

ASYMPTOTIC BIFURCATION SOLUTIONS FOR COMPRESSIONS OF A CLAMPED NONLINEARLY ELASTIC RECTANGLE: TRANSITION REGION AND BARRELLING TO A CORNER-LIKE PROFILE

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Abstract. Buckling and barrelling instabilities in the uniaxial compressions of an elastic rectangle have been studied by many authors under lubricated end conditions. However, in practice it is very difficult to realize such conditions due to friction. In the two experiments by Beatty and his co-authors, it is found that there is a transition region of the aspect ratio which separates buckling and barrelling. Friction, which prevents the lateral movement of the end cross section, might be the cause. Here, we study the compressions of a two-dimensional nonlinearly elastic rectangle under clamped end conditions. One of the purposes is to show, under this setting in which the lateral movement of the end cross section is limited, that there is indeed such a transition region. We achieve this by constructing asymptotic solutions of the field equations. By using combined series-asymptotic expansions, we derive two decoupled nonlinear ordinary differential equations (ODE's). One governs the leading-order axial strain and another governs both the leading-order axial strain and shear strain. By phase plane analysis, the axial strain can be obtained from one of the ODE's as the axial resultant force varies. Then an eigenvalue problem can be formulated from another ODE, which is solved by the WKB method. It is found that when the aspect ratio is relatively large there is only a bifurcation to barrelling which leads to a corner-like profile on the lateral boundaries of the rectangle. When the aspect ratio is relatively small there are only bifurcation points which lead to the buckled profiles. A lower bound of the aspect ratio for barrelling and an upper bound for buckling are found, which implies the existence of the above-mentioned transition region. The critical buckling loads from our asymptotic solutions are also compared with those obtained from the Euler's buckling formula.

Key words. nonlinear elasticity, bifurcation, corner-like profile, barrelling, buckling

AMS subject classifications. 74G10, 74G60, 35B32, 34B16

1. Introduction. Two kinds of instabilities, buckling and barrelling, may occur when a two-dimensional nonlinearly elastic rectangle being compressed uniaxially. The instability of the uniaxial compression of a rod is an old problem which was firstly studied by Euler who gave the Euler's buckling formula to predict the onset of buckling of a rod. However Bernoulli-Euler beam theory does not take into account the effect of transverse shear strain, and the Euler's buckling formula is not valid for relative thick rods. Recently, a more sophisticated beam theory is derived by Russell and White [1] by assuming that the axial displacement is of the second order of the transverse displacement. The existence of solutions is established and the post-buckling solution is computed numerically. In this paper we consider both buckling and barrelling instabilities and present some new analytical post-buckling and post-barrelling results for both thin and moderately thick two-dimensional rectangle under clamped boundary conditions.

Theoretical analysis of this type of compression problems has been carried out by many authors under lubricated boundary conditions, such as [2, 3, 4, 5] and those listed in their references. In [2, 3], Simpson and Spector study barrelling instability

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of compression of a solid circular cylinder. In [2] they obtain the condition of critical compression ratios at which the cylinder will barrel. In [3], for a specific material a detailed relationship between critical compression ratio and modes n are investigated. In [4], both barrelling and buckling instabilities of compression of a two-dimensional elastic rectangle are studied by Davies. It is found that the rectangle will buckle or barrel under some compression ratios. Buckling and barrelling behavior depends on the sign of a parameter which is related to the strain energy function and compression ratio. When this parameter is nonnegative, buckling will occur first. A set of sufficient conditions that barrelling occurs first is given as a theorem. In [5], Davies studies buckling of a square column and barrelling of a circular cylinder. The largest critical compression ratio λ_{BUC} that a square column begins to buckle is compared with the largest critical compression ratio λ_{BAR} that a cylinder begins to barrel. And then some conditions of which instability occurs first are analyzed.

In the above theoretical analysis, lubricated boundary conditions at two ends are used and this condition is also used in many other literatures such as two recent papers [6, 7] concerning buckling and barrelling. In [6], Goriely et. al. study the compression of a three-dimensional incompressible cylindrical tube under axial load. They derive a new and compact formulation of the bifurcation criterion for both barrelling modes and buckling modes. Also, a nonlinear correction is made for the Euler's buckling formula. The asymptotic expression of the critical aspect ratio separating instability by barrelling from buckling is obtained. Simpson and Spector in [7] prove a rectangular elastic rod will buckle under quasistatic compression at two frictionless ends. This is very valuable since seldom results can prove "a second bifurcation of solutions actually bifurcates from a known solution branch when the known branch becomes unstable" (see [7], p. 1). Another proof of the existence of nontrivial bifurcated branch in finite elasticity is given by Healey and Montes-Pizarro [8] and lubricated boundary conditions are also used.

However, in practice and experiments, it is very difficult to realize such lubricated boundary conditions, especially when the external force becomes large. Not so many experimental works on both buckling and barrelling have been carried out. Two well known experiments are done by Beatty and Hook [9] and Beatty and Dadras [10]. In [9], circular rubber bars with different aspect ratios ρ (ratio of diameter with length for cylinders) are compressed at two lubricated ends. It is found that when $\rho < \rho_1 = 0.216$, only buckling can occur. While, when $\rho > \rho_2 = 0.228$, only barrelling can occur. There is a "transition region in which the nature of the strut behavior becomes increasingly ambiguous" (see [9], p. 630). They believe that there should be a limiting aspect ratio value in this transition region which separates instability by barrelling from buckling and this value is estimated by 0.222 graphically. In another experiment of [10], three different elastomers including circular cylindrical bars, rectangular bars and thick-walled tubular columns are also compressed at two lubricated ends. In each case a similar "transition region" is found and similar results are concluded. In these two experiments, their experimental results have good qualitative agreement with the theoretical analysis of Beatty [11]. It should be noted that there exist frictions at two flat ends although they are lubricated in experiments. It is also remarked by Beatty and Hook in [9] that "In spite of lubrication, the material consistently tended to adhere to the end plates of the test machine" ([9], p. 628) and "We are unable to assess in any case the extent to which end effects may have influenced our measurements" ([9], p. 630). So far there is rare literature from either experimental results or theoretical analytical analysis about compressions of bars when friction is taken into account.

Also the effect of frictions on instabilities of bars is not known.

In this paper, due to mathematical difficulties in description of frictions, one of our motivations is to analyze the effect of clamped boundary conditions in instabilities. A key feature shared by a clamped constraint and friction is that they both limit the lateral movement of the end cross section. We aim to reveal, under such a constraint, that there indeed exists a transition region as observed in experiments. As far as we know, it seems that no other work has considered such two or three dimensional clamped boundary conditions in stability and instability analysis. The clamped boundary condition used by others before is mainly a one-dimensional condition which only includes zero lateral displacement and zero slope at ends. Here by using combined series-asymptotic expansions, we can obtain asymptotically approximate boundary conditions for clamped ends.

Another motivation is related to the Euler's buckling formula which gives critical compressive force when a rod begins to buckle under compression. This formula is based on the Bernoulli-Euler constitutive equation. Here we derive a new model equation for the critical stress values where the axial strain is emphasized. Numerical results show that we can give an improvement of the first critical stress value.

The structure of this paper is arranged as follows. In section 2, we present the field equations and traction free boundary conditions. Without loss of generality, we study the isotropic compressible hyperelastic Murnaghan material. Due to the difficulty of nonlinearity, we assume the aspect ratio is small, i.e., we are considering a thin or moderately thick rectangle. Then in section 3, after nondimensionalization of the field equations and traction free boundary conditions, we can find two small parameters and one small variable. Thus in section 4 by using the combined series-asymptotic expansions two decoupled nonlinear ODE's are obtained. One governs the leading order of the axial strain which can be written as a singular ODE system. Another governs both the axial strain and shear strain. These two equations can also be obtained by the variational principle, as shown in section 5. In section 6, by using the combined series-asymptotic expansions, the clamped boundary conditions amount to a set of asymptotically approximate boundary conditions, which will be used to determine the axial strain and shear strain. In section 7, we make some bifurcation analysis. By phase plane analysis, the solution of the axial strain can be obtained under the clamped boundary conditions. An asymptotic solution for the shear strain can be obtained through the WKB method. Then the condition determining the buckling critical stress value is obtained under the clamped boundary conditions. For chosen material constants, numerical calculations show that when the aspect ratio is relatively large (within our assumption of a small aspect ratio), there will be no buckling mode. But there is a bifurcation to the corner-like profile which is a barrelling instability. This corner-like profile is related to Willis instability which is described by Beatty in [12]. In [13,14], Dai and Wang have made some analysis and pointed out that this kind of instability could be caused by the coupling effect of material nonlinearity and geometry size of the rectangle. When the aspect ratio is relatively small, only buckling modes can be found and the thinner of the rectangle the more of the buckling modes. One important difference between our buckling and previous works is that our buckling is not a special kind of buckling while buckling in some of the previous works such as [4, 5] is a Euler-type buckling which is very special. A major finding based on the analytical results is that under clamped boundary conditions there is indeed a transition region such that when the aspect ratio is larger than a critical value the barrelling instability occurs and when it is smaller than another critical value the

buckling instability occurs. This reveals that existence of a transition region, which is in agreement with the experimental results of Beatty and Hook [9] and Beatty and Dadras [10]. In section 8, we will make our conclusions.

2. Field equations. We study the deformation of a two-dimensional rectangle composed of a compressible hyperelastic material. Let the length of the rectangle be l and the thickness be $2a$ and let (X, Y) and (x, y) denote the Cartesian coordinates of a material point in the reference and current configurations respectively. The axial and lateral displacements are denoted by

$$U(X, Y) = x - X, \quad V(X, Y) = y - Y, \quad (2.1)$$

respectively. Then the deformation gradient tensor \mathbf{F} is given by

$$\mathbf{F} = (U_X + 1)\mathbf{e}_x \otimes \mathbf{E}_X + U_Y\mathbf{e}_x \otimes \mathbf{E}_Y + V_X\mathbf{e}_y \otimes \mathbf{E}_X + (V_Y + 1)\mathbf{e}_y \otimes \mathbf{E}_Y, \quad (2.2)$$

where $\mathbf{E}_X, \mathbf{E}_Y$, and $\mathbf{e}_x, \mathbf{e}_y$ represent the orthonormal basis in the reference and current configurations respectively. Here we have chosen $\mathbf{E}_X = \mathbf{e}_x$ and $\mathbf{E}_Y = \mathbf{e}_y$.

Without loss of generality, we assume this rectangle is composed of a Murnaghan material whose strain energy function has the following form

$$\Phi = \frac{\lambda}{2}(\text{Tr}\mathbf{E})^2 + \mu(\text{Tr}\mathbf{E}^2) + \nu_1(\text{Tr}\mathbf{E})(\text{Tr}\mathbf{E}^2) + \frac{\nu_2}{3}(\text{Tr}\mathbf{E})^3 + \frac{\nu_4}{3}(\text{Tr}\mathbf{E}^3), \quad (2.3)$$

where $\mathbf{E} = (\mathbf{F}^T\mathbf{F} - \mathbf{I})/2$ is the Green strain tensor, λ and μ are known as the Lamé constants, ν_1, ν_2 and ν_4 are other constitutive constants, Tr is the trace of a tensor.

The first Piola-Kirchhoff stress tensor Σ containing terms up to the third-order material nonlinearity for an arbitrary strain energy function can be calculated by a formula provided in [15], which is given below:

$$\Sigma_{ij} = a_{jilk}^1 d_{kl} + \frac{1}{2}a_{jilknm}^2 d_{kl}d_{mn} + \frac{1}{6}a_{jilknmqp}^3 d_{kl}d_{mn}d_{pq} + O(|d_{ij}|^4), \quad (2.4)$$

where $\mathbf{d} = \mathbf{F} - \mathbf{I}$, a_{jilk}^1 , a_{jilknm}^2 and $a_{jilknmqp}^3$ are elastic moduli defined by

$$\begin{aligned} a_{jilk}^1 &= \frac{\partial^2 \Phi}{\partial F_{ij} \partial F_{kl}} \Big|_{\mathbf{F}=\mathbf{I}}, \\ a_{jilknm}^2 &= \frac{\partial^3 \Phi}{\partial F_{ij} \partial F_{kl} \partial F_{mn}} \Big|_{\mathbf{F}=\mathbf{I}}, \\ a_{jilknmqp}^3 &= \frac{\partial^4 \Phi}{\partial F_{ij} \partial F_{kl} \partial F_{mn} \partial F_{pq}} \Big|_{\mathbf{F}=\mathbf{I}}, \end{aligned} \quad (2.5)$$

and \mathbf{I} is the identity tensor corresponding to a natural configuration.

Here we study a static problem, and the field equations (neglecting the body force) are given by

$$\frac{\partial \Sigma_{xX}}{\partial X} + \frac{\partial \Sigma_{xY}}{\partial Y} = 0, \quad (2.6)$$

$$\frac{\partial \Sigma_{yX}}{\partial X} + \frac{\partial \Sigma_{yY}}{\partial Y} = 0. \quad (2.7)$$

Substituting (2.4) into (2.6) and (2.7), we have

$$\begin{aligned}
& (\lambda + 2\mu)U_{XX} + \mu U_{YY} + (\lambda + \mu)V_{XY} + (\lambda + 2\nu_1 + 2\nu_2)U_{XX}V_Y \\
& + (\mu + \nu_1 + \frac{1}{2}\nu_4)(U_Y V_{XX} + V_X V_{YY} + 2U_{XY}V_X) \\
& + (\lambda + 2\mu + \nu_1 + \frac{1}{2}\nu_4)(U_X U_{YY} + V_X V_{XX} + U_{YY}V_Y + U_Y V_{YY} + 2U_{XY}U_Y) \\
& + (\lambda + \mu + 3\nu_1 + 2\nu_2 + \frac{1}{2}\nu_4)(U_X V_{XY} + V_{XY}V_Y) \\
& + (3\lambda + 6\mu + 6\nu_1 + 2\nu_2 + 2\nu_4)U_X U_{XX} + H_1 = 0, \tag{2.8}
\end{aligned}$$

$$\begin{aligned}
& (\lambda + \mu)U_{XY} + \mu V_{XX} + (\lambda + 2\mu)V_{YY} + (\lambda + 2\nu_1 + 2\nu_2)U_X V_{YY} \\
& + (\mu + \nu_1 + \frac{1}{2}\nu_4)(U_{XX}U_Y + V_{YY}V_X + 2U_Y V_{XY}) \\
& + (\lambda + 2\mu + \nu_1 + \frac{1}{2}\nu_4)(U_Y U_{YY} + U_{XX}V_X + U_X V_{XX} + V_{XX}V_Y + 2V_X V_{XY}) \\
& + (\lambda + \mu + 3\nu_1 + 2\nu_2 + \frac{1}{2}\nu_4)(U_X U_{XY} + U_{XY}V_Y) \\
& + (3\lambda + 6\mu + 6\nu_1 + 2\nu_2 + 2\nu_4)V_Y V_{YY} + H_2 = 0, \tag{2.9}
\end{aligned}$$

where H_1 and H_2 only include third-order nonlinear terms. Here and hereafter the lengthy expressions of $H_i (i = 1, 2, \dots, 11)$ are omitted and their expressions are given in the appendix.

In addition to the above two governing equations, we assume that there is no lateral distributive loading on the lateral boundaries. Then the stress components Σ_{xY} and Σ_{yY} should vanish on the lateral boundaries. We have the following traction free boundary conditions:

$$\begin{aligned}
\Sigma_{xY} &= \mu U_Y + \mu V_X + (\mu + \nu_1 + \frac{1}{2}\nu_4)(U_X V_X + V_X V_Y) \\
& + (\lambda + 2\mu + \nu_1 + \frac{1}{2}\nu_4)(U_X U_Y + U_Y V_Y) + \frac{1}{2}(\lambda + 2\nu_1 + \nu_4)U_Y V_X^2 \\
& + (2\nu_1 + \nu_4)(\frac{3}{4}U_X^2 V_X + \frac{3}{4}U_Y^2 V_X + \frac{1}{4}V_X^3 + \frac{3}{4}V_X V_Y^2) \\
& + \frac{1}{2}(\lambda + 2\nu_1 + \nu_4)U_Y V_X^2 + (\mu + 2\nu_1 + \nu_4)U_X V_X V_Y \\
& + \frac{1}{2}(\lambda + 2\mu + 7\nu_1 + 2\nu_2 + \frac{5}{2}\nu_4)(U_X^2 U_Y + U_Y V_Y^2) \\
& + (4\nu_1 + 2\nu_2 + \nu_4)U_X U_Y V_Y + \frac{1}{2}(\lambda + 2\mu + 2\nu_1 + \nu_4)U_Y^3 \\
& = 0, \quad \text{at } Y = \pm a, \tag{2.10}
\end{aligned}$$

$$\begin{aligned}
\Sigma_{yY} &= \lambda U_X + (\lambda + 2\mu)V_Y + (\lambda + 2\nu_1 + 2\nu_2)U_X V_Y + \left(\frac{1}{2}\lambda + \nu_1 + \nu_2\right)U_X^2 \\
&\quad + (\mu + \nu_1 + \frac{1}{2}\nu_4)U_Y V_X + \frac{1}{2}(\lambda + 2\mu + \nu_1 + \frac{1}{2}\nu_4)(U_Y^2 + V_X^2) \\
&\quad + \frac{1}{2}(3\lambda + 6\mu + 6\nu_1 + 2\nu_2 + 2\nu_4)V_Y^2 + (\nu_1 + \nu_2)(3U_X V_Y^2 + U_X^3) \\
&\quad + \frac{1}{2}(4\nu_1 + 2\nu_2 + \nu_4)(U_X U_Y^2 + U_X V_X^2) + \frac{1}{2}(\lambda + 4\nu_1 + 4\nu_2)U_X^2 V_Y \\
&\quad + \frac{1}{2}(\lambda + 2\mu + 7\nu_1 + 2\nu_2 + \frac{5}{2}\nu_4)(U_Y^2 V_Y + V_X^2 V_Y) + \frac{3}{2}(2\nu_1 + \nu_4)U_Y V_X V_Y \\
&\quad + (\mu + 2\nu_1 + \nu_4)U_X U_Y V_X + \frac{1}{2}(\lambda + 2\mu + 12\nu_1 + 4\nu_2 + 4\nu_4)V_Y^3 \\
&= 0, \quad \text{at } Y = \pm a. \tag{2.11}
\end{aligned}$$

We will study the bifurcations of the nonlinear PDE's (2.8) and (2.9) under (2.10) and (2.11) and some end conditions.

For the solution obtained if one of its deformation gradients has a finite jump across a curve in the rectangle, this solution should satisfy the jump conditions. These conditions require the force balance and the continuity of the deformation. Namely,

$$\Sigma^+ \mathbf{n} = \Sigma^- \mathbf{n}, \quad \mathbf{F}^+ \mathbf{l} = \mathbf{F}^- \mathbf{l}, \tag{2.12}$$

must hold on the curve of discontinuity. Σ^\pm and \mathbf{F}^\pm denote its limiting values at a point on the curve of discontinuity and \mathbf{n} is the unit normal and \mathbf{l} are all vectors tangent to the curve.

3. Transformed dimensionless equations. It is difficult to study the nonlinear PDE's (2.8) and (2.9) together with traction free boundary conditions (2.10) and (2.11). But we can use the combined series-asymptotic expansion method to deal with this complicated system. This approach adapted here is similar to those used in [14, 16, 17, 18]. First we introduce a new set of dimensionless quantities through the following suitable scalings:

$$U = hu, \quad V = hv, \quad X = \bar{x}l, \quad Y = \bar{y}l, \quad \varepsilon = \frac{h}{l}, \quad \nu = \frac{a^2}{l^2}, \tag{3.1}$$

where h is the characteristic axial displacement which can be regarded as the reduction of the distance between the ends, ε will be treated as a small parameter since here the deformation is considered to be small. We assume the rectangle is thin such that ν is small, say, $\nu < 0.07$; this implies that $a/l < 0.2646$, i.e., the aspect ratio is less than 0.5292. For simplicity of notation, we will drop the bar from \bar{x} and \bar{y} hereafter.

After the above proper scalings, the original field equations and traction free boundary conditions become:

$$\begin{aligned}
&\eta_0 u_{yy} + (1 - \eta_0)v_{xy} + u_{xx} + \varepsilon(\alpha_1 v_y u_{yy} + \alpha_1 u_y v_{yy} + \alpha_1 u_{yy} u_x + \alpha_2 v_{yy} v_x \\
&\quad + 2\alpha_1 u_y u_{xy} + 2\alpha_2 v_x u_{xy} + \alpha_3 v_y v_{xy} + \alpha_3 u_x v_{xy} + \alpha_4 v_y u_{xx} + \alpha_5 u_x u_{xx} \\
&\quad + \alpha_2 u_y v_{xx} + \alpha_1 v_x v_{xx}) + \varepsilon^2 H_3 = 0, \tag{3.2}
\end{aligned}$$

$$\begin{aligned}
&v_{yy} + (1 - \eta_0)u_{xy} + \eta_0 v_{xx} + \varepsilon(\alpha_1 u_y u_{yy} + \alpha_5 v_y v_{yy} + \alpha_4 v_{yy} u_x + \alpha_2 u_{yy} v_x \\
&\quad + \alpha_3 v_y u_{xy} + \alpha_3 u_x u_{xy} + 2\alpha_2 u_y v_{xy} + 2\alpha_1 v_x v_{xy} + \alpha_2 u_y u_{xx} + \alpha_1 v_x u_{xx} \\
&\quad + \alpha_1 v_y v_{xx} + \alpha_1 u_x v_{xx}) + \varepsilon^2 H_4 = 0, \tag{3.3}
\end{aligned}$$

$$\begin{aligned}
& \eta_0 u_y + \eta_0 v_x + \varepsilon(\alpha_1 u_y v_y + \alpha_1 u_y u_x + \alpha_2 v_y v_x + \alpha_2 u_x v_x) + \varepsilon^2 \left((\alpha_1 - \frac{1}{2}) u_y^3 \right. \\
& + \alpha_6 u_y v_y^2 + 2\alpha_7 u_y v_y u_x + \alpha_6 u_y u_x^2 + \frac{3}{2}(\alpha_1 - 1) u_y^2 v_x + \frac{3}{2}(\alpha_1 - 1) v_y^2 v_x + \alpha_8 v_y u_x v_x \\
& \left. + \frac{3}{2}(\alpha_1 - 1) u_x^2 v_x + \alpha_9 u_y v_x^2 + \frac{1}{2}(\alpha_1 - 1) v_x^3 \right) = 0, \quad \text{at } y = \pm\sqrt{\nu}, \quad (3.4)
\end{aligned}$$

$$\begin{aligned}
& v_y + (1 - 2\eta_0) u_x + \varepsilon \left(\frac{1}{2} \alpha_1 u_y^2 + \frac{1}{2} \alpha_5 v_y^2 + \alpha_4 v_y u_x + \frac{1}{2} \alpha_4 u_x^2 + \alpha_2 u_y v_x + \frac{1}{2} \alpha_1 v_x^2 \right) \\
& + \varepsilon^2 (\alpha_7 u_y^2 v_y + (\alpha_5 - \frac{5}{2}) v_y^3 + \alpha_7 u_y^2 u_x + 3\alpha_{10} v_y^2 u_x + \alpha_9 v_y u_x^2 + \alpha_{10} u_x^3 \\
& + 3(\alpha_2 - \eta_0) u_y v_y v_x + \alpha_8 u_y u_x v_x + \alpha_6 v_y v_x^2 + \alpha_7 u_x v_x^2) = 0, \quad \text{at } y = \pm\sqrt{\nu}. \quad (3.5)
\end{aligned}$$

where $\eta_0 = \frac{\mu}{\lambda+2\mu}$, $\eta_1 = \frac{\nu_1}{\lambda+2\mu}$, $\eta_2 = \frac{\nu_2}{\lambda+2\mu}$, $\eta_4 = \frac{\nu_4}{2(\lambda+2\mu)}$ and $\alpha_i (i = 1, \dots, 48)$ are constants related to constitutive constants which are given in the appendix.

It is also hard to analyze the equations (3.2)–(3.5) directly. In the next section by using series expansion and asymptotic reduction, we will obtain much simplified equations.

4. Series and asymptotic reduction. In section 3, we have obtained nondimensionalized field equations and boundary conditions. Since we consider the case that ν is small and $-\sqrt{\nu} \leq y \leq \sqrt{\nu}$, y is a small variable. We can see that the whole problem depends on two small parameters ε and ν , one small variable y and one variable x . Assume that $u(x, y)$ and $v(x, y)$ are sufficiently smooth and then they have the following Taylor expansions in the neighborhood of $y = 0$:

$$u(x, y) = u_0(x) + y^2 u_2(x) + y^4 u_4(x) + \dots + \delta y \cdot (u_1(x) + y^2 u_3(x) + \dots), \quad (4.1)$$

$$v(x, y) = \delta(v_0(x) + y^2 v_2(x) + y^4 v_4(x) + \dots) + y \cdot (v_1(x) + y^2 v_3(x) + \dots), \quad (4.2)$$

where δ is a parameter, which is a measure of deflection of the central axis. Since the purpose here is to deduce the critical loads for buckling when there is a buckling instability and compare the values with those obtained from the Bernoulli-Euler beam theory, we consider the case that δ is small such that $\delta = o(\sqrt{\nu})$.

Substituting (4.1) and (4.2) into the boundary conditions (3.4) and (3.5), we can get the following four equations:

$$\begin{aligned}
& \eta_0 u_1 + \eta_0 v_{0x} + \nu(3\eta_0 u_3 + \eta_0 v_{2x}) + \varepsilon(\alpha_1 u_1 v_1 + \alpha_1 u_1 u_{0x} + \alpha_2 v_1 v_{0x} + \alpha_2 u_{0x} v_{0x} \\
& + \nu(3\alpha_1 u_3 v_1 + 4\alpha_1 u_2 v_2 + 3\alpha_1 u_1 v_3 + 3\alpha_1 u_3 u_{0x} + 2\alpha_1 u_2 u_{1x} + \alpha_1 u_1 u_{2x} + 3\alpha_2 v_3 v_{0x} \\
& + \alpha_2 u_{2x} v_{0x} + \alpha_2 u_{2x} v_{0x} + 2\alpha_2 v_2 v_{1x} + \alpha_2 u_{1x} v_{1x} + \alpha_2 v_1 v_{2x} + \alpha_2 u_{0x} u_{2x})) \\
& + \varepsilon^2 (\alpha_6 u_1 v_1^2 + 2\alpha_7 u_1 v_1 u_{0x} + \alpha_6 u_1 u_{0x}^2 + \frac{3}{2}(\alpha_1 - 1) v_1^2 v_{0x} + \alpha_8 v_1 u_{0x} v_{0x} \\
& + \frac{3}{2}(\alpha_1 - 1) u_{0x}^2 v_{0x}) + O(\nu^2, \varepsilon^2 \nu, \delta^2) = 0, \quad (4.3)
\end{aligned}$$

$$\begin{aligned}
& 2\eta_0 u_2 + \eta_0 v_{1x} + \nu(4\eta_0 u_4 + \eta_0 v_{3x}) + \varepsilon(2\alpha_1 u_2 v_1 + 2\alpha_1 u_2 u_{0x} + \alpha_2 v_1 v_{1x} + \alpha_2 u_{0x} v_{1x} \\
& + \nu(4\alpha_1 u_4 v_1 + 6\alpha_1 u_2 v_3 + 4\alpha_1 u_4 u_{0x} + 2\alpha_1 u_2 u_{2x} + 3\alpha_2 v_3 v_{1x} + \alpha_2 u_{2x} v_{1x} + \alpha_2 v_1 v_{3x} \\
& + \alpha_2 u_{0x} v_{3x})) + \varepsilon^2 (2\alpha_6 u_2 v_1^2 + 4\alpha_7 u_2 v_1 u_{0x} + 2\alpha_6 u_2 u_{0x}^2 + \frac{3}{2}(\alpha_1 - 1) v_1^2 v_{1x} + \alpha_8 v_1 u_{0x} v_{1x} \\
& + \frac{3}{2}(\alpha_1 - 1) u_{0x}^2 v_{1x}) + O(\nu^2, \varepsilon^2 \nu, \delta^2) = 0, \quad (4.4)
\end{aligned}$$

$$\begin{aligned}
& v_1 + (1 - 2\eta_0)u_{0x} + \nu(3v_3 + (1 - 2\eta_0)u_{2x}) + \varepsilon\left(\frac{1}{2}\alpha_5v_1^2 + \alpha_4v_1u_{0x} + \frac{1}{2}\alpha_4u_{0x}^2\right. \\
& \left. + \nu(2\alpha_1u_2^2 + 3\alpha_5v_1v_3 + 3\alpha_4v_3u_{0x} + \alpha_4v_1u_{2x} + \alpha_4u_{0x}u_{2x} + 2\alpha_2u_2v_{1x} + \frac{1}{2}\alpha_1v_{1x}^2)\right) \\
& + \varepsilon^2\left(\alpha_{10}v_1^3 + \frac{3}{2}(\alpha_4 - 1 + 2\eta_0)v_1^2u_{0x} + (\alpha_4 - \frac{1}{2} + \eta_0)v_1u_{0x}^2 + \frac{1}{2}(\alpha_4 - 1 + 2\eta_0)u_{0x}^3\right) \\
& + O(\nu^2, \varepsilon^2\nu, \delta^2) = 0, \tag{4.5}
\end{aligned}$$

$$\begin{aligned}
& 2v_2 + (1 - 2\eta_0)u_{1x} + \nu(4v_4 + (1 - 2\eta_0)u_{3x}) + \varepsilon(2\alpha_1u_1u_2 + 2\alpha_5v_1v_2 \\
& + 2\alpha_4v_2u_{0x} + \alpha_4v_1u_{1x} + \alpha_4u_{0x}u_{1x} + 2\alpha_2u_2v_{0x} + \alpha_2u_1v_{1x} + \alpha_1v_{0x}v_{1x} \\
& + \nu(6\alpha_1u_2u_3 + 4\alpha_1u_1u_4 + 6\alpha_5v_2v_3 + 4\alpha_5v_1v_4 + 4\alpha_4v_4u_{0x} + 3\alpha_4v_3u_{1x} \\
& + 2\alpha_4v_2u_{2x} + \alpha_4u_{1x}u_{2x} + \alpha_4v_1u_{3x} + \alpha_4u_{0x}u_{3x} + 4\alpha_2u_4v_{0x} + 3\alpha_2u_3v_{1x} \\
& + 2\alpha_2u_2v_{2x} + \alpha_1v_{1x}v_{2x} + \alpha_2u_1v_{3x} + \alpha_1v_{0x}v_{0x})) \\
& + \varepsilon^2(4\alpha_6u_1u_2v_1 + 6\alpha_{10}v_1^2v_2 + 4\alpha_7u_1u_2u_{0x} + 6(\alpha_4 - 1 + 2\eta_0)v_1v_2u_{0x} \\
& + (-1 + 2\eta_0 + 2\alpha_4)v_2u_{0x}^2 + \frac{3}{2}(\alpha_1 - 1 + 2\eta_0)v_1^2u_{1x} + (-1 + 2\eta_0 + 2\alpha_4)v_1u_{0x}u_{1x} \\
& + \frac{3}{2}(\alpha_1 - 1 + 2\eta_0)u_{0x}^2u_{1x} + 6(\alpha_1 - 1)u_2v_1v_{0x} + 2\alpha_8u_2u_{0x}v_{0x} \\
& + 3(\alpha_1 - 1)u_1v_1v_{1x} + \alpha_8u_1u_{0x}v_{1x} + 2\alpha_6v_1v_{0x}v_{1x} + 2\alpha_7u_{0x}v_{0x}) \\
& + O(\nu^2, \varepsilon^2\nu, \delta^2) = 0. \tag{4.6}
\end{aligned}$$

In these equations, if $O(\nu^2, \varepsilon^2\nu, \delta^2)$ terms are omitted, there are ten unknowns $u_0, \dots, u_4, v_0, \dots, v_4$. To have a closed system, we need another six equations.

Substituting (4.1) and (4.2) into the field equation (3.2), the left hand side becomes a series in y . All the coefficients of y^n ($n = 0, 1, 2, 3, \dots$) should be zero and as a result we have a set of infinitely-many equations. It turns out that the first three equations ($n = 0, 1, 2$) contain only the above-mentioned 10 unknowns by neglecting proper higher-order terms. These equation are

$$\begin{aligned}
& 2\eta_0u_2 + (1 - \eta_0)v_{1x} + u_{0xx} + \varepsilon(2\alpha_1u_2v_1 + 2\alpha_1u_2u_{0x} + \alpha_3v_1v_{1x} + \alpha_3u_{0x}v_{1x} \\
& + \alpha_4v_1u_{0xx} + \alpha_5u_{0x}u_{0xx}) + \varepsilon^2H_5 + O(\delta^2) = 0, \tag{4.7}
\end{aligned}$$

$$\begin{aligned}
& 6\eta_0u_3 + 2(1 - \eta_0)v_{2x} + u_{1xx} + \varepsilon(6\alpha_1u_3v_1 + 8\alpha_1u_2v_2 + 7\alpha_1u_1v_3 + 6\alpha_1u_3u_{0x} \\
& + 6\alpha_1u_2u_{1x} + 4\alpha_1u_1u_{2x} + 6\alpha_2v_3v_{0x} + 4\alpha_2u_2v_{0x} + 2(\alpha_2 + \alpha_3)v_2v_{1x} \\
& + (2\alpha_2 + \alpha_3)u_{1x}v_{1x} + 2\alpha_3v_1v_{2x} + 2\alpha_3u_{0x}v_{2x} + 2\alpha_4v_2u_{0xx} + \alpha_5u_{1x}u_{0xx} \\
& + \alpha_4v_1u_{1xx} + \alpha_5u_{0x}u_{1xx} + 2\alpha_2u_2v_{0xx} + \alpha_1v_{1x}v_{0xx} + \alpha_2u_1v_{1xx} \\
& + \alpha_2v_{0x}v_{1xx}) + \varepsilon^2H_6 + O(\delta^2) = 0, \tag{4.8}
\end{aligned}$$

$$\begin{aligned}
& 12\eta_0u_4 + 3(1 - \eta_0)v_{3x} + u_{2xx} + \varepsilon(12\alpha_1u_4v_1 + 18\alpha_1u_2v_3 + 12\alpha_1u_4u_{0x} \\
& + 10\alpha_1u_2u_{2x} + 3(2\alpha_2 + \alpha_3)v_3v_{1x} + (4\alpha_2 + \alpha_3)u_{2x}v_{1x} + 3\alpha_3v_1v_{3x} \\
& + 3\alpha_3u_{0x}v_{3x} + 3\alpha_4v_3u_{0xx} + \alpha_5u_{2x}u_{0xx} + \alpha_4v_1u_{2xx} + \alpha_5u_{0x}u_{2xx} \\
& + 2\alpha_2u_2v_{1xx} + \alpha_1v_{1x}v_{1xx}) + \varepsilon^2H_7 + O(\delta^2) = 0. \tag{4.9}
\end{aligned}$$

Similarly, substituting (4.1) and (4.2) into another field equation (3.3) and letting the

coefficients of y^0 , y^1 and y^2 be zero, we have

$$\begin{aligned} & 2v_2 + (1 - \eta_0)u_{1x} + \eta_0v_{0xx} + \varepsilon(2\alpha_1u_1u_2 + 2\alpha_5v_1v_2 + 2\alpha_4v_2u_{0x} + \alpha_3v_1u_{1x} \\ & + \alpha_3u_{0x}u_{1x} + 2\alpha_2u_2v_{0x} + 2\alpha_2u_1v_{1x} + 2\alpha_1v_{0x}v_{1x} + \alpha_2u_1u_{0xx} + \alpha_1v_{0x}u_{0xx} \\ & + \alpha_1v_1v_{0xx} + \alpha_1u_{0x}v_{0xx}) + \varepsilon^2H_8 + O(\delta^2) = 0, \end{aligned} \quad (4.10)$$

$$\begin{aligned} & 6v_3 + 2(1 - \eta_0)u_{2x} + \eta_0v_{1xx} + \varepsilon(4\alpha_1u_2^2 + 6\alpha_5v_1v_3 + 6\alpha_4v_3u_{0x} + 2\alpha_3v_1u_{2x} \\ & + 2\alpha_3u_{0x}u_{2x} + 6\alpha_2u_2v_{1x} + 2\alpha_1v_{1x}^2 + 2\alpha_2u_2u_{0xx} + \alpha_1v_{1x}u_{0xx}) \\ & + \varepsilon^2H_9 + O(\delta^2) = 0, \end{aligned} \quad (4.11)$$

$$\begin{aligned} & 12v_4 + 3(1 - \eta_0)u_{3x} + \eta_0v_{2xx} + \varepsilon(18\alpha_1u_2u_3 + 12\alpha_1u_1u_4 + 18\alpha_5v_2v_3 + 12\alpha_5v_1v_4 \\ & + 12\alpha_4v_4u_{0x} + 3(\alpha_3 + 2\alpha_4)v_3u_{1x} + 2(2\alpha_3 + \alpha_4)v_2u_{2x} + 3\alpha_3u_{1x}u_{2x} + 3\alpha_3v_1u_{3x} \\ & + 3\alpha_3u_{0x}u_{3x} + 12\alpha_2u_4v_{0x} + 12\alpha_2u_3v_{1x} + 10\alpha_2u_2v_{2x} + 6\alpha_1v_{1x}v_{2x} + 6\alpha_2u_1v_{3x} \\ & + 6\alpha_1v_{0x}v_{3x} + 3\alpha_2u_3u_{0xx} + \alpha_1v_{2x}u_{0xx} + 2\alpha_2u_2u_{1xx} + \alpha_1v_{1x}u_{1xx} + \alpha_2u_1u_{2xx} \\ & + \alpha_1v_{0x}u_{2xx} + \alpha_1v_3v_{0xx} + \alpha_1u_{2x}v_{0xx} + 2\alpha_1v_2v_{1xx} + \alpha_1u_{1x}v_{1xx} + \alpha_1u_1v_{2xx} \\ & + \alpha_1v_{0x}v_{2x}) + \varepsilon^2H_{10} + O(\delta^2) = 0. \end{aligned} \quad (4.12)$$

Now the field equations (3.2)–(3.3) and boundary conditions (3.4)–(3.5) are changed into a one-dimensional system of differential equations (4.3)–(4.12) for ten unknowns $u_0, \dots, u_4, v_0, \dots, v_4$ if we neglect $O(\delta^2)$ in (4.7)–(4.12) and $O(\nu^2, \varepsilon^2\nu, \delta^2)$ in (4.3)–(4.6). By using perturbation method, we can solve u_2 from (4.7). Then substituting it into (4.11), we can get v_3 . Further we substitute u_2 and v_3 into (4.9) and we can get u_4 . The results are given below.

$$\begin{aligned} u_2 &= \frac{\eta_0 - 1}{2\eta_0}v_{1x} - \frac{1}{2\eta_0}u_{0xx} + \varepsilon(\alpha_{11}v_1v_{1x} + \alpha_{12}u_{0x}v_{1x} + \alpha_{13}u_{0x}u_{0xx} + \alpha_{14}v_1u_{0xx}) \\ &+ O(\varepsilon^2), \end{aligned} \quad (4.13)$$

$$\begin{aligned} v_3 &= \left(-\frac{1}{3} + \frac{1}{6\eta_0}\right)v_{1xx} + \left(-\frac{1}{6} + \frac{1}{6\eta_0}\right)u_{0xxx} + \varepsilon(\alpha_{15}v_{1x}^2 + \alpha_{16}v_{1x}u_{0xx} + \alpha_{17}u_{0xx}^2 \\ &+ \alpha_{18}v_1v_{1xx} + \alpha_{19}u_{0x}v_{1xx} + \alpha_{20}v_1u_{0xxx} + \alpha_{21}u_{0x}u_{0xxx}) + O(\varepsilon^2), \end{aligned} \quad (4.14)$$

$$\begin{aligned} u_4 &= \left(-\frac{1}{12} + \frac{1}{12\eta_0}\right)v_{1xxx} + \left(-\frac{1}{24} + \frac{1}{12\eta_0}\right)u_{0xxxx} + \varepsilon(\alpha_{22}v_{1x}v_{1xx} + \alpha_{23}u_{0xx}v_{1xx} \\ &+ \alpha_{24}v_{1x}u_{0xxx} + \alpha_{25}u_{0xx}u_{0xxx} + \alpha_{26}v_1v_{1xxx} + \alpha_{27}u_{0x}v_{1xxx} + \alpha_{28}v_1u_{0xxxx} \\ &+ \alpha_{29}u_{0x}u_{0xxxx}) + O(\varepsilon^2). \end{aligned} \quad (4.15)$$

In these equations $O(\varepsilon^2)$ terms are not needed for the final results and their expressions are not written out.

Similarly, we can obtain v_2 , u_3 and v_4 from (4.10), (4.8) and (4.12) respectively. The results are given below.

$$\begin{aligned} v_2 &= \left(-\frac{1}{2} + \frac{\eta_0}{2}\right)u_{1x} - \frac{1}{2}\eta_0v_{0xx} + \varepsilon(-\alpha_1u_1u_2 + \alpha_{30}v_1u_{1x} + \alpha_{31}u_{0x}u_{1x} \\ &- \alpha_2u_2v_{0x} - \alpha_2u_1v_{1x} - \alpha_1v_{0x}v_{1x} - \frac{1}{2}\alpha_2u_1u_{0xx} - \frac{1}{2}\alpha_1v_{0x}u_{0xx} \\ &+ \alpha_{32}v_1v_{0xx} + \alpha_{33}u_{0x}v_{0xx}) + \varepsilon^2H_{11} + O(\varepsilon^3), \end{aligned} \quad (4.16)$$

$$u_3 = \left(-\frac{1}{3} + \frac{\eta_0}{6}\right)u_{1xx} + \left(\frac{1}{6} - \frac{\eta_0}{6}\right)v_{0xxx} + O(\varepsilon), \quad (4.17)$$

$$v_4 = \left(\frac{1}{12} - \frac{\eta_0}{12}\right)u_{1xxx} + \left(-\frac{1}{24} + \frac{\eta_0}{12}\right)v_{0xxxx} + O(\varepsilon). \quad (4.18)$$

In (4.17) and (4.18), $O(\varepsilon)$ terms are not written out as they are not needed in the final results. Now we substitute (4.13), (4.14) and (4.15) into the boundary conditions (4.4) and (4.5) to obtain two equations in terms of u_0 and v_1 if we further omit $O(\varepsilon^2)$. And, we substitute (4.13), (4.14), (4.15), (4.16), (4.17) and (4.18) into the boundary conditions (4.3) and (4.6) to obtain two equations in terms of u_0, u_1, v_0 and v_1 if we further omit $O(\varepsilon\nu)$. Then we can get four equations with four variables u_0, u_1, v_0 and v_1 , which are given below.

$$\begin{aligned} &(-1 + 2\eta_0)v_{1x} - u_{0xx} + \nu\left(\left(\frac{1}{2} - \frac{2\eta_0}{3}\right)v_{1xxx} + \left(\frac{1}{2} - \frac{\eta_0}{3}\right)u_{0xxxx}\right) + \varepsilon\left(-\alpha_4v_1v_{1x}\right. \\ &\left.-\alpha_4u_{0x}v_{1x} - \alpha_4v_1u_{0xx} - \alpha_5u_{0x}u_{0xx} + \nu(\alpha_{34}v_{1x}v_{1xx} + \alpha_{35}u_{0xx}v_{1xx}\right. \\ &\left.+\alpha_{36}v_{1x}u_{0xxx} + \alpha_{37}u_{0xx}u_{0xxx} + \alpha_{38}v_1v_{1xxx} + \alpha_{39}u_{0x}v_{1xxx} + \alpha_{40}v_1u_{0xxxx}\right. \\ &\left.+\alpha_{41}u_{0x}u_{0xxxx})\right) = 0, \end{aligned} \quad (4.19)$$

$$\begin{aligned} &v_1 + (1 - 2\eta_0)u_{0x} + \nu\left(\left(\frac{1}{2} - \eta_0\right)v_{1xx} + \frac{1}{2}u_{0xxx}\right) + \varepsilon\left(\frac{1}{2}\alpha_5v_1^2 + \alpha_4v_1u_{0x}\right. \\ &\left.+\frac{1}{2}\alpha_4u_{0x}^2 + \nu(\alpha_{42}v_{1x}^2 + 2\alpha_{42}v_{1x}u_{0xx} + \alpha_{43}u_{0xx}^2 + \alpha_{42}v_1v_{1xx} + \alpha_{42}u_{0x}v_{1xx}\right. \\ &\left.+\alpha_{42}v_1u_{0xxx} + \alpha_{43}u_{0x}u_{0xxx})\right) = 0, \end{aligned} \quad (4.20)$$

$$\begin{aligned} &\eta_0u_1 + \eta_0v_{0x} + \nu\left(\left(-\frac{3\eta_0}{2} + \eta_0^2\right)u_{1xx} + \left(\frac{\eta_0}{2} - \eta_0^2\right)v_{0xxx}\right) + \varepsilon\left(\alpha_1u_1v_1 + \alpha_1u_1u_{0x}\right. \\ &\left.+\alpha_2v_1v_{0x} + \alpha_2u_{0x}v_{0x}\right) + \varepsilon^2\left(\alpha_6u_1v_1^2 + 2\alpha_7u_1v_1u_{0x} + \alpha_6u_1u_{0x}^2 + \frac{3}{2}(\alpha_1 - 1)v_1^2v_{0x}\right. \\ &\left.+\alpha_8v_1u_{0x}v_{0x} + \frac{3}{2}(\alpha_1 - 1)u_{0x}^2v_{0x}\right) = 0, \end{aligned} \quad (4.21)$$

$$\begin{aligned} &-\eta_0u_{1x} - \eta_0v_{0xx} + \nu\left(\left(\frac{\eta_0}{2} - \frac{\eta_0^2}{3}\right)u_{1xxx} + \left(-\frac{\eta_0}{6} + \frac{\eta_0^2}{3}\right)v_{0xxxx}\right) + \varepsilon\left(-\alpha_2v_1u_{1x}\right. \\ &\left.-\alpha_2u_{0x}u_{1x} - \alpha_2u_1v_{1x} - \alpha_1v_{0x}v_{1x} - \alpha_2u_1u_{0xx} - \alpha_1v_{0x}u_{0xx} - \alpha_1v_1v_{0xx}\right. \\ &\left.-\alpha_1u_{0x}v_{0xx}\right) + \varepsilon^2\left(-\frac{3}{2}(\alpha_1 - 1)v_1^2u_{1x} - \alpha_8v_1u_{0x}u_{1x} - \frac{3}{2}(\alpha_1 - 1)u_{0x}^2u_{1x}\right. \\ &\left.-3(\alpha_1 - 1)u_1v_1v_{1x} - \alpha_8u_1u_{0x}v_{1x} - 2\alpha_6v_1v_{0x}v_{1x} - 2\alpha_7u_{0x}v_{0x}v_{1x} - \alpha_8u_1v_1u_{0xx}\right. \\ &\left.-3(\alpha_1 - 1)u_1u_{0x}u_{0xx} - 2\alpha_7v_1v_{0x}u_{0xx} - 2\alpha_6u_{0x}v_{0x}u_{0xx} - \alpha_6v_1^2v_{0xx}\right. \\ &\left.-2\alpha_7v_1u_{0x}v_{0xx} - \alpha_6u_{0x}^2v_{0xx}\right) = 0. \end{aligned} \quad (4.22)$$

We find that (4.19) and (4.20) are two equations with two variables v_1 and u_0 , (4.21) and (4.22) are two equations with four variables v_1, u_0, u_1 and v_0 . (Note: in these two equations (4.21) and (4.22) we keep ε^2 terms in order to consider the nonlinear effect of the axial strain on the possible deflection.) We also notice that both (4.19)

and (4.22) can be integrated once. Then we obtain the following two equations:

$$\begin{aligned} & (1 - 2\eta_0)v_1 + u_{0x} + \nu\left(-\frac{1}{2} + \frac{2\eta_0}{3}\right)v_{1xx} + \left(-\frac{1}{2} + \frac{\eta_0}{3}\right)u_{0xxx} + \varepsilon\left(\frac{1}{2}\alpha_4v_1^2 + \alpha_4v_1u_{0x}\right. \\ & + \frac{1}{2}\alpha_5u_{0x}^2 + \nu\left((\alpha_{39} - \alpha_{35})u_{0xx}v_{1x} - \alpha_{39}u_{0x}v_{1xx} + \alpha_{40}v_1u_{0xxx} - \frac{1}{2}(\alpha_{34} - \alpha_{38})v_{1x}^2\right. \\ & \left. \left. - \alpha_{38}v_1v_{1xx} - \frac{1}{2}(\alpha_{37} - \alpha_{41})u_{0xx}^2 - \alpha_{41}u_{0x}u_{0xxx}\right)\right) = A, \end{aligned} \quad (4.23)$$

$$\begin{aligned} & \eta_0u_1 + \eta_0v_{0x} + \nu\left(-\frac{\eta_0}{2} + \frac{\eta_0^2}{3}\right)u_{1xx} + \left(\frac{\eta_0}{6} - \frac{\eta_0^2}{3}\right)v_{0xxx} + \varepsilon\left(\alpha_2u_1v_1 + \alpha_2u_1u_{0x}\right. \\ & + \alpha_1v_1u_{0x} + \alpha_1u_{0x}v_{0x}\left.) + \varepsilon^2\left(\frac{3}{2}(\alpha_1 - 1)u_1v_1^2 + \alpha_8u_1v_1u_{0x} + \frac{3}{2}(\alpha_1 - 1)u_1u_{0x}^2\right.\right. \\ & \left. \left. + \alpha_6v_1^2v_{0x} + 2\alpha_7v_1u_{0x}v_{0x} + \alpha_6u_{0x}^2v_{0x}\right) = B, \end{aligned} \quad (4.24)$$

where A and B are two integration constants.

By deleting v_{1xx} terms, from (4.20) and (4.23) and using a perturbation expansion, we obtain

$$v_1 = (-1 + 2\eta_0)u_{0x} + \alpha_{44}\nu u_{0xxx} + \varepsilon(\alpha_{45}u_{0x}^2 + \nu(\alpha_{46}u_{0xx}^2 + \alpha_{47}u_{0x}u_{0xxx})), \quad (4.25)$$

and

$$u_{0x} - \frac{1}{3}\nu u_{0xxx} + \varepsilon(D_1u_{0x}^2 + \nu(-D_2u_{0xx}^2 - 2D_2u_{0x}u_{0xxx})) = \frac{A}{4(1 - \eta_0)\eta_0}, \quad (4.26)$$

where D_1 and D_2 are constants related to constitutive constants and their expressions are given in the appendix.

Similarly, from (4.21) and (4.24), we obtain

$$u_1 = -v_{0x} + 2\nu(\eta_0 - 1)v_{0xxx} + \varepsilon(2(1 - \eta_0)u_{0x}v_{0x}) + \varepsilon^2(\alpha_{48}u_{0x}^2v_{0x}), \quad (4.27)$$

and

$$-\frac{1}{3}\nu v_{0xxx} + \varepsilon u_{0x}v_{0x} + \varepsilon^2(D_1 - 1)u_{0x}^2v_{0x} = \frac{B}{4(1 - \eta_0)\eta_0}. \quad (4.28)$$

In order to find the physical meanings of the two integration constants A and B , we consider the axial resultant force T and the shear resultant force Q . The axial resultant force T is given by

$$T = \int_{-a}^a \Sigma_{xX} dY. \quad (4.29)$$

After expressing Σ_{xX} in terms of u_0 and v_1 and carrying out the integration, we find that

$$\begin{aligned} \frac{T}{2a\varepsilon(\lambda + 2\mu)} &= (1 - 2\eta_0)v_1 + u_{0x} + \nu\left(-\frac{1}{2} + \frac{2\eta_0}{3}\right)v_{1xx} + \left(-\frac{1}{2} + \frac{\eta_0}{3}\right)u_{0xxx} \\ &+ \varepsilon\left(\frac{1}{2}\alpha_4v_1^2 + \alpha_4v_1u_{0x} + \frac{1}{2}\alpha_5u_{0x}^2 + \nu\left((\alpha_{39} - \alpha_{35})u_{0xx}v_{1x} - \alpha_{39}u_{0x}v_{1xx}\right.\right. \\ &+ \alpha_{40}v_1u_{0xxx} - \frac{1}{2}(\alpha_{34} - \alpha_{38})v_{1x}^2 - \alpha_{38}v_1v_{1xx} - \frac{1}{2}(\alpha_{37} - \alpha_{41})u_{0xx}^2 \\ & \left. \left. - \alpha_{41}u_{0x}u_{0xxx}\right)\right). \end{aligned} \quad (4.30)$$

Comparing (4.30) with (4.23), we see that $\frac{T}{2a\varepsilon(\lambda+2\mu)} = A$, i.e., $\frac{T}{2a\varepsilon\hat{E}} = \frac{A}{4(1-\eta_0)\eta_0}$, where $\hat{E} = \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}$ is an elastic modulus. Now we let $\gamma = \frac{T}{2a\hat{E}}$ and hence γ is the nondimensionalized averaged axial resultant force. Then from (4.26), we have

$$\varepsilon u_{0x} - \frac{1}{3}\nu\varepsilon u_{0xxx} + \varepsilon^2\left(D_1 u_{0x}^2 + \nu(-D_2 u_{0xx}^2 - 2D_2 u_{0x} u_{0xxx})\right) = \gamma. \quad (4.31)$$

If we use the original dimensional variable by letting $W(X) = \varepsilon u_{0x}$, we obtain

$$W + D_1 W^2 + a^2\left(-\frac{1}{3}W_{XX} - D_2(W_X^2 + 2WW_{XX})\right) = \gamma. \quad (4.32)$$

Next we consider the shear resultant force Q , which is given by

$$Q = \int_{-a}^a \Sigma_{yX} dY. \quad (4.33)$$

After expressing Σ_{yX} in terms of v_0 and u_1 and carrying out the integration, we find that

$$\begin{aligned} \frac{Q}{2a\varepsilon\delta(\lambda+2\mu)} &= \eta_0 u_1 + \eta_0 v_{0x} + \nu\left(\left(-\frac{\eta_0}{2} + \frac{\eta_0^2}{3}\right)u_{1xx} + \left(\frac{\eta_0}{6} - \frac{\eta_0^2}{3}\right)v_{0xxx}\right) \\ &\quad + \varepsilon\left(\alpha_2 u_1 v_1 + \alpha_2 u_1 u_{0x} + \alpha_1 v_1 u_{0x} + \alpha_1 u_{0x} v_{0x}\right) \\ &\quad + \varepsilon^2\left(\frac{3}{2}(\alpha_1 - 1)u_1 v_1^2 + \alpha_8 u_1 v_1 u_{0x} + \frac{3}{2}(\alpha_1 - 1)u_1 u_{0x}^2\right) \\ &\quad + \alpha_6 v_1^2 v_{0x} + 2\alpha_7 v_1 u_{0x} v_{0x} + \alpha_6 u_{0x}^2 v_{0x}. \end{aligned} \quad (4.34)$$

Comparing (4.34) with (4.24), we see that $\frac{Q}{2a\varepsilon\delta(\lambda+2\mu)} = B$, i.e., $\frac{Q}{2a\varepsilon\delta\hat{E}} = \frac{B}{4(1-\eta_0)\eta_0}$. Now we let $q = \frac{Q}{2a\hat{E}}$ and hence q is the nondimensionalized averaged shear resultant force. Then from (4.28), we have

$$-\frac{1}{3}\varepsilon\delta\nu v_{0xxx} + \varepsilon^2\delta u_{0x} v_{0x} + \varepsilon^3\delta(D_1 - 1)u_{0x}^2 v_{0x} = q. \quad (4.35)$$

Again, if we use the original dimensional variable by letting $W(X) = \varepsilon u_{0x}$ and $G(X) = \varepsilon\delta v_{0x}$, we have

$$(W + (D_1 - 1)W^2)G - \frac{a^2}{3}G_{XX} = q. \quad (4.36)$$

Now we finally obtain two decoupled nonlinear ordinary differential equations (4.32) and (4.36). In the next section, we will derive the same two equations by using the variational principle.

5. Euler-Lagrange Equations. In this section, we will derive exact two same decoupled equations as (4.32) and (4.36) by using the variational principle. From (2.3) we can get the expression of the strain energy Φ up to the fourth-order nonlinearity which implies that the stress components are up to the third-order nonlinearity and

that will conform with our previous derivations:

$$\begin{aligned}
\Phi = & \left(\frac{\lambda}{2} + \mu\right)(U_X^2 + V_Y^2) + \mu\left(\frac{1}{2}U_Y^2 + U_Y V_X + \frac{1}{2}V_X^2\right) + \lambda U_X V_Y + \\
& + \left(\frac{\lambda}{2} + \nu_1 + \nu_2\right)(U_X^2 V_Y + U_X V_Y^2) + \left(\frac{\lambda}{2} + \mu + \frac{\nu_1}{2} + \frac{\nu_4}{4}\right)(U_X U_Y^2 + U_X V_X^2 + U_Y^2 V_Y + V_X^2 V_Y) \\
& + \left(\frac{\lambda}{2} + \mu + \nu_1 + \frac{\nu_2}{3} + \frac{\nu_4}{3}\right)(U_X^3 + V_Y^3) + \left(\mu + \nu_1 + \frac{\nu_4}{2}\right)(U_X U_Y V_X + U_Y V_X V_Y) \\
& + (\nu_1 + \nu_2)(U_X^3 V_Y + U_X V_Y^3) + \left(\frac{\lambda}{4} + \nu_1 + \nu_2\right)U_X^2 V_Y^2 + \left(\frac{\lambda}{8} + \frac{\mu}{4} + \frac{\nu_1}{4} + \frac{\nu_4}{8}\right)(U_Y^4 + V_X^4) \\
& + \left(\frac{\nu_1}{2} + \frac{\nu_4}{4}\right)(U_Y^3 V_X + U_Y V_X^3) + \left(\frac{\lambda}{4} + \frac{\nu_1}{2} + \frac{\nu_4}{4}\right)U_Y^2 V_X^2 + \left(\frac{\lambda}{8} + \frac{\mu}{4} + \frac{3\nu_1}{2} + \frac{\nu_2}{2} + \frac{\nu_4}{2}\right)(U_X^4 + V_Y^4) \\
& + (2\nu_1 + \nu_2 + \frac{\nu_4}{2})(U_X U_Y^2 V_Y + U_X V_X^2 V_Y) + (\mu + 2\nu_1 + \nu_4)U_X U_Y V_X V_Y \\
& + \left(\frac{\lambda}{4} + \frac{\mu}{2} + \frac{7\nu_1}{4} + \frac{\nu_2}{2} + \frac{5\nu_4}{8}\right)(U_X^2 U_Y^2 + U_X^2 V_X^2 + U_Y^2 V_Y^2 + V_X^2 V_Y^2) \\
& + \left(\frac{3\nu_1}{2} + \frac{3\nu_4}{4}\right)(U_X^2 U_Y V_X + U_Y V_X V_Y^2). \tag{5.1}
\end{aligned}$$

The averaged total strain energy per unit length over cross-section is given by

$$\bar{\Phi} = \frac{1}{2a} \int_{-a}^a \Phi dY. \tag{5.2}$$

By using the results in the previous sections, we can express Ψ in terms of u_0, u_1, v_0 and v_1 , which is given by

$$\begin{aligned}
\bar{\Phi} = & \hat{E}\varepsilon^2 \left(\frac{1}{8\eta_0 - 8\eta_0^2} v_1^2 + \frac{1 - 2\eta_0}{4\eta_0 - 4\eta_0^2} v_1 u_{0x} + \frac{1}{8\eta_0 - 8\eta_0^2} u_{0x}^2 + \nu \left(-\frac{(1 - 2\eta_0)^2}{24(-1 + \eta_0)\eta_0^2} v_{1x}^2 \right. \right. \\
& + \frac{-1 + 2\eta_0}{12(-1 + \eta_0)\eta_0^2} v_{1x} u_{0xx} + \frac{1}{4(6 - 6\eta_0)\eta_0^2} u_{0xx}^2 + \frac{1 - 2\eta_0}{24\eta_0 - 24\eta_0^2} v_1 v_{1xx} \\
& + \frac{3 - 4\eta_0}{24(-1 + \eta_0)\eta_0^2} u_{0x} v_{1xx} + \frac{1}{24\eta_0 - 24\eta_0^2} v_1 u_{0xxx} + \frac{3 - 2\eta_0}{24(-1 + \eta_0)\eta_0^2} u_{0x} u_{0xxx} \left. \right) \\
& + \varepsilon \left(\frac{\alpha_5}{24\eta_0 - 24\eta_0^2} v_1^3 + \frac{\alpha_4}{8\eta_0 - 8\eta_0^2} v_1^2 u_{0x} + \frac{\alpha_3}{8\eta_0 - 8\eta_0^2} v_1 u_{0x}^2 + \frac{\alpha_5}{24\eta_0 - 24\eta_0^2} u_{0x}^3 \right. \\
& + \nu (\theta_1 v_1 v_{1x}^2 + \theta_2 u_{0x} v_{1x}^2 + \theta_3 v_1 v_{1x} u_{0xx} + \theta_4 u_{0x} v_{1x} u_{0xx} + \theta_5 v_1 u_{0xx}^2 + \theta_6 u_{0x} u_{0xx}^2 \\
& + \theta_7 v_1^2 v_{1xx} + \theta_8 v_1 u_{0x} v_{1xx} + \theta_9 u_{0x}^2 v_{1xx} + \theta_{10} v_1^2 u_{0xxx} + \theta_{11} v_1 u_{0x} u_{0xxx} + \theta_{12} u_{0x}^2 u_{0xxx} \left. \right) \\
& + \delta^2 \left[\frac{1}{8 - 8\eta_0} u_1^2 + \frac{1}{4 - 4\eta_0} u_1 v_{0x} + \frac{1}{8 - 8\eta_0} v_{0x}^2 + \nu \left(\frac{4 - 3\eta_0}{24 - 24\eta_0} u_{1x}^2 + \frac{3 - 2\eta_0}{-24 + 24\eta_0} u_1 u_{1xx} \right. \right. \\
& + \frac{3 - 2\eta_0}{-24 + 24\eta_0} v_{0x} u_{1xx} + \frac{\eta_0^2}{2(6\eta_0 - 6\eta_0^2)} u_{1x} v_{0xx} + \frac{\eta_0^2}{24\eta_0 - 24\eta_0^2} v_{0xx}^2 + \frac{1 - 2\eta_0}{24 - 24\eta_0} u_1 v_{0xxx} \\
& + \frac{1 - 2\eta_0}{24 - 24\eta_0} v_{0x} v_{0xxx} \left. \right) + \varepsilon \left(\frac{\alpha_1}{8\eta_0 - 8\eta_0^2} u_1^2 v_1 + \frac{\alpha_1}{8\eta_0 - 8\eta_0^2} u_1^2 u_{0x} + \frac{\alpha_2}{4\eta_0 - 4\eta_0^2} u_1 v_1 v_{0x} \right. \\
& + \frac{\alpha_2}{4\eta_0 - 4\eta_0^2} u_1 u_{0x} v_{0x} + \frac{\alpha_1}{8\eta_0 - 8\eta_0^2} v_1 v_{0x}^2 + \frac{\alpha_1}{8\eta_0 - 8\eta_0^2} u_{0x} v_{0x}^2 \left. \right) + \varepsilon^2 \left(\frac{2\alpha_6}{16\eta_0 - 16\eta_0^2} u_1^2 v_1^2 \right. \\
& + \frac{\alpha_7}{4\eta_0 - 4\eta_0^2} u_1^2 v_1 u_{0x} + \frac{2\alpha_6}{16\eta_0 - 16\eta_0^2} u_1^2 u_{0x}^2 - \frac{3(\alpha_1 - 1)}{8(-1 + \eta_0)\eta_0} u_1 v_1^2 v_{0x} \\
& + \frac{\alpha_8}{4\eta_0 - 4\eta_0^2} u_1 v_1 u_{0x} v_{0x} - \frac{3(\alpha_1 - 1)}{8(-1 + \eta_0)\eta_0} u_1 u_{0x}^2 v_{0x} + \frac{2\alpha_6}{16\eta_0 - 16\eta_0^2} v_1^2 v_{0x}^2 \left. \right)
\end{aligned}$$

$$+ \frac{\alpha_7}{4\eta_0 - 4\eta_0^2} v_1 u_{0x} v_{0x}^2 + \frac{2\alpha_6}{16\eta_0 - 16\eta_0^2} u_{0x}^2 v_{0x}^2 \Big] \Big), \quad (5.3)$$

where $\theta_i (i = 1, \dots, 12)$ are constants related to constitutive constants which are given in the appendix.

By further using (4.25) and (4.27) and omitting higher-order terms, we can reduce the above equation to

$$\begin{aligned} \bar{\Phi} &= \varepsilon^2 \hat{E} \left(\frac{1}{2} u_{0x}^2 - \frac{1}{6} \nu u_{0x} u_{0xxx} + \varepsilon \left(\frac{1}{3} D_1 u_{0x}^3 + \nu (F_1 u_{0x} u_{0xx}^2 + F_2 u_{0x}^2 u_{0xxx}) \right) \right. \\ &\quad \left. + \delta^2 \left[\frac{1}{6} \nu v_{0xx}^2 + \frac{1}{2} \varepsilon u_{0x} v_{0x}^2 + \frac{1}{2} (D_1 - 1) \varepsilon^2 u_{0x}^2 v_{0x}^2 \right] \right) \\ &= \hat{E} \left(\frac{1}{2} W^2 - \frac{1}{6} a^2 W W_{XX} + \frac{1}{3} D_1 W^3 + a^2 (F_1 W W_X^2 + F_2 W^2 W_{XX}) \right. \\ &\quad \left. + \frac{1}{6} a^2 G_X^2 + \frac{1}{2} W G^2 + \frac{1}{2} (D_1 - 1) W^2 G^2 \right), \end{aligned} \quad (5.4)$$

where

$$F_1 = \frac{8 - 19\eta_0 + 10\eta_0^2 + 6(-2 + 3\eta_0)D_2 + 2(-6 + 7\eta_0)D_1}{12 - 24\eta_0}, \quad (5.5)$$

$$F_2 = \frac{8 - 19\eta_0 + 10\eta_0^2 - 6\eta_0 D_2 + 2(-6 + 7\eta_0)D_1}{24 - 48\eta_0}. \quad (5.6)$$

The total potential energy per unit area is then given by

$$\Psi = \int_0^l \bar{\Phi} dX - \hat{E} \int_0^l \gamma W dX - \hat{E} \int_0^l q G dX. \quad (5.7)$$

So the Lagrangian is given by

$$L = \bar{\Phi} - \hat{E} \gamma W - \hat{E} q G. \quad (5.8)$$

Further by the variational principle, we have the following Euler-Lagrange equations:

$$\begin{cases} \frac{\partial L}{\partial W} - \frac{d}{dX} \frac{\partial L}{\partial W_X} + \frac{d^2}{dX^2} \frac{\partial L}{\partial W_{XX}} = 0, \\ \frac{\partial L}{\partial G} - \frac{d}{dX} \frac{\partial L}{\partial G_X} = 0. \end{cases} \quad (5.9)$$

If we omit $O(\varepsilon^3 \delta^2)$ and $O(\varepsilon^2 \delta^2)$ in (5.9)₁, we have the following two equations:

$$\begin{cases} W + D_1 W^2 - \frac{a^2}{3} W_{XX} - D_2 a^2 (W_X^2 + 2W W_{XX}) = \gamma, \\ (W + (D_1 - 1)W^2)G - \frac{a^2}{3} G_{XX} = q. \end{cases} \quad (5.10)$$

We find that (5.10)₁ and (5.10)₂ are exactly (4.32) and (4.36) respectively.

These are two decoupled nonlinear ODE's. This system is called the asymptotic normal form equations of the original nonlinear PDE's. After imposing proper end conditions, we will study bifurcations of this system. We can study the bifurcations of the original complicated nonlinear PDE's through the study of the simpler decoupled nonlinear ODE's (5.10).

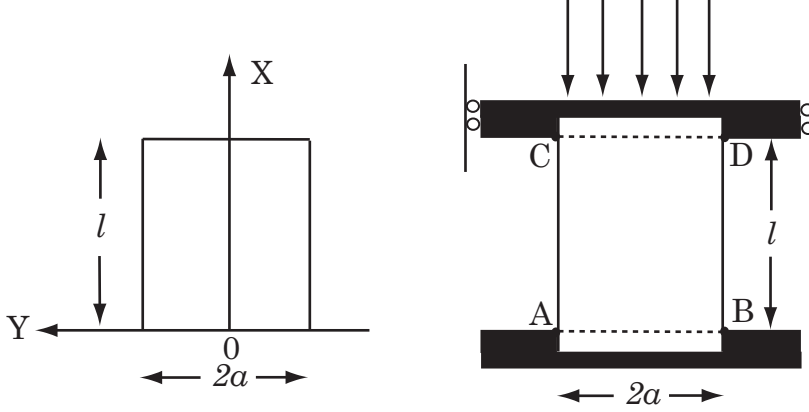


FIG. 6.1. Sketch map.

6. Clamped boundary conditions at two ends. In this section, we consider the simplifications of the clamped boundary conditions. As illustrated in Figure 6.1, two thin parts of the rectangle are clamped by two rigid bodies. The bottom is fixed and the top is slidably supported. We will consider the rectangle between two dashed lines. Without loss of generality, hereafter we take the length of the rectangle to be 1 and then $2a$ is the aspect ratio (ratio of width with length) of the rectangle. Then we can propose the following asymptotic boundary conditions for the clamped ends.

(1) First, for the points A, B, C, D , there is no lateral movement. So we have

$$V(0, \pm a) = 0, \quad V(1, \pm a) = 0. \quad (6.1)$$

From the series expression, we have

$$\begin{aligned} V(0, \pm a) &= hv(0, \pm a) \\ &= h \left[\delta v_0(0) \pm a(v_1(0) + a^2 v_3(0)) \right] + O(h\delta a^2, ha^5) = 0. \end{aligned} \quad (6.2)$$

By omitting $O(h\delta a^2, ha^5)$, we have

$$v_0(0) = 0, \quad (6.3)$$

and

$$v_1(0) + a^2 v_3(0) = 0. \quad (6.4)$$

From the perturbation expansions of v_1 and v_3 and (4.26), we can reduce (6.4) to

$$\left(\frac{1}{2} - 3\alpha_{44}\right)\gamma + \left(-\frac{3}{2} + 2\eta_0 + 3\alpha_{44}\right)(\varepsilon u_{0x}) + \left(D_1\left(-\frac{1}{2} + 3\alpha_{44}\right) + \alpha_{45}\right)(\varepsilon u_{0x})^2 = 0. \quad (6.5)$$

Then we have

$$W(0) = \Delta_1, \quad (6.6)$$

where

$$\Delta_1 = \frac{3 - 4\eta_0 - 6\alpha_{44} - \sqrt{(3 - 4\eta_0 - 6\alpha_{44})^2 + 4\gamma(1 - 6\alpha_{44})(D_1(1 - 6\alpha_{44}) - 2\alpha_{45})}}{2(2\alpha_{45} - D_1(1 - 6\alpha_{44}))}$$

is a small negative number depending on the external force γ and constitutive constants. We can see that $\Delta_1 = 0$ when $\gamma = 0$.

Similarly, we can obtain

$$v_0(1) = 0 \quad \text{and} \quad W(1) = \Delta_1. \quad (6.7)$$

(2) Secondly, for the points A and B , there is no axial movement which leads to the following conditions

$$U(0, \pm a) = 0. \quad (6.8)$$

From the series expansion of U , we have

$$\begin{aligned} U(0, \pm a) &= hu(0, \pm a) \\ &= h \left[u_0(0) \pm \delta a(u_1(0) + a^2 u_3(0)) \right] + O(ha^2, h\delta a^5). \end{aligned} \quad (6.9)$$

By omitting $O(ha^2, h\delta a^5)$, we have

$$u_0(0) = 0, \quad (6.10)$$

and

$$u_1(0) + a^2 u_3(0) = 0. \quad (6.11)$$

From the perturbation expansions of u_1 and u_3 and (4.28), we can reduce (6.11) to

$$G(0) = \Delta_2, \quad (6.12)$$

where

$$\Delta_2 = qK, \quad (6.13)$$

and $K = \frac{\frac{9}{2} - 5\eta_0}{1 - (-\frac{9}{2} + 3\eta_0)\Delta_1 - (\frac{9}{2} - 5\eta_0 + \alpha_{48} + (-\frac{9}{2} + 5\eta_0)D_1)\Delta_1^2}$.

(3) Thirdly, for the points C and D , they should have the same axial displacement, i.e.,

$$U(1, a) = U(1, -a), \quad (6.14)$$

which leads to the following condition

$$G(1) = \Delta_2. \quad (6.15)$$

Now we list the asymptotic end conditions for the asymptotic normal form equations (5.10):

$$v_0 = 0, \quad G = \Delta_2, \quad u_0 = 0, \quad W = \Delta_1 \quad \text{at} \quad X = 0, \quad (6.16)$$

$$v_0 = 0, \quad G = \Delta_2, \quad W = \Delta_1 \quad \text{at} \quad X = 1. \quad (6.17)$$

7. Bifurcation analysis. In this section we will make some bifurcation analysis of the asymptotic normal form equations (5.10). A similar equation as (5.10)₁ has been derived by Dai and Wang in [13, 14] for the compressions of a cylinder and some results are obtained. Here we will use these results to study our problem. It is found that there are two different kinds of bifurcation phenomena for chosen material constants. One is that there is only bifurcation to the corner-like profile when the aspect ratio is relatively large, which is a barrelling instability. Another is that, when the aspect ratio is relatively small, only bifurcation to the buckled profile occurs instead of barrelling to a corner-like profile.

7.1. Barrelling instability: Bifurcation to a corner-like profile. First we can rewrite (5.10)₁ as

$$\begin{cases} W_X = g, \\ g_X = \frac{W + D_1 W^2 - a^2 D_2 g^2 - \gamma}{a^2 \left(\frac{1}{3} + 2D_2 W \right)}. \end{cases} \quad (7.1)$$

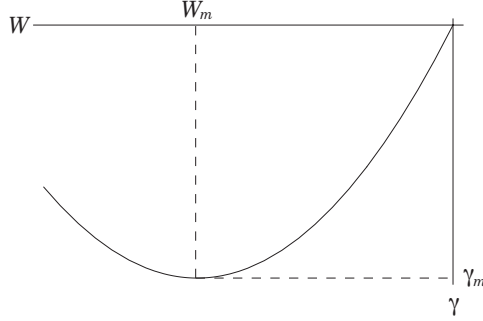


FIG. 7.1. γ - W plot

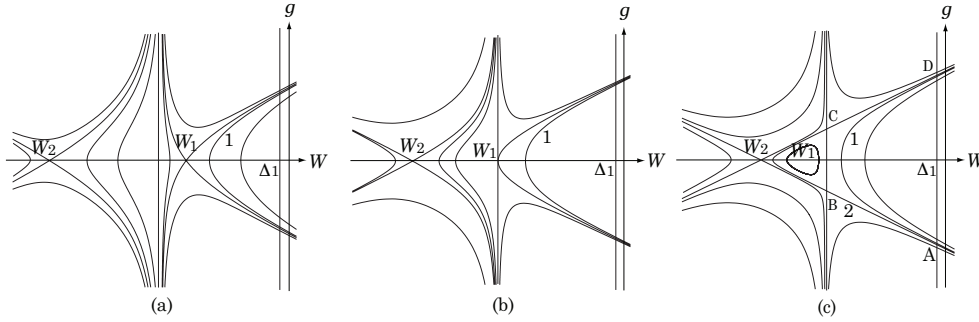


FIG. 7.2. Phase planes for different γ values: (a) $\gamma_p < \gamma < 0$; (b) $\gamma = \gamma_p$; (c) $\gamma_c \leq \gamma < \gamma_p$ and $-\frac{1}{4D_1} < \gamma < \gamma_c$

The vector field of the system (7.1) has a denominator term, which is zero at $W = -\frac{1}{6D_2}$, and thus it is a singular ODE system. Equation (7.1) together with (6.16)₄ and (6.17)₃ form a boundary-value problem of a singular system. In this case, the detailed analysis about this singular dynamical system is almost the same with that in [14]. From (4.26), it can be seen that the arising of the denominator D_2 -term is due to the coupling of the material nonlinearity (measured by ε) and the geometrical size (measured by ν). Here we again point out that it is due to the interaction between the material nonlinearity and geometrical size which causes the bifurcation to the formation of a corner-like profile as noted in [13] and [14].

We concentrate on the case of $3D_2 > D_1 > 0$, for which it is sufficient to illustrate the bifurcation to the corner-like profile. The equilibrium points of the system are given by

$$g = 0, \quad \gamma = W + D_1 W^2. \quad (7.2)$$

In the $W - g$ phase plane, there is also a vertical singular line $W = -\frac{1}{6D_2}$. There are three different phase planes as γ varies, which are shown in Figure 7.2, where γ_c will be defined later and γ_p is the value calculated from (7.2)₂ for $W = -\frac{1}{6D_2}$, i.e.,

$$\gamma_p = D_1 \left(\frac{-1}{6D_2} \right)^2 - \frac{1}{6D_2}. \quad (7.3)$$

The one-dimensional stress strain relationship is given in Figure 7.1. Without loss of generality we let $a = 0.25$, $D_1 = 2.4$, $D_2 = 0.88$, the Poisson ratio $\sigma = \frac{\lambda}{2(\lambda+\mu)} = 0.495$ to get the graphical results. Then $\gamma_p = -0.103306$. For a trajectory in a phase plane to be a solution of the boundary value problem of (7.1), (6.16)₄ and (6.17)₃, a necessary and sufficient condition is that it contacts the vertical line $W = \Delta_1$ twice and its X -interval is exactly equal to 1. Next, we discuss the solution(s) in each phase plane separately.

Case (a) $\gamma_p < \gamma < 0$

In this case there is a unique solution denoted by trajectory 1. This trajectory crosses W -axis at $(W_0, 0)$ in the corresponding phase plane. The solution expression is given in the following by referring to [19].

$$X = \begin{cases} \frac{1}{2} - \frac{1}{\beta} \frac{2}{\sqrt{(W_0 + \frac{1}{6D_2})(E_2 - E_1)}} \left((W_0 - E_2) \Pi \left(\theta, \frac{W_0 - E_1}{E_2 - E_1}, m \right) \right. \\ \quad \left. + (E_2 + \frac{1}{6D_2}) F(\theta, m) \right), & 0 \leq X \leq \frac{1}{2}, \\ \frac{1}{2} + \frac{1}{\beta} \frac{2}{\sqrt{(W_0 + \frac{1}{6D_2})(E_2 - E_1)}} \left((W_0 - E_2) \Pi \left(\theta, \frac{W_0 - E_1}{E_2 - E_1}, m \right) \right. \\ \quad \left. + (E_2 + \frac{1}{6D_2}) F(\theta, m) \right), & \frac{1}{2} \leq X \leq 1, \end{cases} \quad (7.4)$$

where Π and F are the elliptic integral of the third kind and the first kind respectively and

$$E_1 = \frac{-3 - 2D_1W_0 - \sqrt{3(3 + 16\gamma D_1 - 4D_1W_0 - 4D_1^2W_0^2)}}{4D_1}, \quad (7.5)$$

$$E_2 = \frac{-3 - 2D_1W_0 + \sqrt{3(3 + 16\gamma D_1 - 4D_1W_0 - 4D_1^2W_0^2)}}{4D_1}, \quad (7.6)$$

$$\beta = \frac{1}{a} \sqrt{\frac{D_1}{3D_2}}, \quad \theta = \arcsin \sqrt{\frac{(E_2 - E_1)(W - W_0)}{(W_0 - E_1)(W - E_2)}}, \quad m = \sqrt{\frac{(E_2 + \frac{1}{6D_2})(W_0 - E_1)}{(W_0 + \frac{1}{6D_2})(E_2 - E_1)}}. \quad (7.7)$$

As $W = \Delta_1$ at $X = 0$, W_0 is determined by

$$\frac{1}{2} = \frac{1}{\beta} \frac{2}{\sqrt{(W_0 + \frac{1}{6D_2})(E_2 - E_1)}} \left((W_0 - E_2) \Pi \left(\theta_0, \frac{W_0 - E_1}{E_2 - E_1}, m \right) + (E_2 + \frac{1}{6D_2}) F(\theta_0, m) \right), \quad (7.8)$$

where $\theta_0 = \arcsin \sqrt{\frac{(E_2 - E_1)(\Delta_1 - W_0)}{(W_0 - E_1)(\Delta_1 - E_2)}}$.

Case (b) $\gamma = \gamma_p$

In this case, there is a trajectory tangent to the singular line in the phase plane as shown in Figure 7.2 (b). There is a unique solution denoted by trajectory 1 in the

phase plane. Trajectory 1 contacts W -axis at $(W_0, 0)$. The solution expression is given by

$$X = \begin{cases} \frac{1}{2} - \frac{1}{\beta} \frac{2(W_0 + \frac{1}{6D_2})}{\sqrt{(W_0 - E_2)(\frac{-1}{6D_2} - E_1)}} \Pi\left(\theta, \frac{W_0 - E_1}{-\frac{1}{6D_2} - E_1}, m\right), & 0 \leq X \leq \frac{1}{2}, \\ \frac{1}{2} + \frac{1}{\beta} \frac{2(W_0 + \frac{1}{6D_2})}{\sqrt{(W_0 - E_2)(\frac{-1}{6D_2} - E_1)}} \Pi\left(\theta, \frac{W_0 - E_1}{-\frac{1}{6D_2} - E_1}, m\right), & \frac{1}{2} \leq X \leq 1, \end{cases} \quad (7.9)$$

where E_1 and E_2 are the same as (7.5) and (7.6) respectively and

$$\theta = \arcsin \sqrt{\frac{(\frac{-1}{6D_2} - E_1)(W - W_0)}{(W_0 - E_1)(W + \frac{1}{6D_2})}}, \quad m = \sqrt{\frac{(\frac{-1}{6D_2} - E_2)(W_0 - E_1)}{(W_0 - E_2)(\frac{-1}{6D_2} - E_1)}}. \quad (7.10)$$

As $W = \Delta_1$ at $X = 0$, W_0 is determined by

$$\frac{1}{2} = \frac{1}{\beta} \frac{2(W_0 + \frac{1}{6D_2})}{\sqrt{(W_0 - E_2)(\frac{-1}{6D_2} - E_1)}} \Pi\left(\theta_0, \frac{W_0 - E_1}{-\frac{1}{6D_2} - E_1}, m\right), \quad (7.11)$$

where $\theta_0 = \arcsin \sqrt{\frac{(\frac{-1}{6D_2} - E_1)(\Delta_1 - W_0)}{(W_0 - E_1)(\Delta_1 + \frac{1}{6D_2})}}$.

Case (c) $\gamma_c < \gamma < \gamma_p$

There is a unique solution represented by trajectory 1. It contacts W -axis at $(W_0, 0)$. This solution can be expressed in the following form

$$X = \begin{cases} -\frac{1}{\beta} \int_{\Delta_1}^W \sqrt{\frac{W + \frac{1}{6D_2}}{(W - W_0)(W^2 + sW + t)}} dW, & 0 \leq X \leq \frac{1}{2}, \\ \frac{1}{2} + \frac{1}{\beta} \int_{W_0}^W \sqrt{\frac{W + \frac{1}{6D_2}}{(W - W_0)(W^2 + sW + t)}} dW, & \frac{1}{2} \leq X \leq 1, \end{cases} \quad (7.12)$$

where $s = \frac{6\gamma + 3W_0 + 2D_1W_0^2}{2D_1}$ and $t = \frac{3 + 2D_1W_0}{2D_1}$. And W_0 is determined by the following equation

$$\frac{1}{2} = -\frac{1}{\beta} \int_{\Delta_1}^{W_0} \sqrt{\frac{W + \frac{1}{6D_2}}{(W - W_0)(W^2 + sW + t)}} dW. \quad (7.13)$$

Case (d) $\gamma = \gamma_c$

We define γ_c to be the critical stress value such that when $\gamma = \gamma_c$ trajectory 2 in Figure 7.2 (c), which starts from A , goes to B , jumps from B to C and finally arrives at D , is also a solution. In this case, there is another solution indexed by 1 in Figure 7.2 (c). For trajectory 1, it contacts W -axis at $(W_0, 0)$, and its solution also is expressed by (7.12) and W_0 is determined by (7.13). We find that for this solution the value of W_{XX} at $X = 0.5$ is relatively small. Indeed, for $\gamma = \gamma_c$, $W_{XX} = 1.68199$ at $X = 0.5$.

The solution expression corresponding to trajectory 2 is given by

$$W = \begin{cases} E_2 + (E_2 - E_1) \sinh^2 \frac{1}{2} \left(2 \operatorname{arcsinh} \sqrt{\frac{-\frac{1}{6D_2} - E_2}{-E_1 + E_2}} + \beta \left(\frac{1}{2} - X \right) \right), & 0 \leq X \leq \frac{1}{2}, \\ E_2 + (E_2 - E_1) \sinh^2 \frac{1}{2} \left(2 \operatorname{arcsinh} \sqrt{\frac{-\frac{1}{6D_2} - E_2}{-E_1 + E_2}} + \beta \left(X - \frac{1}{2} \right) \right), & \frac{1}{2} \leq X \leq 1, \end{cases} \quad (7.14)$$

where

$$E_1 = \frac{D_1 - 9D_2 - \sqrt{3(-D_1^2 + 27D_2^2 + 6D_1D_2(1 + 24\gamma D_2))}}{12D_1D_2}, \quad (7.15)$$

$$E_2 = \frac{D_1 - 9D_2 + \sqrt{3(-D_1^2 + 27D_2^2 + 6D_1D_2(1 + 24\gamma D_2))}}{12D_1D_2}. \quad (7.16)$$

As $W = \Delta_1$ at $X = 0$, we have

$$\frac{1}{2} = \frac{1}{\beta} \left(2\operatorname{arcsinh} \sqrt{\frac{\Delta_1 - E_2}{E_1 - E_2}} - 2\operatorname{arcsinh} \sqrt{\frac{-\frac{1}{6D_2} - E_2}{-E_1 + E_2}} \right), \quad (7.17)$$

which determines the value γ_c . For the previously-chosen parameters, we find $\gamma_c = -0.1038086$.

Alternatively, (7.14) can be rewritten as

$$W = E_2 + (E_2 - E_1) \sinh^2 \frac{1}{2} \left(2\operatorname{arcsinh} \sqrt{\frac{-\frac{1}{6D_2} - E_2}{-E_1 + E_2}} + \beta \left| \frac{1}{2} - X \right| \right). \quad (7.18)$$

This is a non-smooth solution whose first-order derivative at $X = 0.5$ does not exist. This non-smooth solution is a weak solution of the singular ODE system according to the definition given in [20]. However, for this solution the jump conditions (2.12) cannot be satisfied exactly and we shall not take it as one solution of the field equations. We intend to find the smooth solution which at one point has a very large second-order derivative value.

Case (e) $-\frac{1}{4D_1} < \gamma < \gamma_c$

When $|\gamma|$ is slightly larger than $|\gamma_c|$, there are two solutions, which both can be indexed by 1. Both of them can be expressed by (7.12). We denote the crossing points with the W -axis of the two solution trajectories by $(\tilde{W}_0, 0)$ and $(W_0, 0)$ respectively and \tilde{W}_0 and W_0 can be determined by (7.13). We notice that for one of them the point $(\tilde{W}_0, 0)$ is very close to the singular line, which represents a sharp crest profile with a large second-order derivative W_{XX} at $X = 0.5$.

For $\gamma = -0.103809 < \gamma_c = -0.1038086$, we find that $W_0 = -0.182968$ and $\tilde{W}_0 = -0.189393$ which is very close to the singular line $V = -\frac{1}{6D_2} = -0.189394$, and the corresponding total potential energy values for the two solutions are respectively

$$\Psi = -5.48367 \times 10^{-3} \hat{E}, \quad \tilde{\Psi} = -5.48513 \times 10^{-3} \hat{E} \quad (7.19)$$

which are obtained from (5.7) ($G = 0$ for a barrelling instability).

Trajectory 1 with \tilde{W}_0 has a smaller energy value and is thus an energetically preferred solution. For this solution, we find that $W_{XX} = 115.8$ at $Z = 0.5$, indeed a very large value. The solution curve corresponding to this trajectory is shown in Figure 7.3 (a). The surface profile of the cylinder in the current configuration is shown in Figure 7.3 (b).

By considering the other values of a , we find that when the aspect ratio $0.376 < 2a < 0.5292$ there will be a bifurcation to the corner-like profile. Thus, for the material constants chosen here, 0.376 is a lower bound of the aspect ratio for the barrelling instability. We also point out that for the barrelling solution obtained, we find there is

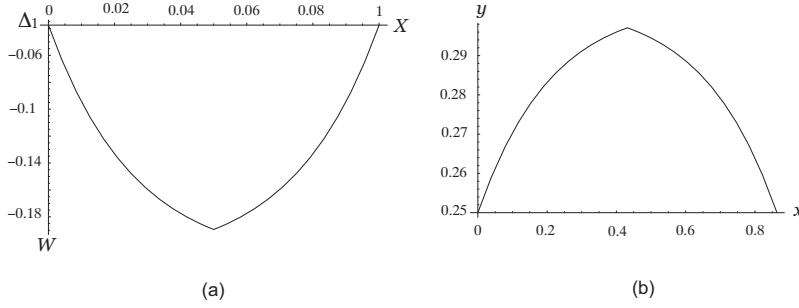


FIG. 7.3. (a) W - X plot; (b) Profile of the lateral boundary of the rectangle (x and y are current coordinates).

no nontrivial solution for $(5.10)_2$ together with boundary conditions $(6.16)_1$, $(6.16)_2$, $(6.17)_1$ and $(6.17)_2$. This implies that no buckling will occur after the barrelling.

Remark: The solutions obtained in this section satisfy $W_m \leq W \leq 0$ (cf. Fig. 7.1), i.e., W is in the “hardening” branch.

7.2. Buckling instability: Bifurcation to a buckled profile. In this subsection, we discuss the bifurcation phenomenon of buckling. Since the solutions for W are discussed in detail in the above, in the following discussion, we concentrate on equation $(5.10)_2$. We will formulate our eigenvalue problem with the clamped boundary conditions. For convenience we rewrite equation $(5.10)_2$ as

$$(W + (D_1 - 1)W^2)G - \frac{a^2}{3}G_{XX} = q. \quad (7.20)$$

This equation together with $(6.16)_1$, $(6.16)_2$, $(6.17)_1$ and $(6.17)_2$ compose an eigenvalue problem where γ , contained in W , is the eigenvalue and G is the corresponding eigenfunction.

Letting $f(X) = W(X) + (D_1 - 1)W(X)^2$ and $\tau = \frac{a}{\sqrt{3}}$, we have

$$\tau^2 G_{XX} - f(X)G = -q. \quad (7.21)$$

Since we set the length $l = 1$ and we are considering a thin rectangle, τ is a small parameter.

We first solve the homogeneous equation

$$\tau^2 G_{XX} - f(X)G = 0 \quad (7.22)$$

by the WKB method, which is a very useful tool for solving bifurcation problems in elastic solids (see, e.g. Fu [21]).

For a small τ , the leading-order solution of the homogeneous equation is

$$\begin{aligned} G(X, \tau) &= C_1 G_1(X, \tau) + C_2 G_2(X, \tau) \\ &= C_1 (-f(X))^{-\frac{1}{4}} \cos \int_0^X \frac{\sqrt{-f(X)}}{\tau} dt + C_2 (-f(X))^{-\frac{1}{4}} \sin \int_0^X \frac{\sqrt{-f(X)}}{\tau} dt, \end{aligned} \quad (7.23)$$

where C_1 and C_2 are two constants.

By the method of variation of parameters, an particular solution (to the leading -order) of (7.21) is found to be

$$G_p(X, \tau) = \frac{q}{\tau} \int_0^X (f(s)f(X))^{-\frac{1}{4}} \sin \int_X^s \frac{\sqrt{-f(t)}}{\tau} dt ds. \quad (7.24)$$

Then the general solution of (7.21), to the leading-order, is given by

$$\begin{aligned} G(X, \tau) &= C_1(-f(X))^{-\frac{1}{4}} \cos \int_0^X \frac{\sqrt{-f(t)}}{\tau} dt + C_2(-f(X))^{-\frac{1}{4}} \sin \int_0^X \frac{\sqrt{-f(t)}}{\tau} dt \\ &\quad + \frac{q}{\tau} \int_0^X (f(s)f(X))^{-\frac{1}{4}} \sin \int_X^s \frac{\sqrt{-f(t)}}{\tau} dt ds. \end{aligned} \quad (7.25)$$

By the condition (6.16)₂, we have $C_1 = \Delta_2(-f(1))^{\frac{1}{4}} = qK(-f(1))^{\frac{1}{4}}$. And from the condition (6.16)₁, we have

$$\int_0^X G(X, \tau) dX = \varepsilon \delta(v_0(x) - v_0(0)) = \varepsilon \delta v_0(x). \quad (7.26)$$

To satisfy the conditions (6.17)₁ and (6.17)₂, we have

$$\begin{cases} C_2 A_1 + \frac{q}{\tau} A_2 = 0, \\ C_2 B_1 + \frac{q}{\tau} B_2 = 0, \end{cases} \quad (7.27)$$

where

$$\begin{cases} A_1 = (-f(1))^{-\frac{1}{4}} \sin \int_0^1 \frac{\sqrt{-f(t)}}{\tau} dt, \\ A_2 = K\tau \left(\cos \int_0^1 \frac{\sqrt{-f(t)}}{\tau} dt - 1 \right) + \int_0^1 (f(s)f(1))^{-\frac{1}{4}} \sin \int_1^s \frac{\sqrt{-f(t)}}{\tau} dt ds, \\ B_1 = \int_0^1 (-f(X))^{-\frac{1}{4}} \sin \int_0^X \frac{\sqrt{-f(t)}}{\tau} dt dX, \\ B_2 = K\tau (-f(1))^{\frac{1}{4}} \int_0^1 (-f(X))^{-\frac{1}{4}} \cos \int_0^X \frac{\sqrt{-f(t)}}{\tau} dt dX \\ \quad + \int_0^1 \int_0^X (f(s)f(X))^{-\frac{1}{4}} \sin \int_X^s \frac{\sqrt{-f(t)}}{\tau} dt ds dX. \end{cases} \quad (7.28)$$

To obtain non-trivial solutions, we need the determinant of the coefficient matrix of (7.27) to be zero, i.e.,

$$A_1 B_2 - A_2 B_1 = 0. \quad (7.29)$$

This is the algebraic equation for determining the eigenvalue γ .

In Figure 7.4, we plot the curves $(A_1 B_2 - A_2 B_1) - \gamma$ for $a = 0.045$ and $a = 0.06$. The stress values at intersection points of the curve with γ axis are the critical stress values. There are three critical stress values when $a = 0.045$ and two when $a = 0.06$. It is expected that more critical stress values will appear when the aspect ratio $2a$ is smaller.

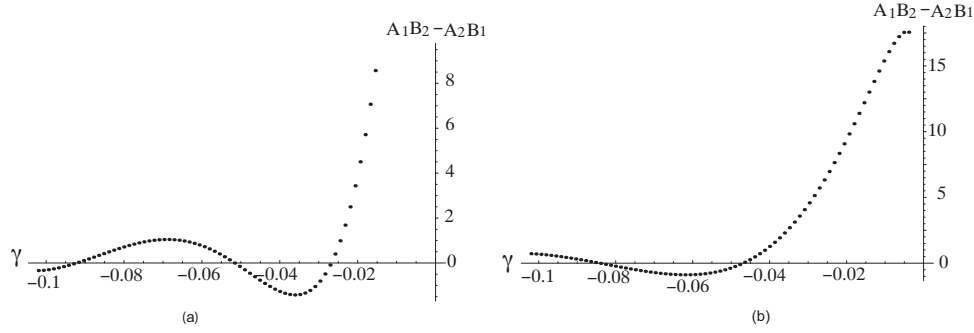


FIG. 7.4. The $(A_1B_2 - A_2B_1)-\gamma$ curve for (a) $a = 0.045$; (b) $a = 0.06$.

In Figure 7.5 we give the curve for $a = 0.25$. It can be seen that there is no bifurcation point to the buckling profile before and after the barrelling to the corner-like profile occurs. Actually, when $a > 0.0955$, there will be no bifurcation to buckling. Thus, for the material constants chosen here, 0.191 is the upper bound of the aspect ratio for the buckling instability.

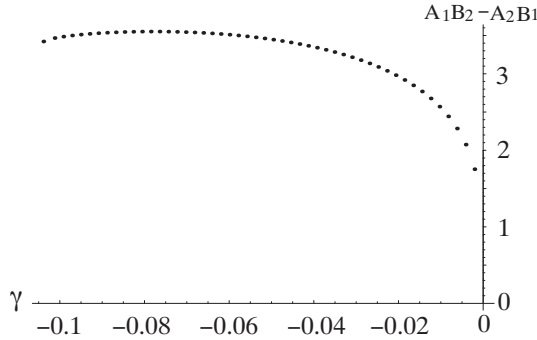


FIG. 7.5. The $(A_1B_2 - A_2B_1)-\gamma$ curve for $a = 0.25$.

In Figure 7.6, we give the eigen-shapes of the rectangle at bifurcation points when $a = 0.045$.

Note that the elastic instability of the compression of a bar at two ends was first discovered by Euler in 1744 and then by Lagrange and Bernoulli. A historical statement about Euler’s derivation can be found in [22] by Komkov. According to the Bernoulli-Euler constitutive equation which says that the bending moment is linear in the change of curvature (see Antman [23]), a basic fourth-order linear ODE for the transverse displacement of a slender rectangle is obtained,

$$\hat{E}Iu^{(4)}(s) + \Lambda u''(s) = 0, \quad 0 < s < 1, \tag{7.30}$$

where u is the transverse displacement of the same material point on the rectangle between the undeformed configuration and the current configuration, $I = \int_{-a}^a Y^2 dY = \frac{2}{3}a^3$ is the moment of inertia and Λ is the absolute value of the compressive applied force.

There is a detailed derivation of (7.30) in [24] by Landau et. al in the studies of small deflections of rod under axial compressions. Here we also can obtain this equa-

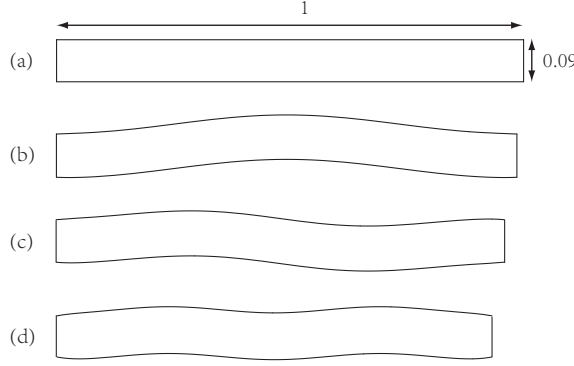


FIG. 7.6. *Eigen-shapes of the rectangle obtained by our model equation for $a = 0.045$: (a) Rectangle in the reference configuration; (b) (c) (d) Buckled rectangle at the first, second and third bifurcation points in the current configuration respectively.*

tion without using the Bernoulli-Euler constitutive equation. First we take derivative to (4.21) and add this equation to (4.22), then we can obtain

$$\begin{aligned}
& \nu \left(\left(\frac{1}{2} \eta_0 - \frac{1}{3} \eta_0^2 \right) u_{1xxx} + \left(-\frac{1}{6} \eta_0 + \frac{1}{3} \eta_0^2 \right) v_{0xxxx} \right) + \varepsilon (\alpha_1 - \alpha_2) (u_{1x} v_1 \\
& + u_{1x} u_{0x} - v_1 v_{0xx} - u_{0x} v_{0xx} + u_1 v_{1x} + u_1 u_{0xx} - v_{1x} v_{0x} - u_{0xx} v_{0x}) \\
& + \varepsilon^2 \left(\left(\frac{1}{2} + \alpha_7 \right) (v_1^2 u_{1x} + u_{0x}^2 u_{1x} + 2u_1 v_1 v_{1x} - 2v_1 v_{0x} v_{1x} + 2u_1 u_{0x} u_{0xx} \right. \\
& \left. - 2u_{0x} v_{0x} u_{0xx} - v_1^2 v_{0xx} - u_{0x}^2 v_{0xx}) + (-\eta_0 + 2\alpha_{10}) (v_1 u_{0x} u_{1x} + u_1 u_{0x} v_{1x} \right. \\
& \left. - u_{0x} v_{0x} v_{1x} + u_1 v_1 u_{0xx} - v_1 v_{0x} u_{0xx} - v_1 u_{0x} v_{0xx}) \right) = 0. \tag{7.31}
\end{aligned}$$

Further the substitution of v_1 and u_1 from (4.25) and (4.27) leads to the following equation if we write in the dimensional variables

$$-IV_{0XXXX} + WV_{0XX} + W_X V_{0X} + (D_1 - 1)(2WW_X V_{0X} + W^2 V_{0XX}) = 0, \tag{7.32}$$

where $W = \varepsilon u_{0x}$ as denoted before and $V_0 = \varepsilon \delta v_0$ which is the transverse displacement of the axis of the rectangle. If we assume that W is a constant and omit higher-order nonlinear terms, we can obtain,

$$\hat{E}IV_{0XXXX} - \hat{E}WV_{0XX} = 0. \tag{7.33}$$

Since $\Lambda = -\hat{E}W$ in the linear case, this equation can be written as

$$\hat{E}IV_{0XXXX} + \Lambda V_{0XX} = 0, \tag{7.34}$$

which is the same as (7.30).

The general solution of (7.30) is given by

$$u(s) = A + Bs + C \sin \lambda s + D \cos \lambda s, \quad 0 < s < 1, \tag{7.35}$$

where $\lambda = \sqrt{\frac{\Lambda}{EI}}$ and A, B, C and D are constants. If we also impose the clamped boundary conditions

$$u(0) = u(1) = 0, \quad u'(0) = u'(1) = 0, \tag{7.36}$$

the eigenvalue λ should satisfy

$$\lambda \sin \lambda + 2 \cos \lambda - 2 = 0. \tag{7.37}$$

Note that the clamped boundary conditions (7.36) are used by many literatures such as Landau and Lifshitz ([24], p. 72) in which the clamped boundary condition is explained as: “The end of the rod is said to be clamped if it cannot move either longitudinally or transversely, and moreover its direction (i.e. the direction of the tangent to the rod) cannot change”. In section 6, we give the asymptotic clamped boundary conditions (6.16) and (6.17) for a two-dimensional rectangle. There are two significant differences between these two clamped boundary conditions. One is that our clamped boundary conditions are related to the axial strain W , see (6.16)₄ and (6.17)₃. Another is that the tangent values of the axis at two ends are not zeros and is related to constitutive constants and the external forces γ and q , see (6.16)₂ and (6.17)₂. From the derivations of the asymptotic clamped boundary conditions, we can see that these differences are caused by the geometry of the rectangle, the effects of the nonuniform axial strain and nonlinearity of the problem.

Under clamped boundary conditions (7.36), the nontrivial solutions of (7.30) are given by

$$u(s) = C \left(\cos \lambda s - 1 + \frac{\cos \lambda - 1}{\lambda - \sin \lambda} (\sin \lambda s - \lambda s) \right), \tag{7.38}$$

where λ is the root of (7.37) and C is constant.

The smallest positive eigenvalue of (7.37) is $\lambda_1 = 2\pi$. Corresponding to this eigenvalue, the external force Λ is given by

$$\Lambda = 4\hat{E}I\pi^2 \tag{7.39}$$

which is the Euler’s buckling formula with two ends clamped. By this Euler’s buckling formula, the first critical stress value (i.e. the smallest absolute critical stress value) is given by $\frac{\Lambda}{2a\hat{E}} = \frac{4}{3}\pi^2 a^2$.

In Figure 7.7 we give the eigen-shapes of the rectangle at critical stress values for this 4th-order linear ODE (7.30).

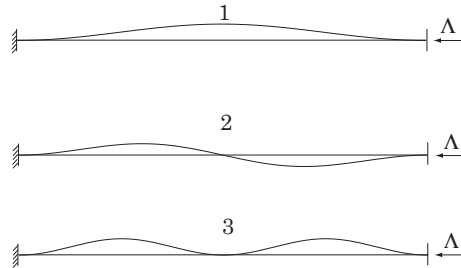


FIG. 7.7. Eigen-shapes of the buckled rectangle for the first three bifurcation points obtained from the Euler’s buckling formula.

If we compare Figure 7.6 with Figure 7.7, we find that the eigen-shapes of rectangle in the buckled state at the first three bifurcation points coincide. Then it is worthy to compare their corresponding critical stress values.

Before this comparison, we first introduce Davies’s work [4]. Davis in [4] studies the buckling and barrelling instabilities in the compression of a two-dimensional elastic

rectangle. In [4], the traction free boundary conditions at the lateral boundaries and the lubricated boundary conditions at two ends are used. At certain compression ratio, the bar is always unstable. There are two kinds of instabilities: barrelling and buckling. Also, numerical computation can give the critical stress values for different barrelling or buckling modes. In the following tables, we compare the critical stress values at the bifurcation points for three different models with the same aspect ratio of the rectangle and the same kind of material used here.

In Table 7.1 and 7.2, we list the absolute values of the first three critical stress values for three different models. For the Euler's buckling formula and our model equations, they have the same corresponding eigen-shapes shown in Figure 7.7. According to the formula provided by Davies [4], the bifurcation points in (C) provided in Table 7.1 and 7.2 correspond to buckling instead of barrelling. From these tables, we find that the absolute critical stress values obtained from [4] are always smaller than those obtained by the other models. It is expected that the rectangle under the lubricated boundary conditions begins to buckle earlier than under the clamped boundary conditions. From the above two tables, we also find that the critical stress values obtained by our model equation are very close to those obtained from (7.30), especially for the first critical stress value.

TABLE 7.1

The averaged nondimensionalized force at the first three bifurcation points when $a = 0.045$ calculated from (A) Euler's buckling formula, (B) our model equation and (C) Davies's method.

	1	2	3
(A) $ \gamma_1 $	0.0266479	0.054515	0.106592
(B) $ \gamma_2 $	0.0268605	0.0516636	0.0922735
(C) $ \gamma_3 $	0.00512409	0.0170796	0.0327633

TABLE 7.2

The averaged nondimensionalized force at the first three bifurcation points when $a = 0.06$ calculated from (A) Euler's buckling formula, (B) our model equation and (C) Davies's method.

	1	2	3
(A) $ \gamma_1 $	0.0473741	0.0969155	0.189496
(B) $ \gamma_2 $	0.047009	0.0842059	
(C) $ \gamma_3 $	0.00850293	0.0273159	0.0492325

In Figure 7.8 (a) we give the plot of the first critical stress values $|\gamma_1|$ and $|\gamma_2|$ as a varies. In Figure 7.8 (b) and (c) we give the plot of the difference of the first critical stress value $|\gamma_2| - |\gamma_1|$ and the relative error $\frac{|\gamma_2| - |\gamma_1|}{|\gamma_2|}$ as a varies respectively.

From Figure 7.8, we find that when $0.03 \leq a < 0.0542$, the first critical stress value obtained by our model equations is larger than that obtained by (7.30). When $a > 0.0542$, the first critical stress value obtained by our model equations is smaller than that obtained by (7.30). So, according to our results, the Euler buckling formula gives an over-estimate of the critical buckling force when the aspect ratio is larger than 0.1084 and under-estimate when it is smaller than 0.1084 (although the error is very small, cf. Fig. 7.8). We also observe that when $a < 0.075$ the relative error is smaller than 5%. Also when $0.03 \leq a < 0.075$, not only can our model equations yield almost the same first critical stress values with those obtained from (7.30), but also can yield corresponding buckled shapes. From the above analysis, we can say that

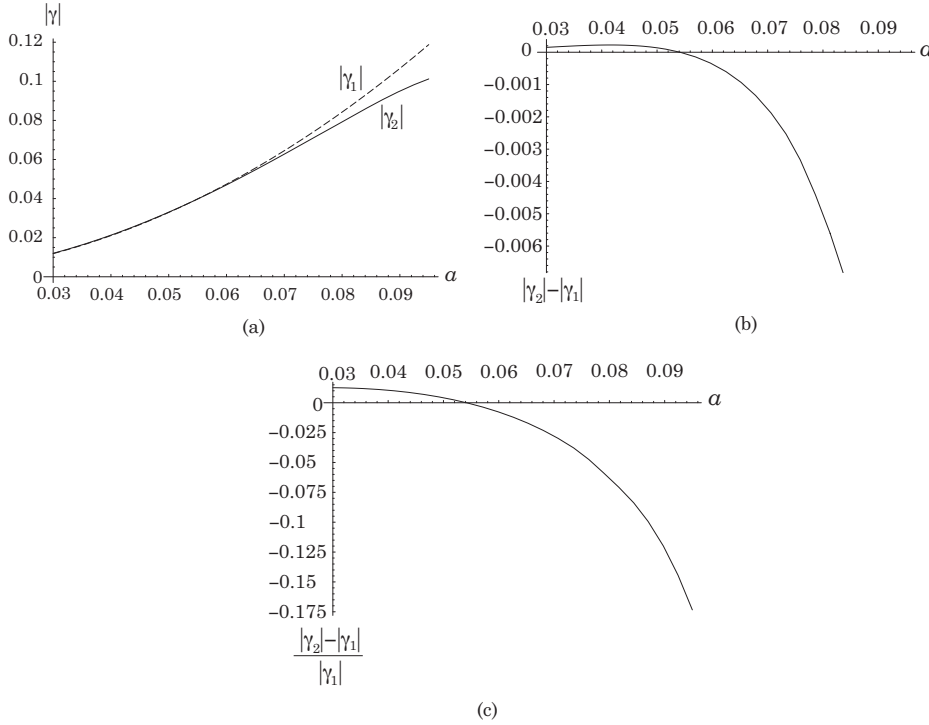


FIG. 7.8. (a) $|\gamma|-a$ plot; (b) $(|\gamma_2| - |\gamma_1|)-a$ plot; (c) $\frac{|\gamma_2| - |\gamma_1|}{|\gamma_1|}-a$ plot.

when $0.03 \leq a < 0.075$, the Bernoulli-Euler constitutive equation induced 4th-order linear ODE can match our model equations well.

However, there are some differences between our model equations and the 4th-order linear ODE. Here we study a typical compressible Murnaghan material. Shearing is taken into account in our derivations. Moreover, our model equations only can yield finite number of bifurcation points when $0.03 \leq a < 0.0955$. While, the 4th-order linear ODE can yield infinite countable number of bifurcation points for slender rectangles.

From the above analysis, we can state that our model equation and the Bernoulli-Euler constitutive equation induced 4th-order linear ODE are good model equations for the uniaxial compression of a rectangle and can match well when $0.03 \leq a < 0.075$. But, the 4th-order linear ODE is not a good model equation for the uniaxial compression of a rectangle when $a > 0.075$, i.e., we have determined an estimated interval $0.06 < 2a < 0.15$ for the aspect ratio of the rectangle that the Euler's formula is valid. Then we believe that the Euler's formula is valid when the aspect ratio $2a$ is less than 0.15. In [25], Levinson investigated the plane strain problem of a rectangular parallelepiped composed of incompressible neo-Hookean material. It is found that his numerical estimate for buckling load can match the Euler's buckling formula when the aspect ratio is smaller than 0.167.

8. Conclusions. For chosen material constants, we have found two types of instability phenomena in the compressions of a clamped rectangle: barrelling and buckling. The barrelling instability leads to a corner-like profile on the lateral boundaries of the rectangle. This occurs only when the aspect ratio is relatively large,

i.e., when the rectangle is thick enough. More specifically, for the material constants chosen here the lower bound of the aspect ratio for barrelling is 0.376 (cf., the last paragraph of subsection 7.1). Our results also reveal that this corner-like profile is caused by the coupling effect of the material nonlinearity and geometrical size of the rectangle. The analytic solution for the post-bifurcation corner-like profile is also obtained. When the rectangle is thin enough, buckling instability occurs instead of barrelling. More specifically, for the material constants chosen here the upper bound of the aspect ratio for buckling is 0.191 (cf. the second paragraph below (7.29)). In the buckling case, we can recover the Bernoulli-Euler constitutive equation induced classical 4th-order linear ODE. Numerical computations show that our model equations and this classical 4th-order linear ODE can yield very close first critical stress value and the same buckling eigen-shapes when the aspect ratio is small. We point out that these two models are both good model equations when the aspect ratio is small. But when the aspect ratio is large enough, the 4th-order linear ODE cannot model the compression of a rectangle.

The discovery of a lower bound of the aspect ratio for barreling and a different upper bound for buckling under clamped ends implies that there indeed exists a transition region, which agrees with the experimental results of Beatty and Hook [9] and Beatty and Dadras [10]. The present analytical study appears to shed certain light on this more than 40-years old mist.

Appendix.

$$\begin{aligned}
\alpha_1 &= 1 + \eta_1 + \eta_4, & \alpha_2 &= \eta_0 + \eta_1 + \eta_4, & \alpha_3 &= 1 - \eta_0 + 3\eta_1 + 2\eta_2 + \eta_4, \\
\alpha_4 &= 1 - 2\eta_0 + 2\eta_1 + 2\eta_2, & \alpha_5 &= 3 + 6\eta_1 + 2\eta_2 + 4\eta_4, & \alpha_6 &= \frac{1}{2} + \frac{7}{2}\eta_1 + \eta_2 + \frac{5}{2}\eta_4, \\
\alpha_7 &= 2\eta_1 + \eta_2 + \eta_4, & \alpha_8 &= \eta_0 + 2\eta_1 + 2\eta_4, & \alpha_9 &= \frac{1}{2} - \eta_0 + \eta_1 + \eta_4, \\
\alpha_{10} &= \eta_1 + \eta_2, & \alpha_{11} &= \frac{1 + \eta_0^2 + \eta_1 + \eta_4 - 2\eta_0(1 + 2\eta_1 + \eta_2 + \eta_4)}{2\eta_0^2}, \\
\alpha_{12} &= \frac{1 + \eta_0^2 + \eta_1 + \eta_4 - 2\eta_0(1 + 2\eta_1 + \eta_2 + \eta_4)}{2\eta_0^2}, \\
\alpha_{13} &= \frac{1 + \eta_1 + \eta_4 - \eta_0(3 + 6\eta_1 + 2\eta_2 + 4\eta_4)}{2\eta_0^2}, \\
\alpha_{14} &= \frac{1 + 2\eta_0^2 + \eta_1 - \eta_0(1 + 2\eta_1 + 2\eta_2) + \eta_4}{2\eta_0^2}, \\
\alpha_{15} &= -\frac{2\eta_0^3 + 2(1 + \eta_1 + \eta_4) + \eta_0^2(3 + 10\eta_1 + 2\eta_2 + 8\eta_4) - \eta_0(5 + 10\eta_1 + 2\eta_2 + 8\eta_4)}{6\eta_0^2}, \\
\alpha_{16} &= \frac{2\eta_0^3 - 4(1 + \eta_1 + \eta_4) - \eta_0^2(3 + 8\eta_1 + 4\eta_2 + 4\eta_4) + \eta_0(7 + 14\eta_1 + 4\eta_2 + 10\eta_4)}{6\eta_0^2}, \\
\alpha_{17} &= -\frac{(-1 + \eta_0)(-1 - \eta_1 - \eta_4 + \eta_0(1 + 3\eta_1 + \eta_2 + 2\eta_4))}{3\eta_0^2}, \\
\alpha_{18} &= -\frac{1 - 2\eta_0^3 + \eta_1 - \eta_0(1 + 2\eta_1 + 2\eta_2) + \eta_4 - 4\eta_0^2(\eta_1 + \eta_4)}{6\eta_0^2}, \\
\alpha_{19} &= -\frac{1 + 2\eta_0^3 + \eta_1 + \eta_4 + \eta_0^2(2 + 4\eta_1 + 4\eta_4) - \eta_0(3 + 6\eta_1 + 2\eta_2 + 4\eta_4)}{6\eta_0^2},
\end{aligned}$$

$$\begin{aligned}
\alpha_{20} &= -\frac{1 - 2\eta_0^3 + \eta_1 + \eta_0^2(1 - 4\eta_1 - 4\eta_4) - 2\eta_0(\eta_2 - \eta_4) + \eta_4}{6\eta_0^2}, \\
\alpha_{21} &= -\frac{1 + 2\eta_0^3 + \eta_1 + \eta_4 - 2\eta_0(2 + 4\eta_1 + \eta_2 + 3\eta_4) + \eta_0^2(1 + 4\eta_1 + 4\eta_4)}{6\eta_0^2}, \\
\alpha_{22} &= -\frac{9 + 2\eta_0^3 + 10\eta_1 + 10\eta_4 + \eta_0^2(7 + 16\eta_1 + 4\eta_2 + 12\eta_4) - 2\eta_0(9 + 19\eta_1 + 6\eta_2 + 13\eta_4)}{24\eta_0^2}, \\
\alpha_{23} &= -\frac{9 + 10\eta_1 + 10\eta_4 + \eta_0^2(9 + 12\eta_1 + 4\eta_2 + 8\eta_4) - 2\eta_0(7 + 15\eta_1 + 6\eta_2 + 9\eta_4)}{24\eta_0^2}, \\
\alpha_{24} &= -\frac{9 - 4\eta_0^3 + 10\eta_1 + \eta_0^2(7 + 4\eta_1 + 4\eta_2) + 10\eta_4 - 2\eta_0(6 + 13\eta_1 + 6\eta_2 + 7\eta_4)}{24\eta_0^2}, \\
\alpha_{25} &= -\frac{9 + 2\eta_0^3 + 10\eta_1 + 10\eta_4 + \eta_0^2(5 + 16\eta_1 + 4\eta_2 + 12\eta_4) - 2\eta_0(10 + 21\eta_1 + 6\eta_2 + 15\eta_4)}{24\eta_0^2}, \\
\alpha_{26} &= \frac{-1 + \eta_0^3 - \eta_1 + \eta_0(1 + 2\eta_1 + 2\eta_2) - \eta_4 + \eta_0^2(-1 + 2\eta_1 + 2\eta_4)}{12\eta_0^2}, \\
\alpha_{27} &= -\frac{1 + \eta_0^3 + \eta_1 + \eta_4 + \eta_0^2(1 + 2\eta_1 + 2\eta_4) - \eta_0(3 + 6\eta_1 + 2\eta_2 + 4\eta_4)}{12\eta_0^2}, \\
\alpha_{28} &= -\frac{1 - \eta_0^3 + \eta_1 - 2\eta_0(\eta_2 - \eta_4) + \eta_4 - 2\eta_0^2(-1 + \eta_1 + \eta_4)}{12\eta_0^2}, \\
\alpha_{29} &= -\frac{1 + \eta_0^3 + \eta_1 + \eta_4 + 2\eta_0^2(\eta_1 + \eta_4) - 2\eta_0(2 + 4\eta_1 + \eta_2 + 3\eta_4)}{12\eta_0^2}, \\
\alpha_{30} &= 1 - \eta_0 + \frac{3\eta_1}{2} - 3\eta_0\eta_1 - \eta_0\eta_2 + \frac{3\eta_4}{2} - 2\eta_0\eta_4, \\
\alpha_{31} &= -\eta_0 + \eta_0^2 - \frac{\eta_1}{2} - \eta_0\eta_1 - \eta_0\eta_2 - \frac{\eta_4}{2}, \\
\alpha_{32} &= -\frac{1}{2} + \frac{3\eta_0}{2} - \frac{\eta_1}{2} + 3\eta_0\eta_1 + \eta_0\eta_2 - \frac{\eta_4}{2} + 2\eta_0\eta_4, \\
\alpha_{33} &= -\frac{1}{2} + \frac{\eta_0}{2} - \eta_0^2 - \frac{\eta_1}{2} + \eta_0\eta_1 + \eta_0\eta_2 - \frac{\eta_4}{2}, \\
\alpha_{34} &= \frac{13}{3} - \frac{11}{6\eta_0} - \frac{8\eta_0}{3} - \frac{2\eta_0^2}{3} + 9\eta_1 - \frac{2\eta_1}{\eta_0} - \frac{16\eta_0\eta_1}{3} + 3\eta_2 - \frac{4\eta_0\eta_2}{3} + 6\eta_4 - \frac{2\eta_4}{\eta_0} - 4\eta_0\eta_4, \\
\alpha_{35} &= 4 - \frac{11}{6\eta_0} - \frac{7\eta_0}{3} + \frac{25\eta_1}{3} - \frac{2\eta_1}{\eta_0} - 4\eta_0\eta_1 + 3\eta_2 - \frac{4\eta_0\eta_2}{3} + \frac{16\eta_4}{3} - \frac{2\eta_4}{\eta_0} - \frac{8\eta_0\eta_4}{3}, \\
\alpha_{36} &= \frac{8}{3} - \frac{11}{6\eta_0} - \frac{7\eta_0}{3} + \frac{4\eta_0^2}{3} + \frac{17\eta_1}{3} - \frac{2\eta_1}{\eta_0} - \frac{4\eta_0\eta_1}{3} + 3\eta_2 - \frac{4\eta_0\eta_2}{3} + \frac{8\eta_4}{3} - \frac{2\eta_4}{\eta_0}, \\
\alpha_{37} &= \frac{16}{3} - \frac{11}{6\eta_0} - \frac{5\eta_0}{3} - \frac{2\eta_0^2}{3} + 11\eta_1 - \frac{2\eta_1}{\eta_0} - \frac{16\eta_0\eta_1}{3} + 3\eta_2 - \frac{4\eta_0\eta_2}{3} + 8\eta_4 - \frac{2\eta_4}{\eta_0} - 4\eta_0\eta_4, \\
\alpha_{38} &= \frac{1}{3} - \frac{1}{6\eta_0} - \frac{2\eta_0}{3} + \frac{2\eta_0^2}{3} + \frac{\eta_1}{3} + \frac{4\eta_0\eta_1}{3} + \eta_2 - \frac{2\eta_4}{3} + \frac{4\eta_0\eta_4}{3}, \\
\alpha_{39} &= \frac{4}{3} - \frac{1}{6\eta_0} - \eta_0 - \frac{2\eta_0^2}{3} + \frac{7\eta_1}{3} - \frac{4\eta_0\eta_1}{3} + \eta_2 + \frac{4\eta_4}{3} - \frac{4\eta_0\eta_4}{3}, \\
\alpha_{40} &= -\frac{1}{6\eta_0} - \eta_0 + \frac{2\eta_0^2}{3} - \frac{\eta_1}{3} + \frac{4\eta_0\eta_1}{3} + \eta_2 - \frac{4\eta_4}{3} + \frac{4\eta_0\eta_4}{3}, \\
\alpha_{41} &= 2 - \frac{1}{6\eta_0} - \frac{\eta_0}{3} - \frac{2\eta_0^2}{3} + \frac{11\eta_1}{3} - \frac{4\eta_0\eta_1}{3} + \eta_2 + \frac{8\eta_4}{3} - \frac{4\eta_0\eta_4}{3},
\end{aligned}$$

$$\begin{aligned}
\alpha_{42} &= 1 - \frac{1}{2\eta_0} - \eta_0 + \eta_1 + \eta_2, & \alpha_{43} &= 2 - \frac{1}{2\eta_0} + 3\eta_1 + \eta_2 + 2\eta_4, \\
\alpha_{44} &= \frac{\eta_0(-3 + 7\eta_0 - 4\eta_0^2)}{3 - 8\eta_0 + 6\eta_0^2}, \\
\alpha_{45} &= -1 - 2\eta_1 - 2\eta_4 - 2\eta_0^2(1 + 6\eta_1 + 2\eta_2 + 4\eta_4) + \eta_0(3 + 8\eta_1 + 8\eta_4), \\
\alpha_{46} &= \frac{1}{(3 - 8\eta_0 + 6\eta_0^2)^3} \left(\eta_0(48\eta_0^6 - 27(1 + 2\eta_1 + 2\eta_4) + 16\eta_0^5(-12 + 9\eta_1 + \eta_2 + 8\eta_4) + 27\eta_0(5 + 12\eta_1 + 12\eta_4) \right. \\
&\quad \left. - 24\eta_0^2(9 + 32\eta_1 + 32\eta_4) - 4\eta_0^4(-52 + 135\eta_1 + 3\eta_2 + 132\eta_4) + \eta_0^3(44 + 900\eta_1 + 900\eta_4) \right), \\
\alpha_{47} &= \frac{1}{(3 - 8\eta_0 + 6\eta_0^2)^2} \left(\eta_0(-36(1 + 2\eta_1 + 2\eta_4) + 48\eta_0^5(1 + 6\eta_1 + 2\eta_2 + 4\eta_4) \right. \\
&\quad \left. - 8\eta_0^4(31 + 129\eta_1 + 35\eta_2 + 94\eta_4) + 3\eta_0(67 + 154\eta_1 + 12\eta_2 + 142\eta_4) + \right. \\
&\quad \left. 2\eta_0^3(239 + 778\eta_1 + 166\eta_2 + 612\eta_4) - \eta_0^2(443 + 1196\eta_1 + 180\eta_2 + 1016\eta_4) \right), \\
\alpha_{48} &= -1 + 2\eta_1 + 2\eta_4 + \eta_0(-1 - 8\eta_1 - 8\eta_4) + \eta_0^2(2 + 12\eta_1 + 4\eta_2 + 8\eta_4).
\end{aligned}$$

$$D_1 = \frac{1}{2(1 - \eta_0)} \left(3 + 6\eta_1 + 6\eta_4 + 4\eta_0^2(3\eta_1 + \eta_2 + 2\eta_4) - 3\eta_0(1 + 4\eta_1 + 4\eta_4) \right),$$

$$\begin{aligned}
D_2 &= \frac{1}{3\eta_0(-1 + \eta_0)} \left(-8\eta_0^4 + \eta_0^3(15 + 2\eta_1 + 6\eta_2 - 4\eta_4) - 2\eta_0^2(3 + \eta_1 + 4\eta_2 - 3\eta_4) \right. \\
&\quad \left. + \eta_0(1 + \eta_1 + \eta_4) - 2(1 + 2\eta_1 + 2\eta_4) \right),
\end{aligned}$$

$$\begin{aligned}
\theta_1 &= \frac{1}{24(-1 + \eta_0)\eta_0^3} \left(-2\eta_0^3 + 2(1 + \eta_1 + \eta_4) + 2\eta_0^2(4 + 7\eta_1 + 3\eta_2 + 4\eta_4) \right. \\
&\quad \left. - \eta_0(7 + 12\eta_1 + 4\eta_2 + 8\eta_4) \right),
\end{aligned}$$

$$\begin{aligned}
\theta_2 &= -\frac{1}{24(-1 + \eta_0)\eta_0^3} \left(4\eta_0^4 - 2(1 + \eta_1 + \eta_4) + 2\eta_0^3(5 + 10\eta_1 + 2\eta_2 + 8\eta_4) \right. \\
&\quad \left. + \eta_0(13 + 18\eta_1 + 4\eta_2 + 14\eta_4) - 2\eta_0^2(11 + 21\eta_1 + 7\eta_2 + 14\eta_4) \right),
\end{aligned}$$

$$\begin{aligned}
\theta_3 &= \frac{1}{12(-1 + \eta_0)\eta_0^3} \left(-2\eta_0^3 + \eta_0^2(3 + 2\eta_1 + 2\eta_2) + 2(1 + \eta_1 + \eta_4) \right. \\
&\quad \left. - 4\eta_0(1 + 2\eta_1 + \eta_2 + \eta_4) \right),
\end{aligned}$$

$$\begin{aligned}
\theta_4 &= \frac{1}{12(-1 + \eta_0)\eta_0^3} \left(2\eta_0^4 + 2(1 + \eta_1 + \eta_4) - 4\eta_0^3(1 + 2\eta_1 + \eta_2 + \eta_4) \right. \\
&\quad \left. + 5\eta_0^2(3 + 6\eta_1 + 2\eta_2 + 4\eta_4) - 2\eta_0(6 + 9\eta_1 + 2\eta_2 + 7\eta_4) \right),
\end{aligned}$$

$$\theta_5 = -\frac{(\eta_0(1 + 4\eta_1 + 4\eta_2) - 2(1 + \eta_1 + \eta_4) + 2\eta_0^2(3\eta_1 + \eta_2 + 2\eta_4))}{24(-1 + \eta_0)\eta_0^3},$$

$$\begin{aligned}
\theta_6 &= \frac{1}{24(-1 + \eta_0)\eta_0^3} \left(2(1 + \eta_1 + \eta_4) - 4\eta_0^3(1 + 3\eta_1 + \eta_2 + 2\eta_4) \right. \\
&\quad \left. + 2\eta_0^2(5 + 11\eta_1 + 3\eta_2 + 8\eta_4) - \eta_0(11 + 18\eta_1 + 4\eta_2 + 14\eta_4) \right),
\end{aligned}$$

$$\theta_7 = \frac{(2\eta_0^2 + 4(1 + \eta_1 + \eta_4) - \eta_0(7 + 14\eta_1 + 6\eta_2 + 8\eta_4))}{48(-1 + \eta_0)\eta_0^2},$$

$$\begin{aligned}
\theta_8 &= \frac{(-1 + 4\eta_0^2 - 2\eta_1 + 2\eta_2 - 4\eta_4 + 8\eta_0(\eta_1 + \eta_4))}{24(-1 + \eta_0)\eta_0}, \\
\theta_9 &= -\frac{(8\eta_0^3 + 4(1 + \eta_1 + \eta_4) + 8\eta_0^2(1 + 2\eta_1 + 2\eta_4) - 5\eta_0(3 + 6\eta_1 + 2\eta_2 + 4\eta_4))}{48(-1 + \eta_0)\eta_0^2}, \\
\theta_{10} &= \frac{(4\eta_0^2 + 4(1 + \eta_1 + \eta_4) - \eta_0(5 + 10\eta_1 + 6\eta_2 + 4\eta_4))}{48(-1 + \eta_0)\eta_0^2}, \\
\theta_{11} &= \frac{(-5 + 4\eta_0^2 - 10\eta_1 + 2\eta_2 - 12\eta_4 + \eta_0(-4 + 8\eta_1 + 8\eta_4))}{24(-1 + \eta_0)\eta_0}, \\
\theta_{12} &= \frac{1}{48(-1 + \eta_0)\eta_0^2}(-8\eta_0^3 - 4(1 + \eta_1 + \eta_4) - 2\eta_0^2(1 + 8\eta_1 + 8\eta_4) \\
&\quad + \eta_0(21 + 42\eta_1 + 10\eta_2 + 32\eta_4)).
\end{aligned}$$

$$\begin{aligned}
H_1 &= \frac{1}{6} \left(9U_X U_{XY} V_X (2\nu_1 + \nu_4) + 9U_Y U_{YY} V_X (2\nu_1 + \nu_4) + \frac{9}{2} U_X^2 V_{XY} (2\nu_1 + \nu_4) \right. \\
&\quad + \frac{9}{2} U_Y^2 V_{XY} (2\nu_1 + \nu_4) + \frac{9}{2} V_X^2 V_{XY} (2\nu_1 + \nu_4) + \frac{9}{2} V_{XY} V_Y^2 (2\nu_1 + \nu_4) \\
&\quad + 9V_X V_Y V_{YY} (2\nu_1 + \nu_4) + 3U_{YY} V_X^2 (\lambda + 2\nu_1 + \nu_4) + 6U_Y V_X V_{XY} (\lambda + 2\nu_1 + \nu_4) \\
&\quad + 6U_{XY} V_X V_Y (\mu + 2\nu_1 + \nu_4) + 6U_X V_{XY} V_Y (\mu + 2\nu_1 + \nu_4) \\
&\quad + 6U_X V_X V_{YY} (\mu + 2\nu_1 + \nu_4) + 9U_Y^2 U_{YY} (\lambda + 2\mu + 2\nu_1 + \nu_4) \\
&\quad + 6U_{XY} U_Y V_Y (4\nu_1 + 2\nu_2 + \nu_4) + 6U_X U_{YY} V_Y (4\nu_1 + 2\nu_2 + \nu_4) \\
&\quad + 6U_X U_Y V_{YY} (4\nu_1 + 2\nu_2 + \nu_4) + 6U_X U_{XY} U_Y \left(\lambda + 2\mu + 7\nu_1 + 2\nu_2 + \frac{5\nu_4}{2} \right) \\
&\quad + 3U_X^2 U_{YY} \left(\lambda + 2\mu + 7\nu_1 + 2\nu_2 + \frac{5\nu_4}{2} \right) + 3U_{YY} V_Y^2 \left(\lambda + 2\mu + 7\nu_1 + 2\nu_2 + \frac{5\nu_4}{2} \right) \\
&\quad \left. + 6U_Y V_Y V_{YY} \left(\lambda + 2\mu + 7\nu_1 + 2\nu_2 + \frac{5\nu_4}{2} \right) \right) + \frac{1}{6} \left(18U_X^2 V_{XY} (\nu_1 + \nu_2) \right. \\
&\quad + 36U_X U_{XX} V_Y (\nu_1 + \nu_2) + 18V_{XY} V_Y^2 (\nu_1 + \nu_2) + 6U_X V_{XY} V_Y (\lambda + 4\nu_1 + 4\nu_2) \\
&\quad + 3U_{XX} V_Y^2 (\lambda + 4\nu_1 + 4\nu_2) + 9U_X U_{XY} V_X (2\nu_1 + \nu_4) + 9U_{XX} U_Y V_X (2\nu_1 + \nu_4) \\
&\quad + 9U_X U_Y V_{XX} (2\nu_1 + \nu_4) + 6U_Y V_X V_{XY} (\mu + 2\nu_1 + \nu_4) + 6U_{XY} V_X V_Y (\mu + 2\nu_1 + \nu_4) \\
&\quad + 6U_Y V_{XX} V_Y (\mu + 2\nu_1 + \nu_4) + 3U_Y^2 V_{XY} (4\nu_1 + 2\nu_2 + \nu_4) + 3V_X^2 V_{XY} (4\nu_1 + 2\nu_2 + \nu_4) \\
&\quad + 6U_{XY} U_Y V_Y (4\nu_1 + 2\nu_2 + \nu_4) + 6V_X V_{XX} V_Y (4\nu_1 + 2\nu_2 + \nu_4) \\
&\quad + 6U_X U_{XY} U_Y \left(\lambda + 2\mu + 7\nu_1 + 2\nu_2 + \frac{5\nu_4}{2} \right) + 3U_{XX} U_Y^2 \left(\lambda + 2\mu + 7\nu_1 + 2\nu_2 + \frac{5\nu_4}{2} \right) \\
&\quad + 3U_{XX} V_X^2 \left(\lambda + 2\mu + 7\nu_1 + 2\nu_2 + \frac{5\nu_4}{2} \right) + 6U_X V_X V_{XX} \left(\lambda + 2\mu + 7\nu_1 + 2\nu_2 + \frac{5\nu_4}{2} \right) \\
&\quad \left. + 9U_X^2 U_{XX} (\lambda + 2\mu + 12\nu_1 + 4\nu_2 + 4\nu_4) \right),
\end{aligned}$$

$$\begin{aligned}
H_2 &= \frac{1}{6} \left(\frac{9}{2} U_X^2 U_{XY} (2\nu_1 + \nu_4) + 9U_X U_{XX} U_Y (2\nu_1 + \nu_4) + \frac{9}{2} U_{XY} U_Y^2 (2\nu_1 + \nu_4) \right. \\
&\quad \left. + \frac{9}{2} U_{XY} V_X^2 (2\nu_1 + \nu_4) + 9U_Y V_X V_{XX} (2\nu_1 + \nu_4) + 9U_Y V_{XY} V_Y (2\nu_1 + \nu_4) \right)
\end{aligned}$$

$$\begin{aligned}
& +\frac{9}{2}U_{XY}V_Y^2(2\nu_1+\nu_4)+6U_{XY}U_YV_X(\lambda+2\nu_1+\nu_4)+3U_Y^2V_{XX}(\lambda+2\nu_1+\nu_4) \\
& +6U_XU_YV_{XY}(\mu+2\nu_1+\nu_4)+6U_XU_{XY}V_Y(\mu+2\nu_1+\nu_4)+6U_{XX}U_YV_Y(\mu+2\nu_1+\nu_4) \\
& +9V_X^2V_{XX}(\lambda+2\mu+2\nu_1+\nu_4)+6U_XV_XV_{XY}(4\nu_1+2\nu_2+\nu_4) \\
& +6U_{XX}V_XV_Y(4\nu_1+2\nu_2+\nu_4)+6U_XV_{XX}V_Y(4\nu_1+2\nu_2+\nu_4) \\
& +6U_XU_{XX}V_X\left(\lambda+2\mu+7\nu_1+2\nu_2+\frac{5\nu_4}{2}\right)+3U_X^2V_{XX}\left(\lambda+2\mu+7\nu_1+2\nu_2+\frac{5\nu_4}{2}\right) \\
& +6V_XV_{XY}V_Y\left(\lambda+2\mu+7\nu_1+2\nu_2+\frac{5\nu_4}{2}\right)+3V_{XX}V_Y^2\left(\lambda+2\mu+7\nu_1+2\nu_2+\frac{5\nu_4}{2}\right) \\
& +\frac{1}{6}\left(18U_X^2U_{XY}(\nu_1+\nu_2)+18U_{XY}V_Y^2(\nu_1+\nu_2)+36U_XV_YV_{YY}(\nu_1+\nu_2)\right) \\
& +6U_XU_{XY}V_Y(\lambda+4\nu_1+4\nu_2)+3U_X^2V_{YY}(\lambda+4\nu_1+4\nu_2)+9U_{YY}V_XV_Y(2\nu_1+\nu_4) \\
& +9U_YV_{XY}V_Y(2\nu_1+\nu_4)+9U_YV_XV_{YY}(2\nu_1+\nu_4)+6U_{XY}U_YV_X(\mu+2\nu_1+\nu_4) \\
& +6U_XU_{YY}V_X(\mu+2\nu_1+\nu_4)+6U_XU_YV_{XY}(\mu+2\nu_1+\nu_4)+3U_{XY}U_Y^2(4\nu_1+2\nu_2+\nu_4) \\
& +6U_XU_YU_{YY}(4\nu_1+2\nu_2+\nu_4)+3U_{XY}V_X^2(4\nu_1+2\nu_2+\nu_4) \\
& +6U_XV_XV_{XY}(4\nu_1+2\nu_2+\nu_4)+6U_YU_{YY}V_Y\left(\lambda+2\mu+7\nu_1+2\nu_2+\frac{5\nu_4}{2}\right) \\
& +6V_XV_{XY}V_Y\left(\lambda+2\mu+7\nu_1+2\nu_2+\frac{5\nu_4}{2}\right)+3U_Y^2V_{YY}\left(\lambda+2\mu+7\nu_1+2\nu_2+\frac{5\nu_4}{2}\right) \\
& +3V_X^2V_{YY}\left(\lambda+2\mu+7\nu_1+2\nu_2+\frac{5\nu_4}{2}\right)+9V_Y^2V_{YY}(\lambda+2\mu+12\nu_1+4\nu_2+4\nu_4),
\end{aligned}$$

$$\begin{aligned}
H_3 &= 6u_xu_{xx}v_y(\eta_1+\eta_2)+u_{xx}v_y^2\left(\frac{1}{2}-\eta_0+2\eta_1+2\eta_2\right)+6u_xu_{xy}v_x(\eta_1+\eta_4) \\
& +3u_{xx}u_yv_x(\eta_1+\eta_4)+3u_yu_{yy}v_x(\eta_1+\eta_4)+3u_xu_yv_{xx}(\eta_1+\eta_4)+3v_xv_yv_{yy}(\eta_1+\eta_4) \\
& +u_{yy}v_x^2\left(\frac{1}{2}-\eta_0+\eta_1+\eta_4\right)+4u_{xy}u_yv_y(2\eta_1+\eta_2+\eta_4)+2u_xu_{yy}v_y(2\eta_1+\eta_2+\eta_4) \\
& +2v_xv_{xx}v_y(2\eta_1+\eta_2+\eta_4)+2u_xu_yv_{yy}(2\eta_1+\eta_2+\eta_4)+\frac{3}{2}u_x^2v_{xy}(3\eta_1+2\eta_2+\eta_4) \\
& +\frac{3}{2}v_{xy}v_y^2(3\eta_1+2\eta_2+\eta_4)+u_xv_{xy}v_y(1-\eta_0+6\eta_1+4\eta_2+2\eta_4) \\
& +u_y^2v_{xy}\left(\frac{7\eta_1}{2}+\eta_2+\frac{5\eta_4}{2}\right)+v_x^2v_{xy}\left(\frac{7\eta_1}{2}+\eta_2+\frac{5\eta_4}{2}\right)+u_y^2u_{yy}\left(\frac{3}{2}+3\eta_1+3\eta_4\right) \\
& +u_yv_xv_{xy}(1-\eta_0+4\eta_1+4\eta_4)+2u_xu_{xy}u_y(1+7\eta_1+2\eta_2+5\eta_4) \\
& +\frac{1}{2}u_{xx}u_y^2(1+7\eta_1+2\eta_2+5\eta_4)+\frac{1}{2}u_x^2u_{yy}(1+7\eta_1+2\eta_2+5\eta_4) \\
& +\frac{1}{2}u_{xx}v_x^2(1+7\eta_1+2\eta_2+5\eta_4)+u_xv_xv_{xx}(1+7\eta_1+2\eta_2+5\eta_4) \\
& +\frac{1}{2}u_{yy}v_y^2(1+7\eta_1+2\eta_2+5\eta_4)+u_yv_yv_{yy}(1+7\eta_1+2\eta_2+5\eta_4) \\
& +u_x^2u_{xx}\left(\frac{3}{2}+18\eta_1+6\eta_2+12\eta_4\right)+2u_{xy}v_xv_y(\eta_0+2(\eta_1+\eta_4)) \\
& +u_yv_{xx}v_y(\eta_0+2(\eta_1+\eta_4))+u_xv_xv_{yy}(\eta_0+2(\eta_1+\eta_4)),
\end{aligned}$$

$$\begin{aligned}
H_4 = & 6u_x v_y v_{yy} (\eta_1 + \eta_2) + u_x^2 v_{yy} \left(\frac{1}{2} - \eta_0 + 2\eta_1 + 2\eta_2 \right) + 3u_x u_{xx} u_y (\eta_1 + \eta_4) \\
& + 3u_y v_x v_{xx} (\eta_1 + \eta_4) + 3u_{yy} v_x v_y (\eta_1 + \eta_4) + 6u_y v_{xy} v_y (\eta_1 + \eta_4) + 3u_y v_x v_{yy} (\eta_1 + \eta_4) \\
& + u_y^2 v_{xx} \left(\frac{1}{2} - \eta_0 + \eta_1 + \eta_4 \right) + 2u_x u_y u_{yy} (2\eta_1 + \eta_2 + \eta_4) + 4u_x v_x v_{xy} (2\eta_1 + \eta_2 + \eta_4) \\
& + 2u_{xx} v_x v_y (2\eta_1 + \eta_2 + \eta_4) + 2u_x v_{xx} v_y (2\eta_1 + \eta_2 + \eta_4) + \frac{3}{2} u_x^2 u_{xy} (3\eta_1 + 2\eta_2 + \eta_4) \\
& + \frac{3}{2} u_{xy} v_y^2 (3\eta_1 + 2\eta_2 + \eta_4) + u_x u_{xy} v_y (1 - \eta_0 + 6\eta_1 + 4\eta_2 + 2\eta_4) \\
& + u_{xy} u_y^2 \left(\frac{7\eta_1}{2} + \eta_2 + \frac{5\eta_4}{2} \right) + u_{xy} v_x^2 \left(\frac{7\eta_1}{2} + \eta_2 + \frac{5\eta_4}{2} \right) + v_x^2 v_{xx} \left(\frac{3}{2} + 3\eta_1 + 3\eta_4 \right) \\
& + u_{xy} u_y v_x (1 - \eta_0 + 4\eta_1 + 4\eta_4) + u_x u_{xx} v_x (1 + 7\eta_1 + 2\eta_2 + 5\eta_4) \\
& + \frac{1}{2} u_x^2 v_{xx} (1 + 7\eta_1 + 2\eta_2 + 5\eta_4) + u_y u_{yy} v_y (1 + 7\eta_1 + 2\eta_2 + 5\eta_4) \\
& + 2v_x v_{xy} v_y (1 + 7\eta_1 + 2\eta_2 + 5\eta_4) + \frac{1}{2} v_{xx} v_y^2 (1 + 7\eta_1 + 2\eta_2 + 5\eta_4) \\
& + \frac{1}{2} u_y^2 v_{yy} (1 + 7\eta_1 + 2\eta_2 + 5\eta_4) + \frac{1}{2} v_x^2 v_{yy} (1 + 7\eta_1 + 2\eta_2 + 5\eta_4) \\
& + v_y^2 v_{yy} \left(\frac{3}{2} + 18\eta_1 + 6\eta_2 + 12\eta_4 \right) + u_x u_{yy} v_x (\eta_0 + 2(\eta_1 + \eta_4)) \\
& + 2u_x u_y v_{xy} (\eta_0 + 2(\eta_1 + \eta_4)) + u_{xx} u_y v_y (\eta_0 + 2(\eta_1 + \eta_4)),
\end{aligned}$$

$$\begin{aligned}
H_5 = & (1 + 7\eta_1 + 2\eta_2 + 5\eta_4) u_2 v_1^2 + 4(2\eta_1 + \eta_2 + \eta_4) u_2 v_1 u_{0x} \\
& + (1 + 7\eta_1 + 2\eta_2 + 5\eta_4) u_2 u_{0x}^2 + \frac{3}{2} (3\eta_1 + 2\eta_2 + \eta_4) v_1^2 v_{1x} \\
& + (1 - \eta_0 + 6\eta_1 + 4\eta_2 + 2\eta_4) v_1 u_{0x} v_{1x} + \frac{3}{2} (3\eta_1 + 2\eta_2 + \eta_4) u_{0x}^2 v_{1x} \\
& + \left(\frac{1}{2} - \eta_0 + 2\eta_1 + 2\eta_2 \right) v_1^2 u_{0xx} + 6(\eta_1 + \eta_2) v_1 u_{0x} u_{0xx} \\
& + \left(\frac{3}{2} + 18\eta_1 + 6\eta_2 + 12\eta_4 \right) u_{0x}^2 u_{0xx},
\end{aligned}$$

$$\begin{aligned}
H_6 = & 8 \left(\frac{3}{2} + 3\eta_1 + 3\eta_4 \right) u_1 u_2^2 \\
& + 3(1 + 7\eta_1 + 2\eta_2 + 5\eta_4) u_3 v_1^2 + 8(1 + 7\eta_1 + 2\eta_2 + 5\eta_4) u_2 v_1 v_2 \\
& + 6(1 + 7\eta_1 + 2\eta_2 + 5\eta_4) u_1 v_1 v_3 + 12(2\eta_1 + \eta_2 + \eta_4) u_3 v_1 u_{0x} \\
& + 16(2\eta_1 + \eta_2 + \eta_4) u_2 v_2 u_{0x} + 12(2\eta_1 + \eta_2 + \eta_4) u_1 v_3 u_{0x} \\
& + 3(1 + 7\eta_1 + 2\eta_2 + 5\eta_4) u_3 u_{0x}^2 + 12(2\eta_1 + \eta_2 + \eta_4) u_2 v_1 u_{1x} \\
& + 6(1 + 7\eta_1 + 2\eta_2 + 5\eta_4) u_2 u_{0x} u_{1x} + 8(2\eta_1 + \eta_2 + \eta_4) u_1 v_1 u_{2x} \\
& + 4(1 + 7\eta_1 + 2\eta_2 + 5\eta_4) u_1 u_{0x} u_{2x} + 12(\eta_1 + \eta_4) u_2^2 v_{0x} \\
& + 18(\eta_1 + \eta_4) v_1 v_3 v_{0x} + 6(\eta_0 + 2(\eta_1 + \eta_4)) v_3 u_{0x} v_{0x} \\
& + 4(\eta_0 + 2(\eta_1 + \eta_4)) v_1 u_{2x} v_{0x} + 12(\eta_1 + \eta_4) u_{0x} u_{2x} v_{0x} \\
& + 6(\eta_1 + \eta_4) u_1 u_2 v_{1x} + 4 \left(\frac{7\eta_1}{2} + \eta_2 + \frac{5\eta_4}{2} \right) u_1 u_2 v_{1x}
\end{aligned}$$

$$\begin{aligned}
& +6(\eta_1 + \eta_4)v_1v_2v_{1x} + 6(3\eta_1 + 2\eta_2 + \eta_4)v_1v_2v_{1x} \\
& +2(1 - \eta_0 + 6\eta_1 + 4\eta_2 + 2\eta_4)v_2u_{0x}v_{1x} + 2(\eta_0 + 2(\eta_1 + \eta_4))v_2u_{0x}v_{1x} \\
& + (1 - \eta_0 + 6\eta_1 + 4\eta_2 + 2\eta_4)v_1u_{1x}v_{1x} + 2(\eta_0 + 2(\eta_1 + \eta_4))v_1u_{1x}v_{1x} \\
& +6(\eta_1 + \eta_4)u_{0x}u_{1x}v_{1x} + 3(3\eta_1 + 2\eta_2 + \eta_4)u_{0x}u_{1x}v_{1x} \\
& +4\left(\frac{1}{2} - \eta_0 + \eta_1 + \eta_4\right)u_2v_{0x}v_{1x} + 2(1 - \eta_0 + 4\eta_1 + 4\eta_4)u_2v_{0x}v_{1x} \\
& + (1 - \eta_0 + 4\eta_1 + 4\eta_4)u_1v_{1x}^2 + 2\left(\frac{7\eta_1}{2} + \eta_2 + \frac{5\eta_4}{2}\right)v_{0x}v_{1x}^2 \\
& +3(3\eta_1 + 2\eta_2 + \eta_4)v_1^2v_{2x} + 2(1 - \eta_0 + 6\eta_1 + 4\eta_2 + 2\eta_4)v_1u_{0x}v_{2x} \\
& +3(3\eta_1 + 2\eta_2 + \eta_4)u_{0x}^2v_{2x} + 2(1 + 7\eta_1 + 2\eta_2 + 5\eta_4)u_1u_2u_{0xx} \\
& +4\left(\frac{1}{2} - \eta_0 + 2\eta_1 + 2\eta_2\right)v_1v_2u_{0xx} + 12(\eta_1 + \eta_2)v_2u_{0x}u_{0xx} \\
& +6(\eta_1 + \eta_2)v_1u_{1x}u_{0xx} + 2\left(\frac{3}{2} + 18\eta_1 + 6\eta_2 + 12\eta_4\right)u_{0x}u_{1x}u_{0xx} \\
& +6(\eta_1 + \eta_4)u_2v_{0x}u_{0xx} + 3(\eta_1 + \eta_4)u_1v_{1x}u_{0xx} \\
& + (1 + 7\eta_1 + 2\eta_2 + 5\eta_4)v_{0x}v_{1x}u_{0xx} + \left(\frac{1}{2} - \eta_0 + 2\eta_1 + 2\eta_2\right)v_1^2u_{1xx} \\
& +6(\eta_1 + \eta_2)v_1u_{0x}u_{1xx} + \left(\frac{3}{2} + 18\eta_1 + 6\eta_2 + 12\eta_4\right)u_{0x}^2u_{1xx} \\
& +2(\eta_0 + 2(\eta_1 + \eta_4))u_2v_1v_{0xx} + 6(\eta_1 + \eta_4)u_2u_{0x}v_{0xx} \\
& +2(2\eta_1 + \eta_2 + \eta_4)v_1v_{1x}v_{0xx} + (1 + 7\eta_1 + 2\eta_2 + 5\eta_4)u_{0x}v_{1x}v_{0xx} \\
& + (\eta_0 + 2(\eta_1 + \eta_4))u_1v_1v_{1xx} + 3(\eta_1 + \eta_4)u_1u_{0x}v_{1xx} \\
& +2(2\eta_1 + \eta_2 + \eta_4)v_1v_{0x}v_{1xx} + (1 + 7\eta_1 + 2\eta_2 + 5\eta_4)u_{0x}v_{0x}v_{1xx}, \\
H_7 = & 8\left(\frac{3}{2} + 3\eta_1 + 3\eta_4\right)u_2^3 + 6(1 + 7\eta_1 + 2\eta_2 + 5\eta_4)u_4v_1^2 \\
& +18(1 + 7\eta_1 + 2\eta_2 + 5\eta_4)u_2v_1v_3 + 24(2\eta_1 + \eta_2 + \eta_4)u_4v_1u_{0x} \\
& +36(2\eta_1 + \eta_2 + \eta_4)u_2v_3u_{0x} + 6(1 + 7\eta_1 + 2\eta_2 + 5\eta_4)u_4u_{0x}^2 \\
& +20(2\eta_1 + \eta_2 + \eta_4)u_2v_1u_{2x} + 10(1 + 7\eta_1 + 2\eta_2 + 5\eta_4)u_2u_{0x}u_{2x} \\
& +12(\eta_1 + \eta_4)u_2^2v_{1x} + (14\eta_1 + 4\eta_2 + 10\eta_4)u_2^2v_{1x} + 18(\eta_1 + \eta_4)v_1v_3v_{1x} \\
& +9(3\eta_1 + 2\eta_2 + \eta_4)v_1v_3v_{1x} + 3(1 - \eta_0 + 6\eta_1 + 4\eta_2 + 2\eta_4)v_3u_{0x}v_{1x} \\
& +6(\eta_0 + 2(\eta_1 + \eta_4))v_3u_{0x}v_{1x} + (1 - \eta_0 + 6\eta_1 + 4\eta_2 + 2\eta_4)v_1u_{2x}v_{1x} \\
& +4(\eta_0 + 2(\eta_1 + \eta_4))v_1u_{2x}v_{1x} + 12(\eta_1 + \eta_4)u_{0x}u_{2x}v_{1x} \\
& +3(3\eta_1 + 2\eta_2 + \eta_4)u_{0x}u_{2x}v_{1x} + 2\left(\frac{1}{2} - \eta_0 + \eta_1 + \eta_4\right)u_2v_{1x}^2 \\
& +2(1 - \eta_0 + 4\eta_1 + 4\eta_4)u_2v_{1x}^2 + \left(\frac{7\eta_1}{2} + \eta_2 + \frac{5\eta_4}{2}\right)v_{1x}^3 \\
& +\frac{9}{2}(3\eta_1 + 2\eta_2 + \eta_4)v_1^2v_{3x} + 3(1 - \eta_0 + 6\eta_1 + 4\eta_2 + 2\eta_4)v_1u_{0x}v_{3x} \\
& +\frac{9}{2}(3\eta_1 + 2\eta_2 + \eta_4)u_{0x}^2v_{3x} + 2(1 + 7\eta_1 + 2\eta_2 + 5\eta_4)u_2^2u_{0xx} \\
& +6\left(\frac{1}{2} - \eta_0 + 2\eta_1 + 2\eta_2\right)v_1v_3u_{0xx} + 18(\eta_1 + \eta_2)v_3u_{0x}u_{0xx}
\end{aligned}$$

$$\begin{aligned}
& +6(\eta_1 + \eta_2) v_1 u_{2x} u_{0xx} + 2 \left(\frac{3}{2} + 18\eta_1 + 6\eta_2 + 12\eta_4 \right) u_{0x} u_{2x} u_{0xx} \\
& +6(\eta_1 + \eta_4) u_2 v_{1x} u_{0xx} + \frac{1}{2} (1 + 7\eta_1 + 2\eta_2 + 5\eta_4) v_{1x}^2 u_{0xx} \\
& + \left(\frac{1}{2} - \eta_0 + 2\eta_1 + 2\eta_2 \right) v_1^2 u_{2xx} + 6(\eta_1 + \eta_2) v_1 u_{0x} u_{2xx} \\
& + \left(\frac{3}{2} + 18\eta_1 + 6\eta_2 + 12\eta_4 \right) u_{0x}^2 u_{2xx} + 2(\eta_0 + 2(\eta_1 + \eta_4)) u_2 v_1 v_{1xx} \\
& +6(\eta_1 + \eta_4) u_2 u_{0x} v_{1xx} + 2(2\eta_1 + \eta_2 + \eta_4) v_1 v_{1x} v_{1xx} \\
& + (1 + 7\eta_1 + 2\eta_2 + 5\eta_4) u_{0x} v_{1x} v_{1xx},
\end{aligned}$$

$$\begin{aligned}
H_8 = & 2(1 + 7\eta_1 + 2\eta_2 + 5\eta_4) u_1 u_2 v_1 + 2 \left(\frac{3}{2} + 18\eta_1 + 6\eta_2 + 12\eta_4 \right) v_1^2 v_2 \\
& +4(2\eta_1 + \eta_2 + \eta_4) u_1 u_2 u_{0x} + 12(\eta_1 + \eta_2) v_1 v_2 u_{0x} \\
& +2 \left(\frac{1}{2} - \eta_0 + 2\eta_1 + 2\eta_2 \right) v_2 u_{0x}^2 + \frac{3}{2} (3\eta_1 + 2\eta_2 + \eta_4) v_1^2 u_{1x} \\
& + (1 - \eta_0 + 6\eta_1 + 4\eta_2 + 2\eta_4) v_1 u_{0x} u_{1x} + \frac{3}{2} (3\eta_1 + 2\eta_2 + \eta_4) u_{0x}^2 u_{1x} \\
& +6(\eta_1 + \eta_4) u_2 v_1 v_{0x} + 2(\eta_0 + 2(\eta_1 + \eta_4)) u_2 u_{0x} v_{0x} \\
& +6(\eta_1 + \eta_4) u_1 v_1 v_{1x} + 2(\eta_0 + 2(\eta_1 + \eta_4)) u_1 u_{0x} v_{1x} \\
& +2(1 + 7\eta_1 + 2\eta_2 + 5\eta_4) v_1 v_{0x} v_{1x} + 4(2\eta_1 + \eta_2 + \eta_4) u_{0x} v_{0x} v_{1x} \\
& +(\eta_0 + 2(\eta_1 + \eta_4)) u_1 v_1 u_{0xx} + 3(\eta_1 + \eta_4) u_1 u_{0x} u_{0xx} \\
& +2(2\eta_1 + \eta_2 + \eta_4) v_1 v_{0x} u_{0xx} + (1 + 7\eta_1 + 2\eta_2 + 5\eta_4) u_{0x} v_{0x} u_{0xx} \\
& +\frac{1}{2} (1 + 7\eta_1 + 2\eta_2 + 5\eta_4) v_1^2 v_{0xx} + 2(2\eta_1 + \eta_2 + \eta_4) v_1 u_{0x} v_{0xx} \\
& +\frac{1}{2} (1 + 7\eta_1 + 2\eta_2 + 5\eta_4) u_{0x}^2 v_{0xx},
\end{aligned}$$

$$\begin{aligned}
H_9 = & 4(1 + 7\eta_1 + 2\eta_2 + 5\eta_4) u_2^2 v_1 + 6 \left(\frac{3}{2} + 18\eta_1 + 6\eta_2 + 12\eta_4 \right) v_1^2 v_3 \\
& +8(2\eta_1 + \eta_2 + \eta_4) u_2^2 u_{0x} + 36(\eta_1 + \eta_2) v_1 v_3 u_{0x} \\
& +6 \left(\frac{1}{2} - \eta_0 + 2\eta_1 + 2\eta_2 \right) v_3 u_{0x}^2 + 3(3\eta_1 + 2\eta_2 + \eta_4) v_1^2 u_{2x} \\
& +2(1 - \eta_0 + 6\eta_1 + 4\eta_2 + 2\eta_4) v_1 u_{0x} u_{2x} + 3(3\eta_1 + 2\eta_2 + \eta_4) u_{0x}^2 u_{2x} + \\
& 18(\eta_1 + \eta_4) u_2 v_1 v_{1x} + 6(\eta_0 + 2(\eta_1 + \eta_4)) u_2 u_{0x} v_{1x} \\
& +2(1 + 7\eta_1 + 2\eta_2 + 5\eta_4) v_1 v_{1x}^2 + 4(2\eta_1 + \eta_2 + \eta_4) u_{0x} v_{1x}^2 + \\
& 2(\eta_0 + 2(\eta_1 + \eta_4)) u_2 v_1 u_{0xx} + 6(\eta_1 + \eta_4) u_2 u_{0x} u_{0xx} \\
& +2(2\eta_1 + \eta_2 + \eta_4) v_1 v_{1x} u_{0xx} + (1 + 7\eta_1 + 2\eta_2 + 5\eta_4) u_{0x} v_{1x} u_{0xx} \\
& +\frac{1}{2} (1 + 7\eta_1 + 2\eta_2 + 5\eta_4) v_1^2 v_{1xx} + 2(2\eta_1 + \eta_2 + \eta_4) v_1 u_{0x} v_{1xx} \\
& +\frac{1}{2} (1 + 7\eta_1 + 2\eta_2 + 5\eta_4) u_{0x}^2 v_{1xx},
\end{aligned}$$

$$H_{10} = 18(1 + 7\eta_1 + 2\eta_2 + 5\eta_4) u_2 u_3 v_1 + 12(1 + 7\eta_1 + 2\eta_2 + 5\eta_4) u_1 u_4 v_1$$

$$\begin{aligned}
& +12(1+7\eta_1+2\eta_2+5\eta_4)u_2^2v_2+18(1+7\eta_1+2\eta_2+5\eta_4)u_1u_2v_3 \\
& +36\left(\frac{3}{2}+18\eta_1+6\eta_2+12\eta_4\right)v_1v_2v_3+12\left(\frac{3}{2}+18\eta_1+6\eta_2+12\eta_4\right)v_1^2v_4 \\
& +36(2\eta_1+\eta_2+\eta_4)u_2u_3u_{0x}+24(2\eta_1+\eta_2+\eta_4)u_1u_4u_{0x} \\
& +108(\eta_1+\eta_2)v_2v_3u_{0x}+72(\eta_1+\eta_2)v_1v_4u_{0x} \\
& +12\left(\frac{1}{2}-\eta_0+2\eta_1+2\eta_2\right)v_4u_{0x}^2+8(2\eta_1+\eta_2+\eta_4)u_2^2u_{1x} \\
& +4\left(\frac{7\eta_1}{2}+\eta_2+\frac{5\eta_4}{2}\right)u_2^2u_{1x}+36(\eta_1+\eta_2)v_1v_3u_{1x} \\
& +9(3\eta_1+2\eta_2+\eta_4)v_1v_3u_{1x}+12\left(\frac{1}{2}-\eta_0+2\eta_1+2\eta_2\right)v_3u_{0x}u_{1x} \\
& +3(1-\eta_0+6\eta_1+4\eta_2+2\eta_4)v_3u_{0x}u_{1x}+4(2\eta_1+\eta_2+\eta_4)u_1u_2u_{2x} \\
& +8\left(\frac{7\eta_1}{2}+\eta_2+\frac{5\eta_4}{2}\right)u_1u_2u_{2x}+12(\eta_1+\eta_2)v_1v_2u_{2x} \\
& +12(3\eta_1+2\eta_2+\eta_4)v_1v_2u_{2x}+4\left(\frac{1}{2}-\eta_0+2\eta_1+2\eta_2\right)v_2u_{0x}u_{2x} \\
& +4(1-\eta_0+6\eta_1+4\eta_2+2\eta_4)v_2u_{0x}u_{2x} \\
& +3(1-\eta_0+6\eta_1+4\eta_2+2\eta_4)v_1u_{1x}u_{2x}+9(3\eta_1+2\eta_2+\eta_4)u_{0x}u_{1x}u_{2x} \\
& +\frac{9}{2}(3\eta_1+2\eta_2+\eta_4)v_1^2u_{3x}+3(1-\eta_0+6\eta_1+4\eta_2+2\eta_4)v_1u_{0x}u_{3x} \\
& +\frac{9}{2}(3\eta_1+2\eta_2+\eta_4)u_{0x}^2u_{3x}+36(\eta_1+\eta_4)u_4v_1v_{0x} \\
& +54(\eta_1+\eta_4)u_2v_3v_{0x}+12(\eta_0+2(\eta_1+\eta_4))u_4u_{0x}v_{0x} \\
& +4(1-\eta_0+4\eta_1+4\eta_4)u_2u_{2x}v_{0x}+2(\eta_0+2(\eta_1+\eta_4))u_2u_{2x}v_{0x} \\
& +36(\eta_1+\eta_4)u_3v_1v_{1x}+48(\eta_1+\eta_4)u_2v_2v_{1x} \\
& +36(\eta_1+\eta_4)u_1v_3v_{1x}+12(\eta_0+2(\eta_1+\eta_4))u_3u_{0x}v_{1x} \\
& +2(1-\eta_0+4\eta_1+4\eta_4)u_2u_{1x}v_{1x}+6(\eta_0+2(\eta_1+\eta_4))u_2u_{1x}v_{1x} \\
& +2(1-\eta_0+4\eta_1+4\eta_4)u_1u_{2x}v_{1x}+2(\eta_0+2(\eta_1+\eta_4))u_1u_{2x}v_{1x} \\
& +12(1+7\eta_1+2\eta_2+5\eta_4)v_3v_{0x}v_{1x}+4(2\eta_1+\eta_2+\eta_4)u_{2x}v_{0x}v_{1x} \\
& +4\left(\frac{7\eta_1}{2}+\eta_2+\frac{5\eta_4}{2}\right)u_{2x}v_{0x}v_{1x} \\
& +5(1+7\eta_1+2\eta_2+5\eta_4)v_2v_{1x}^2+4(2\eta_1+\eta_2+\eta_4)u_{1x}v_{1x}^2 \\
& +\left(\frac{7\eta_1}{2}+\eta_2+\frac{5\eta_4}{2}\right)u_{1x}v_{1x}^2+30(\eta_1+\eta_4)u_2v_1v_{2x} \\
& +10(\eta_0+2(\eta_1+\eta_4))u_2u_{0x}v_{2x}+6(1+7\eta_1+2\eta_2+5\eta_4)v_1v_{1x}v_{2x} \\
& +12(2\eta_1+\eta_2+\eta_4)u_{0x}v_{1x}v_{2x}+18(\eta_1+\eta_4)u_1v_1v_{3x} \\
& +6(\eta_0+2(\eta_1+\eta_4))u_1u_{0x}v_{3x}+6(1+7\eta_1+2\eta_2+5\eta_4)v_1v_{0x}v_{3x} \\
& +12(2\eta_1+\eta_2+\eta_4)u_{0x}v_{0x}v_{3x}+3(\eta_0+2(\eta_1+\eta_4))u_3v_1u_{0xx} \\
& +4(\eta_0+2(\eta_1+\eta_4))u_2v_2u_{0xx}+3(\eta_0+2(\eta_1+\eta_4))u_1v_3u_{0xx} \\
& +9(\eta_1+\eta_4)u_3u_{0x}u_{0xx}+6(\eta_1+\eta_4)u_2u_{1x}u_{0xx}+3(\eta_1+\eta_4)u_1u_{2x}u_{0xx} \\
& +6(2\eta_1+\eta_2+\eta_4)v_3v_{0x}u_{0xx}+(1+7\eta_1+2\eta_2+5\eta_4)u_{2x}v_{0x}u_{0xx} \\
& +4(2\eta_1+\eta_2+\eta_4)v_2v_{1x}u_{0xx}+(1+7\eta_1+2\eta_2+5\eta_4)u_{1x}v_{1x}u_{0xx}
\end{aligned}$$

$$\begin{aligned}
& +2(2\eta_1 + \eta_2 + \eta_4) v_1 v_{2x} u_{0xx} + (1 + 7\eta_1 + 2\eta_2 + 5\eta_4) u_{0x} v_{2x} u_{0xx} \\
& +2(\eta_0 + 2(\eta_1 + \eta_4)) u_2 v_1 u_{1xx} + 6(\eta_1 + \eta_4) u_2 u_{0x} u_{1xx} \\
& +2(2\eta_1 + \eta_2 + \eta_4) v_1 v_{1x} u_{1xx} + (1 + 7\eta_1 + 2\eta_2 + 5\eta_4) u_{0x} v_{1x} u_{1xx} \\
& +(\eta_0 + 2(\eta_1 + \eta_4)) u_1 v_1 u_{2xx} + 3(\eta_1 + \eta_4) u_1 u_{0x} u_{2xx} \\
& +2(2\eta_1 + \eta_2 + \eta_4) v_1 v_{0x} u_{2xx} + (1 + 7\eta_1 + 2\eta_2 + 5\eta_4) u_{0x} v_{0x} u_{2xx} \\
& +4\left(\frac{1}{2} - \eta_0 + \eta_1 + \eta_4\right) u_2^2 v_{0xx} + 3(1 + 7\eta_1 + 2\eta_2 + 5\eta_4) v_1 v_3 v_{0xx} \\
& +6(2\eta_1 + \eta_2 + \eta_4) v_3 u_{0x} v_{0xx} + 2(2\eta_1 + \eta_2 + \eta_4) v_1 u_{2x} v_{0xx} \\
& + (1 + 7\eta_1 + 2\eta_2 + 5\eta_4) u_{0x} u_{2x} v_{0xx} + 6(\eta_1 + \eta_4) u_2 v_{1x} v_{0xx} \\
& + \left(\frac{3}{2} + 3\eta_1 + 3\eta_4\right) v_{1x}^2 v_{0xx} + 4\left(\frac{1}{2} - \eta_0 + \eta_1 + \eta_4\right) u_1 u_2 v_{1xx} \\
& +2(1 + 7\eta_1 + 2\eta_2 + 5\eta_4) v_1 v_2 v_{1xx} + 4(2\eta_1 + \eta_2 + \eta_4) v_2 u_{0x} v_{1xx} \\
& +2(2\eta_1 + \eta_2 + \eta_4) v_1 u_{1x} v_{1xx} + (1 + 7\eta_1 + 2\eta_2 + 5\eta_4) u_{0x} u_{1x} v_{1xx} \\
& +6(\eta_1 + \eta_4) u_2 v_{0x} v_{1xx} + 3(\eta_1 + \eta_4) u_1 v_{1x} v_{1xx} \\
& +2\left(\frac{3}{2} + 3\eta_1 + 3\eta_4\right) v_{0x} v_{1x} v_{1xx} + \frac{1}{2}(1 + 7\eta_1 + 2\eta_2 + 5\eta_4) v_1^2 v_{2xx} \\
& +2(2\eta_1 + \eta_2 + \eta_4) v_1 u_{0x} v_{2xx} + \frac{1}{2}(1 + 7\eta_1 + 2\eta_2 + 5\eta_4) u_{0x}^2 v_{2xx},
\end{aligned}$$

$$\begin{aligned}
H_{11} &= (-1 - \eta_1 - \eta_4) u_1 u_2 + \left(1 - \eta_0 + \frac{3\eta_1}{2} - 3\eta_0\eta_1 - \eta_0\eta_2 + \frac{3\eta_4}{2} - 2\eta_0\eta_4\right) v_1 u_{1x} \\
& + \left(-\eta_0 + \eta_0^2 - \frac{\eta_1}{2} - \eta_0\eta_1 - \eta_0\eta_2 - \frac{\eta_4}{2}\right) u_{0x} u_{1x} + (-\eta_0 - \eta_1 - \eta_4) u_2 v_{0x} \\
& + (-\eta_0 - \eta_1 - \eta_4) u_1 v_{1x} + (-1 - \eta_1 - \eta_4) v_{0x} v_{1x} \\
& + \left(-\frac{\eta_0}{2} - \frac{\eta_1}{2} - \frac{\eta_4}{2}\right) u_1 u_{0xx} + \left(-\frac{1}{2} - \frac{\eta_1}{2} - \frac{\eta_4}{2}\right) v_{0x} u_{0xx} \\
& + \left(-\frac{1}{2} + \frac{3\eta_0}{2} - \frac{\eta_1}{2} + 3\eta_0\eta_1 + \eta_0\eta_2 - \frac{\eta_4}{2} + 2\eta_0\eta_4\right) v_1 v_{0xx} \\
& + \left(-\frac{1}{2} + \frac{\eta_0}{2} - \eta_0^2 - \frac{\eta_1}{2} + \eta_0\eta_1 + \eta_0\eta_2 - \frac{\eta_4}{2}\right) u_{0x} v_{0xx}.
\end{aligned}$$

REFERENCES

- [1] D. L. RUSSELL AND L. W. WHITE, *An Elementary Nonlinear Beam Theory with Finite Buckling Deformation Properties*, SIAM J. Appl. Math., 62 (2002), pp. 1394–1413.
- [2] H. C. SIMPSON AND S. J. SPECTOR, *On barrelling instabilities in finite elasticity*, J. Elasticity, 14 (1984), pp. 103–125.
- [3] H. C. SIMPSON AND S. J. SPECTOR, *On barrelling for a special material in finite elasticity*, Quart. Appl. Math., 42 (1984), pp. 99–111.
- [4] P. J. DAVIES, *Buckling and barrelling instabilities in finite elasticity*, J. Elasticity, 21 (1989), pp. 147–192.
- [5] P. J. DAVIES, *Buckling and barrelling instabilities of nonlinearly elastic columns*, Quart. Appl. Math., 49 (1991), pp. 407–426.
- [6] A. GORIELY, R. VANDIVER AND M. DESTRADE, *Nonlinear Euler buckling*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 464 (2008), pp. 3003–3019.
- [7] H. C. SIMPSON AND S. J. SPECTOR, *On bifurcation in finite elasticity: Buckling of a rectangular rod*, J. Elasticity, 92 (2008), pp. 277–326.
- [8] T. J. HEALEY AND E. L. MONTES-PIZARRO, *Global Bifurcation in Nonlinear Elasticity with an Application to Barrelling States of Cylindrical Columns*, J. Elasticity, 71 (2003), pp. 33–58.
- [9] M. F. BEATTY AND D. E. HOOK, *Some experiments on the elastic stability of circular rubber bars under end thrust*, Internat. J. Solids Structures, 4 (1968), pp. 623–635.
- [10] M. F. BEATTY AND P. DADRAS, *Some experiments on the elastic stability of some highly elastic bodies*, Internat. J. Engrg. Sci., 14 (1976), pp. 233–238.
- [11] M. F. BEATTY, *A theory of elastic stability for incompressible, hyperelastic bodies*, Internat. J. Solids Structures, 3 (1967), 23–37.
- [12] M. F. BEATTY, *Topics in finite elasticity*, Appl. Mech. Rev., 40 (1987), pp. 1699–1734.
- [13] H. H. DAI AND F. F. WANG, *Corner instabilities in a slender nonlinearly elastic cylinder: analytical solutions and formation mechanism*, C. R. Math. Acad. Sci. Paris, 345 (2007), pp. 55–58.
- [14] H. H. DAI AND F. F. WANG, *Bifurcation to a corner-like formation in a slender nonlinearly elastic cylinder: asymptotic solution and mechanism*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 464 (2008), pp. 1587–1613.
- [15] Y. B. FU AND R. W. OGDEN, *Nonlinear stability analysis of pre-stressed elastic bodies*, Contin. Mech. Thermodyn., 11 (1999), pp. 141–172.
- [16] H. H. DAI AND Y. HUO, *Asymptotically approximate model equations for nonlinear dispersive waves in incompressible elastic rods*, Acta Mech., 157 (2002), pp. 97–112.
- [17] H. H. DAI AND X. J. FAN, *Asymptotically approximate model equations for weakly nonlinear long waves in compressible elastic rods and their comparisons with other simplified model equations*, Math. Mech. Solids, 9 (2004), pp. 61–79.
- [18] H. H. DAI AND Z. X. CAI, *Phase transitions in a slender cylinder composed of an incompressible elastic material. I. Asymptotic model equation*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 462 (2006), pp. 75–95.
- [19] I. S. GRADSHTEYN AND I. M. RYZHIK, *Table of Integrals, Series, and Products*, edited by A. Jeffrey, Academic Press, New York, 1980.
- [20] H. H. DAI, D. B. HUANG, AND Z. R. LIU, *Singular Dynamics with Application to Singular Waves in Physical Problems*, J. Phys. Soc. Japan, 73 (2004), pp. 1151–1155.
- [21] Y. B. FU AND M. S. POUR, *WKB Method with Repeated Roots and Its Application to the Buckling Analysis of an Everted Cylindrical Tube*, SIAM J. Appl. Math., 62 (2002), pp. 1856–1871.
- [22] V. KOMKOV, *Euler’s buckling formula and Wirtinger’s inequality*, Internat. J. Math. Ed. Sci. Tech., 14 (1983), pp. 661–668.
- [23] S. S. ANTMAN, *Nonlinear Problems of Elasticity*, Springer-Verlag, New York, 1995.
- [24] L. D. LANDAU AND E. M. LIFSHITZ, *Theory of elasticity 3rd Edition*, translated from Sykes, J.B. and Reid, W.H., revised and enlarged by Lifshitz, E. M. and Kosevich, A. M. and Pitaevskii, L. P., Oxford, Pergamon Press, 1986.
- [25] M. LEVINSON, *Stability of a compressed neo-Hookean rectangular parallelepiped*, J. Mech. Phys. Solids, 16 (1968), pp. 403–415.