

# Invariant manifolds for analytic difference equations

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November 11, 2018

## Abstract

We use a modification of the *parameterization method* to study invariant manifolds for difference equations. We establish existence, regularity, smooth dependence on parameters and study several singular limits, even if the difference equations do not define a dynamical system. This method also leads to efficient algorithms that we present with their implementations. The manifolds we consider include not only the classical strong stable and unstable manifolds but also manifolds associated to non-resonant spaces.

When the difference equations are the Euler-Lagrange equations of a discrete variational we present sharper results. Note that, if the Legendre condition fails, the Euler-Lagrange equations can not be treated as a dynamical system. If the Legendre condition becomes singular, the dynamical system may be singular while the difference equation remains regular. We present numerical applications to several examples in the physics literature: the Frenkel-Kontorova model with long-range interactions and the Heisenberg model of spin chains with a perturbation. We also present extensions to finite differentiable difference equations.

## 1 Introduction

### 1.1 Difference equations

In this paper we generalize some notions that have played an important role in dynamics, namely invariant manifolds, to the more general context of implicit difference equations. We also present algorithms to compute this object and apply them to several problems in the physics literature.

Consider a smooth manifold  $M$  of dimension  $d$ . Let  $Z : M^{N+1} \rightarrow \mathbb{R}^d$  be an analytic function. The function  $Z$  represents the following difference equation of order  $N$

$$Z(\theta_k, \theta_{k+1}, \dots, \theta_{k+N}) = 0. \quad (1)$$

The solutions of the difference equation (1) are sequences  $(\theta_k)_{k \geq 0}$  which satisfy (1), for all values of  $k \geq 0$ . A recurrence

$$\theta_{k+N} = F(\theta_k, \dots, \theta_{k+N-1}) \quad (2)$$

is a particular case of (1) taking  $Z(\theta_k, \theta_{k+1}, \dots, \theta_{k+N}) = F(\theta_k, \dots, \theta_{k+N-1}) - \theta_{k+N}$ . In some situations, we can consider (1) as defining  $\theta_{k+N}$  in terms of the other variables, so that one can transform (1) into a recurrence of the form (2). However, there are instances in which it is impossible to do so and therefore problem (1) is more general than (2). Recurrences have been studied mainly as dynamical systems and they have a rich geometric study.

The goal of this paper is to show that some familiar constructions in the theory of dynamical systems have rather satisfactory generalizations in the context of implicit difference equations. We will use the so-called parameterization method [CFdIL03a, CFdIL03b, CFdIL05] to show existence, regularity and smooth dependence on parameters of these invariant manifolds. Note that other methods such as the graph transform, which depend very much on the dynamical formulation do not seem to generalize to the context of implicit difference equations.

The Legendre transform –if it exists– makes a difference equation into a dynamical system. In some situations, a family of Legendre transforms exist but becomes singular. We study some of these singular limits, in which some invariant manifolds continue to exist through the singularity of this Legendre transformation and the implicit equation remains smooth, even if the dynamical system does not.

We will also describe and implement some rather efficient numerical algorithms to compute these objects with high accuracy. We point out that some of the algorithms are novel, even in the dynamical system case since we study not only the classical stable (and strong stable) manifolds but also the weak stable manifolds as well as singular limits. We believe that this is interesting because it shows a new way of approximating stable and unstable manifolds. In order to show that the method is robust we perform some explicit calculations that can be found in the code that supplements this paper.

*Remark 1.1.* We note that the relation between difference equations in implicit form (1) and the explicit recurrence is similar to the relation of Differential-Algebraic Equations (DAE) of the form  $Z(y, y', \dots, y^{(n)}) = 0$  and explicit Differential equations  $y^{(n)} = F(y, \dots, y^{(n-1)})$ . It seems that the methods presented here can be extended to DAE, but one needs extra technicalities. We hope to come back to this question. For more information on DAE, see [KM06].

## 1.2 Some examples and applications

### 1.2.1 Variational principles

One important source of the problems of the form (1), to which we pay special attention, is discrete variational problems (which appear in physics, economics, dynamic programming, etc.). A mathematical review of these discrete variational problems can be found in [Ves91, Gol01].

Let  $S : M^{N+1} \rightarrow \mathbb{R}$  be analytic. When studying variational problems one is interested –among other things– in solutions to Euler-Lagrange equations  $Z(\theta_k, \dots, \theta_{k+2N}) = 0$ , with

$$Z(\theta_0, \dots, \theta_{2N}) \equiv \sum_{j=0}^N \partial_j S(\theta_{N-j}, \theta_{N-j+1}, \dots, \theta_{2N-j}) = 0, \quad (3)$$

which are of the form (1) and of order  $2N$ . The Euler-Lagrange equations appear when one finds critical points of the formal variational principle based on  $\mathcal{J}(\theta) \equiv \sum_k S(\theta_k, \theta_{k+1}, \dots, \theta_{k+N})$ . Indeed, (3) is formally just  $\frac{\partial}{\partial \theta_k} \mathcal{J}(\theta) = 0$ , for all  $k$ .

As we mentioned, there are well known conditions (Legendre conditions) which allow to transform the Euler-Lagrange equations into recurrences, often called discrete Hamilton equations or twist maps. In situations where Legendre conditions fail –or become singular– the Euler-Lagrange equations cannot be transformed into Hamiltonian equations. See the examples below and those treated in more detail in Section 6.

Even if we will not explore it in detail, we note that the invariant manifolds considered here have applications to dynamic programming. We plan to come back to these issues in future investigations.

### 1.2.2 The Heisenberg XY model of magnetism

There are many examples of Lagrangian systems that can not be transformed in dynamical systems. Consider, for instance, the Heisenberg XY model. In this model, one is interested in solutions of

$$\sin(\theta_{k+1} - \theta_k) + \sin(\theta_{k-1} - \theta_k) - \varepsilon \sin \theta_k = 0, \quad (4)$$

where each  $\theta_k$  is an angle that represents the spin state of a particle in position  $k \in \mathbb{Z}$ . The parameter  $\varepsilon$  corresponds to the strength of an external magnetic field. The equation (4) appears as the Euler-Lagrange equation of an energy functional  $\mathcal{J}(\theta) = \sum_k \cos(\theta_{k+1} - \theta_k) + \varepsilon \cos(\theta_k)$  and has Lagrangian function  $S(\theta_0, \theta_1) = \cos(\theta_1 - \theta_0) + \varepsilon \cos(\theta_0)$ .

The dynamical interpretation of (4) is problematic because in order to get  $\theta_{k+1}$  in terms of  $\theta_k$  and  $\theta_{k-1}$ , we need to have

$$|\sin(\theta_k - \theta_{k-1}) + \varepsilon \sin \theta_k| < 1.$$

This condition is not invariant under the dynamics and having it for one value of  $k$  does not guarantee to have it for others. However,  $\theta_k \equiv 0$  is a solution of (4). In fact, there could exist many solutions, defined for  $k \geq 0$ , that converge to the fixed point. So, it is interesting and useful to identify these solutions and understand their geometry. We will present algorithms illustrating our general results in this model in Section 6.4.

### 1.2.3 The Frenkel-Kontorova model with non-nearest interaction

The Frenkel-Kontorova model was introduced to describe dislocations [FK39], but it has also found interest in the description of deposition [AL83, BK04]. One simplified version of the model (more general versions will be discussed in Section 6.3) leads to the study of solutions of non-nearest interactions. For instance, we have

$$\varepsilon (\theta_{k+2} + \theta_{k-2} - 2\theta_k) + (\theta_{k+1} + \theta_{k-1} - 2\theta_k) + V'(\theta_k) = 0. \quad (5)$$

Note that for  $\varepsilon = 0$ , equation (5) can be transformed into a second order recurrence; i.e. a dynamical system in 2 dimensions. Indeed, this 2-D map is the famous *standard map*, [Mat93, Gol01], whereas for  $\varepsilon \neq 0$  – no matter how small – one is lead to a dynamical system in 4 dimensions.

It turns out that some terms in this 4–dimensional system blow up as  $\varepsilon \rightarrow 0$ . Hence, the perturbation introduced by the  $\varepsilon$  term is singular in the Hamiltonian formalism. Nevertheless, we observe that the singularity appears only when we try to get  $\theta_{k+2}$  as a function of  $\theta_{k+1}, \theta_k, \theta_{k-1}, \theta_{k-2}$ . The equations (5) themselves depend smoothly on parameters.

Indeed, in section 5.1, we will show that the invariant manifolds of (5) that we construct, are smooth across  $\varepsilon = 0$ . We note that in the applications to solid state physics, the regime of small  $\varepsilon$  is the physically relevant, one expects that there are interactions with even longer range which become smaller with the distance [CDFM07, Suz71].

### 1.2.4 Dependence of parameters of invariant manifolds and Melnikov theory

In many situations, the solutions of a difference equation change dramatically when parameters are introduced. In the case of dynamical systems the transverse intersection of the stable and unstable manifolds is associated with chaos, and gave rise to the famous horseshoe construction of Smale. The Poincaré-Melnikov method is a widely used technique for detecting such intersections starting from a situation when the manifolds agree. One can assume that a system has pair of saddles and a degenerate heteroclinic or saddle connection between them. The classical Melnikov theory computes the rate at which the distance between the manifolds changes with a perturbation. In this context, it is important to understand dependence of parameters in invariant objects that appear in a Lagrangian setting. In particular, in the *XY* and *XYZ* models, dynamics is not longer useful and a purely variational formalism is needed. In this paper, we show that the variational theory is robust and there is smooth dependence on parameters, so our theory could lead to a variational formulation of Melnikov’s theory. Previous work in this direction was done in [Tab95, Lom97] in the context of twist maps.

## 1.3 Organization of the paper

The method of analysis is inspired by the study in [CFdlL03a, CFdlL03b, CFdlL05] and is organized as follows. First, in Section 3 we will study the linearized problem and generalize the notion of characteristic polynomial, spectrum and spectral subspaces. They give necessary conditions so that one can even consider solving (8).

The Main Theorem 4.1 is stated in Subsection 4.1. After we make the choices of invariant spaces in the linear approximation, we will show that, provided that these spaces satisfy some non-resonance conditions –automatically satisfied by the (strong) stable spaces considered in the classical literature– the solutions of the linear problem lead to solutions of the full problem.

The main tool is some appropriate implicit function theorems in Banach spaces. These implicit function theorems some times require that we consider approximate solutions of order higher than the first. In Appendix A we review the theory of Banach spaces of analytic functions and describe how it fits in our problem. In Subsection 4.4 we prove the Main Theorem and in 4.5, we consider the systematic computation of higher order approximations. Besides being useful in the proof of theorems, the higher order approximations are the basis of efficient numerical algorithms presented in Section 6. In particular, in Example 6.2, we show that the method can be used to approximate slow manifolds, even in the presence of a singular limit.

Some more refined analysis also leads to the study of singular limits in Section 5.1. Since the main tool is the implicit function theorem, we can obtain easily smooth dependence on parameters (see Subsection 5.2).

## 2 General Setup

### 2.1 Parameterized solutions

We are interested in extending the theory of invariant manifolds associated to a hyperbolic fixed point to the more general context of difference equations. The extension of fixed point is clear.

**Definition 2.1.** *If  $\theta^* \in M$  satisfies  $Z(\theta^*, \dots, \theta^*) = 0$ , then we will say that  $\theta_k \equiv \theta^*$  is a fixed point solution.*

The key to the generalization of the invariant manifolds from dynamical systems to difference equations is to observe that the invariant manifolds of a dynamical systems are just manifolds of orbits. This formulation makes sense in the context of difference equations. Note also that the dynamics restricted to the invariant manifold is semi-conjugate to the dynamics in the whole manifold (in the language of ergodic theory, it is a *factor*). Hence, we define:

**Definition 2.2** (Parameterized stable solution). *Let  $\theta^*$  be a fixed point solution to the difference equation (1). Let  $\mathcal{D}$  be an open disk of  $\mathbb{R}^m$  around the origin. We will say that a smooth function  $P : \mathcal{D} \rightarrow M$  is a stable parameterization of dimension  $m$  with internal dynamics  $h : \mathcal{D} \rightarrow h(\mathcal{D})$  when:*

- a)  $P(0) = \theta^*$ .
- b)  $0$  is an attracting fixed point of  $h$ .
- c) If  $z \in \mathcal{D}$  then

$$Z\left(P\left(h^k(z)\right), P\left(h^{k+1}(z)\right), \dots, P\left(h^{k+N}(z)\right)\right) = 0, \quad (6)$$

for all  $k \geq 0$ .

*Remark 2.1.* Notice that, in the definition above, if we let  $z_0 \in \mathcal{D}$  and  $\theta_k = P(h^k(z_0))$ , then the sequence  $(\theta_k)_{k \geq 0}$  satisfies the difference equation (1) and also  $\theta_k \rightarrow \theta^*$ , as  $k \rightarrow \infty$ .

In some situations it might be useful to consider a geometric object associated to the parameterization. As a consequence of definition 2.2, we have the following result.

**Proposition 2.3.** *Let  $\theta^*$  be a fixed point solution and  $P : \mathcal{D} \subset \mathbb{R}^m \rightarrow M$  a parameterization of dimension  $m$  with internal dynamics  $h : \mathcal{D} \rightarrow h(\mathcal{D})$ . If  $P'(0)$  has rank equal to the dimension of  $\mathcal{D}$ , then there exists an open disk  $\widehat{\mathcal{D}} \subset \mathcal{D}$  around the origin such that*

$$\mathcal{W} = \{(P(z), P(h(z)), \dots, P(h^{N-1}(z))) : z \in \widehat{\mathcal{D}}\} \quad (7)$$

is an embedded submanifold  $\mathcal{W} \hookrightarrow M^N$  of dimension  $m$ . In such case, we will say that a manifold  $\mathcal{W}$  as in (7) is a local stable manifold with parameterization  $P$ .

We note that the parameterization  $P$  and the internal dynamics  $h$  satisfy

$$[\Phi(P)](z) := Z(P(z), P(h(z)), \dots, P(h^N(z))) = 0, \quad (8)$$

together with the normalizations obtained by a change of origin

$$P(0) = \theta^*, \quad h(0) = 0. \quad (9)$$

The equation (8) is the centerpiece of our analysis. Using methods of functional analysis we can show that, under appropriate conditions, (8) has solutions. In order to do this, we will find it convenient to think of (8) as the equation  $\Phi(P) = 0$ , defined on a suitable function space.

The solutions of (8) thus produced generalize the familiar stable and strong stable manifolds but also include some other invariant manifolds associated to non-resonant spaces including, in some cases, the *slow manifolds*.

## 2.2 An important simplification

If  $L : \widetilde{\mathcal{D}} \rightarrow \mathcal{D}$  is a diffeomorphism and we define  $\widetilde{P} = P \circ L$  and  $\widetilde{h} = L^{-1} \circ h \circ L$ , then  $(\widetilde{P}, \widetilde{h})$  solves (8) on the domain  $\widetilde{\mathcal{D}}$ . In other words, the  $P$  and  $h$  solving (8) is not unique. Nevertheless, the range of  $P$  and  $\widetilde{P}$  is unique.

One can take advantage of this lack of uniqueness of (8) to impose extra normalization on the maps  $h$ . Recall that, by the theorem of Sternberg [Ste55, Ste58, Ste59], under non-resonance conditions on the spectrum,<sup>1</sup> any analytic one dimensional invariant curve tangent to a contracting eigenvector has dynamics which is analytically conjugate to the linear one. So that, we can take  $h$  to be linear if we assume non-resonance or if we deal with 1-dimensional manifolds.

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<sup>1</sup>This is automatically satisfied for the strong stable and strong unstable manifolds and includes the classical stable and unstable manifolds.

In this paper, we will not consider the resonant case. This justifies that in (8) we do not consider  $h$  as part of the unknowns, since it will be linear. In the cases considered here, it will suffice to consider  $h$  to be linear, we will write  $h(z) = \Lambda z$ . Hence, instead of (8), we will consider

$$[\Phi(P)](z) := Z(P(z), P(\Lambda z), \dots, P(\Lambda^N z)) = 0, \quad (10)$$

where  $\Lambda$  is a matrix that will be determined from a linearized equation near the fixed point. In fact,  $\Lambda$  is a matrix that depends on the roots of a polynomial. The determination of  $\Lambda$  and  $P'(0)$  will be studied in Section 3. We will call  $\Phi$  the parameterization operator. The domain of the operator  $\Phi$  is a function space that depends on the analytic properties of  $Z$  and will be studied in detail in Section 4.

In summary, the basic *ansatz* that we propose is that there are solutions to the difference equation that are of the form

$$\theta_k = P\left(\Lambda^k z\right),$$

where  $\Lambda$  is determined by the linearization at the fixed point. In addition,  $\Lambda$  and  $P$  are chosen so that  $\theta_k \rightarrow \theta^*$  as  $k \rightarrow \infty$ . The smoothness of  $P$  depends on the smoothness of  $Z$ .

### 2.3 Unstable manifolds

With the parameterization method we can also study unstable manifolds.

**Definition 2.4** (Parameterized unstable solution). *Let  $\theta^*$  be a fixed point solution to the difference equation (1). Let  $\mathcal{D}$  be an open disk of  $\mathbb{R}^m$  around the origin. We will say that a smooth function  $P : \mathcal{D} \rightarrow M$  is a unstable parameterization of dimension  $m$  with internal dynamics  $h : \mathcal{D} \rightarrow h(\mathcal{D})$  when:*

- a)  $P(0) = \theta^*$ .
- b)  $0$  is an attracting fixed point of  $h$ .
- c) If  $z \in \mathcal{D}$  then

$$Z\left(P\left(h^{k+N}(z)\right), P\left(h^{k+N-1}(z)\right), \dots, P\left(h^k(z)\right)\right) = 0, \quad (11)$$

for all  $k \geq 0$ .

*Remark 2.2.* As in the stable case, each parameterization produces sequences that satisfy the original difference equation. In this case, if we let  $z_0 \in \mathcal{D}$  and  $\theta_{-k} = P(h^k(z_0))$ , then the sequence  $(\theta_k)_{k \leq 0}$  satisfies the difference equation (1) and also  $\theta_k \rightarrow \theta^*$ , as  $k \rightarrow -\infty$ .

This corresponds to an extension of the original difference  $Z(\theta_k, \dots, \theta_{k+N}) = 0$ , to negative values  $k \leq 0$ . Alternatively, we can write the difference equation for negative values in terms of a dual problem

$$\tilde{Z}\left(\tilde{\theta}_k, \tilde{\theta}_{k+1}, \dots, \tilde{\theta}_{k+N}\right) = Z\left(\tilde{\theta}_{k+N}, \tilde{\theta}_{k+N-1}, \dots, \tilde{\theta}_k\right) = 0.$$

We interpret the new variable as  $\tilde{\theta}_k = \theta_{N-k}$ . In this way, the unstable case is reduced to the stable case.

## 2.4 Explicit examples

**Example 2.1.** Let  $\eta > 0$ . Consider the function  $Z : \mathbb{R}^2 \rightarrow \mathbb{R}$  given by

$$Z(\theta_0, \theta_1, \theta_2) = \theta_2 + \theta_0 - \frac{2 \cosh(\eta)\theta_1}{\theta_1^2 + 1}.$$

The resulting difference equation  $Z(\theta_k, \theta_{k+1}, \theta_{k+2}) = 0$  [McM71] is an explicit recurrence called the McMillan map. It is *integrable* in the sense that  $J(x, y) = x^2y^2 + x^2 + y^2 - 2 \cosh(\eta)xy$  satisfies  $J(\theta_k, \theta_{k+1}) = J(\theta_{k+1}, \theta_{k+2})$ . In [DRR98], it was shown that  $P(z) = 2 \sinh(\eta)z/(z^2 + 1)$  satisfies

$$Z(P(z), P(\lambda z), P(\lambda^2 z)) = P(\lambda^2 z) + P(z) - \frac{2 \cosh(\eta)P(\lambda z)}{[P(\lambda z)]^2 + 1} \equiv 0,$$

provided  $\lambda = e^{-\eta}$ . Therefore  $P$  is a parameterized solution with internal dynamics  $h(z) = \lambda z$ . Notice that, in the case of explicit recurrences, the parameterization can follow the manifold even if it folds and ceases to be a graph.

**Example 2.2.** Let  $M = (-\infty, 1)$ . On  $M \times M$ , we define the following the Lagrangian  $S(\theta_0, \theta_1) = \frac{1}{2}(\theta_0 - f(\theta_1))^2$ , where  $f(\theta) = 2\theta/(1 - \theta)$ . The corresponding Euler-Lagrange difference equation can be written as  $Z(\theta_k, \theta_{k+1}, \theta_{k+2}) = 0$  where  $Z : M^3 \rightarrow \mathbb{R}$  is the function given by

$$Z(\theta_0, \theta_1, \theta_2) = (f(\theta_1) - \theta_0) f'(\theta_1) + (\theta_1 - f(\theta_2)). \quad (12)$$

Clearly,  $\theta^* = 0$  is a fixed point solution.

Let  $P(z) = z/(1 - z)$ . Since  $P(z) \in M$ , the maximal disk in which  $P$  can be defined is  $\mathcal{D} = (-1/2, 1/2)$ . If we let  $\lambda = \frac{1}{2}$ , then one can check that  $P$  satisfies  $f(P(\lambda z)) = P(z)$ , for all  $z \in \mathcal{D}$ . After substitution, we verify that  $Z(P(z), P(\lambda z), P(\lambda^2 z)) \equiv 0$ . Therefore,  $P$  is a stable parameterized solution for  $\theta^* = 0$ .

However, the difference equation (12) is not a dynamical system because it can not be inverted, unless  $f$  is a diffeomorphism of  $M$ . In other words, we can find a parameterization solution even if the system is not dynamic.

**Example 2.3.** Suppose that  $G : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is a map, possibly non-invertible, with  $G(0) = 0$ . We would like to parameterize the stable and unstable manifolds of 0, if they exist. For instance, we can solve the following pair of one-dimensional problems.

- Stable manifold problem: if  $0 < \lambda < 1$  is an eigenvalue of  $DG(0)$ , find a function  $P$  such that  $P(\lambda z) = G(P(z))$ .
- Unstable manifold problem: if  $\mu > 1$  is an eigenvalue of  $DG(0)$ , find a function  $P$  such that  $P(\mu z) = G(P(z))$ .

Both problems can be solved using the parameterization method for difference equations. The same ideas can be used for higher dimensional objects. For examples, for  $d = 3$ , two-dimensional stable and unstable manifolds were found in [MJL10]. As we will see, our analysis covers not only one-dimensional stable and unstable manifolds, but also other non-resonant manifolds.



### 3 Linear analysis

Since we are interested in solutions of (10) it is natural to study the behavior of the linearization. As it happens in other situations, this will lead to certain choices. In subsequent sections, we will show that the once these choices are made (satisfying some mild conditions), then there is indeed a manifold which agrees with these constraints.

Suppose that there exist differentiable  $P$ , and a matrix  $\Lambda$  solving (10). Taking derivatives of (8) with respect to  $z$ , and evaluating at  $z = 0$ , we get that

$$\sum_{i=0}^N B_i V \Lambda^i = 0, \quad (13)$$

where  $B_i = \partial_i Z(\theta^*, \dots, \theta^*)$  and  $V = P'(0)$ . Our first goal is to understand conditions that allow to solve equation (13) which is a necessary condition for the existence of differentiable solutions for (10).

*Remark 3.1.* The dimensions of the matrices are determined by the type of parametrization that we have. Notice that each  $B_i$  is  $d \times d$ ,  $\Lambda$  is  $m \times m$  and  $V$  is  $d \times m$ .

**Definition 3.1** (Characteristic polynomial). *Let  $\theta^*$  a fixed point solution. If we let*

$$B_i = \partial_i Z(\theta^*, \dots, \theta^*),$$

*then the characteristic polynomial of the fixed point is defined to be:*

$$\mathcal{F}(\lambda) := \det \left( \sum_{i=0}^N \lambda^i B_i \right).$$

*If  $\lambda$  is a root of  $\mathcal{F}$ , then we will say that it is an eigenvalue. The set of eigenvalues is the spectrum, denoted by  $\sigma(\theta^*)$ , of the fixed point. If  $\lambda \in \sigma(\theta^*)$ , then any vector  $v \in \mathbb{C}^d \setminus \{0\}$  that satisfies*

$$\sum_{i=0}^N \lambda^i B_i v = 0$$

*will be called an eigenvector of  $\lambda$ . In addition, if  $V$  is a  $d \times m$  matrix with non-zero columns and  $\Lambda$  is a  $m \times m$  matrix that satisfy the linear relation (13), then we will say that the pair  $(V, \Lambda)$  is an eigensolution of dimension  $m$ .*

*Remark 3.2.* If the difference equation is of the form (3) and is the Euler-Lagrange of a variational principle then the characteristic polynomial is of the form  $\mathcal{F}(\lambda) = \lambda^{Nd} \mathcal{L}(\lambda)$  where  $\mathcal{L}$  is given by

$$\mathcal{L}(\lambda) := \det \left( \sum_{i,j=0}^N \lambda^{j-i} A_{ij} \right), \quad (14)$$

and  $A_{ij} = \partial_{ij} S(\theta^*, \dots, \theta^*)$ . Now, since  $A_{ij}^T = A_{ji}$ , we have that if  $\lambda$  is in the spectrum, then  $1/\lambda$  is also in the spectrum. This is a well known symmetry for the spectrum of symplectic matrices. In [Ves91] one can find a proof that, when the Euler-Lagrange equation (3) defines a dynamical system, it becomes symplectic.

*Remark 3.3.* Let  $(V, \Lambda)$  be an eigensolution as above. Suppose that  $w$  is an eigenvector of  $\Lambda$  with eigenvalue  $\lambda$ . Then  $v = \Lambda w$  is an eigenvector with eigenvalue  $\lambda$  in the sense of definition 3.1.

**Proposition 3.2.** *Let  $\{v_1, \dots, v_m\}$  be a linearly independent set of eigenvectors with distinct eigenvalues  $\lambda_1, \dots, \lambda_m$ . Let  $V$  be the  $d \times m$  matrix  $V = (v_1 v_2 \cdots v_m)$  and  $\Lambda$  be the  $m \times m$  matrix  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_m)$ . Then  $(V, \Lambda)$  is an eigensolution and satisfies equation (13). In addition, if  $Q$  is an invertible matrix,  $\tilde{V} = VQ$  and  $\tilde{\Lambda} = Q^{-1}\Lambda Q$  then  $(\tilde{V}, \tilde{\Lambda})$  is also an eigensolution.*

*Remark 3.4.* Suppose that  $\lambda = \mu + i\nu$  is a root of the characteristic polynomial  $\mathcal{F}$  of a fixed point  $\theta^*$ . Let  $v = r + is$  be an eigenvector of  $\lambda$ . If we want to avoid the use of complex numbers, we can use  $\lambda$  and  $v$  in order to find a solution with dimension  $m = 2$ . Let  $V$  be the  $d \times 2$  matrix given by  $V = (r \ s)$  and

$$\Lambda = \begin{pmatrix} \mu & -\nu \\ \nu & \mu \end{pmatrix}.$$

It is easy to see that  $(V, \Lambda)$  is an eigensolution of dimension 2.

*Remark 3.5.* We notice that  $\mathcal{F}(\lambda)$  is always a polynomial of degree at most  $2Nd$ . If the matrix  $B_0$  is non-singular then the degree of  $\mathcal{F}(\lambda)$  is exactly  $Nd$ . If  $\lambda$  is an eigenvalue, then there is only a finite number of values of  $n$  for which  $\mathcal{F}(\lambda^n) = 0$ . These considerations motivate the following.

**Definition 3.3** (Non-singularity condition). *We will say that a fixed point  $\theta^*$  is non-singular if the corresponding characteristic polynomial satisfies:  $\mathcal{F}(0) \neq 0$ .*

**Definition 3.4** (Non-resonance condition). *We will say that an eigenvalue  $\lambda \in \sigma(\theta^*)$  is non-resonant if  $\mathcal{F}(\lambda^n) \neq 0$ , for all  $n \geq 2$ .*

More generally, we will consider non-resonant sets of eigenvalues. In what follows, we will be using the multi-index notation where, as usual, if  $z = (z_1, \dots, z_m) \in \mathbb{C}^m$  and  $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{Z}_+^m$  is a multi-index then  $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_m^{\alpha_m}$ .

**Definition 3.5.** *We will say that  $\lambda = (\lambda_1, \dots, \lambda_m)$  is a non-resonant vector of eigenvalues if, for all multi-indices  $\alpha \in \mathbb{Z}_+^m$ ,*

- a)  $\mathcal{F}(\lambda^\alpha) = 0$  if  $|\alpha| = 1$ ,
- b)  $\mathcal{F}(\lambda^\alpha) \neq 0$  if  $|\alpha| > 1$ .

*If  $\lambda = (\lambda_1, \dots, \lambda_m)$  is a non-resonant vector of eigenvalues then we will also say that the set  $\{\lambda_1, \dots, \lambda_m\}$  is non-resonant.*

**Definition 3.6** (Hyperbolicity). *Let  $\theta^*$  be a non-singular fixed point solution of an analytic difference equation  $Z$  and suppose that  $\mathcal{F}$  is its characteristic polynomial. We will say that  $\theta^*$  is hyperbolic if none of the eigenvalues of  $\mathcal{F}$  are on the unit circle. Similarly, we will say that a vector of eigenvalues  $\lambda = (\lambda_1, \dots, \lambda_m)$  is stable if  $|\lambda_i| < 1$ , for all  $i = 1, \dots, m$ .*

*Remark 3.6.* Let  $\theta^*$  be a non-singular fixed point solution and  $\sigma(\theta^*)$  its spectrum. In other words, all elements of  $\sigma(\theta^*)$  are non-zero. Notice that, even if the condition of non-resonance involves infinitely many conditions, for stable sets all except a finite number of them are automatic. Suppose that  $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)$  is a stable vector of eigenvalues. If  $n \in \mathbb{N}$  is such that

$$\left( \max_{\lambda_i \in \boldsymbol{\lambda}} |\lambda_i| \right)^n \leq \min_{\lambda \in \sigma(\theta^*)} |\lambda|,$$

then we have that  $\boldsymbol{\lambda}^\alpha$  cannot be in the spectrum if  $|\alpha| \geq n$ . So that there are only a finite number of conditions to check and the set of non-resonant  $\boldsymbol{\lambda}$  is an open-dense, full measure set among the stable ones.

All this analysis shows that there are obstructions to the computation of invariant manifolds that appear using just the linear approximation. These obstructions are a generalization of the observation, in the dynamical systems case, that the only invariant manifolds have to have tangent spaces that are invariant under the linearization.

The goal of the rest of the paper is to show that if we choose a subset of the spectrum and an invariant subspace, which also satisfies the non-resonance conditions, then there is a solution to the parameterization problem. The analysis will also show that the non-resonance conditions are needed to obtain a general result.

## 4 Existence and analyticity of solutions

As indicated before, we will first obtain a formal approximation and then use an implicit function theorem. The proof of Theorem 4.1 is an application of the implicit function theorem in a Banach space of analytic functions (see Subsection 4.4). This will lead rather quickly to the analytic dependence on parameters, and the possibility of getting approximate solutions (see Subsection 4.5).

### 4.1 Statement of the Main Theorem

**Theorem 4.1.** *Let  $Z$  be an analytic difference equation function with a fixed point at 0 and such that  $B_0 = \partial_0 Z(0)$  is a non-singular matrix. Let  $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)$  be a stable non-resonant vector of eigenvalues, with corresponding eigenvectors  $v_1, \dots, v_m$ . Let  $(V, \Lambda)$  be the corresponding eigensolution of the linearized problem (13) with  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_m)$  and  $V = (v_1 \cdots v_m)$ .*

*Then, there exist an analytic function  $P$  such that its derivative at  $z = 0$  is  $P'(0) = V$  and satisfies (9) and (10). The solution is unique among the solutions of the equation.*

*Remark 4.1.* The radius of convergence of the resulting analytic function is not specified. In the proof of 4.1, we will consider an equivalent result, in which the radius of convergence is 1, but the lengths of the vectors  $v_1, \dots, v_m$  are modified.

*Remark 4.2.* If  $Z$  is the Euler-Lagrange equation (3) that corresponds to a generating function  $S$ , then  $B_0 = \partial_{0,N} S(0)$  and hence the non-singularity condition is  $\det(\partial_{0,N} S(0)) \neq 0$ .

In Section 5.1, we will weaken the assumption  $\det(B_0) \neq 0$ , which is tantamount to a local version of the Legendre condition. This includes, in particular, the extended Frenkel Kontorova models with singularities.

## 4.2 Analyticity of the parameterization operator

In appendix A, we give a general definition of analyticity and introduce spaces of analytic functions. In particular, given  $\rho > 0$  we define

$$A_\rho(\mathbb{C}^\ell, \mathbb{C}^d) = \left\{ f : \mathbb{C}^\ell(\rho) \rightarrow \mathbb{C}^d \left| f(z) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} z^\alpha \eta_\alpha, \|f\|_\rho < \infty \right. \right\},$$

where

$$\|f\|_\rho = \sum_{k=0}^{\infty} \left( \sum_{|\alpha|=k} \|\eta_\alpha\|_\infty \right) \rho^k,$$

$\|\cdot\|_\infty$  is the uniform norm (45), and  $\mathbb{C}^\ell(\rho) = \{z \in \mathbb{C}^\ell : \|z\|_\infty \leq \rho\}$ .

Let  $d, N \in \mathbb{N}$  and  $\ell = d(N+1)$ . We will consider  $\mathbb{C}^\ell \simeq (\mathbb{C}^d)^{N+1}$ . As an standing assumption, the function that defines the difference equation satisfies  $Z \in A_\rho(\mathbb{C}^\ell, \mathbb{C}^d)$  for some  $\rho > 0$ ; this is,  $Z$  is analytic near the origin.

In this section we will show that the operator  $\Phi$ , defined in (10), is an analytic operator defined on spaces of analytic functions in  $\mathbb{C}^m$ . Let  $\Lambda$  be an  $m \times m$  matrix such that, if  $\|z\|_\infty < 1$  then  $\|\Lambda z\|_\infty < 1$ . We define the linear function  $\mathcal{G} : A_1(\mathbb{C}^m, \mathbb{C}^d) \rightarrow A_1(\mathbb{C}^m, \mathbb{C}^\ell)$  by

$$\mathcal{G}(P) = (P, P \circ \Lambda, \dots, P \circ \Lambda^N).$$

Besides being linear, the functional  $\mathcal{G}$  has the property that  $\mathcal{G}(A_1^\rho(\mathbb{C}^m, \mathbb{C}^d)) \subset A_1^\rho(\mathbb{C}^m, \mathbb{C}^\ell)$ , for all  $\rho > 0$ . Let

$$X = A_1(\mathbb{C}^m, \mathbb{C}^d) \tag{15}$$

and define the open set

$$\mathcal{U} = \{f \in X : \|f\|_1 < \rho\}. \tag{16}$$

For a correct application of the implicit function theorem, it suffices  $\Phi$  to be  $C^1$ . However, we next show if  $\Phi$  is analytic. This will give analytic dependence on parameters of the invariant manifolds.

Notice that  $\mathcal{G}(\mathcal{U}) \subset A_1^\rho(\mathbb{C}^m, \mathbb{C}^\ell)$ . Since  $Z \in A_\rho(\mathbb{C}^\ell, \mathbb{C}^d)$ , we can write the parameterization operator as a composition of analytic functions:  $\Phi = \mathcal{C}_Z \circ \mathcal{G}$ , where  $\mathcal{C}_Z$  is the composition operator  $\mathcal{C}_Z(g) = Z \circ g$ . Consequently, using Lemma A.4, we get the following:

**Proposition 4.2.** *Let  $\mathcal{U} \subset X$  as in (15) and (16). Then the operator  $\Phi : \mathcal{U} \rightarrow X$  is analytic.*

### 4.3 Fréchet derivative of the parameterization operator

Consider the parameterization operator  $\Phi$  defined in (10), in which  $\Lambda$  is a diagonal matrix. As we have seen, this operator is a function  $\Phi : \mathcal{U} \rightarrow X$ , where  $X$  and  $\mathcal{U}$  are defined in (15) and (16) respectively.

We have the following.

**Lemma 4.3.** *Let  $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)$  be a vector such that  $\|\boldsymbol{\lambda}\|_\infty < 1$ . Let  $\Phi : \mathcal{U} \rightarrow X$  be the parameterization operator defined in (10) where  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_m)$  and define*

$$T(\boldsymbol{\lambda}) := \sum_{i=0}^N \lambda^i B_i, \quad (17)$$

where  $B_i = \partial_i Z(0, \dots, 0)$ . Let  $\varphi \in X$  be of the form  $\varphi(z) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} z^\alpha \varphi_\alpha$ . Then  $D\Phi(0)\varphi \in X$  and

$$[D\Phi(0)\varphi](z) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} z^\alpha T(\boldsymbol{\lambda}^\alpha) \varphi_\alpha. \quad (18)$$

*Proof.* We notice the conditions on  $\boldsymbol{\lambda}$  imply that  $\varphi \circ \Lambda^i \in X$  for all  $i = 0, \dots, N$ . We also have that the Fréchet derivative of  $\Phi$  satisfies

$$[D\Phi(0)\varphi](z) = \sum_{i=0}^N B_i \varphi(\Lambda^i z), \quad (19)$$

for all  $\|z\|_\infty \leq 1$ . This implies that  $D\Phi(0)\varphi \in X$ . Clearly,  $(\Lambda^i z)^\alpha = z^\alpha (\boldsymbol{\lambda}^\alpha)^i$  and therefore

$$\varphi(\Lambda^i z) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} z^\alpha (\boldsymbol{\lambda}^\alpha)^i \eta_\alpha.$$

Combining the last equation with (17) and (19), we get (18).  $\square$

Let  $H$  be the Banach subspace of analytic functions in the unit disk, that vanish at the origin along with their first derivatives.

$$H = \left\{ P \in X \left| P(z) = \sum_{k=2}^{\infty} \sum_{|\alpha|=k} z^\alpha P_\alpha; \|P\|_1 = \sum_{k=2}^{\infty} \sum_{|\alpha|=k} \|P_\alpha\|_\infty < \infty \right. \right\}. \quad (20)$$

*Remark 4.3.* In the notation of the appendix A, this subspace is just  $H = \{P \in A_1(\mathbb{C}^m, \mathbb{C}^d) : P(0) = 0, P'(0) = 0\}$ .

Clearly, (18) implies that  $H$  is invariant under  $D\Phi(0)$ . In addition, we get the following result.

**Lemma 4.4.** *Let  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_m)$ , where  $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)$  is a non-resonant stable vector of eigenvalues. If  $B_0$  is a non-singular matrix, then*

a)  $D\Phi(0)$  is invertible in  $H$ , with bounded inverse.

b) If  $\varphi \in H$  is such that  $\varphi(z) = \sum_{k=2}^{\infty} \sum_{|\alpha|=k} z^\alpha \varphi_\alpha$  and  $\eta = D\Phi(0)^{-1}\varphi$  can be written as  $\eta(z) = \sum_{k=2}^{\infty} \sum_{|\alpha|=k} z^\alpha \eta_\alpha$ , then  $\eta_\alpha = T(\boldsymbol{\lambda}^\alpha)^{-1}\varphi_\alpha$ , for every multi-index  $\alpha$  such that  $|\alpha| \geq 2$ .

*Proof.* Clearly, as  $k \rightarrow \infty$  we get that  $T(\lambda^k) \rightarrow B_0$ , a matrix that is invertible. This implies that there exists a constant  $C_0 > 0$  and a radius  $0 < \delta < 1$  such that, if  $|\lambda| < \delta$  then  $\|T(\lambda)^{-1}\| \leq C_0$ . In particular, if  $|\lambda^\alpha| < \delta$  then  $\|T(\lambda^\alpha)^{-1}\| \leq C_0$ .

We know that, since  $\lambda$  is non-resonant, the matrices  $T(\lambda^\alpha)$  are invertible, whenever  $|\alpha| \geq 2$ . Since  $\lambda$  is stable there is only a finite number of elements in the set

$$\{\lambda^\alpha \in \mathbb{C} : |\alpha| \geq 2, |\lambda^\alpha| \geq \delta\}.$$

Let  $C > 0$  be a constant such that  $C \geq C_0$  and

$$C \geq \max\{\|T(\lambda^\alpha)^{-1}\| : |\alpha| \geq 2, |\lambda^\alpha| \geq \delta\}.$$

This implies that  $\|T(\lambda^\alpha)^{-1}\| \leq C$ , for all  $|\alpha| \geq 2$ .

From (18) we get that  $D\Phi(0)$  is injective. Let  $\varphi \in H$  with  $\varphi(z) = \sum_{k=2}^{\infty} \sum_{|\alpha|=k} z^\alpha \varphi_\alpha$ . Define  $\eta(z) = \sum_{k=2}^{\infty} \sum_{|\alpha|=k} \eta_\alpha z^\alpha$ , with  $\eta_\alpha = T(\lambda^\alpha)^{-1} \varphi_\alpha$ , for every multi-index  $|\alpha| \geq 2$ . From (18), we have that  $D\Phi(0)\eta = \varphi$  and therefore  $D\Phi(0)$  is invertible in  $H$ . In addition,

$$\|D\Phi(0)^{-1}\varphi\|_1 = \|\eta\|_1 = \sum_{k=2}^{\infty} \sum_{|\alpha|=k} \|T(\lambda^\alpha)^{-1}\varphi_\alpha\|_\infty \leq C \sum_{k=2}^{\infty} \sum_{|\alpha|=k} \|\varphi_\alpha\|_\infty = C\|\varphi\|_1.$$

We conclude that  $D\Phi(0)$  is invertible in  $H$  and  $\|D\Phi(0)^{-1}\| \leq C$ .  $\square$

#### 4.4 Proof of Theorem 4.1.

Let  $H$  as in (20). When we consider the linear part  $V$  of the parameterization  $P$ , it is convenient to choose the scale sufficiently small. Choosing this scale is tantamount to choosing the radius of convergence of the solution  $P$ . Let  $\tau = (\tau_1, \dots, \tau_m)$ , where  $\tau_1, \dots, \tau_m$  represent scaling factors for the columns of the linear part. We will use the notation  $\tau \cdot V = V \text{diag}(\tau_1, \dots, \tau_m)$ .

Since any possible solution of (10) has to match the lower order terms found, it is natural to consider a new decomposition  $P = \tau \cdot V + P^>$  where  $P^>$  is an analytic function which vanishes to order 2 and depends on the size of the scale  $\tau$ . Because of the change of variables, we can seek for  $P^>$  among analytic functions of radius 1 that vanish to first order, i.e.  $P^> \in H$ .

Hence we write the equation (10) as

$$\Psi(\tau, P^>) \equiv \Phi(\tau \cdot V + P^>) = 0. \quad (21)$$

It is important to note, since  $V$  is known, that we only need to find the appropriate scale  $\tau$ , and the function  $P^>$ . Furthermore, since  $P^>$  vanishes to order 1,  $\Psi(\tau, P^>)$  also vanishes to order 1. In other words, we can choose  $H$  to be the codomain of  $\Psi$ . In addition, the coefficients of  $P^>$  are small if  $\tau$  is small.

We notice that there exists an open subset  $\mathcal{V} \subset \mathbb{R}^m \times H$  defined by the property that if  $(\tau, P^>) \in \mathcal{V}$  then  $\tau \cdot V + P^> \in \mathcal{U}$ , the domain of  $\Phi$ . In this way, we can restrict the domain of  $\Psi$  and consider  $\Psi : \mathcal{V} \rightarrow H$ . It is clear that this is a neighborhood of the origin  $(0, 0)$  in which the operator  $\Psi$  is defined and is analytic.

Taking the derivative with respect to the second variable, it is clear that  $D_2\Psi(0,0) = D\Phi(0)$  and therefore, by Lemma 4.4, we have that the operator

$$D_2\Psi(0,0) : H \rightarrow H$$

is invertible with bounded inverse. The implicit function theorem in Banach spaces [HS91] implies that there exists  $\delta > 0$  and a function  $\tau \mapsto P_\tau^<$  such that if  $|\tau| < \delta$ , then  $\Psi(\tau, P_\tau^>) = 0$  and the solution is unique if we require  $\|P^>\|_1 < \delta$ .

Fix a solution of (21) of the form  $(\tau, P_\tau^>)$  such that  $\tau$  has positive entries and  $\Psi(\tau, P_\tau^>) \equiv 0$ . Then  $\tilde{P}(z) = \tau \cdot Vz + P_\tau^>(z)$  is a solution of the parameterization problem. By construction,  $\tilde{P}$  is an analytic function with radius of convergence 1. If we want to have  $V$  as the linear part of the solution, we modify  $\tilde{P}$  in the following way. Let  $P = \tilde{P} \circ \text{diag}(\tau_1, \dots, \tau_m)^{-1}$  or

$$P(z) = Vz + P_\tau^>(\text{diag}(\tau_1, \dots, \tau_m)^{-1}z).$$

Using a linear change of variables as in Section 2.2, we conclude that  $P$  is also a solution of the parameterization problem. However, the radius of convergence is not longer 1 but depends on the choice of  $\tau$ . If we let  $r = \min\{\tau_1, \dots, \tau_m\}$ , then  $r > 0$ , and  $\|z\|_\infty \leq r$  implies that  $\|\text{diag}(\tau_1, \dots, \tau_m)^{-1}z\|_\infty \leq 1$ . We conclude that  $P \in A_r(\mathbb{C}^m, \mathbb{C}^d)$  and has radius of convergence equal to  $r$ .  $\square$

#### 4.5 Formal approximations to higher order

Once  $P'(0) = V$  and  $\Lambda$  are chosen, the solution  $P$  of the parameterization problem can be approximated with the first terms of the power series. Due to analyticity, we can write the solution  $P \in A_\rho(\mathbb{C}^m, \mathbb{C}^d)$  of the parameterization problem as a sum of homogeneous polynomials like in (50):

$$P(z) = \sum_{\ell=1}^{\infty} \sum_{|\alpha|=\ell} z^\alpha P_\alpha,$$

where  $P_\alpha \in \mathbb{C}^d$ , and has  $P$  radius of convergence  $\rho$ .

For each multi-index  $\alpha \in \mathbb{Z}_+^m$ , we will use the notation  $[\cdot]_\alpha$  for the coefficient vector of the term that corresponds to  $z^\alpha$ . Clearly, if  $f$  is an analytic function at the origin, then this coefficient can be written as  $[f]_\alpha = \partial^\alpha f(0)/\alpha!$ . In the case of  $P$  above, we get that  $[P]_\alpha = P_\alpha$ .

For each  $n \in \mathbb{N}$ , let  $P^{\leq n}$  be the polynomial

$$P^{\leq n}(z) = \sum_{\ell=1}^n \sum_{|\alpha|=\ell} z^\alpha P_\alpha.$$

It is clear that  $\|P - P^{\leq n}\|_\rho \rightarrow 0$  as  $n \rightarrow \infty$ . Notice also that, for any  $|\alpha| \leq n$ ,

$$[\Phi(P^{\leq n})]_\alpha = [\Phi(P)]_\alpha = 0, \tag{22}$$

We will describe how to construct the polynomials  $P^{\leq n}$  recursively, provided that the eigenvalues are non-resonant.

A simple computation shows that  $[\Phi(P^{\leq 1})]_{\alpha} = 0$  for  $|\alpha| = 1$ . Let  $\mathcal{N}(P) = D\Phi(0)P - \Phi(P)$ . Then, it turns out that  $[\mathcal{N}(P^{\leq n})]_{\alpha} = [\mathcal{N}(P^{\leq n+1})]_{\alpha}$ , for all  $|\alpha| = n + 1$ . Lemma 4.3 implies that  $[D\Phi(0)P]_{\alpha} = T(\boldsymbol{\lambda}^{\alpha})P_{\alpha}$ , for all  $|\alpha| > 1$ . Therefore, we conclude that

$$\begin{aligned} 0 &= [\Phi(P^{\leq n+1})]_{\alpha} = [D\Phi(0)P^{\leq n+1} + \mathcal{N}(P^{\leq n+1})]_{\alpha} \\ &= [D\Phi(0)P^{\leq n+1}]_{\alpha} + [\mathcal{N}(P^{\leq n+1})]_{\alpha} \\ &= T(\boldsymbol{\lambda}^{\alpha})P_{\alpha} + [\mathcal{N}(P^{\leq n})]_{\alpha}, \end{aligned}$$

From this we find the expression

$$P_{\alpha} = T(\boldsymbol{\lambda}^{\alpha})^{-1} [\mathcal{N}(P^{\leq n})]_{\alpha} = T(\boldsymbol{\lambda}^{\alpha})^{-1} [\Phi(P^{\leq n})]_{\alpha}, \quad (23)$$

for all  $|\alpha| = n + 1$ . The polynomial  $P^{\leq n+1}$  can be found from  $P^{\leq n}$  and recursion (23).

It is important to note that the choice of  $P'(0)$  determines the tangent space to the manifold. Hence  $V$  is determined once we choose this space. On the other hand,  $P'(0)$  is determined only up to the size of its columns. These multiples will not be too crucial for the mathematical analysis, but it will be important in the numerical calculations in Section 6.

**Lemma 4.5.** *With the notations above, assume that  $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)$  is a stable non-resonant vector of eigenvalues, and that (13) is satisfied with  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_m)$  and for some  $V = P'(0)$  of maximal rank. Then, for every  $|\alpha| \geq 2$ , we can find a unique  $P_{\alpha}$  such that (22) holds. Furthermore, we can make all  $P_{\alpha}$  arbitrarily small by making the columns of  $P'(0)$  sufficiently small.*

*Remark 4.4.* The assumption that the matrix  $\Lambda$  is diagonal can be eliminated. Following the discussion in Section 2.2, it suffices that  $\Lambda$  is diagonalizable.

*Remark 4.5.* As we will see in Section 6 the proof in this section can be turned into an efficient algorithm using the methods of “automatic differentiation” [JZ05, Nei10, BCH<sup>+</sup>06] which allow a fast evaluation of the coefficients  $P_{\alpha}$ , specially in the case that the manifolds are 1-dimensional. More details, including an implementation in examples, are given in Section 6.

*Remark 4.6.* Notice that the main theorem is proved by a contraction mapping theorem. The formal solutions are indeed an approximate solution. Indeed, in practical problems – see Section 6 – it is possible to produce solutions that have an error comparable to round off. These bounds can be proved rigorously using interval arithmetic.

Given some bounds on the contraction properties of the operator, one concludes bounds on the distance between the approximate solution and the true solution. Hence, the proof presented here gives a strategy to lead to computer assisted proofs.

## 5 Singular case and dependence on parameters

### 5.1 Singular case

In this section, we show how the results can be extended to the case in which  $B_0$  is singular. The key will be to generalize Lemma 4.4. This can be done by estimating the singularity of the matrix  $T(\boldsymbol{\lambda})$



that was defined in (17).

**Example 5.1.** Consider the Lagrangian function  $S : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$  given by

$$S(\theta_0, \theta_1) = -\theta_0^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \theta_1 + \frac{1}{2} \theta_0^T \begin{pmatrix} 1 & 1 \\ 1 & 6 \end{pmatrix} \theta_0.$$

Let  $Z$  be the difference equation that arises from the Euler-Lagrange equation (3). The point  $\theta^* = (0, 0)$  gives a fixed point solution. As before, we define  $T(\lambda)$  as in equation (17). Using definition (14) in remark 3.2, it is possible to verify that the characteristic polynomial is of the form  $\mathcal{F}(\lambda) = \lambda(-2\lambda + 1)(\lambda - 2)$ . Therefore,  $\theta^*$  is singular. We notice that the degree is strictly less than the maximum  $Nd = 4$ ,  $\mathcal{F}(0) = 0$  and  $\lambda$  divides  $\mathcal{F}(\lambda)$ .

In order to deal with the singular case, we have the following result, that will lead to a generalization of Lemma 4.4.

**Lemma 5.1.** *Let  $\theta^*$  be a singular fixed point solution with characteristic polynomial  $\mathcal{F}$ . Let  $e$  be the greatest integer  $e \in \mathbb{Z}_+$  such that  $\lambda^e$  divides  $\mathcal{F}(\lambda)$ , i.e., the polynomial  $\mathcal{F}$  is of the form  $\mathcal{F}(\lambda) = \lambda^e g(\lambda)$ , where  $g$  is a polynomial such that  $g(0) \neq 0$ . Let  $T$  be as in (17) and  $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)$  be a stable non-resonant vector of eigenvalues none of which is zero. Then there exists a constant  $C > 0$  such that*

$$\|T(\boldsymbol{\lambda}^\alpha)^{-1}\| \leq C (\boldsymbol{\lambda}^\alpha)^{-e}, \quad (24)$$

for all multi-indices  $|\alpha| \geq 2$ .

*Proof.* The inverse  $T(\lambda)^{-1}$  is a rational function of the form

$$T(\lambda)^{-1} = \frac{1}{\lambda^e g(\lambda)} Q(\lambda),$$

where  $Q$  is a polynomial matrix and  $g(0) \neq 0$ . Then  $\lambda^e T(\lambda)^{-1}$  is also a rational function, but the limit  $\lim_{\lambda \rightarrow 0} \lambda^e T(\lambda)^{-1}$  exists.

As in the proof of Lemma 4.4, we can argue that since  $\boldsymbol{\lambda}$  is non-resonant and stable, the matrices  $T(\boldsymbol{\lambda}^\alpha)$  are invertible and  $(\boldsymbol{\lambda}^\alpha)^e T(\boldsymbol{\lambda}^\alpha)^{-1}$  are uniformly bounded for all multi-indices  $|\alpha| \geq 2$ . Therefore, there exists a constant  $C > 0$  such that  $\|(\boldsymbol{\lambda}^\alpha)^e T(\boldsymbol{\lambda}^\alpha)^{-1}\| \leq C$ .  $\square$

Lemma 5.1 tells us that the derivative  $D\Phi(0)^{-1}$  is an operator, but it might not be well defined for all the elements of  $H$ . We will introduce a new Banach space. For each  $\mu > 0$ , let  $D(\mu) = \mathbb{C}^m(\mu) = \{z \in \mathbb{C}^m : \|z\|_\infty \leq \mu\}$  be the complex disk around the origin of complex radius  $\mu$ . Using  $D(\mu)$ , we define

$$H(\mu) := \left\{ P^> : D(\mu) \rightarrow \mathbb{C}^d \left| P^>(z) = \sum_{k=2}^{\infty} \sum_{|\alpha|=k} z^\alpha P_\alpha; \|P^>\|_\mu = \sum_{k=2}^{\infty} \sum_{|\alpha|=k} \mu^n \|P_\alpha\|_\infty < \infty \right. \right\}.$$

*Remark 5.1.* Notice that, in the notation of the appendix,

$$H(\mu) = \{P \in A_\mu(\mathbb{C}^m, \mathbb{C}^d) : P(0) = 0, P'(0) = 0\}.$$

If  $\mu_1 < \mu_2$  then  $D(\mu_1) \subset D(\mu_2)$  and  $H(\mu_2)$  can be regarded as a subspace of  $H(\mu_1)$  through the standard inclusion  $H(\mu_2) \hookrightarrow H(\mu_1)$  given by

$$f \mapsto f|_{D(\mu_1)}.$$

In particular, we have that  $H = H(1)$  and if  $0 < \mu < 1$  then we have the inclusion  $H \hookrightarrow H(\mu)$ . In that case, we define an operator  $\Delta_\mu : H(\mu) \rightarrow H$  by

$$\Delta_\mu[\varphi](z) = \varphi(\mu z).$$

Clearly,  $\Delta_\mu$  is always bounded. Functions in  $H(\mu)$  *gain analyticity* through  $\Delta_\mu$ , and the inverse  $\Delta_\mu^{-1}$  is also bounded but *destroys analyticity*. Following the proof of Lemma 4.4, as a corollary of Lemma 5.1 we have the following result.

**Corollary 5.2.** *Let  $\theta^*$  be a fixed point solution and  $\lambda = (\lambda_1, \dots, \lambda_m)$  be a stable non-resonant vector of eigenvalues. Let  $e$  be the greatest integer  $e \in \mathbb{Z}_+$  such that  $\lambda^e$  divides  $\mathcal{F}(\lambda)$ .*

*Suppose that  $0 < \mu \leq \min\{|\lambda_1|^e, \dots, |\lambda_m|^e\}$ . Then  $D\Phi(0)^{-1}$  is a bounded linear operator  $D\Phi(0)^{-1} : H \rightarrow H(\mu)$ . In addition, the composition  $\Delta_\mu D\Phi(0)^{-1} : H \rightarrow H$  is bounded and*

$$\|\Delta_\mu D\Phi(0)^{-1}\| < C, \tag{25}$$

where  $C$  is any constant that satisfies (24) in Lemma 5.1.

## 5.2 Dependence on parameters

Suppose that the difference equation depends smoothly on  $q$  parameters that, for simplicity belong to an open set  $\mathcal{E} \subset \mathbb{R}^q$  around the origin. A special difficulty arises when there is a parameter in which the equation becomes singular. We would like to regularize the singular limit.

**Example 5.2.** Consider the Lagrangian  $S$  defined on  $\mathbb{R}^3$  and given by

$$S(\theta_0, \theta_1, \theta_2) = \frac{\varepsilon}{2} (\theta_2 - \theta_0)^2 + \frac{1}{2} (\theta_1 - \theta_0)^2 + W(\theta_0),$$

where  $\varepsilon$  is a small parameter. From this, we get the Euler-Lagrange equation expressed as (3). If  $W'(0) = 0$ , then the point  $\theta^* = (0, 0)$  is a fixed point solution. The characteristic polynomial for this point is

$$\mathcal{F}(\lambda) = \varepsilon\lambda^4 + \lambda^3 - (2\varepsilon + 2 + W''(0))\lambda^2 + \lambda + \varepsilon.$$

According to Definition 3.3, the fixed point  $\theta^* = (0, 0)$  is non-singular if  $\mathcal{F}(0) \neq 0$  and this happens if and only if  $\varepsilon \neq 0$ .

As before, let  $X = A_1(\mathbb{C}^m, \mathbb{C}^d)$ . As an assumption, suppose that there is an open set  $\mathcal{U} \subset X$  such that the equation  $\Phi : \mathcal{E} \times \mathcal{U} \rightarrow \mathbb{C}^d$  is analytically defined. Suppose that  $\theta^* = 0 \in \mathcal{U}$  is a fixed point solution for all parameters. On  $\mathcal{E} \times \mathcal{U}$ , we define the *nonlinearity* at  $\theta^*$  as

$$\mathcal{N}(\varepsilon, P) = \Phi(\varepsilon, P) - D_2\Phi(\varepsilon, \theta^*)P,$$

for all  $(\varepsilon, P) \in \mathcal{E} \times \mathcal{U}$ .

*Remark 5.2.* In general, for each value of  $\varepsilon \in \mathcal{E}$ , the corresponding characteristic polynomial  $\mathcal{F}_\varepsilon(\lambda)$  is of the form  $\mathcal{F}_\varepsilon(\lambda) = \lambda^{e(\varepsilon)}g_\varepsilon(\lambda)$ , where  $e(\varepsilon)$  is an integer that depends on  $\varepsilon$  and  $g_\varepsilon(0) \neq 0$ . The function  $e(\varepsilon)$  does not need to be continuous and this constitutes a potential difficulty. However, the theorem below only requires that one can find a constant  $\mu$  for which the matrices  $\mu^{|\alpha|}T(\boldsymbol{\lambda}(0)^\alpha)^{-1}$  are uniformly bounded and the nonlinearity  $\mathcal{N}$  vanishes at higher order.

The solution of the stable manifold problem is a local issue, so we can consider  $\mathcal{U}$  and  $\mathcal{E}$  as small neighborhoods that not necessarily cover the largest possible domain for  $\Phi$ . Now we can prove a more general theorem that includes parameters.

**Theorem 5.3.** *Let  $\Phi, \theta^*, \mathcal{N}, \mathcal{E}$  and  $\mathcal{U}$  as above. Suppose that there exists analytic functions  $\lambda_i : \mathcal{E} \rightarrow \mathbb{C}$ ,  $v_i : \mathcal{E} \rightarrow \mathbb{C}^d$ , for  $i = 1, \dots, m$  and constants  $C, \mu > 0$  such that the following conditions are satisfied, for all  $\varepsilon \in \mathcal{E}$ .*

- a) *Each  $\lambda_i(\varepsilon)$  is a non-resonant eigenvalue with eigenvector  $v_i(\varepsilon)$ .*
- b)  *$\boldsymbol{\lambda}(\varepsilon) = (\lambda_1(\varepsilon), \dots, \lambda_m(\varepsilon))$  is a stable non-resonant vector of eigenvalues.*
- c)  *$\|T(\boldsymbol{\lambda}(0)^\alpha)^{-1}\| < C\mu^{-|\alpha|}$ , for all multi-indices such that  $|\alpha| \geq 2$ .*
- d) *There exists an open neighborhood of the origin  $\mathcal{U}_0 \subset \mathcal{U}$  such that the operator  $\mathcal{R}(\varepsilon, P) = \mathcal{N}(\varepsilon, \Delta_\mu^{-1}P)$  can be defined as a function  $\mathcal{R} : \mathcal{E} \times \mathcal{U}_0 \rightarrow X$ .*

For each  $\varepsilon \in \mathcal{E}$ , let  $V(\varepsilon) = (v_1(\varepsilon) \cdots v_m(\varepsilon))$  and  $\Lambda(\varepsilon) = \text{diag}(\lambda_1(\varepsilon), \dots, \lambda_m(\varepsilon))$ . Then, there exist a function  $P_\varepsilon(z)$ , analytic in  $z$  and  $\varepsilon$  such that  $P_\varepsilon(0) = 0$ , its derivative at  $z = 0$  is  $\partial_z P'_\varepsilon(0) = V(\varepsilon)$  and

$$Z_\varepsilon(P_\varepsilon(z), P_\varepsilon(\Lambda(\varepsilon)z), \dots, P_\varepsilon(\Lambda(\varepsilon)^N z)) \equiv 0,$$

for all  $z$  in a neighborhood of the origin. The solution is unique among the solutions of the equation.

*Proof.* For simplicity, let the fixed point solution be  $\theta^* = 0$ . We will use the notation  $\Phi_\varepsilon = \Phi(\varepsilon, \cdot)$ .

Let  $H$  as in (20). The main step of the proof is to show that there exists  $\delta > 0$  and an analytic function  $(\tau, \varepsilon) \mapsto Q_{\tau, \varepsilon}^>$  defined for all  $\|\tau\|_\infty + \|\varepsilon\|_\infty < \delta$  such that

$$\Phi(\varepsilon, \tau \cdot V(\varepsilon) + Q_{\tau, \varepsilon}^>) \equiv 0,$$

with  $Q_{\tau, \varepsilon}^> \in H$  and the solution is unique provided  $\|Q_{\tau, \varepsilon}^>\|_1 < \delta$ .

Now, since  $0 \in \mathcal{U}_0$ , there exists an open set  $\mathcal{V} \subset \mathbb{R}^m \times \mathcal{E} \times H$  that contains  $(0, 0, 0)$  such that if  $(\tau, \varepsilon, Q^>) \in \mathcal{V}$  then  $\tau \cdot (\mu V(\varepsilon)) + Q^> \in \mathcal{U}_0$ . From equation (25), we get that  $D\Phi(0)^{-1}$  is a bounded linear operator  $D\Phi(0)^{-1} : H \rightarrow H(\mu)$  and  $\Delta_\mu D\Phi_0(0)^{-1} : H \rightarrow H$  is uniformly bounded.

Let  $\mathcal{R}(\varepsilon, P) = \mathcal{N}(\varepsilon, \Delta_\mu^{-1}P)$ . Notice that, if  $(\tau, \varepsilon, Q^>) \in \mathcal{V}$  then  $\mathcal{R}(\varepsilon, \tau \cdot (\mu V(\varepsilon)) + Q^>) \in H$ . Let  $\Psi : \mathcal{V} \rightarrow H$  be the operator defined by

$$\Psi(\tau, \varepsilon, Q^>) = D\Phi_0(0)\Delta_\mu^{-1}Q^> + \mathcal{R}(\varepsilon, \tau \cdot (\mu V(\varepsilon)) + Q^>).$$

We notice that  $\Psi(0, 0, 0) = 0$  and  $D_3\Psi(0, 0, 0) = D\Phi_0(0)\Delta_\mu^{-1}$ , that by construction is invertible with bounded inverse. Using the implicit function theorem of Banach spaces, we can find  $\delta > 0$  and a function  $(\tau, \varepsilon) \mapsto Q_{\tau, \varepsilon}^> \in H$  defined for  $\|\tau\|_\infty + \|\varepsilon\|_\infty < \delta$  such that  $\Psi(\tau, \varepsilon, Q_{\tau, \varepsilon}^>) = 0$  and the solution is unique if  $\|Q_{\tau, \varepsilon}^>\| < \delta$ . For each such  $(\tau, \varepsilon)$ , define  $P_{\tau, \varepsilon}^> = \Delta_\mu^{-1}Q_{\tau, \varepsilon}^>$ . This implies that

$$D\Phi_\varepsilon(0)P_{\tau, \varepsilon}^> + \mathcal{N}(\varepsilon, \tau \cdot V(\varepsilon) + P_{\tau, \varepsilon}^>) \equiv 0.$$

Since  $D\Phi_\varepsilon(0)V(\varepsilon) \equiv 0$ , we have that  $P_{\tau, \varepsilon} = \tau \cdot V(\varepsilon) + P_{\tau, \varepsilon}^>$  is a solution and satisfies

$$\Phi_\varepsilon(P_{\tau, \varepsilon}) = \Phi_\varepsilon(\tau \cdot V(\varepsilon) + P_{\tau, \varepsilon}^>) \equiv 0.$$

The proof is finished as in the proof of Theorem 4.1, with a change of variables. Suppose that  $P_{\tau, \varepsilon}^>$  is a solution such that the vector  $\tau = (\tau_1, \dots, \tau_m)$  satisfies  $\|\tau\|_\infty < r$  and its entries are positive. Let

$$P_\varepsilon = V(\varepsilon) + P_{\tau, \varepsilon}^> \circ \text{diag}(\tau_1, \dots, \tau_m)^{-1}.$$

Then  $P_\varepsilon(0) = 0$ ,  $\partial_z P'_\varepsilon(0) = V(\varepsilon)$ , and has a radius of convergence  $r$ , where  $r$  is given by

$$r = \min\{\tau_1, \dots, \tau_m\}.$$

□

## 6 Examples of numerical algorithms

In this section we describe efficient numerical methods to compute the parameterizations  $P$  described in the previous sections. We will present algorithms for

- A.** Standard map model with several harmonics.
- B.** Frenkel-Kontorova models with long range interactions.<sup>2</sup>
- C.** Heisenberg  $XY$  models.
- D.** Invariant manifolds in Froeschlé maps

For simplicity, the invariant manifolds are 1–dimensional. System **A** is a twist map, **B** contains a singular limit, **C** does not define a map. In **D** is a 4–dimensional and we study both strong stable and slow invariant manifolds.

### 6.1 Standard map model with $K$ –harmonics

Let  $C_1, \dots, C_K$  be given numbers. Consider the Lagrangian  $S : \mathbb{R}^2 \rightarrow \mathbb{R}$  given by  $S(\theta_0, \theta_1) = \frac{1}{2}(\theta_0 - \theta_1)^2 + W(\theta_0)$ , with

$$W(\theta) = - \sum_{j=0}^K \frac{C_j}{j} \cos(j\theta).$$

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<sup>2</sup>We will have the C code available in the web.

The corresponding Euler-Lagrange difference equation can be written as  $Z(\theta_0, \theta_1, \theta_2) \equiv 0$ , where  $Z : \mathbb{R}^3 \rightarrow \mathbb{R}$  is the function given by  $Z(\theta_0, \theta_1, \theta_2) = \theta_2 - 2\theta_1 + \theta_0 - W'(\theta_1)$ .

Many authors have treated the original ( $K = 1$ ) standard map, also known as Chirikov model. The bi-harmonic model ( $K = 2$ ) was studied by [BM94, LC06]. The parameterization problem to solve is

$$P(\lambda^2 z) - 2P(\lambda z) + P(z) - \sum_{j=0}^K C_j \sin(jP(\lambda z)) = 0, \quad (26)$$

where  $\lambda$  solves  $\mathcal{F}(\lambda) = \lambda^2 + \left(-2 + \sum_{j=0}^K j C_j\right) \lambda + 1 = 0$ , and  $|\lambda| < 1$ . Sometimes, it is more convenient to write (26) as

$$P(\lambda z) - 2P(z) + P(\lambda^{-1} z) - \sum_{j=0}^K C_j \sin(jP(z)) = 0. \quad (27)$$

We solve (27) by equating coefficients of like terms. The orders 0 and 1 are special. Clearly, we can take  $P_0 = 0$ . This corresponds to choosing the fixed point  $\theta^* = 0$ . Equating terms of first order in (27), we obtain:

$$\left( \lambda + \lambda^{-1} - 2 + \sum_{j=0}^K j C_j \right) P_1 = 0. \quad (28)$$

Since we choose  $\lambda$  so that the term in parenthesis vanishes, we obtain that  $P_1$  is arbitrary. Once we have chosen  $\lambda$  so that it solves the quadratic equation, any  $P_1$  will lead to a solution of (28).

Any choice of  $P_1$  is equivalent from the mathematical point of view as they correspond to the choice of scale of the parameterization. However, from the numerical point of view it is convenient to choose  $P_1$  in such a way that the subsequent coefficients have comparable sizes so that the round of error is minimized. In practice, to find a good choice of  $P_1$  we perform a trial run of low order which gives an idea of the exponential growth or (decay) of the coefficients  $P_n$  and then fix  $P_1$  so that the coefficients  $P_n$  neither grow nor decay too much.

The equation for  $\lambda$  is quadratic. The product of its roots is 1 so, when

$$\left| -2 + \sum_{j=1}^K j C_j \right| > 2,$$

we get two roots  $\lambda_1$  and  $\lambda_2$  of the characteristic polynomial  $\mathcal{F}$  such that  $|\lambda_1| < 1$ ,  $|\lambda_2| > 1$ . We choose the stable eigenvalue  $\lambda_1$ .

Since  $\lambda^n + \lambda^{-n} - 2 + \sum_{j=1}^K j C_j \neq 0$  ( $\lambda^n$  is not a root of  $\mathcal{F}$ ), we get

$$P_n = \left( \lambda^n + \lambda^{-n} - 2 + \sum_{j=1}^K j C_j \right)^{-1} \left[ \sum_{j=1}^K C_j \sin(jP^{\leq(n-1)}) \right]_n. \quad (29)$$

Note that the right hand side can be evaluated if we know  $P^{\leq(n-1)}$  and hence we can recursively compute  $P_n$ . In each step, the coefficients can be found using algorithms explained below which will be also used in other sections.

## 6.2 Efficient evaluation of trigonometric functions

Given a series,  $P(z) = \sum_{n=0}^{\infty} P_n z^n$ , we often want to compute the power series expansions of  $\sin(P(z))$  and  $\cos(P(z))$ . The following algorithm is taken from [Knu97]. Denote  $S(z) = \sin(P(z))$  and  $C(z) = \cos(P(z))$ . Then, we have

$$S'(z) = C(z)P'(z), \quad C'(z) = -S(z)P'(z). \quad (30)$$

Suppose that we can write these functions as  $S(z) = \sum_{n=0}^{\infty} S_n z^n$  and  $C(z) = \sum_{n=0}^{\infty} C_n z^n$ . For each  $n \in \mathbb{N}$ , we will denote  $S^{\leq n}(z) = \sum_{k=0}^n S_k z^k$ ,  $C^{\leq n}(z) = \sum_{k=0}^n C_k z^k$  and  $P^{\leq n}(z) = \sum_{k=0}^n P_k z^k$ . Also,  $[\cdot]_n$  will represent the coefficient of order  $n$  of an analytic function.

Equating terms of order  $n$  in (30), we obtain

$$\begin{aligned} (n+1)S_{n+1} &= \sum_{j=0}^n C_{n-j}(j+1)P_{j+1}, \\ (n+1)C_{n+1} &= -\sum_{j=0}^n S_{n-j}(j+1)P_{j+1}. \end{aligned} \quad (31)$$

The recursion (31) allows to compute the pair of coefficients  $S_{n+1}$ ,  $C_{n+1}$  provided that we know the coefficients  $S_0, \dots, S_n$  and  $C_0, \dots, C_n$ . We note that, obviously  $S_0, C_0$  are straightforward to compute. From this, we also make the obvious observation that

$$\begin{aligned} S_{n+1} &= C_0 P_{n+1} + \frac{1}{n+1} [\cos(P^{\leq n}) Q^{\leq n}]_{n+1}, \\ C_{n+1} &= -S_0 P_{n+1} - \frac{1}{n+1} [\sin(P^{\leq n}) Q^{\leq n}]_{n+1}, \end{aligned} \quad (32)$$

where  $Q^{\leq n}(z) = \sum_{k=0}^n (k+1)P_{k+1}z^k$ . In particular, if  $P_0 = 0$ , then we conclude that

$$\begin{aligned} [S]_{n+1} &= P_{n+1} + [\sin(P^{\leq n})]_{n+1}, \\ [C]_{n+1} &= [\cos(P^{\leq n})]_{n+1}. \end{aligned}$$

This recursion allows us to get the expansion to order  $n$  of  $\sin(P(z))$  and  $\cos(P(z))$ , given the expansion of  $P$  to order  $n$ . Furthermore, we observe that if we change  $P_n$ —the coefficient of order  $n$  of  $P$ —this only affects the coefficients of  $\sin(P(z))$  and  $\cos(P(z))$  of order  $n$  or higher.

The practical arrangement of the calculation of the coefficients in the standard map with  $K$  harmonics is to keep different polynomials  $S_\ell^{\leq n}$  and  $C_\ell^{\leq n}$  that correspond to the series expansions of  $\sin(\ell P)$  and  $\cos(\ell P)$  up to order  $n$ . If the polynomial  $P$  is computed to order  $n-1$  and the  $S_\ell^{\leq n}$  and  $C_\ell^{\leq n}$  corresponding to  $P^{\leq (n-1)}$  are computed up to order  $n$ , we can compute the coefficient  $P_n$  using (29). Then, we can compute the corresponding  $S_\ell^{\leq (n+1)}$  and  $C_\ell^{\leq (n+1)}$  up to order  $n+1$  using (31). We note that similar algorithms can be deduced for  $e^{P(z)}$ ,  $\log P(z)$ ,  $P(z)^\gamma$  or indeed the composition of  $P$  with any function that solves a simple differential equation.

## 6.3 Frenkel-Kontorova model with extended interactions

### 6.3.1 Set up

Consider the Frenkel-Kontorova model with long range interactions. A particle interacts not only with its nearest neighbors, but with other neighbors that are far away. Let  $N \geq 2$ . We consider the Lagrangian function  $S : \mathbb{R}^{N+1} \rightarrow \mathbb{R}$  given by

$$S(\theta_0, \dots, \theta_N) = \frac{1}{2} \sum_{L=1}^N \gamma_L (\theta_L - \theta_0)^2 + W(\theta_0).$$

The corresponding Euler-Lagrange equilibrium equations are expressions of  $2N + 1$  variables, that in this case have the form:

$$\sum_{L=1}^N \gamma_L (\theta_{k+L} - 2\theta_k + \theta_{k-L}) - W'(\theta_k) = 0.$$

These equations represent a difference equation of order  $2N$ .

Suppose that  $W'(0) = 0$ . Then the system has a fixed point solution at 0 and the corresponding characteristic polynomial is of the form  $\mathcal{F}(\lambda) = \lambda^N \mathcal{L}(\lambda)$ , where

$$\mathcal{L}(\lambda) = \sum_{L=1}^N \gamma_L (\lambda^L - 2 + \lambda^{-L}) - W''(0).$$

In addition, the one-dimensional parameterization equations of the point can be written as

$$\sum_{L=1}^N \gamma_L (P(\lambda^{N+L}z) - 2P(\lambda^N z) + P(\lambda^{N-L}z)) - W'(P(\lambda^N z)) = 0,$$

where  $\lambda$  is a non-resonant stable root of the characteristic function  $\mathcal{L}$ .

*Remark 6.1.* We can simplify the characteristic polynomial above. Notice that, if we let  $\omega = (\lambda + \lambda^{-1})/2$ , then

$$\frac{\lambda^L + \lambda^{-L}}{2} = \mathcal{T}_L(\omega),$$

where  $\mathcal{T}_L$  is the  $L$ -th Tchebychev polynomial. Let  $r(\omega)$  be the polynomial of degree  $N$  given by

$$r(\omega) = \sum_{L=1}^N \gamma_L (\mathcal{T}_L(\omega) - 1) - \frac{1}{2} W''(0). \quad (33)$$

Then, characteristic polynomial  $\mathcal{F}(\lambda)$ , can be written as  $\mathcal{F}(\lambda) = 2\lambda^N r((\lambda + \lambda^{-1})/2)$ . In addition,  $\mathcal{F}(\lambda)$  has no zeroes on the unit circle if and only if  $r(\omega)$  has no roots on the segment  $[-1, 1] \subset \mathbb{C}$ . For each root  $\omega$  of  $r$ , we get a pair of eigenvalues. If  $\omega$  is real and  $|\omega| > 1$ , then these eigenvalues are a pair of real numbers

$$\lambda^{s,u} = \omega \pm \sqrt{\omega^2 - 1}$$

that satisfy  $0 < |\lambda^s| < 1 < |\lambda^u|$ .

### 6.3.2 Singular limit and slow manifolds

In many situations the long-range interactions of the particles in the model are small. We could ask the question of what happens in the limit. It turns out that the system becomes singular and the usual dynamical systems approach fails to be useful. However, certain stable manifolds persist, as in Theorem 5.3. We illustrate this difficulty with an example.

**Example 6.1.** Consider a Frenkel-Kontorova equation with  $\gamma_1 = 1$  and  $\gamma_2 = \varepsilon$ . In this case, the auxiliary polynomial in (33) is

$$r(\omega) = \varepsilon(2\omega^2 - 2) + \omega - \beta,$$

where  $\beta = 1 + \frac{1}{2}W''(0)$ .

Solving for  $\omega(\varepsilon)$ , we get that

$$\omega^\pm(\varepsilon) = \frac{2(\beta + 2\varepsilon)}{1 \pm \sqrt{1 + 8\varepsilon(\beta + 2\varepsilon)}}.$$

If  $\varepsilon \rightarrow 0$  then we have a singular limit. We notice that, as  $\varepsilon \rightarrow 0$ , the two roots of the polynomial have two different limits  $\omega^+(\varepsilon) \rightarrow \beta$  and  $\omega^-(\varepsilon) \rightarrow \infty$ . In terms of the stability of the fixed point, the limit  $\omega^-(\varepsilon) \rightarrow \infty$  corresponds to a pair of eigenvalues  $\lambda^s, \lambda^u$  that are very hyperbolic in the sense that  $\lambda^s \lambda^u = 1$  and  $\lambda^s \rightarrow 0$  and  $\lambda^u \rightarrow \infty$  as  $\varepsilon \rightarrow 0$ .

Fortunately, the other pair of eigenvalues can be continued through the singularity  $\varepsilon = 0$ . This family is smooth and will be denoted by  $\lambda^s(\varepsilon), \lambda^u(\varepsilon)$ . They satisfy  $0 < \lambda^s(\varepsilon) < 1 < \lambda^u(\varepsilon)$ ,  $\lambda^s(\varepsilon)\lambda^u(\varepsilon) = 1$  and  $\lambda^s(0) + \lambda^u(0) = 2\beta$ .

*Remark 6.2.* In general, if we let  $\beta = 1 + W''(0)/(2\gamma_1)$ , then there is a family of roots  $\omega$  of  $r(\omega)$  such that  $\omega \rightarrow \beta$  as  $(\gamma_2, \dots, \gamma_N) \rightarrow 0$ . It follows that, if  $|\beta| > 1$  and the coefficients  $\gamma_2, \dots, \gamma_N$  are small enough, then the fixed point  $\theta^* = 0$  is hyperbolic. This occurs, for instance, when 0 is a minimum of the potential  $W$ ,  $\gamma_1 > 0$ , and the long range interactions are weak.

We are interested in the persistence of slow manifolds in the Frenkel-Kontorova model with long-range interactions. We can consider that the interactions are small. Suppose that we have  $N$  long-range interactions represented by small coefficients  $\gamma_2(\varepsilon), \dots, \gamma_N(\varepsilon)$  that depend analytically on the parameter  $\varepsilon$ . Assume that  $\gamma_2(0) = \dots = \gamma_N(0) = 0$ ,  $\gamma'_N(0) \neq 0$  and, without loss of generality, that  $\gamma_1(\varepsilon) \equiv 1$ .

For each  $\varepsilon$ , the characteristic polynomial  $\mathcal{F}_\varepsilon$  of the fixed point  $\theta^* = 0$  is of degree at most  $Nd$ . From the implicit function theorem, we can argue that there exists a number  $\varepsilon_0 > 0$  and a smooth function  $\omega(\varepsilon)$  such that, if  $|\varepsilon| \leq \varepsilon_0$  then  $r_\varepsilon(\omega(\varepsilon)) \equiv 0$ , where  $\omega(0) = \beta$  and

$$r_\varepsilon(\omega) = \sum_{L=1}^N \gamma_L(\varepsilon)(\mathcal{T}_L(\omega) - 1) - \frac{1}{2}W''(0).$$

All the other roots of  $r_\varepsilon(\omega)$  diverge as  $\varepsilon \rightarrow \infty$ . For each non-singular root  $\omega(\varepsilon)$  of  $r_\varepsilon$ , we get the following stable eigenvalue

$$\lambda^s(\varepsilon) = \omega(\varepsilon) - \sqrt{\omega(\varepsilon)^2 - 1}.$$



The family  $\lambda^s(\varepsilon)$  can be continued through the singularity  $\varepsilon = 0$  and, for all  $|\varepsilon| \leq \varepsilon_0$ , it corresponds to a slow manifold, i.e. the invariant manifold with the largest stable eigenvalue. In fact,  $\lambda^s(\varepsilon)$  is analytic near  $\varepsilon = 0$ .

For each  $|\varepsilon| \leq \varepsilon_0$ , there exists a non-negative integer  $e(\varepsilon)$  such that  $\mathcal{F}_\varepsilon(\lambda) = \lambda^{e(\varepsilon)}g_\varepsilon(\lambda)$ . By construction,  $e(\varepsilon) = 0$  if  $\varepsilon \neq 0$  and  $e(\varepsilon) = N - 2$  if  $\varepsilon = 0$ . Using the notation of Theorem 5.3, we have that the nonlinearity of the parameterization operator at  $\theta^*$  is precisely

$$[\mathcal{N}(\varepsilon, P)](z) = W''(0)P(\lambda^s(\varepsilon)^N z) - W'(P(\lambda^s(\varepsilon)^N z)).$$

Let  $\mu$  be a number such that

$$\lambda^s(0)^N < \mu < \lambda^s(0)^{N-2}.$$

Restricting  $\varepsilon_0$  further, it is possible to assume that  $\mu$  also satisfies  $\lambda^s(\varepsilon)^N < \mu < \lambda^s(\varepsilon)^{N-2}$ , for all  $|\varepsilon| \leq \varepsilon_0$ . Using Lemma 5.1, we conclude that there exists a constant  $C$  such that the matrix norm satisfies

$$\|\lambda^s(0)^{(N-2)k}T(\lambda^s(0)^k)^{-1}\| \leq C,$$

for all  $k \geq 2$ . Furthermore, we have that

$$\|\mu^k T(\lambda^s(0))^{-1}\| \leq C\mu^k \lambda^s(0)^{-(N-2)k} < C,$$

for all  $k \geq 2$ . Now, the condition on the nonlinearity is that  $\mathcal{R}(\varepsilon, P) = \mathcal{N}(\varepsilon, \Delta_\mu^{-1}P) \in X$ , for all  $P \in X$  in a neighborhood of the origin. However, the nonlinearity satisfies

$$[\mathcal{R}(\varepsilon, P)](z) = [\mathcal{N}(\varepsilon, \Delta_\mu^{-1}P)](z) = W''(0)P(\mu^{-1}\lambda^s(\varepsilon)^N z) - W'(P(\mu^{-1}\lambda^s(\varepsilon)^N z)).$$

Since  $|\mu^{-1}\lambda^s(\varepsilon)^N| < 1$ , we conclude that there exists an open neighborhood  $\mathcal{U}_0$  of 0 in which the nonlinearity can be defined as an operator. From these considerations we conclude that the Theorem 5.3 applies. Therefore, there exists a family of analytic solutions  $P_\varepsilon$  of the parameterization problem. This is illustrated in the numerical example that follows.

### 6.3.3 Some numerics

If we consider the  $K$ -harmonic potential  $W(\theta) = -\delta \sum_{j=1}^K \frac{C_j}{j} \cos(j\theta)$ , then the equilibrium equations are

$$\sum_{L=1}^N \gamma_L(\theta_{k+L} - 2\theta_k + \theta_{k-L}) + \delta \sum_{j=1}^K C_j \sin(j\theta_k) = 0. \quad (34)$$

The parameterization equations of a 1-dimensional stable manifold can be written as

$$[\Phi(P)](z) = \sum_{L=1}^N \gamma_L(P(\lambda^L z) - 2P(z) + P(\lambda^{-L} z)) + \delta \sum_{j=1}^K C_j \sin(jP(z)) = 0, \quad (35)$$

where  $\lambda$  is a non-resonant stable eigenvalue. By symmetry, if  $P$  parameterizes a stable manifold it also parameterizes an unstable one. The characteristic polynomial is  $\mathcal{F}(\lambda) = \lambda^N \mathcal{L}(\lambda)$ , where

$$\mathcal{L}(\lambda) = \sum_{L=1}^N \gamma_L(\lambda^L + \lambda^{-L} - 2) + \delta \sum_{j=1}^K j C_j.$$

We will find a stable parameterization  $P$  corresponding to the fixed point solution 0. Suppose that  $P$  is of the form  $P(z) = \sum_{k=0}^{\infty} z^k P_k$ . We set, therefore,  $P_0 = 0$ . Also, we chose a solution  $\lambda$  of  $\mathcal{L}(\lambda) = 0$ , which amounts to choosing the stable manifold we want to study, and set  $P_1$  so that the numerical error is minimized. When  $n \geq 2$ , matching coefficients of  $z^n$  in (35), we obtain

$$\mathcal{L}(\lambda^{n+1})P_{n+1} + \delta \left[ \sum_{j=1}^K C_j \sin(jP^{\leq n}) \right]_{n+1} = 0. \quad (36)$$

For a generic set of values of  $\gamma_1, \dots, \gamma_N$  and  $C_1, \dots, C_K$ , we have that  $\mathcal{L}(\lambda^{n+1}) \neq 0$ , for all  $n \in \mathbb{N}$ . Therefore, we can solve the equation (36) and get a non-resonant eigenvalue.

We keep the polynomials  $P, S^j, C^j$  as in Section 6.2 and assume that we know  $P^{\leq(n-1)}$  and the  $S^j, C^j$  corresponding to  $P^{\leq(n-1)}$  up to order  $n$ . We use (36) to compute  $P_n$  and then (31) to compute the  $S^j, C^j$  corresponding to  $n$  up to order  $n - 1$ . The only difference with the short range case is that, when solving the recursion, we need to divide by a slightly different factor.

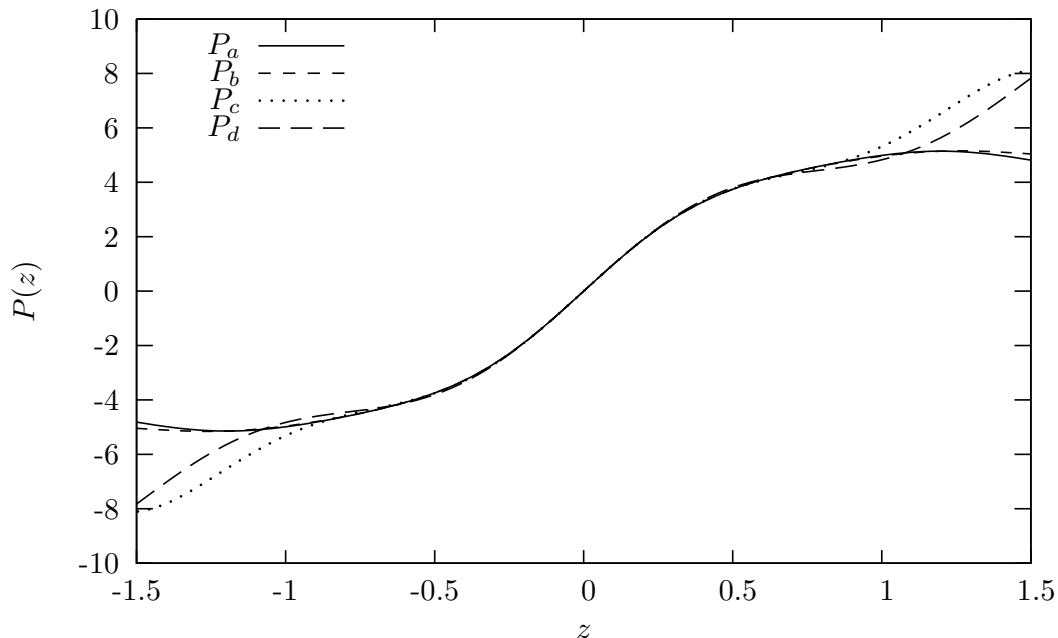


Figure 1: Four parameterizations for the Frenkel-Kontorova model of example 6.2 with parameters given in table 1.

**Example 6.2.** Consider an specific singular limit. Let the Frenkel-Kontorova model with  $N = 3$ ,  $K = 1$ ,  $\delta = 0.4$ , and  $C_1 = 1$ . We fix the values of  $\gamma_1, \gamma_2, \gamma_3$  in four examples.

Using the algorithm proposed of this section, we find the first 100 coefficients of the Taylor series expansion of the parameterizations corresponding to the following values. In each case, the computed eigenvalue  $\lambda$  corresponds to the slow manifold.

The solution of the parameterization problem (35) is, in fact, a family of functions that depend on the size of the derivative. We find uniqueness only when the derivative  $P'(0)$  is fixed. This value is

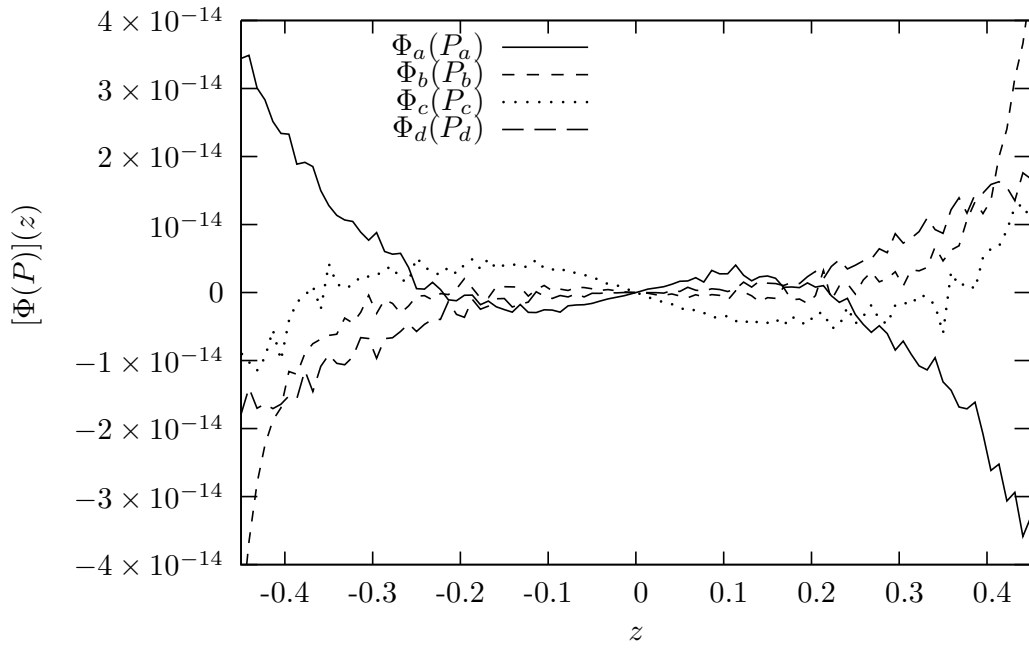


Figure 2: Error function  $\Phi(P)$  for the approximation to the parameterization solution of the Frenkel-Kontorova model of example 6.2 with the parameters given in table 1.

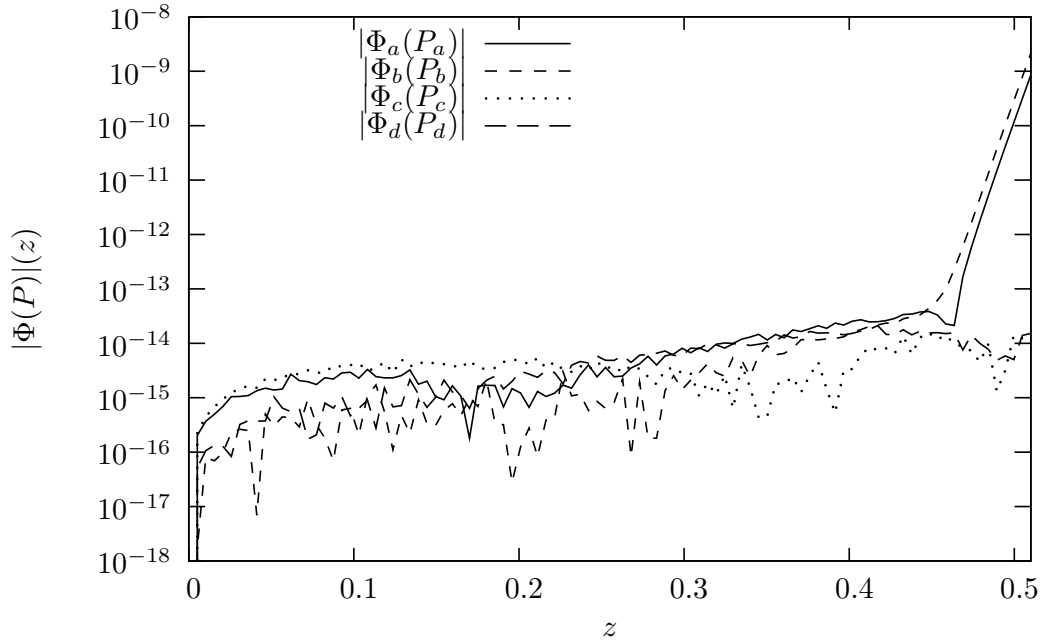


Figure 3: These graphs illustrate the absolute value  $|\Phi(P)|$  of the error function for the Frenkel-Kontorova model of example 6.2 with parameters given in table 1. Logarithmic scale is used in the vertical axis.

Parameterization	$\gamma_1$	$\gamma_2$	$\gamma_3$	$\delta$	$P'(0)$	$\lambda$
$P_a$	1	0.1	0.00	0.4	10.0	0.592583231399561
$P_b$	1	0.14	0.00	0.4	10.0	0.609158827181520
$P_c$	1	0.1	0.01	0.4	10.0	0.603202338024902
$P_d$	1	0.1	0.03	0.4	10.0	0.621569001269222

Table 1: Parameters for the Frenkel-Kontorova examples.

determined so that the coefficients are of order 1. As input we use the parameters given in table 1 and, as output, we get the approximation  $P^{\leq 100}$ .

In the example, parameterizations of the slow manifold are computed. The dimension of the problem changes but the method allows the continuation of the solution through the singularity. In other words, if we regard the difference equation as a dynamical system, then cases  $a$  and  $b$  would be maps in  $\mathbb{R}^4$  and cases  $c$  and  $d$  would be maps in  $\mathbb{R}^6$ . This collapse in the dimension is problematic if one uses a dynamical system point of view, but is manageable when the Lagrangian point of view is used.

We notice that the four parameterization functions  $P_a$ ,  $P_b$ ,  $P_c$ , and  $P_d$  are similar for small values of  $z$ . However, the difference equation is singular for the parameters of the first and second examples. The numerical results are illustrated in Figure 1. In Figures 2 and 3, we can see an approximation to the value of  $\Phi(P)$  near  $z = 0$ . In each case, we provide a graph of  $\Phi(P^{\leq 100})$ . These graphs quantify the error in the approximation.

## 6.4 The Heisenberg $XY$ model

Consider the difference equation mentioned in the introduction and given by (4). The characteristic polynomial of the fixed point  $\theta^* = 0$  is  $\mathcal{F}(\lambda) = \lambda^2 - (2 + \varepsilon)\lambda + 1$ . The corresponding parameterization equations can be written as

$$\sin(P(\lambda z) - P(z)) + \sin(P(\lambda^{-1}z) - P(z)) - \varepsilon \sin P(z) = 0, \quad (37)$$

where  $\lambda$  is a stable root of  $\mathcal{F}$ .

Equating terms of order  $n$  in (37) we obtain that, for  $n = 0$ , the choice of  $P_0 = 0$  corresponds to choosing the fixed point solution we are studying. The term of order  $n = 1$  amounts to choosing the manifold and setting the numerical scale at which we are working.

The equations obtained matching order  $n \geq 2$  are

$$\begin{aligned} \mathcal{F}(\lambda^{n+1})P_{n+1} &= [\sin(P^{\leq n}(z) - P^{\leq n}(\lambda z))]_{n+1} \\ &+ [\sin(P^{\leq n}(z) - P^{\leq n}(\lambda^{-1}z))]_{n+1} + \varepsilon [\sin(P^{\leq n}(z))]_{n+1}. \end{aligned} \quad (38)$$

The only difference with the Chirikov model or standard map is that we also have to compute  $\bar{S}(z) = \sin(P(z) - P(\lambda z))$  and  $\bar{C}(z) = \cos(P(z) - P(\lambda z))$ . Of course,  $\bar{S}(\lambda^{-1}z) = \sin(P(\lambda^{-1}z) - P(z))$  and  $\bar{C}(\lambda^{-1}z) = \cos(P(\lambda^{-1}z) - P(z))$ . In the inductive step we assume that we know  $P^{\leq (n-1)}$  and the  $S, C, \bar{S}, \bar{C}$  corresponding to  $P^{\leq (n-1)}$  up to order  $n$ . Using (38), we can compute  $P_n$  and then use (31) to compute  $S, C, \bar{S}, \bar{C}$  corresponding to  $P^{\leq n}$  up to order  $n + 1$  and the induction can continue.

## 6.5 Non resonant invariant manifolds in Froeschlé maps

The Froeschlé map is a popular model [Fro72, OV94] of higher dimensional twist maps. It is designed to be a model of the behavior of a double resonance. In the Lagrangian formulation, the Lagrangian of the model is a function  $S : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$  that is given by

$$S(\theta_0, \theta_1) = \frac{1}{2}(\theta_1 - \theta_0)^2 + W(\theta_0),$$

where  $W : \mathbb{R}^2 \rightarrow \mathbb{R}$  is a potential functions such that  $W(\theta + k) = W(\theta)$ , for all  $k \in \mathbb{Z}^2$ . The resulting Euler-Lagrange equations are given by

$$\theta_{k+1} - 2\theta_k + \theta_{k-1} + \nabla W(\theta_k) = 0. \quad (39)$$

If  $\nabla W(0) = 0$ , then  $\theta^* = 0$  is a fixed point solution and the characteristic function is

$$\mathcal{L}(\lambda) = \det((\lambda + \lambda^{-1} - 2)I + D^2W(0)).$$

The particular example used by Froeschlé is

$$W(x_1, x_2) = a \cos(2\pi x_1) + b \cos(2\pi x_2) + c \cos(2\pi(x_1 - x_2)).$$

The matrix  $I - \frac{1}{2}D^2W(0)$  is a  $2 \times 2$  symmetric matrix. Typically, it has two real eigenvalues  $\omega_1$  and  $\omega_2$ . It turns out that there are four roots of  $\mathcal{L}(\lambda)$ , that constitute the spectrum of the fixed point. They are given by the solutions of

$$\lambda + \lambda^{-1} - 2\omega_i = 0.$$

It is easy to see that these four solutions are given by  $\omega_i \pm \sqrt{\omega_i^2 - 1}$ , for  $i = 1, 2$ . From this, we conclude that the solutions are of the form  $\lambda_1, \lambda_2, \lambda_1^{-1}, \lambda_2^{-1}$ , where  $|\lambda_1| \leq |\lambda_2| \leq 1$ . We have to consider three possibilities:

- a)**  $0 < \lambda_1 \leq \lambda_2 < 1$ .                      **b)**  $0 < \lambda_1 < 1, |\lambda_2| = 1$ .                      **c)**  $|\lambda_1| = |\lambda_2| = 1$ .

The classical theory of invariant manifolds allows to associate an invariant one dimensional manifold with  $\lambda_1$  in cases a) and b), and a two dimensional invariant manifold in case a). See also [dlL97] for the case  $0 < \lambda_1 = \lambda_2 < 1$ .

The parameterization method allows also to make sense of each manifold tangent to the space corresponding to  $\lambda_2$  provided  $\lambda_1^k \neq \lambda_2$  and  $\lambda_2^k \neq \lambda_1$ , for  $k \geq 2, k \in \mathbb{N}$ . The calculations are remarkably similar to those of the stable manifolds for the Chirikov map studied before.

We now indicate the algorithm. Since we are considering the fixed point  $\theta^* = 0$ , we make  $P_0 = 0$ . As before, the first coefficient of the parameterization is an eigenvector associated with the eigenvalue  $\lambda_2$ . Again, we note that the size of  $P_1$  corresponds to different scales of the parameterization, and does not affect the mathematical considerations. On the other hand, choosing an appropriate scale is crucial in order to minimize round-off errors.

When  $W$  is a trigonometric polynomial, as in the original Froeschlé model, we can compute the components of  $[\nabla W (P^{\leq n-1})]_n$ . We conclude that  $n$ -th coefficient of the parameterization satisfies

$$((\lambda^n + \lambda^{-n} - 2)I + D^2W(0)) P_n = - [\nabla W (P^{\leq n-1})]_n. \quad (40)$$

## 7 The continuously differentiable case

The approach to invariant manifolds of this paper, also applies when the map  $Z$  is finite differentiable. Many of the techniques presented above can be adapted to the finite differentiable case. Here, we will just present an analogue of Theorem 4.1, but we leave to the reader other issues such as singular limits or dependence on parameters.

**Theorem 7.1.** *Assume that  $Z$  is  $C^{r+1}$ . Let  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_m)$ , where  $\lambda = (\lambda_1, \dots, \lambda_m)$  is a stable non-resonant vector. Assume that  $B_0$  is invertible and that  $r$  is sufficiently large, depending only on  $\Lambda$ . Then, we can find a  $C^r$   $P$  solving (10),  $P(0) = 0$  and such that the range of  $P_1$  is the space corresponding to the eigenvalues in  $\Lambda$ .*

The conditions on  $r$  assumed in Theorem 7.1 will be made explicit in Lemma 7.3. It will be clear from the proof of Theorem 7.1 that, if we assume more regularity in  $Z$ , we can obtain differentiability with respect to parameters (keeping  $\Lambda$  fixed).

We still study the equation  $\Phi(P) = 0$  by implicit function methods but now  $P$  ranges over a space of finite differentiable functions. To apply the implicit function theorem, we just need the differentiability of the functional  $\Phi$  and to show that  $D\Phi(0)$  is invertible with bounded inverse. One complication is that we cannot show that  $D\Phi(0)$  is boundedly invertible by matching powers. Nevertheless, we will show that  $D\Phi(0)$  is invertible for functions that vanish at high order. Hence, we will use that we can find power series solutions so that we can write  $P = P^{<} + P^{\geq}$  where  $P^{<}$  is a polynomial we will assume known and  $P^{\geq}$  are functions vanishing to high order.

The functional analysis properties of  $\Phi$  will be taken care of in the following lemmas.

**Lemma 7.2.** *If  $Z \in C^{r+\ell}$   $r, \ell \in \mathbb{N}$ , then there exists an open subset  $\mathcal{U} \subset C^r(D, \mathbb{R}^d)$  such that the operator  $\Phi : \mathcal{U} \rightarrow C^r(D, \mathbb{R}^d)$  is  $C^\ell$ . Furthermore, if  $\ell \geq 1$ , we have*

$$D\Phi(P)\varphi = \sum_{j=0}^N \partial_j Z(P, P \circ \Lambda, \dots, P \circ \Lambda^N) \varphi \circ \Lambda^j. \quad (41)$$

In particular,

$$D\Phi(0)\varphi = \sum_{j=0}^n B_j(\varphi \circ \Lambda^j) \quad (42)$$

where, as before,  $B_i = \partial_i Z(0, \dots, 0)$ .

The Lemma 7.2 is a direct consequence of the results of [dlLO99] on the composition operator. Of course, the formula (42) is easy to guess heuristically by noticing that it is the leading term in changes in  $P$ . We emphasize that we are considering  $\Lambda$  fixed. The differentiability properties of the operator with respect to  $\Lambda$  are much more subtle.

Let  $D = \mathbb{R}^m(\rho)$ , be the closed disk, in  $\mathbb{R}^m$ , around the origin of radius  $\rho$ . As before,  $\|\cdot\|_\infty$  is the uniform norm defined in (45). To study  $C^r$  spaces, it is convenient to recall that higher-order derivatives are multilinear maps. Given a  $C^r$  function  $g : \mathbb{R}^m \rightarrow \mathbb{R}^d$  and  $k \leq r$ , then the  $k$ -th derivative  $D^k g(a)$

is a  $k$ -linear map  $D^k g(a) : \mathbb{R}^m \times \cdots \times \mathbb{R}^m \rightarrow \mathbb{R}^d$ , symmetric under permutation of the order of the arguments. Recall also that there is a natural norm on the vector space of  $k$ -linear maps  $B$  given by

$$\|B\| = \sup\{\|B(\xi_1, \dots, \xi_k)\|_\infty : \|\xi_1\|_\infty = \cdots = \|\xi_k\|_\infty = 1\}.$$

Once we fix a system of coordinates,  $D^k g$  can be expressed in terms of the partial derivatives

$$\frac{1}{k!} D^k g(a)(h, \dots, h) = \sum_{|\alpha|=k} \frac{1}{\alpha!} \partial^\alpha g(a) h^\alpha.$$

The usual norm in  $C^r(D, \mathbb{R}^d)$  spaces is  $\|g\|_{C^r} = \max_{i \leq r} \sup_{x \in D} \|D^i g(x)\|$ .

To study our problem, we define the closed subspace of  $C^r(D, \mathbb{R}^d)$ .

$$H_r = \left\{ P \in C^r(D, \mathbb{R}^d) : D^i P(0) = 0 \quad 0 \leq i \leq r-1 \right\}.$$

Clearly  $P \in H_r$  if and only if  $\partial^\alpha P(0) = 0$ , for all  $|\alpha| \leq r-1$ . The space  $H_r$  is a Banach space if endowed with the norm  $\|P\|'_r = \sup\{\|D^r P(x)\| : x \in D\}$ . Because  $P$  vanishes with its derivatives at 0 and we are considering a bounded domain, this norm is equivalent to the standard  $C^r$  norm.

Assume that  $B_0$  is non-singular. Then, we can rewrite (42) as:

$$D\Phi(0)\varphi = B_0 \left[ \varphi + \sum_{j=1}^N B_0^{-1} B_j (\varphi \circ \Lambda^j) \right] = B_0 (\text{Id} + L) \varphi, \quad (43)$$

where  $L$  is the operator defined by:

$$L(\varphi) = \sum_{i=1}^N B_0^{-1} B_i (\varphi \circ \Lambda^i).$$

**Lemma 7.3.** *Let  $\mu$  be a real number such that:*

- a)  $\|\Lambda\|^r \leq \mu < 1$ ,
- b)  $\sum_{i=1}^N \mu^i \|B_0^{-1} B_i\| < 1$ .

*Then the linear operator  $D\Phi(0) : H_r \rightarrow H_r$  is invertible with bounded inverse.*

We emphasize that if  $\lambda$  is a stable non-resonant vector of eigenvalues, the conditions of Lemma 7.3 are satisfied for sufficiently large  $r$ . This is the condition alluded to in Theorem 7.1.

*Proof.* Since  $D\Phi(0) = B_0^{-1}(\text{Id} + L)$  it will suffice to prove that  $L$  is a contraction in the  $\|\cdot\|'_r$  norm. We note that if  $\varphi \in H_r$ , then  $\varphi \circ \Lambda^j \in H_r$  and  $L\varphi \in H_r$ . In addition,

$$[D^r (\varphi \circ \Lambda^j)(x)] (\xi_1, \dots, \xi_r) = [(D^r \varphi)(\Lambda^j x)] (\Lambda^j \xi_1, \dots, \Lambda^j \xi_r).$$

Hence, if  $x \in D$  then

$$\|D^r (\varphi \circ \Lambda^j)(x)\| \leq \|(D^r \varphi)(\Lambda^j x)\| \|\Lambda^j\|^r \leq \|(D^r \varphi)(\Lambda^j x)\| \mu^j,$$

and, since  $\Lambda D \subset D$ , we get that  $\|\varphi \circ \Lambda^j\|'_r \leq \mu^j \|\varphi\|'_r$ . From this we conclude

$$\|L(\varphi)\|'_r \leq \sum_{j=1}^N \|B_0^{-1} B_j\| \|\varphi \circ \Lambda^j\|'_r \leq \left( \sum_{i=1}^N \mu^i \|B_0^{-1} B_i\| \right) \|\varphi\|'_r.$$

This shows that  $L$  is a contraction. □

To complete the proof of Theorem 7.1, we argue in a similar manner as in the applications of the implicit function theorem before. Following the method in Section 4.5, and fixing  $P_1$  to be an embedding on the space, we can find a unique polynomial  $P^<$  of degree  $r - 1$  in such a way that  $P^<(0) = 0$ ,  $(P^<)'(0) = P_1$  and  $D^j \Phi(P^<)(0) = 0$ , for all  $0 \leq j \leq r - 1$ . We recall that the coefficients of  $P^<$  are obtained by matching coefficients. In addition, if we choose  $P_1$  small, they will also be small.

Furthermore, we also note that if  $P^\geq \in H_r$ , we have that

$$D^j \Phi(P^< + P^\geq)(0) = 0; \quad 0 \leq j \leq r - 1.$$

The above remarks can be formulated in terms of functional analysis as saying that the operator  $\tilde{\Phi}(P^\geq) = \Phi(P^< + P^\geq)$  maps an open subset of  $H_r$  into  $H_r$ .

If we take  $P_1$  small without changing its range (this is the same as the change of scale that we considered before), we have that  $\tilde{\Phi}(0)$  will be as small as desired,  $D\tilde{\Phi}(0) = D\Phi(P^<)$  will approach  $D\Phi(0)$  and, in particular will be invertible. The differentiability properties of  $\tilde{\Phi}$  will remain uniformly differentiable. Hence, we can deduce Theorem 7.1 from the implicit function theorem in Banach spaces.

More explicitly, consider the fixed point problem for  $P^\geq$  given by:

$$-D\Phi(0)^{-1} \Phi(P^< + P^\geq) + P^\geq = P^\geq,$$

as acting on  $H_r$ . When  $P^<$  is chosen corresponding to a specific small  $P_1$ , the left hand side will be a contraction and map a ball in  $H_r$  near the origin into itself.

## 8 Final comments

We have studied invariant objects in the context of difference equations. The goal was to find the analog of stable and unstable manifolds that are associated to fixed point solutions of hyperbolic type. We generalized the method proposed in [CFdlL03a, CFdlL03b, CFdlL05] and used the implicit function theorem in Banach spaces to perform this task.

The method is not only theoretical but can be implemented numerically as it was shown in example 6.2. The method is robust in the sense that it can deal with certain singular limits. In particular, slow manifolds can be detected and approximated even in the presence of a singularity.

For future work we plan to consider the finite differentiable case. Also we can use the tools to consider the case of conformally symplectic maps and applications to economics. In addition, we showed that the variational theory has a smooth dependence on parameters and is robust. We plan to explore variational formulations of Melnikov's theory.



## A Banach spaces of analytic functions

In this appendix we study spaces of analytic functions (taking values and having range in Banach spaces) and, in particular, the composition operator  $\mathcal{C}_f(g) = f \circ g$  between analytic functions. We show that the composition operator is itself analytic when defined in spaces of analytic functions.

We call attention to the paper [Mey75] which carried out a similar study and showed that the operator  $\Gamma(f, g) = f \circ g$  was  $C^\infty$ . The paper [Mey75] showed also that many problems in the theory of dynamical systems –invariant manifolds, limit cycles, conjugacies– could be reduced to problems involving the composition operator. Using the result of analyticity presented here, most of the regularity results in [Mey75] can be improved from  $C^\infty$  to analytic.

There are of course, specialized books which contain much more material than we need. For example [Hof88, Nac69].

### A.1 Analytic functions in general Banach spaces

Let  $E$  and  $F$  be Banach spaces. We define  $\mathcal{S}_k(E, F)$  as the linear space of bounded symmetric  $k$ –linear functions from  $E$  to  $F$ . For each  $a_k \in \mathcal{S}_k(E, F)$ , the notation  $a_k(x^{\otimes k})$  denotes the  $k$ –homogeneous function  $E \rightarrow F$  given by

$$a_k(x^{\otimes k}) = a_k(x, \dots, x).$$

On  $\mathcal{S}_k(E, F)$ , we require to have a norm  $\|\cdot\|_{\mathcal{S}_k(E, F)}$  such that

$$\|a_k(x_1, \dots, x_k)\|_F \leq \|a_k\|_{\mathcal{S}_k(E, F)} \|x_1\|_E \cdots \|x_k\|_E, \quad (44)$$

for all  $a_k \in \mathcal{S}_k(E, F)$ . In particular, this implies that

$$\left\| a_k(x^{\otimes k}) \right\|_F \leq \|a_k\|_{\mathcal{S}_k(E, F)} \|x\|_E^k.$$

*Remark A.1.* Of course, a natural choice of norms is

$$\|a_k\| = \sup\{\|a_k(x_1, \dots, x_k)\|_\delta : \|x_1\|_E = \cdots = \|x_k\|_E = 1\},$$

but there are others. In the specific case  $E = \mathbb{C}^\ell$  and  $F = \mathbb{C}^d$ , we have found that the norm defined in (48) is useful and well adapted to analytic functions of complex variables.

Once the norms on the spaces  $\mathcal{S}_k(E, F)$  are fixed, we can define analytic functions from  $E$  to  $F$ . Throughout the rest of the appendix,  $E(\delta) = \{\|x\|_E \leq \delta\}$  will denote the closed ball in  $E$  centered at the origin with radius  $\delta$ .

**Definition A.1.** Let  $\delta > 0$ . The space of analytic functions from  $E$  to  $F$  with radius of convergence  $\delta$  is the set of functions  $A_\delta(E, F)$  given by

$$A_\delta(E, F) = \left\{ f : E(\delta) \rightarrow F \left| f(x) = \sum_{k=0}^{\infty} a_k(x^{\otimes k}), a_k \in \mathcal{S}_k(E, F), \|f\|_\delta < \infty \right. \right\},$$

where  $\|f\|_\delta := \sum_{k=0}^{\infty} \|a_k\|_{\mathcal{S}_k(E, F)} \delta^k$ . The a norm  $\|\cdot\|_\delta$  makes  $A_\delta(E, F)$  into a Banach space. Given  $\rho > 0$ , we will denote  $A_\delta^\rho(E, F)$  the closed ball centered at the origin with radius  $\rho$ .

It is possible to check that if  $g \in A_\delta^\rho(E, F)$  and  $f \in A_\rho(F, G)$ , then  $f \circ g \in A_\delta(E, G)$ . In fact, it is proved in [Mey75] that the function  $\Gamma : A_\delta^\rho(E, F) \times A_\rho(F, G) \rightarrow A_\delta(E, G)$  given by  $\Gamma(g, f) = f \circ g$  is  $C^\infty$ . In particular, this is true if we consider the Banach spaces of analytic functions that are used in the text. Further details are given below for some specific situations that are relevant to our problem. In subsection A.3, we will show that an operator similar to  $\Gamma$  is analytic.

## A.2 Analytic functions of complex variables

We are interested in some concrete Banach spaces of analytic functions. We will define specific norms for the following spaces of  $k$ -linear functions:

- $\mathcal{S}_k(\mathbb{C}^\ell, \mathbb{C}^d)$ ,
- $\mathcal{S}_k(E, F)$  where  $E = A_\delta(\mathbb{C}^m, \mathbb{C}^\ell)$  and  $F = A_\delta(\mathbb{C}^m, \mathbb{C}^d)$ .

First, consider the Euclidean spaces  $\mathbb{C}^r$  with uniform norm

$$\|\eta\|_\infty = \|(\eta_1, \dots, \eta_r)\|_\infty = \max\{|\eta_1|, \dots, |\eta_r|\}, \quad (45)$$

for each  $\eta = (\eta_1, \dots, \eta_r) \in \mathbb{C}^r$ . We notice that this norm satisfies

$$|z^\alpha| \leq (\|z\|_\infty)^{|\alpha|}, \quad (46)$$

for all multi-indices  $\alpha \in \mathbb{Z}_+^r$ , and  $z \in \mathbb{C}^r$ .

Let  $\mathcal{S}_k(\mathbb{C}^\ell, \mathbb{C}^d)$  be the space of bounded symmetric  $k$ -linear functions from  $\mathbb{C}^\ell$  to  $\mathbb{C}^d$ . Clearly, the spaces  $\mathcal{S}_k(\mathbb{C}^\ell, \mathbb{C}^d)$  consist of polynomial functions in  $\ell$  complex variables and  $d$  coordinates. It is well known that  $\mathcal{S}_k(\mathbb{C}^\ell, \mathbb{C}^d)$  and the space  $\mathcal{H}_k(\mathbb{C}^\ell, \mathbb{C}^d)$  of homogeneous polynomials of degree  $k$  are linearly isomorphic.

In particular, given  $q \in \mathcal{S}_k(\mathbb{C}^\ell, \mathbb{C}^d)$ , the polynomial  $q(z^{\otimes k})$  is an element of  $\mathcal{H}_k(\mathbb{C}^\ell, \mathbb{C}^d)$ . Conversely, for each homogeneous polynomial  $g \in \mathcal{H}_k(\mathbb{C}^\ell, \mathbb{C}^d)$ , there exists an unique  $q \in \mathcal{S}_k(\mathbb{C}^\ell, \mathbb{C}^d)$  such that  $q(z^{\otimes k}) = g(z)$ . Below, in the proof of Lemma A.2, we will describe how such  $q$  can be found.

We need to provide each  $\mathcal{S}_k(\mathbb{C}^\ell, \mathbb{C}^d)$  with a norm that satisfies condition (44). Consider  $q \in \mathcal{S}_k(\mathbb{C}^\ell, \mathbb{C}^d)$ . Clearly, the homogeneous function  $q(z^{\otimes k})$  is of the form

$$q(z^{\otimes k}) = \sum_{|\alpha|=k} z^\alpha \eta_\alpha, \quad (47)$$

where  $\eta_\alpha$  are constant vectors of coefficients in  $\mathbb{C}^d$ . Using these coefficients, we define the norm of as

$$\|q\|_{\mathcal{S}_k(\mathbb{C}^\ell, \mathbb{C}^d)} := \sum_{|\alpha|=k} \|\eta_\alpha\|_\infty. \quad (48)$$

If it is clear from the context, we will write the previous norm as simply  $\|q\|$ .

**Lemma A.2.** *The norm on  $\mathcal{S}_k(\mathbb{C}^\ell, \mathbb{C}^d)$  defined in (48) satisfies condition (44).*

*Proof.* Fixing  $k > 0$ , we consider a multi-index  $\alpha \in \mathbb{Z}_+^\ell$  with  $|\alpha| = k$ . We will give an useful description of the unique multi-linear symmetric polynomial function  $\kappa_\alpha \in \mathcal{S}_k(\mathbb{C}^\ell, \mathbb{C})$  that satisfies  $\kappa_\alpha(z^{\otimes k}) = z^\alpha$ .

Let  $\Pi_k$  be the permutation group of  $k$  elements. Let  $\beta_1, \dots, \beta_k \in \mathbb{Z}_+^\ell$  be multi-indices such that  $\beta_1 + \dots + \beta_k = \alpha$  and  $|\beta_i| = 1$ . Using these vectors, we define the function  $\kappa_\alpha \in \mathcal{S}_k(\mathbb{C}^\ell, \mathbb{C})$  as

$$\kappa_\alpha(z_1, \dots, z_k) := \frac{1}{k!} \sum_{\pi \in \Pi_k} z_{\pi(1)}^{\beta_1} z_{\pi(2)}^{\beta_2} \cdots z_{\pi(k)}^{\beta_k}, \quad (49)$$

where  $z_1, \dots, z_k \in \mathbb{C}^\ell$ . It is clear that  $\kappa_\alpha$  is a well-defined multi-linear function and  $\kappa_\alpha(z^{\otimes k}) = \kappa_\alpha(z, z, \dots, z) = z^\alpha$ . In addition, using (46), we get that  $\kappa_\alpha$  satisfies

$$\begin{aligned} |\kappa_\alpha(z_1, \dots, z_k)| &\leq \frac{1}{k!} \sum_{\pi \in \Pi_k} |z_{\pi(1)}^{\beta_1}| |z_{\pi(2)}^{\beta_2}| \cdots |z_{\pi(k)}^{\beta_k}| \\ &\leq \frac{1}{k!} \sum_{\pi \in \Pi_k} \|z_{\pi(1)}\|_\infty \|z_{\pi(2)}\|_\infty \cdots \|z_{\pi(k)}\|_\infty = \|z_1\|_\infty \cdots \|z_k\|_\infty. \end{aligned}$$

From this, we conclude that if  $q \in \mathcal{S}_k(\mathbb{C}^\ell, \mathbb{C}^d)$  and  $q$  satisfies (47) then

$$q(z_1, \dots, z_k) = \sum_{|\alpha|=k} \kappa_\alpha(z_1, \dots, z_k) \eta_\alpha,$$

and therefore

$$\|q(z_1, \dots, z_k)\|_\infty \leq \sum_{|\alpha|=k} |\kappa_\alpha(z_1, \dots, z_k)| \|\eta_\alpha\|_\infty \leq \|q\|_{\mathcal{S}_k(\mathbb{C}^\ell, \mathbb{C}^d)} \|z_1\|_\infty \cdots \|z_k\|_\infty.$$

The last inequality implies that the norm defined in (48) satisfies (44).  $\square$

From the discussion above, we also conclude that if  $g \in A_\rho(\mathbb{C}^\ell, \mathbb{C}^d)$  then  $g$  is of the form

$$g(z) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} z^\alpha \eta_\alpha. \quad (50)$$

Therefore, with the norms defined in (48) on the space of  $k$ -symmetric functions  $\mathcal{S}_k(\mathbb{C}^\ell, \mathbb{C}^d)$ , we get the following norm, for any  $g \in A_\rho(\mathbb{C}^\ell, \mathbb{C}^d)$ :

$$\|g\|_\rho = \sum_{k=0}^{\infty} \left( \sum_{|\alpha|=k} \|\eta_\alpha\|_\infty \right) \rho^k. \quad (51)$$

This norm has the following property. If  $g \in A_\rho(\mathbb{C}^\ell, \mathbb{C})$  and  $\eta \in \mathbb{C}^d$  then  $g \cdot \eta \in A_\rho(\mathbb{C}^\ell, \mathbb{C}^d)$  and

$$\|g \cdot \eta\|_\rho = \|g\|_\rho \|\eta\|_\infty. \quad (52)$$

We also have the following.

**Lemma A.3.** *If  $g_1, g_2 \in A_\rho(\mathbb{C}^r, \mathbb{C})$ , then  $g_1 g_2 \in A_\rho(\mathbb{C}^r, \mathbb{C})$  and*

$$\|g_1 g_2\|_\rho \leq \|g_1\|_\rho \|g_2\|_\rho.$$

*Proof.* Let  $g_1, g_2 \in A_\rho(\mathbb{C}^r, \mathbb{C})$ . Then, they are of the form

$$g_i(z) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} z^\alpha \eta_\alpha^i,$$

for  $i = 1, 2$ . This implies that

$$\begin{aligned} g_1(z)g_2(z) &= \sum_{\ell_1, \ell_2=0}^{\infty} \sum_{|\alpha|=\ell_1} \sum_{|\beta|=\ell_2} z^{\alpha+\beta} \eta_\alpha^1 \eta_\beta^2 \\ &= \sum_{\ell=0}^{\infty} \sum_{|\alpha|+|\beta|=\ell} z^{\alpha+\beta} \eta_\alpha^1 \eta_\beta^2 = \sum_{\ell=0}^{\infty} \sum_{|\gamma|=\ell} z^\gamma \left( \sum_{\alpha+\beta=\gamma} \eta_\alpha^1 \eta_\beta^2 \right), \end{aligned}$$

for all  $z \in \mathbb{C}^r(\rho)$ . Notice that the coefficients  $\eta_\alpha^i$  are complex numbers and therefore they satisfy  $|\eta_\alpha^1 \eta_\beta^2| = |\eta_\alpha^1| |\eta_\beta^2|$ . Using (51), we get that

$$\begin{aligned} \|g_1 g_2\|_\rho &= \sum_{\ell=0}^{\infty} \sum_{|\gamma|=\ell} \left| \sum_{\alpha+\beta=\gamma} \eta_\alpha^1 \eta_\beta^2 \right| \rho^\ell \leq \sum_{\ell=0}^{\infty} \sum_{|\gamma|=\ell} \sum_{\alpha+\beta=\gamma} |\eta_\alpha^1| |\eta_\beta^2| \rho^\ell \\ &= \left( \sum_{\ell_1=0}^{\infty} \sum_{|\alpha|=\ell_1} |\eta_\alpha^1| \rho^{\ell_1} \right) \left( \sum_{\ell_2=0}^{\infty} \sum_{|\beta|=\ell_2} |\eta_\beta^2| \rho^{\ell_2} \right) = \|g_1\|_\rho \|g_2\|_\rho. \end{aligned}$$

This also shows that the series involved in  $\|g_1 g_2\|_\rho$  converges. □

The previous result also shows that  $A_\rho(\mathbb{C}^r, \mathbb{C})$  is a Banach algebra. Furthermore, if  $g \in A_\rho(\mathbb{C}^r, \mathbb{C}^s)$  and  $\alpha \in \mathbb{Z}_+^s$  is a multi-index, then  $g^\alpha \in A_\rho(\mathbb{C}^r, \mathbb{C})$  and

$$\|g^\alpha\|_\rho \leq \|g\|_\rho^{|\alpha|}. \quad (53)$$

### A.3 The composition operator

Let  $E = A_\delta(\mathbb{C}^m, \mathbb{C}^\ell)$  and  $F = A_\delta(\mathbb{C}^m, \mathbb{C}^d)$  be two Banach spaces of analytic functions. We will define norms on the corresponding spaces of  $k$ -symmetric linear and bounded functions. For each  $b_k \in \mathcal{S}_k(E, F)$  we define the norm

$$\|b_k\|_{\mathcal{S}_k(E, F)} := \sup\{\|b_k(g_1, \dots, g_k)\|_\delta : \|g_1\|_\delta = \dots = \|g_k\|_\delta = 1\}. \quad (54)$$

Notice that this norm satisfies condition (44). With these norms, we can show that each analytic function in  $A_\rho(\mathbb{C}^\ell, \mathbb{C}^d)$  can be extended to an analytic operator in  $A_\rho(E, F)$ , acting on spaces of analytic functions.

**Lemma A.4.** *Let  $f \in A_\rho(\mathbb{C}^\ell, \mathbb{C}^d)$  and  $\delta > 0$ . Let  $E = A_\delta(\mathbb{C}^m, \mathbb{C}^\ell)$  and  $F = A_\delta(\mathbb{C}^m, \mathbb{C}^d)$  be two Banach spaces of analytic functions. If  $\mathcal{C}_f : E(\rho) \rightarrow F$  is the operator given by  $\mathcal{C}_f(g) = f \circ g$ , then  $\mathcal{C}_f$  is well defined and analytic.<sup>3</sup> In addition, with the norms on the spaces  $\mathcal{S}_k(E, F)$  defined in (54), we have that  $\|\mathcal{C}_f\|_{A_\rho(E, F)} \leq \|f\|_\rho$ .*

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<sup>3</sup>This means that  $\mathcal{C}_f \in A_\rho(E, F)$ .

*Proof.* We know that  $f$  is of the form

$$f(z) = \sum_{k=0}^{\infty} a_k \left( z^{\otimes k} \right),$$

with  $a_k \in \mathcal{S}_k(\mathbb{C}^\ell, \mathbb{C}^d)$ , and

$$\|f\|_\rho = \sum_{k=0}^{\infty} \|a_k\|_{\mathcal{S}_k(\mathbb{C}^\ell, \mathbb{C}^d)} \rho^k < \infty.$$

Each multi-linear function  $a_k \in \mathcal{S}_k(\mathbb{C}^\ell, \mathbb{C}^d)$  can be extended to a function  $\tilde{a}_k \in \mathcal{S}_k(E, F)$  by

$$[\tilde{a}_k(g_1, \dots, g_k)](z) := a_k(g_1(z), \dots, g_k(z)),$$

for each  $z \in \mathbb{C}^m$  such that  $\|z\|_\infty \leq \delta$ , and all  $g_1, \dots, g_k \in E$ .

As in the proof of Lemma A.2, we can assume that each  $a_k$  is of the form

$$a_k(z_1, \dots, z_k) = \sum_{|\alpha|=k} \kappa_\alpha(z_1, \dots, z_k) \eta_\alpha.$$

In particular, each polynomial  $\kappa_\alpha$  can be extended to a function  $\tilde{\kappa}_\alpha \in \mathcal{S}_k(E, F)$ . This implies that

$$\tilde{a}_k(g_1, \dots, g_k) = \sum_{|\alpha|=k} \tilde{\kappa}_\alpha(g_1, \dots, g_k) \eta_\alpha.$$

In addition, using (49) and inequality (53), we get that

$$\|\tilde{\kappa}_\alpha(g_1, \dots, g_k)\|_\delta \leq \|g_1\|_\delta \cdots \|g_k\|_\delta.$$

Using (52),

$$\|\tilde{a}_k(g_1, \dots, g_k)\|_\delta \leq \left( \sum_{|\alpha|=k} \|\eta_\alpha\|_\infty \right) \|g_1\|_\delta \cdots \|g_k\|_\delta.$$

Therefore, for all  $k \geq 0$ , the norm (54) on  $\mathcal{S}_k(E, F)$  is such that

$$\|\tilde{a}_k\|_{\mathcal{S}_k(E, F)} \leq \|a_k\|_{\mathcal{S}_k(\mathbb{C}^\ell, \mathbb{C}^d)}.$$

Finally, we notice that the operator  $\mathcal{C}_f$  can be written as

$$\mathcal{C}_f(g) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \tilde{a}_k \left( g^{\otimes k} \right),$$

and conclude that

$$\|\mathcal{C}_f\|_{A_\rho(E, F)} = \sum_{k=0}^{\infty} \|\tilde{a}_k\|_{\mathcal{S}_k(E, F)} \rho^k \leq \sum_{k=0}^{\infty} \|a_k\|_{\mathcal{S}_k(\mathbb{C}^\ell, \mathbb{C}^d)} \rho^k = \|f\|_\rho < \infty.$$

□

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