

For the regularized problem we find $\tau_\varepsilon = 1$. Differently from the unregularized problem, the solution to the regularized problem is unique. The optimal control and trajectory are given by

$$(u_1, u_2) = (-1, -.5) \text{ with } (x_1, x_2) = (1 - t, .5(1 - t)).$$

In this particular example the solution of the regularized problem does not depend on ε. Note that this solution is also one of the minimum norm solutions of the unregularized problem. The adjoint is $p_\varepsilon = (1 + \frac{3\varepsilon}{8}, \frac{\varepsilon}{2})$. It satisfies

$$u = -\sigma_\varepsilon(p_\varepsilon^T) = \begin{pmatrix} -1 \\ -\frac{1}{2} \end{pmatrix}$$

and the transversality condition $1 + \frac{\varepsilon}{2}|u|^2 + p_\varepsilon^T B u = 0$.

3. Semismooth Newton method. In this section the semismooth Newton method for solving the regularized optimality system (2.24) is described and analyzed. It will allow (2.24) to be solved efficiently in spite of the fact that σ_ε is not differentiable.

Throughout we assume (H1) to hold. We fix $\varepsilon > 0$ and denote by $(x_\varepsilon, u_\varepsilon, \tau_\varepsilon) \in W^{1,2}(0,1) \times L^2(0,1) \times \mathbb{R}$ a solution to (P_ε) with associated adjoint $p_\varepsilon \in W^{1,2}(0, 1)$. It is assumed that

$$\text{(H2)} \qquad \text{there exists } \bar{s} \in (0,1) \text{ such that } \left| \frac{1}{\varepsilon} b_{i^*}^T p_\varepsilon(\bar{s}) \right| = |(u_\varepsilon)_{i^*}(\bar{s})| < 1,$$

and

$$\text{(H3)} \qquad |b_i^T p_\varepsilon(1)| \neq \varepsilon \text{ for all } i = 1, \dots, m.$$

Assumption (H2) corresponds to (2.16), where we now use the fact that, as a consequence of Theorem 2.4, the control u_ε is continuous. With (H2) and (H3) holding, there exists a neighborhood $\mathcal{U}_{p_\varepsilon}$ of p_ε in $W^{1,2}(0,1;\mathbb{R}^n)$, $\bar{t} \in (0,1)$, and a nontrivial interval $(\alpha, \alpha+\delta) \subset (0,1)$ such that for $p \in \mathcal{U}_{p_\varepsilon}$ we have

$$|b_i^T\, p(t)| \neq \varepsilon \text{ for all } t \in [\bar{t},\, 1] \text{ and } i = 1, \ldots, m$$

and

$$|b_{i^*}^T p(t)| < \varepsilon \text{ for } t \in (\alpha, \alpha+\delta). \tag{3.1}$$

We set $U = \{u \in L^2(0,\, 1; \mathbb{R}^m) : u|[\bar{t},\, 1] \in W^{1,2}(\bar{t}, 1; \mathbb{R}^m)\}$, endowed with the norm

$$|u|_U = (|u|^2_{L^2(0,1)} + |\dot{u}|^2_{L^2(\bar{t},1)})^{\frac{1}{2}},$$

and introduce

$$F : D_F \subset X \to L^2(0,1;\mathbb{R}^n) \times L^2(0,1;\mathbb{R}^n) \times U \times \mathbb{R}^n \times \mathbb{R},$$

where

$$D_F = W^{1,2}(0,1) \times \mathcal{U}_{p_\varepsilon} \times U \times \mathbb{R},$$

$$X = W^{1,2}(0,1;\mathbb{R}^n) \times W^{1,2}(0,1;\mathbb{R}^n) \times U \times \mathbb{R},$$

and

$$F(x,\, p,\, u,\, \tau) = \begin{pmatrix} \dot{x} - \tau A x - \tau B u, \\ -\dot{p} - \tau A^T p, \\ u + \sigma_\varepsilon(B^T p), \\ x(1) - x_1, \\ 1 + \frac{\varepsilon}{2}|u(1)|^2 + p(1)^T\,(Ax(1) + Bu(1)) \end{pmatrix}. \tag{3.2}$$

Note that $F = (F_1, \ldots, F_5)$ is well defined. This is obvious for F_1, F_2, and F_3. For F_4, F_5 it follows from the fact that $W^{1,2}(0,1)$ embeds continuously into $C(0,1)$. Moreover, $F(x_\varepsilon,\, p_\varepsilon,\, u_\varepsilon,\, \tau_\varepsilon) = 0$. We shall keep $x_\varepsilon(0) = x_0$ as an explicit constraint.

Remark 3.1. The need for introducing U in such a way that its elements are more regular at 1 is due to the fact that we use here the pointwise transversality condition rather than the integrated form (2.23). Condition (H3) will be needed to prove superlinear convergence of the Newton-iteration.

Applying Newton's method to $F = 0$ is impeded by the nondifferentiability of σ_ε. We use

$$G\sigma_\varepsilon(s) := \begin{cases} \frac{1}{\varepsilon} & \text{if } |s| < \varepsilon, \\ 0 & \text{if } |s| \geq \varepsilon \end{cases} \tag{3.3}$$

as a generalized derivative and argue that the resulting Newton-iteration is semismooth and hence locally superlinearly convergent. The Newton-iteration step is given by

$$DF(x,\, p,\, u,\, \tau)(\delta x,\, \delta p,\, \delta u,\, \delta\tau) = -F(x,\, p,\, u,\, \tau), \tag{3.4}$$

where $\delta x(0) = 0$ and DF denotes the Frechet-derivative in all terms of F except for $p \to \sigma_\varepsilon(B^T p)$, for which the generalized derivative is taken according to (3.3). For further reference we give the detailed form of (3.4):

$$\begin{cases} \frac{d}{dt}\delta x - \tau A\delta x - \tau B\,\delta u - \delta\tau(Ax + Bu) = -F_1, \quad \delta x(0) = 0, \\ -\frac{d}{dt}\delta p - \tau A^T\delta p - \delta\tau A^T p = -F_2, \\ \delta u + G\sigma_\varepsilon(B^T p)B^T\delta p = -F_3, \\ \delta x(1) = -F_4, \\ p(1)^T(A\,\delta x(1) + B\delta u(1)) + \delta p(1)^T(Ax(1) + Bu(1)) \\ \qquad\qquad + \varepsilon u(1)^T\delta u(1) = -F_5, \end{cases} \tag{3.5}$$

where the coordinates of $G\sigma_\varepsilon(B^T p)B^T\delta p$ are given by $G\sigma_\varepsilon((B^T p)_i)(B^T\delta p)_i$.

A possible initialization may consist of choosing $((u)_0, \tau_0)$, setting $(x)_0$ as the linear interpolation between x_0 and x_1, and determining $(p)_0$ such that the transversality condition and the adjoint equation are satisfied.

We now briefly summarize those facts from semismooth Newton methods which are relevant for this paper. Let X and Z be Banach spaces, and let $F : D_F \subset X \to Z$ be a nonlinear mapping with open domain D_F.

Definition 3.1. *The mapping $F : D_F \subset X \to Z$ is called Newton-differentiable on an open subset $\mathcal{U} \subset D_F$ if for each $\mathbf{x} \in \mathcal{U}$ there exists a generalized derivative $DF(\mathbf{x}) \in \mathcal{L}(X, Z)$ and*

$$\lim_{h\to 0} \frac{1}{|h|_X} |F(\mathbf{x}+h) - F(\mathbf{x}) - DF(\mathbf{x}+h)h|_Z = 0. \tag{3.6}$$

Theorem 3.2. *Suppose that $\mathbf{x}^* \in \mathcal{U}$ is a solution to $F(\mathbf{x}) = 0$ and that F is Newton-differentiable in an open set $\mathcal{U}$ containing $\mathbf{x}^*$. Further, if $\{\|DF(\mathbf{x})^{-1}\| : \mathbf{x} \in \mathcal{U}\}$ is bounded, then the Newton-iteration $\mathbf{x}_{k+1} = \mathbf{x}_k - DF(\mathbf{x}_k)^{-1}F(\mathbf{x}_k)$ converges q-superlinearly to $\mathbf{x}^*$, provided that $|\mathbf{x}_0 - \mathbf{x}^*|_X$ is sufficiently small.*

For the statement and proof of superlinear convergence of the time-optimal control problem, some further notation is required. For $(x, p, u, \tau) \in D_F$ we define $\mathcal{A} \in \mathbb{R}^{(n+1)\times(n+1)}$ by

$$\mathcal{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{pmatrix},$$

where

$$A_{11} = \varepsilon^{-1}\tau \int_0^1 e^{\tau A(1-t)} B\chi_I B^T\, e^{\tau A^T(1-t)}\, dt \in \mathbb{R}^{n\times n}, \tag{3.7}$$

$$lA_{12} = \varepsilon^{-1}\tau \int_0^1 e^{\tau A(1-t)} B\chi_I B^T \int_t^1 e^{-\tau A^T(t-s)} A^T p(s)\, ds\, dt \\ - \int_0^1 e^{\tau A(1-t)}(Ax + Bu)\, dt \in \mathbb{R}^n, \tag{3.8}$$

$$A_{21} = (Ax(1) + Bu(1))^T - (p^T(1)B + \varepsilon u^T(1))\, G\sigma_\varepsilon(B^T p(1))B^T \in (\mathbb{R}^n)^T, \tag{3.9}$$

with $\chi_I = \mathrm{diag}(\chi_{I_1}, \dots, \chi_{I_m})$ and χ_{I_i} the characteristic function of the set

$$I_i = I_i(p) = \left\{ t : |(B^T p)_i| < \frac{1}{\varepsilon} \right\}, \quad i = 1, \dots, m,$$

which is nonempty for $p \in \mathcal{U}_{p_\varepsilon}$ and $i = i^*$. The controllability assumption (H1), together with (H2), implies that the symmetric matrix A_{11} is invertible with uniformly bounded inverse with respect to $p \in \mathcal{U}_{p_\varepsilon}$ and τ in compact subsets of $(0, \infty)$. In fact, since $I_{i^*}(p) \supset (\alpha,\, \alpha + \delta)$, we obtain for some $\bar{c} > 0$

$$\begin{aligned}
A_{11} &= \varepsilon^{-1}\tau \int_0^1 e^{\tau A(1-t)} \sum_{i=1}^m b_i \chi_{I_i} b_i^T e^{\tau A^T(1-t)}\, dt \\
&\geq \varepsilon^{-1}\tau \int_\alpha^{\alpha+\delta} e^{\tau A(1-t)}\, b_{i^*} b_{i^*}^T\, e^{\tau A^T(1-t)}\, dt \\
&= \varepsilon^{-1}\tau e^{\tau A(1-\alpha)} \int_0^\delta e^{-\tau A t}\, b_{i^*} b_{i^*}^T\, e^{-\tau A^T t} dt\, e^{\tau A^T(1-\alpha)}\, dt > \bar{c},
\end{aligned}$$

where we use that the controllability Gramian,

$$\int_0^\delta e^{-\tau A t} b_{i^*} b_{i^*}^T e^{-\tau A^T t}\, dt,$$

is uniformly positive definite for τ in compact subsets of $(0, \infty)$.

For our analysis we shall utilize the fact that the Schur complement

$$A_{21}\, A_{11}^{-1}\, A_{12} \in \mathbb{R}$$

of $\mathcal{A}$ for $(x,\, p,\, u,\, \tau)$ in a neighborhood of $(x_\varepsilon,\, p_\varepsilon,\, u_\varepsilon,\, \tau_\varepsilon)$ is nontrivial. If $A_{21}\, A_{11}^{-1}\, A_{12} \neq 0$ at $(x_\varepsilon,\, p_\varepsilon,\, u_\varepsilon,\, \tau_\varepsilon)$, we cannot conclude that $A_{21}\, A_{11}^{-1}\, A_{12} \neq 0$ for $(x,\, p,\, u,\, \tau)$ in a neighborhood of $(x_\varepsilon,\, p_\varepsilon,\, u_\varepsilon,\, \tau_\varepsilon)$ since, with (H2) holding, A_{21} is continuous with respect to $(x,\, p,\, u,\, \tau) \in X$. This is not the case for A_{11}^{-1} and A_{12} due to the term χ_I. We therefore assume that

$$\text{(H4)} \qquad \begin{cases}
\text{there exists a bounded neighborhood} \\
\mathcal{U} \subset D_F \subset X \text{ of } (x_\varepsilon,\, p_\varepsilon,\, u_\varepsilon,\, \tau_\varepsilon) \text{ and } c > 0 \text{ such that} \\
|A_{21}\, A_{11}^{-1}\, A_{12}| \geq c \ \text{ for all } \ (x,\, p,\, u,\, \tau) \in \mathcal{U}.
\end{cases}$$

THEOREM 3.3. *If* (H1)–(H4) *hold and* $(x_\varepsilon,\, u_\varepsilon,\, \tau_\varepsilon)$ *denotes a solution to* (P_ε) *with associated adjoint* p_ε*, then the semismooth Newton algorithm converges superlinearly, provided that the initialization is sufficiently close to* $(x_\varepsilon, p_\varepsilon,\, u_\varepsilon,\, \tau_\varepsilon)$.

For the proof we require the following lemma.

LEMMA 3.4. *Suppose that* (H1)–(H4) *hold. Then there exists a constant* C *such that for every* $(x,\, p,\, u,\, \tau) \in \mathcal{U}$*, and every* $F \in L^2(0,\, 1) \times L^2(0,\, 1) \times U \times \mathbb{R}$*,*

$$DF(x,\, p,\, u,\, \tau)(\delta x,\, \delta p, \delta u,\, \delta \tau) = -F$$

admits a unique solution $(\delta x,\, \delta p, \delta u,\, \delta\tau) \in X$ *and*

$$|(\delta x,\, \delta p, \delta u,\, \delta\tau)|_X \le C\, |F|_{L^2\times L^2\times U\times\mathbb{R}}. \tag{3.10}$$

Proof. Let $(x,\, p,\, u, \tau) \in \mathcal{U}$, and note that system (3.5) is equivalent to

$$\begin{cases} \delta x(t) = \int_0^t e^{\tau A(t-s)}(-F_1 + \tau b\,\delta u + \delta\tau(Ax+Bu))\, ds, \\ \delta p(t) = e^{\tau A^T(1-t)}\delta p(1) + \int_t^1 e^{-\tau A^T(t-s)}(\delta\tau A^T p - F_2)\, ds, \\ \delta u + G\sigma_\varepsilon(B^T p)B^T\delta p = -F_3, \\ \delta x(1) = -F_4, \\ p(1)^T(-AF_4 + B\,\delta u(1)) + \delta p(1)^T(Ax(1)+Bu(1)) + \varepsilon\delta u(1) = -F_5. \end{cases} \tag{3.11}$$

On I_i we have $(G\sigma_\varepsilon(B^T p))_i = \varepsilon^{-1}$. The third equation in (3.11) can be expressed as

$$\delta u = -F_3 - \varepsilon^{-1}\chi_I\, B^T\delta p \quad \text{a.e. in } (0,1). \tag{3.12}$$

Let us set

$$\hat{F}_1 = -\int_0^1 e^{\tau A(1-t)}F_1(t)\, dt, \quad \hat{F}_2 = \varepsilon^{-1}\tau\int_0^1\int_t^1 e^{\tau A(1-t)}B\chi_I B^T\, e^{-\tau A^T(t-s)}F_2(s)\, ds\, dt,$$

$$\hat{F}_3 = -\tau B\int_0^1 e^{\tau A(1-t)}F_3(t)\, dt.$$

From (3.12) and the first and fourth equations in (3.11), we have

$$\begin{aligned} -F_4 &= \delta x(1) \\ &= -\varepsilon^{-1}\tau\int_0^1 e^{\tau A(1-t)}B\chi_I B^T\,\delta p\, dt + \delta\tau\int_0^1 e^{\tau A(1-t)}(Ax+Bu)\, dt + \hat{F}_1 + \hat{F}_3. \end{aligned} \tag{3.13}$$

Replacing δp by the second equation in (3.11), we find

$$\begin{aligned} -F_4 = {}& -\varepsilon^{-1}\tau\int_0^1 e^{\tau A(1-t)}B\chi_I B^T\, e^{\tau A^T(1-t)}\delta p(1)\, dt \\ & - \varepsilon^{-1}\tau\delta\tau\int_0^1 e^{\tau A(1-t)}B\chi_I B^T\int_t^1 e^{-\tau A^T(t-s)}A^T p(s)\, ds\, dt + \hat{F}_2 \\ & + \delta\tau\int_0^1 e^{\tau A(1-t)}(Ax+Bu)\, dt + \hat{F}_1 + \hat{F}_3, \end{aligned}$$

which involves $\delta p(1)$ and $\delta\tau$ as unknowns and can be expressed as

$$A_{11}\delta p(1) + A_{12}\delta\tau = \hat{F}_1 + \hat{F}_2 + \hat{F}_3 + F_4 =: r_1. \tag{3.14}$$

Eliminating $\delta u(1)$ from the last equation in (3.11) by means of the third equation implies

$$A_{21}\delta p(1) = -F_5 + F_3(1)^T(B^T p(1) + \varepsilon u(1)) + p(1)^T AF_4 =: r_2. \tag{3.15}$$

Combining (3.14) and (3.15), we obtain the following linear system for $(\delta p(1),\, \delta\tau)$:

$$\mathcal{A}\begin{pmatrix} \delta p(1) \\ \delta\tau \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}. \tag{3.16}$$

By (H1), (H2), and (H4) its unique solution is given by

$$\delta\tau = (A_{21}\, A_{11}^{-1}\, A_{12})^{-1}(A_{21}\, A_{11}^{-1}\, r_1 - r_2), \qquad \delta p(1) = A_{11}^{-1}(r_1 - A_{12}\delta\tau). \tag{3.17}$$

Moreover, there exists a constant $C = C(\tau,\, |x|_{C(0,1)},\, |p|_{C(0,1)},\, |u|_{L^2(0,1)},\, |u(1)|)$ such that

$$|\delta p(1)| + |\delta\tau| \le C\, |F|_{L^2\times L^2\times U\times\mathbb{R}}.$$

From (3.5) and (3.11), C can also be chosen such that

$$|(\delta x,\, \delta p,\, \delta u,\, \delta\tau)|_X \le C\, |F|_{L^2\times L^2\times U\times\mathbb{R}}. \qquad \square$$

Proof of Theorem 3.3. We apply Theorem 3.2 with $\mathbf{x}^* = (x_\varepsilon,\, p_\varepsilon,\, u_\varepsilon,\, \tau_\varepsilon)$. Lemma 3.4 implies the required uniform bound of the generalized inverses DF in the neighborhood $\mathcal{U} \subset D_F$ of $(x_\varepsilon,\, p_\varepsilon,\, u_\varepsilon,\, \tau_\varepsilon)$. Therefore it suffices to argue for Newton-differentiability of F in D_F. This is obvious for all coordinates of F, except for F_3, and specifically for the mapping

$$\mathcal{F} : p \to \sigma_\varepsilon(B^T p)$$

from $\mathcal{U}_{p_\varepsilon} \subset W^{1,2}(0,\,1;\,\mathbb{R}^n) \to U$. Utilizing the definitions of $\mathcal{U}_{p_\varepsilon}$ and σ_ε, it suffices to consider the restriction of $\mathcal{F}$ from $W^{1,2}(0,\,\bar{t};\,\mathbb{R}^n)$ to $L^2(0,\,\bar{t};\,\mathbb{R}^1)$, which we again denote by $\mathcal{F}$. Note that $\mathcal{F}$ can be decomposed as

$$\mathcal{F} = \mathcal{F}_3 \circ \mathcal{F}_2 \circ \mathcal{F}_1,$$

where

$$\mathcal{F}_1 : W^{1,2}(0,\,\bar{t};\,\mathbb{R}^n) \to W^{1,2}(0,\,\bar{t};\,\mathbb{R}), \quad \mathcal{F}_2 : W^{1,2}(0,\,\bar{t};\,\mathbb{R}) \to L^4(0,\,\bar{t};\,\mathbb{R}),$$
$$\mathcal{F}_3 : L^4(0,\,\bar{t};\,\mathbb{R}) \to L^2(0,\,\bar{t};\,\mathbb{R})$$

are given by

$$\mathcal{F}_1(u) = B^T u, \quad \mathcal{F}_2(v) = \max\left(-1,\, \frac{v}{\varepsilon}\right), \quad \mathcal{F}_3(v) = \min(1, v).$$

In [HIK, IK2] it was shown that $v \to \max(0, v)$ is Newton-differentiable from $L^p(\Omega)$ to $L^q(\Omega)$ if $\infty \ge p > q \ge 1$, if Ω is a bounded domain. Since $\min(1, v) = 1 + \min(0,\, v-1)$, this implies that $\mathcal{F}_3$ and similarly that $\mathcal{F}_2$ are Newton-differentiable. From the chain rule for Newton-differentiable mapping in [HK] it follows that $\mathcal{F}_3 \circ \mathcal{F}_2$ is Newton-differentiable. The chain rule for a linear mapping, here $\mathcal{F}_1$, followed by the Newton-differentiable mappings $\mathcal{F}_3 \circ \mathcal{F}_2$ [IK1], implies that $\mathcal{F}$ is Newton-differentiable in D_F. $\square$

4. A numerical example. The semismooth Newton method is used to solve a classical time-optimal problem related to the harmonic oscillator with three switching points. We consider

$$
\text{(4.1)} \qquad \begin{cases} \min_{\tau\geq 0} \int_0^\tau dt \\ \text{subject to} \\ \frac{d}{dt}x(t) = Ax(t) + Bu(t),\ |u(t)| \leq 1,\ x(0) = x_0,\ x(\tau) = x_1, \end{cases}
$$

where

$$
A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad x_0 = \begin{pmatrix} -5 \\ 5 \end{pmatrix}, \qquad x_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
$$

The optimal minimal time for the continuous problem is known to be $\tau^* = 10.5871$. To solve (4.1) numerically, a time discretization based on the Crank–Nicolson method with equidistant grid points was applied to (3.5). The initialization for the state was chosen as a semicircle connecting x_0 and x_1. Then $u(1)$ was chosen to be active, and p was chosen so that the transversality condition and the adjoint equation hold. With respect to the choice of the parameter $c = \frac{1}{\varepsilon}$ we utilized a continuation procedure, starting with a small value and increasing it, using the solution from the smaller value of c as initialization for the next larger c-value. Certainly this procedure can be automated, as has been done elsewhere, but this is not the focus of this paper. In Table 1 we show the number of iterates of the Netwon-iteration (outer loop) that was required for this continuation procedure with respect to c. The Newton-iteration was stopped when the residual of the optimality system in the L^2-norm was below 10^{-8}. Also in Table 1 we depict the optimal minimal times $\tau^*(c)$. These results are obtained for meshsize $h = \frac{1}{32}$.

Then the results for $c = 1$ are interpolated to the finer grid $h = \frac{1}{128}$, and the continuation procedure with respect to c is repeated. The results are depicted in Table 2. The graphs for the corresponding controls are given in Figure 1.

The same procedure with $h = 1/512$ and $c = 100$ gives the optimal time 10.588. In some cases, typically at the beginning of the iterations and for the lowest values of c, the full Newton step was too large. Therefore we used a one-dimensional line search based on a quadratic polynomial interpolation for the L^2-norm of the residual combined with an Armijo rule.

Table 3 depicts the quotients $\frac{|u^{k+1} - u^*(c)|_{L^2}}{|u^k - u^*(c)|_{L^2}}$, where $u^*(c)$ is the solution to the discretized version of (2.17) for $c = 50$. It shows that the algorithm is in fact superlinearly convergent.

TABLE 1

c	1	5	10	20
No. of iterations	8	8	4	7
Final time	11.26515	10.84455	10.82977	10.81781

TABLE 2

c	1	10	50	100	200
No. of iterations	5	46	4	4	3
Final time	11.1088	10.6092	10.6034	10.6033	10.6031

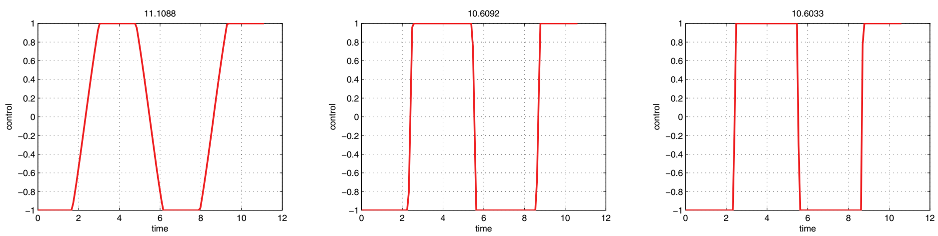

FIG. 1. $N = 128$ *and* $c = 1$*(left),* 10*(middle), and* 100*(right).*

TABLE 3

No. of iterations	1	2	3	4
c_k	0.94138	0.00037	0.00001	0.00000

In this paper we chose to regularize σ by the ramp functions σ_ε with increasing slopes as $\varepsilon \to 0^+$. Certainly other alternatives are possible as, for instance, $\tilde{\sigma}_c(s) = \frac{2}{\pi}\, a\tan(c\, s)$. This family of C^∞-functions also has the property that it converges to σ as $c \to 0$, but it appears to be less apt for the purpose of approximating the discontinuous switching structure of the optimal controls since c has to be taken significantly larger for $\tilde{\sigma}_c$ than for $\sigma_{\frac{1}{c}}$ to obtain comparable results.

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