On the Existence of General Factors in Regular Graphs

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Abstract

Let G be a graph, and $H\colon V(G) \to 2^{\mathbb{N}}$ a set function associated with G. A spanning subgraph F of G is called an H-factor if the degree of any vertex v in F belongs to the set $H(v)$. This paper contains two results on the existence of H-factors in regular graphs. First, we construct an r-regular graph without some given H^*-factor. In particular, this gives a negative answer to a problem recently posed by Akbari and Kano. Second, by using Lovász's characterization theorem on the existence of (g, f)-factors, we find a sharp condition for the existence of general H-factors in $\{r,\, r+1\}$-graphs, in terms of the maximum and minimum of H. The result reduces to Thomassen's theorem for the case that $H(v)$ consists of the same two consecutive integers for all vertices v, and to Tutte's theorem if the graph is regular in addition.

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1 Introduction

Let $G = (V(G),\, E(G))$ be a simple graph, where $V(G)$ and $E(G)$ denote the set of vertices and edges of G respectively. For any vertex v, denote the degree of v by $d_G(v)$. Let $2^{\mathbb{N}}$ denote the collection of sets of nonnegative integers. We call

$$H\colon V(G) \to 2^{\mathbb{N}}$$

a *set function associated with* G if $H(v) \subseteq \{0,\, 1,\, \ldots,\, d_G(v)\}$. A spanning subgraph F of G is called an *H-factor* if $d_F(v) \in H(v)$ for all v. It is often that $H(v)$ coincides with some set H' for all v. In this case, we call H' a *set associated with* G, and call F an H'*-factor* without confusion. Let

$$g,\, f\colon V(G) \to \mathbb{Z}$$

be two functions such that $g(v) \le f(v)$ for all v. An H-factor is called a (g, f)*-factor* if $H(v)$ is the interval $[g(v), f(v)]$ for all v. A (g, f)-factor is called an $[a, b]$*-factor* if $g(v) = a$ and $f(v) = b$ for all v. An $[a, b]$-factor F is called an (a, b)*-parity-factor* if

$$d_F(v) \equiv a \equiv b \pmod 2 \qquad \text{for every vertex } v.$$

In particular, F is called a k*-factor* if $a = b = k$.

A graph is said to be r*-regular* if every vertex has degree r. This paper is concerned with the existence of H-factors in regular graphs. The study on the existence of factors in regular graphs was started, to the best of our knowledge, from Petersen [9].

Theorem 1.1 (Petersen)**.** *Let r and k be even integers such that $1 \le k \le r$. Then any r-regular graph has a k-factor.*

In contrast with even-factors in Theorem 1.1, Gallai [6] obtained the next result for odd-factors. For any graph G, we call the number $|V(G)|$ of vertices the *order* of G, denoted alternatively by $|G|$ as usual.

Theorem 1.2 (Gallai)**.** *Let r, k and m be integers such that r is even, k is odd and*

$$\frac{r}{m} \le k \le r\left(1 - \frac{1}{m}\right).$$

Then any m-edge-connected r-regular graph of even order has a k-factor.

It is clear that having an odd-factor implies that the order of the graph must be even. So the "even order" condition in Theorem 1.2 is not a real restriction. Removing the parity conditions for both r and k, Tutte [12] gave the following theorem.

Theorem 1.3 (Tutte)**.** *Let $1 \le k \le r - 1$. Then any r-regular graph has a $\{k,\, k+1\}$-factor.*

A graph G is said to be an $\{r,\, r+1\}$*-graph* if every vertex of G has degree r or $r + 1$. Thomassen [11] generalized Theorem 1.3 by considering $\{r,\, r+1\}$-graphs.

Theorem 1.4 (Thomassen)**.** *Let $1 \le k \le r - 1$. Then any $\{r,\, r+1\}$-graph has a $\{k,\, k+1\}$-factor.*

For more results along this line, the reader is referred to Akiyama and Kano's book [3]. Recently, Akbari and Kano [2] considered the existence of $\{k,\, r-k\}$-factors in r-regular graphs.

Theorem 1.5 (Akbari-Kano)**.** *Let r and k be integers such that r is odd, k is even and $1 \le k \le r$. Then any r-regular graph has a $\{k,\, r-k\}$-factor.*

By Theorems 1.1, 1.3 and 1.5, any r-regular graph has a $\{k,\, r-k\}$-factor as if k is even. For odd k, Akbari and Kano [2] posed the next problem for the case r is even, and a conjecture for the case that r is odd.

Problem 1.6 (Akbari-Kano)**.** *Let r and k be integers such that r is even, k is odd and $1 \le k \le r/2 - 1$. Is it true that every connected r-regular simple graph of even order has a $\{k,\, r-k\}$-factor?*

Again, the "even order" condition is not a real restriction. On the other hand, any r-regular graph of even order has an $r/2$-factor. This can be seen immediately from Theorem 1.2 if one notices that any even-regular graph is 2-edge connected. Therefore, the condition $1 \le k \le r/2 - 1$ is not a real restriction either.

The first aim of this paper is to give a negative answer to Problem 1.6. In Section 2, we construct an r-regular graph G^* without $\{k,\, r-k\}$-factors for all $1 \le k \le r/2 - 2$, and deal with the case $k = r/2 - 1$ by using the following Lovász's characterization [8] (see also [3, Theorem 6.1]) on parity-factors. For any two subsets S and T of $V(G)$, denote by $E_G(S,T)$ the set of edges with one end in S and the other end in T. Denote

$$e_G(S,T) = |E_G(S,T)|.$$

Theorem 1.7 (Lovász)**.** *Let G be a graph, and $g, f\colon V(G) \to \mathbb{Z}$ be functions such that $g(v) \le f(v)$ and $g(v) \equiv f(v) \pmod 2$ for all vertices v. Then G has a (g,f)-parity-factor if and only if*

$$\eta(S,T) = \sum_{s\in S} f(s) + \sum_{t\in T}\big(d_G(t) - g(t)\big) - e_G(S,T) - q(S,T) \ge 0 \tag{1.1}$$

for all disjoint subsets S and T of $V(G)$, where $q(S,T)$ denotes the number of components C of the graph $G - S - T$ such that

$$\sum_{c\in V(C)} f(c) + e_G(V(C),\, T) \equiv 1 \pmod 2. \tag{1.2}$$

In fact, Lovász [8] presented a structural description for the degree constrained subgraph problem for the case that no two consecutive integers are missed in $H(v)$ for every v. He also showed that the problem without this restriction is NP-complete. In particular, the next theorem, which is due to Lovász [7] (see also [3, Theorem 4.1]), will be used in our deduction.

Theorem 1.8 (Lovász)**.** *Let G be a graph, and $g,\, f\colon V(G) \to \mathbb{Z}$ be functions such that $g(v) \le f(v)$ for all vertices v. Then G has a (g,f)-factor if and only if*

$$\gamma(S,T) = \sum_{s\in S} f(s) + \sum_{t\in T}\big(d_G(t) - g(t)\big) - e_G(S,T) - q^*(S,T) \ge 0$$

for all disjoint subsets S and T of $V(G)$, where $q^*(S,T)$ denotes the number of components C of the graph $G-S-T$ satisfying (1.2), and $g(v)=f(v)$ for all $v \in V(C)$.

By using Alon's combinatorial nullstellensatz [4], Shirazi and Verstraëte [10] established the following brief result for general H-factors, which was originally posed by Addario-Berry et al. [1] as a conjecture.

Theorem 1.9 (Shirazi-Verstraëte)**.** *Let G be a graph with an associated set function H. If*

$$|H(v)| > \left\lceil \frac{d_G(v)}{2} \right\rceil \qquad \text{for all } v \in V(G), \tag{1.3}$$

then G has an H-factor.

Frank et al. [5] found an elementary proof for Theorem 1.9 by using the next result on directed graphs. For any directed graph G, denote by $d_G^-(v)$ the in-degree of v.

Theorem 1.10 (Frank et al.)**.** *Let G be a graph with an associated set function H. If G has an orientation for which*

$$d_G^-(v) \ge |\{0,\,1,\,\ldots,\,d_G(v)\}\backslash H(v)| \qquad \text{for all } v \in V(G), \tag{1.4}$$

then G has an H-factor.

It seems that the existence of H-factors in regular graphs has not been extensively investigated yet. Let G be a graph, and H a set function associated with G. Denote

$$mH = \min_{v\in G} \min H(v),$$

$$MH = \max_{v\in G} \max H(v).$$

Here is the second result of this paper.

Theorem 1.11. *Let G be an $\{r,\,r+1\}$-graph with an associated set function H. If $mH \ge 1$, $MH \le r$ and*

$$|H(v)| \ge \frac{MH - mH + 3}{2} \quad \text{for all } v \in V(G), \tag{1.5}$$

then G has an H-factor.

The proof of Theorem 1.11 will be given in Section 3. As will be seen, the condition (1.5) is sharp. For the case

$$H(v) = \{k,\,k+1\} \qquad \text{for all } v \in V(G),$$

where $1 \le k \le r-1$, Theorem 1.11 reduces to Theorem 1.4. Moreover, as a result restricting to $\{r,\,r+1\}$-graphs, Theorem 1.11 is stronger than Theorem 1.9 because the condition (1.3) implies (1.5) for $\{r,\,r+1\}$-graphs.

2 Answer to Akbari-Kano's problem

This section is concerned with Problem 1.6. Note that $1 \le k \le r/2 - 1$. The following theorem deal with the case $k \le r/2 - 2$. For any integer n, denote by $[n]_{\mathrm{odd}}$ the set of positive odd integers less than or equal to n. For any vertex v in any graph G, denote by $N_G(v)$ the neighborhood of v in G.

Theorem 2.1. *For any even integer r, there exists an r-regular graph G^* of even order such that G^* has no H^*-factors where*

$$H^* = [r]_{\mathrm{odd}} \Big\backslash \Big\{ \frac{r}{2} - 1,\ \frac{r}{2},\ \frac{r}{2} + 1 \Big\}.$$

In particular, G^* has no $\{k,\, r-k\}$-factors for any odd integer k such that $1 \le k \le r/2 - 2$.

Proof. Let J be the graph obtained by removing an edge from the complete graph K_{r+1}. Let $J_1,\, J_2,\, \ldots,\, J_r$ be pairwise disjoint copies of J. In each copy J_i, let a_i and b_i be the ends of the edge that removed from K_{r+1}. Let G^* be the graph consisting of the copies $J_1,\, J_2,\, \ldots,\, J_r$, together with two new vertices u and v, such that

$$N_{G^*}(u) = \big\{ a_1,\ b_1,\ a_2,\ b_2,\ \ldots,\ a_{\frac{r}{2}-1},\ b_{\frac{r}{2}-1},\ a_{r-1},\ a_r \big\}, \tag{2.1}$$

$$N_{G^*}(v) = \big\{ a_{\frac{r}{2}},\ b_{\frac{r}{2}},\ a_{\frac{r}{2}+1},\ b_{\frac{r}{2}+1},\ \ldots,\ a_{r-2},\ b_{r-2},\ b_{r-1},\ b_r \big\}.$$

Then G^* is an r-regular graph of the even order $r(r+1)+2$.

Now we shall show that G^* has no H^*-factors. Suppose to the contrary that F is an H^*-factor of G^*. Let $1 \le i \le r$. Since $d_F(w)$ is odd for all $w \in J_i$, and the order $|J_i|$ is odd, we find

$$\sum_{w \in J_i} d_F(w) \equiv 1 \pmod 2. \tag{2.2}$$

Let F_i be the subgraph of F induced by the vertices in J_i. By the Handshaking theorem, we have

$$\sum_{w \in J_i} d_{F_i}(w) \equiv 0 \pmod 2. \tag{2.3}$$

Taking the difference between (2.2) and (2.3), we obtain

$$e_F(J_i,\, \{u, v\}) = \sum_{w \in J_i} \big(d_F(w) - d_{F_i}(w) \big) \equiv 1 \pmod 2.$$

Since $e_{G^*}(J_i,\, u) = 2$ and $e_{G^*}(J_i,\, v) = 0$ for $1 \le i \le r/2 - 1$, we derive

$$e_F(J_i,\, u) = 1 \qquad \text{for } 1 \le i \le \frac{r}{2} - 1.$$

By the definition (2.1) of $N_{G^*}(u)$, we get

$$d_F(u) \in \Big\{ \frac{r}{2} - 1,\ \frac{r}{2},\ \frac{r}{2} + 1 \Big\},$$

contradicting the definition of H^*. This completes the proof. ∎

The graph G^* constructed above will be used to explain the sharpness of the condition (1.5) in the next section. Now we cope with the case $k = r/2 - 1$.

Theorem 2.2. *Let r be an even integer such that $r/2$ is even. Then any connected r-regular graph of even order has an $\{r/2-1,\, r/2+1\}$-factor.*

Proof. We shall apply Theorem 1.7 by setting $g(v) = r/2 - 1$ and $f(v) = r/2 + 1$ for all vertices v. Let G be a connected r-regular graph of even order. Let S and T be disjoint subsets of $V(G)$. First, we claim that

$$e_G(S \cup T,\, V(G)\backslash S\backslash T) \geq 2\, q(S,T). \tag{2.4}$$

In fact, if $S \cup T \in \{\emptyset, G\}$, then $q(S,T) = 0$, and (2.4) follows immediately. Otherwise, let C be a component of $G - S - T$. Then both $S \cup T$ and C are nonempty. Note that any even-regular graph is 2-edge-connected. So G is 2-edge-connected. In particular, we have

$$e_G(S \cup T,\, C) \geq 2.$$

Summing the above inequality over all components C, we get the desired inequality (2.4). Hence,

$$\begin{aligned}
\eta(S,T) &= \Big(\frac{r}{2}+1\Big)\big(|S|+|T|\big) - e_G(S,T) - q(S,T)\\
&\geq \frac{r}{2}\big(|S|+|T|\big) - e_G(S,T) - \frac{1}{2}e_G(S\cup T,\, V(G)\backslash S\backslash T)\\
&= e_G(S,S) + e_G(T,T) \geq 0.
\end{aligned}$$

By Theorem 1.7, G has an $\{r/2-1,\, r/2+1\}$-factor. ∎

Combining Theorems 2.1 and 2.2, we obtain a negative answer to Problem 1.6.

3 The existence of H-factors in regular graphs

This section is devoted to establish Theorem 1.11. A subset U of $V(G)$ is called *independent* if any two vertices in U are not adjacent in G. We need the following lemma to prove Theorem 1.11.

Lemma 3.1. *Let r and k be positive integers such that $1 \leq k \leq r-1$. Let G be an $\{r,\, r+1\}$-graph and*

$$U = \{v \in V(G) \mid d_G(v) = r+1\}.$$

If U is independent, then G has a $\{k,\, k+1\}$-factor F such that

$$d_F(u) = k+1 \qquad \textit{as if} \quad u \in U.$$

Proof. Let $f(v) = k+1$ for all vertices v, and

$$g(v) = \begin{cases} k+1, & \text{if } v \in U, \\ k, & \text{otherwise.} \end{cases}$$

It suffices to show that G has a (g, f)-factor. Suppose to the contrary that G has no (g, f)-factors. By Theorem 1.8, we have

$$\gamma(S, T) < 0 \qquad \text{for some } S, T \subseteq V(G).$$

Let S and T be disjoint subsets of $V(G)$ such that $\gamma(S, T) < 0$ and the set $S \cup T$ is maximal. We claim that $q^*(S, T) = 0$.

Suppose to the contrary that $q^*(S, T) \geq 1$. Let C be a component of $G - S - T$ counted by $q^*(S, T)$. It follows that

$$e_G(C,\, G - S - T) = 0. \tag{3.1}$$

By the definition of $q^*(S, T)$, we have

$$g(v) = f(v) = k+1 \qquad \text{for all } v \in V(C). \tag{3.2}$$

So $V(C) \subseteq U$. But U is independent, we deduce that C is a single vertex, say, $V(C) = \{a\}$. Let $S' = S \cup \{a\}$ and $T' = T \cup \{a\}$. Then (3.1) implies

$$q^*(S', T) = q^*(S, T) - 1, \tag{3.3}$$

$$q^*(S, T') = q^*(S, T) - 1. \tag{3.4}$$

Note that the condition (1.2) implies $e_G(a, T) \neq k+1$. If $e_G(a, T) \leq k$, then (3.1) and (3.2) yield

$$d_G(a) - e_G(a, S) = e_G(a, T) \leq g(a) - 1.$$

Together with (3.4), we have

$$\gamma(S, T') - \gamma(S, T) = d_G(a) - g(a) - e_G(S, a) - q^*(S, T') + q^*(S, T) \leq 0.$$

So $\gamma(S, T') < 0$, contradicting the maximality of $S \cup T$. Otherwise $e_G(a, T) \geq k+2$. By (3.3), we deduce

$$\gamma(S', T) - \gamma(S, T) = f(a) - e_G(a, T) - q^*(S', T) + q^*(S, T) \leq 0.$$

So $\gamma(S', T) < 0$, contradicting, again, the maximality of $S \cup T$. Thus the claim is true.

Now we can deduce

$$\begin{aligned}
\gamma(S,T) &= \sum_{s\in S} d_G(s)\frac{f(s)}{d_G(s)} + \sum_{t\in T} d_G(t)\left(1-\frac{g(t)}{d_G(t)}\right) - e_G(S,T)\\
&\ge \sum_{\substack{s\in S,\ t\in T\\ st\in E(G)}} \left(\frac{f(s)}{d_G(s)} + \left(1-\frac{g(t)}{d_G(t)}\right)\right) - e_G(S,T)\\
&= \sum_{\substack{s\in S,\ t\in T\\ st\in E(G)}} \left(\frac{k+1}{d_G(s)} - \frac{g(t)}{d_G(t)}\right)\\
&\ge \sum_{\substack{x\in S,\ y\in T\\ xy\in E(G)}} \left(\frac{k+1}{r+1} - \max\left(\frac{k}{r}, \frac{k+1}{r+1}\right)\right) = 0,
\end{aligned}$$

contradicting the hypothesis $\gamma(S,T) < 0$. This completes the proof. ∎

We remark that Lemma 3.1 is a generalization of Theorem 1.3. Now we are in a position to prove Theorem 1.11.

Proof. Write $m = mH$ and $M = MH$ for short. By Theorem 1.4, we can suppose that F is an $\{M,\ M+1\}$-factor of G with the minimum number of edges. It follows that any two vertices of degree $M+1$ in F, if they exist, are not adjacent. By Lemma 3.1, F has an $\{m-1,\ m\}$-factor, say, F', such that

$$d_{F'}(v) = m \quad \text{as if} \quad d_F(v) = M+1. \tag{3.5}$$

Let F'' be the complemented graph of F' in F. In view of (3.5), we have

$$d_{F''}(v) \in \{M-m,\ M-m+1\} \qquad \text{for all } v. \tag{3.6}$$

We observe that F'' has an orientation such that

$$d^-_{F''}(v) \ge \left\lfloor \frac{d_{F''}(v)}{2} \right\rfloor \qquad \text{for all } v. \tag{3.7}$$

This can be seen by orienting an eulerian tour of the graph that obtained from F'' by adding a new vertex and joining it to all vertices of odd degree in F''. Let

$$H'(v) = \{h - d_{F'}(v) \mid h \in H(v)\} \qquad \text{for all } v.$$

Then the condition (1.5) reads

$$|H'(v)| = |H(v)| \ge \frac{M-m+3}{2}. \tag{3.8}$$

By (3.6), (3.7) and (3.8), it is easy to verify that

$$|\{0,\ 1,\ \ldots,\ d_{F''}(v)\}\backslash H'(v)| \le d^-_{F''}(v) \qquad \text{for all } v.$$

By Theorem 1.10, the graph F'' has an H'-factor, say, G'. Hence, the graph induced by the edge set $E(F') \cup E(G')$ is an H-factor of G. This completes the proof. ∎

In fact, the condition (1.5) is sharp. For instance, when r is even, let G^* be the graph constructed in the proof of Theorem 2.1. Consider a set H of the form

$$H = \{m,\, m+2,\, m+4,\, \ldots,\, M\},$$

where both m and M are odd, and $M \le r/2 - 2$. On one hand, G^* has no H-factors by Theorem 2.1. On the other hand, it is straightforward to compute

$$|H| = \frac{M - m + 2}{2}.$$

Comparing it with the condition (1.5), we deduce the latter one is sharp. For other possibilities of the associated set H, for example, $mH + MH$ is odd, we mention that it is also not hard to find r-regular graphs without H-factors such that

$$|H(v)| = \left\lfloor \frac{MH - mH + 2}{2} \right\rfloor \qquad \text{for all } v \in V(G).$$

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