UNCONDITIONAL UNIQUENESS OF THE CUBIC GROSS-PITAEVSKII HIERARCHY WITH LOW REGULARITY

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Abstract. In this paper, we establish the unconditional uniqueness of solutions to the cubic Gross-Pitaevskii hierarchy on $\mathbb{R}^d$ in a low regularity Sobolev type space. More precisely, we reduce the regularity s down to the currently known regularity requirement for unconditional uniqueness of solutions to the cubic nonlinear Schrödinger equation ($s \geqslant \frac{d}{6}$ if $d = 1, 2$ and $s > s_c = \frac{d-2}{2}$ if $d \geqslant 3$). In such a way, we extend the recent work of Chen-Hainzl-Pavlović-Seiringer [3].

1. Introduction

1.1. **Background.** The cubic Gross-Pitaevskii (GP) hierarchy in $\mathbb{R}^d$ is an infinite system of coupled linear equations given by

$$i\partial_t \gamma^{(k)} = (-\Delta_{\underline{x}_k} + \Delta_{\underline{x}'_k})\gamma^{(k)} + \lambda B_{k+1}\gamma^{(k+1)}, \quad \forall k \in \mathbb{N}, \tag{1.1}$$

where $\gamma^{(k)} = \gamma^{(k)}(t, \underline{x}_k, \underline{x}'_k) : I \times \mathbb{R}^{dk} \times \mathbb{R}^{dk} \to \mathbb{C}$, $I \subset \mathbb{R}$ is a time interval and $\lambda = \pm 1$. Here, we denote d-dimensional k-spatial variables $(x_1, x_2, ..., x_k)$ by $\underline{x}_k$, and the corresponding Laplace operator by $\Delta_{\underline{x}_k} = \sum_{j=1}^k \Delta_{x_j}$, and similarly for the primed variables. For each $k \in \mathbb{N}$, $\gamma^{(k)}$ is a bosonic density matrix on $L^2_{sym}(\mathbb{R}^{dk})$ which is hermitian,

$$\gamma^{(k)}(t, \underline{x}_k, \underline{x}'_k) = \overline{\gamma^{(k)}(t, \underline{x}'_k, \underline{x}_k)},$$

and is symmetric in all components of $\underline{x}_k$, and in all components of $\underline{x}'_k$, respectively,

$$\gamma^{(k)}(t, x_{\sigma(1)}, \cdots, x_{\sigma(k)}, x'_{\sigma'(1)}, \cdots, x'_{\sigma'(k)}) = \gamma^{(k)}(t, \underline{x}_k, \underline{x}'_k)$$

for any permutations σ, σ' on k elements. The equations in (1.1) are coupled by the *contraction operator* B_{k+1},

$$B_{k+1} = \sum_{j=1}^k B_{j;k+1} = \sum_{j=1}^k (B^+_{j;k+1} - B^-_{j;k+1}),$$

where each $B^+_{j;k+1}$ contracts the triple x_j, x_{k+1}, x'_{k+1},

$$\begin{aligned}\left(B^+_{j;k+1}\gamma^{(k+1)}\right)(t, \underline{x}_k, \underline{x}'_k) &= \int dx_{k+1}dx'_{k+1}\delta(x_j - x_{k+1})\delta(x_j - x'_{k+1})\gamma^{(k+1)}(t, \underline{x}_{k+1}; \underline{x}'_{k+1}) \\ &= \gamma^{(k+1)}(t, \underline{x}_k, x_j, \underline{x}'_k, x_j)\end{aligned}$$

and each $B^-_{j;k+1}$ contracts the triple x'_j, x_{k+1}, x'_{k+1},

$$\begin{aligned}\left(B^-_{j;k+1}\gamma^{(k+1)}\right)(t, \underline{x}_k, \underline{x}'_k) &= \int dx_{k+1}dx'_{k+1}\delta(x'_j - x_{k+1})\delta(x'_j - x'_{k+1})\gamma^{(k+1)}(t, \underline{x}_{k+1}; \underline{x}'_{k+1}) \\ &= \gamma^{(k+1)}(t, \underline{x}_k, x'_j, \underline{x}'_k, x'_j).\end{aligned}$$

The cubic GP hierarchy is called *focusing* (*defocusing*, respectively) if $\lambda = 1$ ($\lambda = -1$, respectively).

Date: October 23, 2018.

The cubic GP hierarchy is an infinite hierarchy of equations modeling a Bose-Einstein condensate. For the mathematical study of Bose-Einstein condensation (BEC) in systems of interacting bosons in the stationary case, we refer to the fundamental works [30, 33, 32, 31] and the references therein. To study the dynamics of Bose-Einstein condensates, one considers N bosonic particles whose quantum mechanical wave function $\psi_N \in L^2_{sym}(\mathbb{R}^{dN})$ satisfies the N-body Schrödinger equation

$$i\partial_t \psi_N = H_N \psi_N, \tag{1.2}$$

where

$$H_N = \sum_{j=1}^{N} (-\Delta_{x_j}) + \frac{1}{N} \sum_{1 \leqslant i < j \leqslant N} V_N(x_i - x_j)$$

and $V_N(x) = N^{d\beta} V(N^\beta x)$ with $\beta \in (0,1)$ (we remark that the case $\beta = 1$ is much more difficult to control [11, 12, 13, 14]). The pair interaction potential V is assumed to be rotationally symmetric, and to satisfy certain regularity properties. The cubic GP hierarchy is then formally obtained from a limit of the BBGKY hierarchy of marginal density matrices associated to (1.2) as $N \to \infty$. In this limit, V_N converges weakly to $(\int V(x)dx)\delta$, where δ denotes the delta distribution. In this sense, the cubic GP hierarchy describes a Bose gas of infinitely many particles with repulsive or attractive two-body delta interactions.

In the special case of factorized initial data $\gamma_0^{(k)}(\underline{x}_k, \underline{x}'_k) = \prod_{j=1}^{k} \phi_0(x_j)\overline{\phi_0}(x'_j)$ in (1.1), the state of a Bose-Einstein condensate can be simply described by the cubic nonlinear Schrödinger equation (NLS). Indeed, in this case, the cubic GP hierarchy admits a solution

$$\gamma^{(k)}(t, \underline{x}_k, \underline{x}'_k) = \prod_{j=1}^{k} \phi(t, x_j)\overline{\phi}(t, x'_j),$$

preserving the factorization property as time evolves, if ϕ solves the cubic NLS

$$i\partial_t \phi = -\Delta\phi + \lambda|\phi|^2\phi, \ \phi(0) = \phi_0. \tag{1.3}$$

In this way, the cubic NLS is derived as a dynamical mean field limit of the many body quantum dynamics of an interacting Bose gas, provided that given initial data, a solution to the GP hierarchy is unique. We call this formal derivation the BBGKY approach. In his fundamental works [28, 29], Lanford had employed the BBGKY hierarchy to study N-body systems in classical mechanics in the limit $N \to \infty$.

Research efforts aimed at providing a rigorous derivation of nonlinear dispersive equations as mean field limits of N-body Schrödinger dynamics have a long and rich history. The first results on the derivation of nonlinear Hartree equations (NLH) were due to Hepp [22], and Ginibre and Velo [16, 17]. Their techniques are based on embedding the N-body Schrödinger equation into the second quantized Fock-space representation. In [37] Spohn gives the first derivation of NLH by use of the BBGKY hierarchy. More recently, Erdös, Schlein and Yau further developed the BBGKY approach, and gave the first derivation of NLS in their celebrated works [11, 12, 13, 14]. In [35], Rodnianski and Schlein proved estimates on the convergence rate of the evolution in the mean field limit using the Fock space approach. Their results were extended with second-order corrections in the two-body interaction setting by Grillakis, Machedon and Margetis [19, 20], and three-body interaction setting by X. Chen [8].

The derivation of the cubic NLS in $\mathbb{R}^3$, via the BBGKY approach, due to Erdös, Schlein and Yau [11, 12, 13, 14], comprises the following two main parts:

(i) Derivation of the GP hierarchy as the limit of the N-body BBGKY hierarchy as $N \to \infty$.
(ii) Establishing the uniqueness of solutions to the GP hierarchy. In particular, it is proved that for factorized initial data, the solutions to the GP hierarchy are determined by a cubic NLS.

In this program, the proof of the uniqueness theorem (part (ii)) is very involved, one of the difficulties being the factorial growth of the number of terms from iterated Duhamel expansions. The authors give a sophisticated combinatorial argument that settled this problem by a clever re-grouping of Feynman graph expansions.

Later, in [27], Klainerman and Machedon gave a shorter proof of uniqueness of solutions to the 3D cubic GP hierarchy in a different solution space, provided that solutions obey a priori bound,

$$\int_0^T \|R^{(k)} B_{j;k+1}\gamma^{(k+1)}(t,\cdot,\cdot)\|_{L^2(\mathbb{R}^{dk}\times\mathbb{R}^{dk})} dt < C^k, \quad \forall k \in \mathbb{N}, \tag{1.4}$$

where $R_j = (-\Delta_{x_j})^{1/2}$, $R_j' = (-\Delta_{x_j'})^{1/2}$ and $R^{(k)} = \prod_{j=1}^k R_j \prod_{j=1}^k R_j'$. The approach is in part motivated by the authors' previous work on the space-time estimates [26]. In [27], Klainerman and Machedon gave a concise reformulation of the Erdös-Schlein-Yau combinatorial method [11, 12, 13, 14], and presented it as an elegant board game argument. The uniqueness theorem of [27] is *conditional* due to the hypothesis (1.4). Since the work [25] for the cubic GP hierarchy on two dimensional Euclidean space as well as the 2-dimensional torus, the approach of Klainerman and Machedon was used in various recent works for the derivation of the NLS from interacting Bose gases [5, 6, 9, 10, 25, 7, 39]. The method also inspired the analysis of the Cauchy problem for the GP hierarchy, which was initiated in [4] and continued e.g. in [18, 7].

We will call the uniqueness of solutions to the GP hierarchy *unconditional* if it holds without assuming any a priori bound of the form (1.4). Recently, in [3], Chen-Hainzl-Pavlović-Seiringer presented a new, simpler proof of the unconditional uniqueness of solutions to the 3D cubic GP hierarchy, which is equivalent to the uniqueness result of Erdös-Schlein-Yau [12]. The authors employed the quantum de Finetti theorem (Theorem 1.2 and 1.3) combined with the Erdös-Schlein-Yau combinatorial method [11, 12, 13, 14] in board game representation as presented by Klainerman-Machedon in [27].

1.2. **Main result.** In this paper, we investigate the unconditional uniqueness of solutions to the cubic GP hierarchy in a low regularity setting.

To state the main theorem, we first introduce the following definitions. Let $\{\gamma^{(k)}\}_{k\in\mathbb{N}}$ be a sequence of bosonic density matrices on $L^2_{sym}(\mathbb{R}^{dk})$. We say that $\{\gamma^{(k)}\}_{k\in\mathbb{N}}$ is *admissible* if $\gamma^{(k)}$ is a non-negative trace class operator on $L^2_{sym}(\mathbb{R}^{dk})$ and $\gamma^{(k)} = \mathrm{Tr}(\gamma^{(k+1)})$ for all $k \in \mathbb{N}$. We call a sequence $\{\gamma^{(k)}\}_{k\in\mathbb{N}}$ a *limiting hierarchy* if there is a sequence $\{\gamma_N^{(N)}\}_{N\in\mathbb{N}}$ of non-negative density matrices on $L^2_{sym}(\mathbb{R}^{dN})$ with $\mathrm{Tr}(\gamma_N^{(N)}) = 1$ such that $\gamma^{(k)}$ is the weak-* limit of the k-particle marginals of $\gamma_N^{(N)}$ in the trace class on $L^2_{sym}(\mathbb{R}^{dk})$, that is,

$$\gamma_N^{(k)} := \mathrm{Tr}_{k+1,\cdots,N}(\gamma_N^{(N)}) \rightharpoonup^* \gamma^{(k)} \text{ as } N \to \infty.$$

For $s \in \mathbb{R}$, we define the function space $\mathfrak{H}^s$ by the collection of sequences $\{\gamma^{(k)}\}_{k\in\mathbb{N}}$ of density matrices on $L^2_{sym}(\mathbb{R}^{dk})$ such that

$$\mathrm{Tr}(|S^{(k,s)}\gamma^{(k)}|) < M^{2k} \quad \forall k \in \mathbb{N} \text{ for some constant } M > 0,$$

where

$$S^{(k,s)} := \prod_{j=1}^k (1-\Delta_{x_j})^{\frac{s}{2}}(1-\Delta_{x_j'})^{\frac{s}{2}}.$$

We say that $\{\gamma^{(k)}(t)\}_{k\in\mathbb{N}}$ is a *mild solution*, in the space $L^\infty_{t\in[0,T)}\mathfrak{H}^s$, to the cubic GP hierarchy with initial data $\{\gamma^{(k)}(0)\}_{k\in\mathbb{N}}$ if it solves the integral equation

$$\gamma^{(k)}(t) = U^{(k)}(t)\gamma^{(k)}(0) + i\lambda\int_0^t U^{(k)}(t-s)B_{k+1}\gamma^{(k+1)}(s)ds,$$

where $U^{(k)}(t) := e^{it(\Delta_{\underline{x}_k}-\Delta_{\underline{x}'_k})}$, and satisfies the bound

$$\sup_{t\in[0,T)} \operatorname{Tr}(|S^{(k,s)}\gamma^{(k)}(t)|) < M^{2k} \quad \forall k\in\mathbb{N} \text{ for some constant } M>0.$$

Our main theorem states that any mild solution to the cubic GP hierarchy, which is either admissible or a limiting hierarchy, is unconditionally unique in $L^\infty_{t\in[0,T)}\mathfrak{H}^s$ for small s.

Theorem 1.1 (Unconditional uniqueness)**.** *Let*

$$\begin{cases} s \geqslant \frac{d}{6} & \text{if } d=1,2, \\ s > s_c & \text{if } d \geqslant 3, \end{cases} \tag{1.5}$$

where $s_c = \frac{d-2}{2}$*. If* $\{\gamma^{(k)}(t)\}_{k\in\mathbb{N}}$ *is a mild solution in* $L^\infty_{t\in[0,T)}\mathfrak{H}^s$ *to the (de)focusing cubic GP hierarchy with initial data* $\{\gamma^{(k)}(0)\}_{k\in\mathbb{N}}$*, which is either admissible or a limiting hierarchy for each* t*, then it is the only such solution for the given initial data.*

Our theorem reduces the regularity requirement for unconditional uniqueness for the GP hierarchy in [3]. We remark that the regularity assumption in (1.5) is the same as in the currently known unconditional uniqueness results for the cubic NLS

$$i\partial_t\phi + \Delta\phi - \lambda|\phi|^2\phi = 0,\ \phi(0) = \phi_0 \in H^s.$$

For NLS, by unconditional uniqueness, we mean uniqueness of solutions in the Sobolev space H^s itself, while uniqueness in the intersection of the Sobolev space and auxiliary spaces is called conditional. By the contraction mapping argument with auxiliary Strichartz spaces, the conditional uniqueness is proved in H^s for $s \geqslant \max(s_c, 0)$, where $s_c = \frac{d-2}{2}$ (see [1]). However, the unconditional uniqueness is proved in H^s only for s in (1.5), and it is an open problem to push s down to zero in one and two dimensions [23, 15, 36, 38, 21].

Our proof uses the Klainerman-Machedon board game formulation [27] of the combinatorial argument of Erdös-Schlein-Yau [11, 12, 13, 14], and the method of Chen-Hainzl-Pavlović-Seiringer [3] via the quantum de Finetti theorem.

The quantum de Finetti theorem is a quantum analogue of the Hewitt-Savage theorem in probability theory. We state its strong and weak versions in the formulation of [34].

Theorem 1.2 (Strong quantum de Finetti theorem)**.** *If a sequence* $\{\gamma^{(k)}\}_{k\in\mathbb{N}}$ *of bosonic density matrices on* $L^2_{sym}(\mathbb{R}^{dk})$ *is admissible, then there exists a unique Borel probability measure* μ*, supported on the unit sphere* $S \subset L^2(\mathbb{R}^d)$ *and invariant under multiplication of* $\phi\in L^2(\mathbb{R}^d)$ *by complex numbers of modulus one, such that*

$$\gamma^{(k)} = \int d\mu(\phi)(|\phi\rangle\langle\phi|)^{\otimes k} \quad k\in\mathbb{N}. \tag{1.6}$$

Theorem 1.3 (Weak quantum de Finetti theorem)**.** *If a sequence* $\{\gamma^{(k)}\}_{k\in\mathbb{N}}$ *of bosonic density matrices on* $L^2_{sym}(\mathbb{R}^{dk})$ *is a limiting hierarchy, then there exists a unique Borel probability measure* μ*, supported on the unit ball* $\mathcal{B} \subset L^2(\mathbb{R}^d)$ *and invariant under multiplication of* $\phi \in L^2(\mathbb{R}^d)$ *by complex numbers of modulus one, such that* (1.6) *holds.*

The crucial advantage of using the quantum de Finetti theorem is that it provides a factorized representation of solutions to the GP hierarchy in the integral form (see (2.10)). This structure allows us to make use of techniques of NLS theory to analyze solutions to the GP hierarchies (see [3] and [2]).

As described in Section 6.1.1 of [3], the main difficulty in lowering regularity is from the last cubic term $\||\phi|^2\phi\|_{L^2} = \|\phi\|_{L^6}^3$ in the distinguished tree. Indeed, this last term can be controlled by the Sobolev norm $\|\phi\|_{H^s}^3$ only for $s \geqslant 1$ in $\mathbb{R}^3$. We solve this problem by using the dispersive estimate

$$\|e^{it\Delta}f\|_{L^{\frac{6}{1+2\epsilon}}} \lesssim |t|^{-(1-\epsilon)}\|f\|_{L^{\frac{6}{5-2\epsilon}}}$$

in $\mathbb{R}^3$, for instance. Indeed, if one applies the dispersive estimate and the endpoint Strichartz estimate to the factorized representation of the solution in the framework of [3], one gets a better last cubic term $\||\phi|^2\phi\|_{L^{\frac{6}{5-2\epsilon}}} = \|\phi\|^3_{L^{\frac{18}{5-2\epsilon}}}$, and it allows us to reduce s down to $\frac{2}{3}+\epsilon$. The regularity requirement in the classical Kato's work on the unconditional uniqueness for the 3D cubic NLS [23] can be covered in this way. We further push s almost down to the critical regularity by employing negative order Sobolev norms (Lemma A.3), which are well-known tools in the literature on unconditional uniqueness for NLS. Combining the dispersive estimate, the Strichartz estimates and negative Sobolev norms, we formulate the key trilinear estimates (Lemma 2.6) in our proof.

Organization of the paper. We prove Theorem 1.1 in Section 2, by reducing it to the main Lemma 2.5. In Section 3, we present an example calculation to explain the ingredients involved in the proof of Lemma 2.5. In Section 4, we introduce tree graphs for the organization of iterated Duhamel expansions, and give properties of the associated kernels. Finally, we prove the main Lemma 2.5 in Section 5. We prove the crucial trilinear estimates in Lemma 2.6 in Appendix A.

2. Proof of the Main Theorem

In this section, we prove the main theorem. First, in §2.1, we present the setup of the proof. In §2.2 we review Klainerman-Machedon's board game formulation [27] of the combinatorial argument of Erdös-Schlein-Yau [11, 12, 13, 14]. In §2.3, we reduce the proof of the main theorem to the key lemma (Lemma 2.5), via the quantum de Finetti theorem. The rest of the paper is then devoted to the proof of the lemma.

2.1. **Setup of the proof.** The setup of the proof is similar to that of Chen-Hainzl-Pavlović-Seiringer [3], but we use a negative order Sobolev type norm to lower the regularity.

Let $\{\gamma_1^{(k)}(t)\}_{k\in\mathbb{N}}$ and $\{\gamma_2^{(k)}(t)\}_{k\in\mathbb{N}}$ be two mild solutions in $L^\infty_{t\in[0,T)}\mathfrak{H}^s$ to the cubic GP hierarchy with the same initial data, which are either admissible or limiting hierarchies. For uniqueness, it is enough to show that their difference $\{\gamma^{(k)}(t)\}_{k\in\mathbb{N}}$, given by

$$\gamma^{(k)}(t) := \gamma_1^{(k)}(t) - \gamma_2^{(k)}(t), \qquad k\in\mathbb{N},$$

vanishes for all k in a certain norm.

Due to the linearity of the GP hierarchy, it follows that the difference $\{\gamma^{(k)}(t)\}_{k\in\mathbb{N}}$ solves the GP hierarchy with zero initial data. Hence, each $\gamma^{(k)}(t)$ satisfies the integral equation

$$\gamma^{(k)}(t) = i\lambda\int_0^t U^{(k)}(t-t_1)B_{k+1}\gamma^{(k+1)}(t_1)dt_1.$$

Now fix k. Iterating this integral equation $(n-1)$ times, we write

$$\gamma^{(k)}(t) = (i\lambda)^n\int_{t_n\leqslant\cdots\leqslant t_1\leqslant t} U^{(k)}(t-t_1)B_{k+1}\cdots U^{(k+n-1)}(t_{n-1}-t_n)B_{k+n}\gamma^{(k+n)}(t_n)dt_1\cdots dt_n.$$

For notational convenience, we denote $(k+1)$-temporal variables $(t_0, t_1, \cdots, t_n)$ by $\underline{t}_n$ with $t_0 = t$, and the linear propagator $U^{(i)}(t_j - t_{j'})$ by $U^{(i)}_{j,j'}$. Then, we rewrite $\gamma^{(k)}(t)$ in a compact form as

$$\gamma^{(k)}(t) = (i\lambda)^n \int_{t_n \leqslant \cdots \leqslant t_1 \leqslant t} J^k(\underline{t}_n) d\underline{t}_n, \tag{2.1}$$

where

$$J^k(\underline{t}_n) := U^{(k)}_{0,1} B_{k+1} U^{(k)}_{1,2} B_{k+2} \cdots U^{(k+n-1)}_{n-1,n} B_{k+n} \gamma^{(k+n)}(t_n).$$

By density, our uniqueness theorem follows from uniqueness in an even weaker norm.

Proposition 2.1. *For all $t \in [0, T)$ with $T > 0$ small enough, the trace norm of $S^{(k,-d)}$(2.1) vanishes as $n \to \infty$ uniformly in k, that is*

$$\operatorname{Tr}(|S^{(k,-d)} \gamma^{(k)}(t)|) = 0, \quad \forall k, \tag{2.2}$$

where $d > 0$ is the dimension.

2.2. **Erdös-Schlein-Yau Combinatorial method in board-game form.** One obstacle in showing uniqueness is the number of terms in $J^k(\underline{t}_n)$. Indeed, each B_{k+i} is a sum of $(k+i-1)$ terms. Thus, in the expansion of $J^k(\underline{t}_n)$, there are a total of $k(k+1)\cdots(k+n-1) = \mathcal{O}(n!)$ terms for fixed k. We solve this problem by using the powerful combinatorial methods of Erdös-Schlein-Yau [11, 12, 13, 14] in the board-game formulation of Klainerman-Machedon [27].

The key idea of the *board game* arguments is that, by grouping the large number of integral terms into equivalence classes in which we have control, we can avoid estimating the rapidly increasing number of terms one by one. Throughout this section, we present a few lemmas that will help us group these terms and derive bounds on certain equivalence classes.

Let μ be a map from $\{k+1, k+2, \cdots, k+n\}$ to $\{1, 2, \cdots, k+n-1\}$ such that $\mu(2) = 1$ and $\mu(j) < j$ for all j. Denotes by $\mathcal{M}_{k,n}$ the set of all such maps.

We express the operators B_{k+i} and J^k in terms of map μ. We have

$$B_{k+i} = \sum_{j=1}^{k+i-1} B_{j;k+i} = \sum_{\mu \in \mathcal{M}_{k,n}} B_{\mu(k+i);k+i}$$

and

$$J^k(\underline{t}_n) = \sum_{\mu \in \mathcal{M}_{k,n}} J^k(\underline{t}_n; \mu), \tag{2.3}$$

where

$$J^k(\underline{t}_n; \mu) = U^{(k)}(t - t_1) B_{\mu(k+1);k+1} U^{(k+1)}(t_1 - t_2) \cdots U^{(k+n-1)}(t_{n-1} - t_n) B_{\mu(k+n);k+n} \gamma^{(k+n)}(t_n).$$

By the definition of μ, we can represent μ by highlighting exactly one nonzero entry $B_{\mu(k+l),k+l}$ (l-th column, $\mu(k+l)$-th row) in each column of a $(k+n-1) \times n$ matrix. Since $\mu(k+l) < k+l$, we set the remaining entries of the matrix equal to 0.

$$\begin{pmatrix} \mathbf{B_{1;k+1}} & B_{1;k+2} & \cdots & \mathbf{B_{1;k+n}} \\ B_{2;k+1} & B_{2;k+2} & \cdots & B_{2;k+n} \\ \cdots & \cdots & \cdots & \cdots \\ B_{k;k+1} & \mathbf{B_{k;k+2}} & \cdots & B_{k;k+n} \\ 0 & B_{k+1;k+2} & \cdots & B_{k+1;k+n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & B_{k+n-1;k+n} \end{pmatrix} \tag{2.4}$$

Henceforth, we can rewrite (2.1) as

$$\gamma^{(k)}(t) = \int_0^t \cdots \int_0^{t_n} \sum_{\mu \in \mathcal{M}_{k,n}} J^k(\underline{t}_{k+n}; \mu) dt_1 \ldots dt_n. \tag{2.5}$$

Here the time domain $\{t_n \leqslant t_{n-1} \leqslant \cdots \leqslant t\} \subset [0,t]^n$ is the same for all $\mu \in \mathcal{M}_{k,n}$. We now consider the terms $I(\mu,\sigma)$ in the sum $\gamma^{(k)}(t) = \sum_{\mu\in\mathcal{M}_{k,n}} I(\mu,\sigma)$. We have

$$I(\mu,\sigma) = \int_{t_{\sigma(n)}\leqslant t_{\sigma(n-1)}\leqslant\cdots\leqslant t} J^k(\underline{t}_{k+n};\mu)dt_1\dots dt_n, \tag{2.6}$$

where σ is a permutation of $1,2,\dots,n$. We associate the integral $I(\mu,\sigma)$ the following $(k+n)\times n$ matrix. We may also use it to visualize $B_{\mu(k+j);k+j}$ that correspond to a highlighted entry.

$$\begin{pmatrix} t_{\sigma^{-1}(1)} & t_{\sigma^{-1}(2)} & \cdots & t_{\sigma^{-1}(n)} \\ \mathbf{B_{1;k+1}} & B_{1;k+2} & \cdots & \mathbf{B_{1;k+n}} \\ B_{2;k+1} & B_{2;k+2} & \cdots & B_{2;k+n} \\ \cdots & \cdots & \cdots & \cdots \\ B_{k;k+1} & \mathbf{B_{k;k+2}} & \cdots & B_{k;k+n} \\ 0 & B_{k+1;k+2} & \cdots & B_{k+1;k+n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & B_{k+n} \end{pmatrix} \tag{2.7}$$

The columns of matrix (2.7) are labeled 1 through n, and the rows are labeled 0 through $k+n-1$.

Each term (2.6) corresponds to a unique matrix of form (2.7). A key observation is that two matrices of this form can have to the same value for $I(\mu;\sigma)$ given that one matrix can be transformed to another under the so called *acceptable moves.*

In the following paragraph, we will present a few key lemmas to help us with the combinatorial reduction. For the proof of these lemmas, we refer the reader to [11, 12, 13, 14, 27, 5, 39].

2.2.1. *Acceptable Moves.* If $\mu(k+j+1) < \mu(k+j)$, we take the following steps at the same time

- exchange the highlights in columns j and $j+1$
- exchange the highlights in rows $k+j$ and $k+j+1$
- exchange $t_{\sigma^{-1}(j)}$ and $t_{\sigma^{-1}(j+1)}$

The exchange only happens when there is a highlight, if there is no highlight we will skip that step. The following lemma highlights the necessity to introduce *equivalence classes.*

Lemma 2.2. *Let (μ,σ) be transformed into (μ',σ') by an acceptable move. Then, for the corresponding integrals* (2.6)*, we have $I(\mu,\sigma) = I(\mu',\sigma')$*

2.2.2. *Equivalence Class.* Consider the subset $\{\mu_s\} \subset \mathcal{M}_{k,n}$ of *special upper echelon* matrices in which each highlighted element of a higher row is to the left of each highlighted element of a lower row. An example of a special upper echelon matrix (with $k=1, n=4$) is

$$\begin{pmatrix} \mathbf{B_{1;2}} & \mathbf{B_{1;3}} & B_{1;4} & B_{1;5} \\ 0 & B_{2;3} & B_{2;4} & B_{2;5} \\ 0 & 0 & \mathbf{B_{3;4}} & B_{3;5} \\ 0 & 0 & 0 & \mathbf{B_{4;5}} \end{pmatrix}$$

Lemma 2.3. *For each element of $\mathcal{M}_{k,n}$ there is a finite number of acceptable moves which brings the matrix to upper echelon form.*

Lemma 2.4. *Let $C_{k,n}$ be the number of $(k+n-1)\times n$ special upper echelon matrices of the type discussed above. Then $C_{k,n} \leqslant 2^{k+2n-2}$.*

Let μ_s be a special upper echelon matrix. We say μ is in the equivalence class of μ_s: $\mu\sim\mu_s$ if μ can be transformed to μ_s in finitely many acceptable moves.

Theorem 2.1. *There exists a subset D of $[0,t]^n$ such that*

$$\sum_{\mu\sim\mu_s}\int_0^t\cdots\int_0^{t_{n-1}} J^k(\underline{t}_n;\mu)dt_1\dots dt_n = \int_D J^k(\underline{t}_n;\mu)dt_1\dots dt_n. \tag{2.8}$$

Proof. We perform finitely many acceptable moves on the matrix associated to the integral

$$I(\mu, id) = \int_0^t \dots \int_0^{t_{n-1}} J^k(\underline{t}_n; \mu) dt_1 \dots dt_n.$$

Let $I(\mu, id)$ be the integral associated to the upper echelon matrix obtained. By Lemma 2.2

$$I(\mu, id) = I(\mu_s, \sigma).$$

Assume that (μ_1, id) and (μ_2, id) with $\mu_1 \neq \mu_2$ yield the same echelon form μ_s. Then the corresponding permutations σ_1 and σ_2 must be different. Therefore, D can be chosen to be the union of all $\{t \geqslant t_{\sigma(1)} \geqslant t_{\sigma(2)} \geqslant \dots \geqslant t_{\sigma(n)}\}$ for all permutations σ which occur in a given equivalence class of some μ_s. □

With the above theorem, we are able to reduce the sum of $\mathcal{O}(n!)$ terms into a sum of $\mathcal{O}(C^n)$ terms:

$$\gamma^{(k)}(t) = \sum_{\sigma \in \mathcal{M}_{k,n}} \int_{D_{\sigma,t}} d\underline{t}_n J^k(\underline{t}_n; \sigma), \tag{2.9}$$

which we can afford.

2.3. **Proof of the main theorem.** As mentioned above, it suffices to show Proposition 2.1. For the proof, we uses the framework of Chen-Hainzl-Pavlović-Seiringer [3] via the quantum de Finetti theorem.

Applying the strong or the weak quantum de Finetti theorem, we write

$$\gamma^{(k)}(t) = \int d\tilde{\mu}_t(\phi)(|\phi\rangle\langle\phi|)^{\otimes k}, \quad \forall k \in \mathbb{N}, \tag{2.10}$$

where $\tilde{\mu}_t = \mu_t^{(1)} - \mu_t^{(2)}$ with

$$\gamma_i^{(k)}(t) = \int d\mu_t^{(i)}(\phi)(|\phi\rangle\langle\phi|)^{\otimes k}, \quad i = 1, 2.$$

Plugging (2.10) into $J^k(\underline{t}_n; \sigma)$ in the reduced Duhamel expansion (2.9), we obtain a new expression

$$\gamma^{(k)}(t) = \sum_{\sigma \in \mathcal{M}_{k,n}} \int_{D_{\sigma,t}} d\underline{t}_n \int d\tilde{\mu}_{t_n}(\phi) J^k(\underline{t}_n; \sigma), \tag{2.11}$$

where

$$J^k(\underline{t}_n; \sigma) = U_{0,1}^{(k)} B_{\sigma(k+1);k+1} U_{1,2}^{(k+1)} B_{\sigma(k+2);k+2} \cdots U_{n-1,n}^{(k+n-1)} B_{\sigma(k+n);k+n} (|\phi\rangle\langle\phi|)^{\otimes(k+n)}. \tag{2.12}$$

Then, we formulate the following key lemma that implies Proposition 2.1 (and thus the main theorem).

Lemma 2.5 (Key lemma)**.** *There exists a uniform constant $C > 0$ such that for arbitrarily small $\epsilon > 0$, we have*

$$\int_{[0,T)^{n-1}} d\underline{t}_{n-1} \mathrm{Tr}(|S^{(k,-d)} J^k(\underline{t}_n; \sigma)|) \leqslant \begin{cases} (CT^\epsilon)^{n-1} \|\phi\|_{H^{s_\epsilon}}^{2(k+n)} & \text{if } d \geqslant 3 \\ (CT^{1/3})^{n-1} \|\phi\|_{H^{1/3}}^{2(k+n)} & \text{if } d = 2 \\ (CT^{1/2})^{n-1} \|\phi\|_{H^{1/6}}^{2(k+n)} & \text{if } d = 1, \end{cases} \tag{2.13}$$

where $s_\epsilon = \frac{d-2}{2} + \epsilon$.

Proof of Theorem 1.1, assuming Lemma 2.5. We present the proof for the case $d \geqslant 3$ only. Indeed, when $d = 1$ ($d = 2$, resp), it can be proved in an analogous way by replacing the H^{s_c} norm with the $H^{1/6}$ norm (the $H^{1/3}$ norm, resp).

Let $\{\gamma^{(k)}(t)\}_{k\in\mathbb{N}}$ be as above. The goal is to show that $\mathrm{Tr}(|S^{(k,-d)}\gamma^{(k)}(t)|) = 0$ for all $k \in \mathbb{N}$. Applying the triangle inequality and Lemma 2.5, we write

$$\begin{aligned}\mathrm{Tr}(|S^{(k,-d)}\gamma^{(k)}(t)|) &\leqslant \sum_{i=1,2}\sum_{\sigma\in\mathcal{M}_{k,n}}\int_{[0,T)^n} d\underline{t}_n \int d\mu^{(i)}_{t_n}(\phi)\mathrm{Tr}(|S^{(k,-d)}J^k(\underline{t}_n;\sigma)|)\\ &\leqslant (CT^\epsilon)^{n-1}T\sum_{i=1,2}\sum_{\sigma\in\mathcal{M}_{k,n}}\sup_{t_n\in[0,T)}\int d\mu^{(i)}_{t_n}(\phi)\|\phi\|^{2(k+n)}_{H^{s_\epsilon}}.\end{aligned} \tag{2.14}$$

We claim that there exists $M > 0$ such that

$$\|\phi\|_{H^{s_\epsilon}} \leqslant M \quad \text{a.s. } \mu^{(i)}_t, \quad \forall t \in [0,T). \tag{2.15}$$

Indeed, since $\{\gamma^{(k)}(t)\}_{k\in\mathbb{N}} \in L^\infty_{t\in[0,T)}\mathfrak{H}^s$, there exists $M > 0$ such that

$$\int d\mu^{(i)}_t(\phi)\|\phi\|^{2k}_{H^s} = \mathrm{Tr}(|S^{(k,s)}\gamma^{(k)}(t)|) < M^{2k}, \quad \forall k \in \mathbb{N}. \tag{2.16}$$

Hence, it follows from the Chebyshev inequality that for $\lambda > M$,

$$\mu^{(i)}_t\big(\{\phi \in L^2 : \|\phi\|_{H^s} > \lambda\}\big) \leqslant \frac{1}{\lambda^{2k}}\int d\mu^{(i)}_t(\phi)\|\phi\|^{2k}_{H^s} < \Big(\frac{M}{\lambda}\Big)^{2k} \to 0 \quad \text{as } k \to \infty. \tag{2.17}$$

Returning to (2.14), by (2.15) and Lemma 2.4, we prove that

$$\mathrm{Tr}(|S^{(k,-d)}\gamma^{(k)}(t)|) \leqslant (CT^\epsilon)^{n-1}T\cdot 2\cdot 2^{k+2n-2}\cdot M^{2(k+n)} = \frac{M^{2k}2^{k-1}T}{CT^\epsilon}(4CT^\epsilon M^2)^n \to 0 \text{ as } n \to \infty. \tag{2.18}$$

for $T < (4CM^2)^{-1/\epsilon}$. □

The remainder of our paper will be devoted to proving Lemma 2.5. We remark that our proof heavily relies on the following trilinear estimates which combine the dispersive estimate, the Strichartz estimates and negative Sobolev norms. The proof of these trilinear estimates is given in the appendix.

Lemma 2.6 (Trilinear estimates)**.** *We define the trilinear form T by*

$$T(f,g,h) = (e^{i(t-t_1)\Delta}f)(e^{i(t-t_2)\Delta}g)(e^{i(t-t_3)\Delta}h).$$

(i) $d \geqslant 3$. For small $\epsilon > 0$, we have

$$\|T(f,g,h)\|_{L^1_{t\in[0,T)}W_x^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \lesssim T^\epsilon\|f\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}}\|g\|_{H^{s_\epsilon}}\|h\|_{H^{s_\epsilon}}, \tag{2.19}$$

$$\|T(f,g,h)\|_{L^1_{t\in[0,T)}H_x^{s_\epsilon}} \lesssim T^\epsilon\|f\|_{H^{s_\epsilon}}\|g\|_{H^{s_\epsilon}}\|h\|_{H^{s_\epsilon}}, \tag{2.20}$$

where $s_\epsilon = s_c + \epsilon = \frac{d-2}{2} + \epsilon$, $r_\epsilon = \frac{2d}{d+2(1-\epsilon)}$.
(ii) $d = 2$. For small $\epsilon > 0$, we have

$$\|T(f,g,h)\|_{L^1_{t\in[0,T)}W_x^{-(\frac{1}{3}-\frac{\epsilon}{2}),\frac{2}{2-\epsilon}}} \lesssim T^\epsilon\|f\|_{W^{-(\frac{1}{3}-\frac{\epsilon}{2}),\frac{2}{2-\epsilon}}}\|g\|_{H^{1/3}}\|h\|_{H^{1/3}}, \tag{2.21}$$

$$\|T(f,g,h)\|_{L^1_{t\in[0,T)}H_x^{1/3}} \lesssim T^{1/3}\|f\|_{H^{1/3}}\|g\|_{H^{1/3}}\|h\|_{H^{1/3}}. \tag{2.22}$$

(ii) $d = 1$. We have

$$\|T(f,g,h)\|_{L^1_{t\in[0,T)}L^1_x} \lesssim T^{1/2}\|f\|_{L^1}\|g\|_{L^2}\|h\|_{L^2}, \tag{2.23}$$

$$\|T(f,g,h)\|_{L^1_{t\in[0,T)}L^2_x} \lesssim T^{1/2}\|f\|_{L^2}\|g\|_{L^2}\|h\|_{L^2}. \tag{2.24}$$

We will prove Lemma 2.5 in the following sections. To this end, we will proceed as in [3] and use *binary tree graphs* to help organize the terms in $J^k(\underline{t}_n, \sigma)$ (see (2.12)). For the reader's convenience, before proving the lemma, we give an example calculation in Section 3. We remark that the trilinear estimates in Lemma 2.6 are the key estimates, and will be applied recursively in general case (see Section 5).

3. An Example

In this section, we illustrate the ideas of the proof of Lemma 2.5 via an example.

Let $d \geqslant 3$, $k = 2$ and $n = 4$ in Lemma 2.5. We investigate the example

$$\int_{[0,T)^3} d\underline{t}_3 \operatorname{Tr}(|S^{(2,-d)} J^2(\underline{t}_4; \sigma)|) \tag{3.1}$$

with a specific map σ represented by the matrix

$$\begin{pmatrix} \mathbf{B_{1;3}} & B_{1;4} & B_{1;5} & B_{1;6} \\ B_{2;3} & \mathbf{B_{2;4}} & B_{2;5} & B_{2,6} \\ 0 & B_{3;4} & \mathbf{B_{3;5}} & \mathbf{B_{3,6}} \\ 0 & 0 & B_{4;5} & B_{4,6} \end{pmatrix}. \tag{3.2}$$

In other words,

$$J^2 = J^2(\underline{t}_4; \sigma) = U^{(2)}_{0,1} B_{1,3} U^{(3)}_{1,2} B_{2,4} U^{(4)}_{2,3} B_{3,5} U^{(5)}_{3,4} B_{3,6} (|\phi\rangle\langle\phi|)^{\otimes 6}. \tag{3.3}$$

To this end, in §3.1-3.2, we organize the terms in $J^2(\underline{t}_4, \sigma)$. Then, in §3.3, we estimate the example by the trilinear estimates (Lemma 2.6).

3.1. Factorization of J^2.

We will decompose J^2 into two one-particle density matrices by examining the effect of the contraction operators starting with the last one on the RHS of (3.3). We denote each factor in the last term $(|\phi\rangle\langle\phi|)^{\otimes 6}$ by u_i, ordered by increasing index i, so that $(|\phi\rangle\langle\phi|)^{\otimes 6} = \otimes_{i=1}^{6} u_i$.

First of all, in (3.3), the last interaction operator $B_{3,6}$ contracts the factor u_3 and u_6, and leaves all other factors unchanged,

$$B_{3,6}(\otimes_{i=1}^{6} u_i) = u_1 \otimes u_2 \otimes \Theta_4 \otimes u_4 \otimes u_5. \tag{3.4}$$

where

$$\Theta_4 := B_{1,2}(u_3 \otimes u_6).$$

The index α in Θ_α associates Θ_α to the α-th interaction operator from the left in (3.3). Since we only run the expansion to the n-th level, we have $1 \leqslant \alpha \leqslant n$. In this specific case, $n = 4$, the 4th interaction operator is $B_{3,6}$.

Next, $B_{3,5}$ contracts $U^{(1)}_{3,4}\Theta_4$ and $U^{(1)}_{3,4}u_5$,

$$B_{3,5} U^{(5)}_{3,4}((3.4)) = (U^{(2)}_{3,4}(u_1 \otimes u_2)) \otimes \Theta_3 \otimes (U^{(1)}_{3,4} u_4), \tag{3.5}$$

where

$$\Theta_3 := B_{1,2}((U^{(1)}_{3,4}\Theta_4) \otimes (U^{(1)}_{3,4} u_5)).$$

Then, by the semigroup property, $U^{(i)}_{2,3} U^{(i)}_{3,4} = U^{(i)}_{2,4}$. The operator $B_{2,4}$ contracts $U^{(1)}_{2,4}u_2$ with $U^{(1)}_{2,4}u_4$, which correspond to the 2nd and 5th factors in (3.5). The other factors are left invariant.

$$B_{2,4} U^{(4)}_{2,3}((3.5)) = (U^{(1)}_{2,4} u_1) \otimes \Theta_2 \otimes (U^{(1)}_{2,3}\Theta_3), \tag{3.6}$$

where

$$\Theta_2 = B_{1,2}(U^{(2)}_{2,4}(u_2 \otimes u_4)).$$

Finally, $B_{1,3}$ contracts $(U^{(1)}_{1,4}u_1)$ and $(U^{(1)}_{1,3}\Theta_3)$ and leaves other factors unchanged.

$$B_{1,3}U^{(3)}_{1,2}((3.6)) = \Theta_1 \otimes (U^{(1)}_{1,2}\Theta_2), \tag{3.7}$$

where

$$\Theta_1 = B_{1,2}((U^{(1)}_{1,4}u_1) \otimes (U^{(1)}_{1,3}\Theta_3)).$$

Therefore, J^2 can be factorized as

$$J^2 = (U^{(1)}_{0,1}\Theta_1) \otimes (U^{(1)}_{0,2}\Theta_2) := J^1_1 \otimes J^1_2. \tag{3.8}$$

In the above expression we may write the factors J^1_j (for $j \leqslant k = 2$) as one-particle matrices and substitute with $u_i = |\phi\rangle\langle\phi|$, for $i \leqslant k + n = 6$. Thus, it follows that

$$J^1_1 = U^{(1)}_{0,1}B_{1,2}U^{(2)}_{1,3}B_{2,3}U^{(3)}_{3,4}B_{2,4}(|\phi\rangle\langle\phi|)^{\otimes 4} \tag{3.9}$$

where we relabel the index in operators $B_{\sigma_1(r),r}$ such that the interaction operators in (3.9) correspond to $B_{1,3}, B_{3,5}, B_{3,6}$ respectively, and most importantly keep the connectivity structure between them. The relabeling function σ_1 (see the notation in (2.12)) take values: $\sigma_1(2) = 1, \sigma_1(3) = 2, \sigma_1(4) = 3$. Moreover, for $j = 1$, we perform the relabeling in the same spirit find that

$$J^1_2 = U^{(1)}_{0,2}B_{1,2}U^{(2)}_{2,4}(|\phi\rangle\langle\phi|)^{\otimes 2} \tag{3.10}$$

where $\sigma_2(2) = 1$.

We note that for any $l < l'$, the interaction operators $B_{\sigma(l),l}$ and $B_{\sigma(l'),l'}$ in J^2 (associated to the matrix (3.2)) belong to the same factor J^1_j if either $\sigma(l) = \sigma(l')$ or $\sigma(l') = l$. In such cases, we consider them as being *connected*. This connectivity structure is exactly the key point of the Duhamel terms that we want to illustrate using binary tree graphs. Each σ_j can be viewed as the restriction of σ to J^1_j. We call factors that have a free propagator applied to each ϕ (like J^1_2) *regular* and factors that involve the contractions of $(|\phi\rangle\langle\phi|)^{\otimes 2}$ without free propagator in between (like J^1_1) *distinguished*.

3.2. **Recursive determination of contraction structure.** Next, repeating the argument in §3.1, we express the kernel of each factor explicitly.

Consider the distinguished factor J^1_1. For $\alpha = 1, 2, 3$, we denote by Θ_α the kernel obtained after contracting a two particle density matrix to a one particle matrix via the interaction operator. We will determine Θ_α recursively in the normal form

$$\Theta_\alpha(x, x') = \sum_{\beta_\alpha} c^\alpha_{\beta_\alpha}\psi^\alpha_{\beta_\alpha}(x)\overline{\chi^\alpha_{\beta_\alpha}}(x'),\ c^\alpha_{\beta_\alpha} = \pm 1 \tag{3.11}$$

from the last interaction operator. First, contracting variables by $B_{2,4}$, we get

$$B_{2,4}(|\phi\rangle\langle\phi|)^{\otimes 4} = (|\phi\rangle\langle\phi|) \otimes \Theta_3 \otimes (|\phi\rangle\langle\phi|) \tag{3.12}$$

with

$$\Theta_3(x, x') = |\phi|^2\phi(x)\overline{\phi}(x') - \phi(x)\overline{|\phi|^2\phi}(x') = \sum_{\beta_3=1}^{2} c^3_{\beta_3}\psi^3_{\beta_3}(x)\overline{\chi^3_{\beta_3}}(x').$$

Next, contracting variables by $B_{2,3}$,

$$B_{2,3}U^{(3)}_{3,4}(3.12) = (|U_{3,4}\phi\rangle\langle U_{3,4}\phi|) \otimes \Theta_2, \tag{3.13}$$

where $U_{i,j} := e^{i(t_i - t_j)\Delta}$ and

$$\Theta_2(x, x') = \sum_{\beta_3=1}^{2} c^3_{\beta_3}\Big(U_{3,4}\psi^3_{\beta_3}|U_{3,4}\phi|^2\Big)(x)\overline{U_{3,4}\chi^3_{\beta_3}}(x') - c^3_{\beta_3}U_{3,4}\psi^3_{\beta_3}(x)\Big(\overline{U_{3,4}\psi^3_{\beta_3}}|U_{3,4}\chi|^2\Big)(x')$$

$$=: \sum_{\beta_2=1}^{4} c^2_{\beta_2}\psi^2_{\beta_2}(x)\overline{\chi^2_{\beta_2}}(x').$$

Finally, by the first interaction operator $B_{1,2}$,

$$B_{1,2}U^{(2)}_{1,3}(3.13) = B_{1,2}\Big(|U_{1,4}\phi\rangle\langle U_{1,4}\phi| \otimes \sum_{\beta_2=1}^{4} c^2_{\beta_2}|U_{1,3}\psi^2_{\beta_2}\rangle\langle U_{1,3}\chi^2_{\beta_2}|\Big) = \Theta_1,$$

where $\Theta_1(x,x')$ is given by

$$\begin{aligned}&\sum_{\beta_2=1}^{4} c^2_{\beta_2}\Big(U_{1,4}\phi U_{1,3}\psi^2_{\beta_2}\overline{U_{1,3}\chi^2_{\beta_2}}\Big)(x)\overline{U_{1,4}\phi}(x') - c^2_{\beta_2}U_{1,4}\phi(x)\Big(\overline{U_{1,4}\phi U_{1,3}\psi^2_{\beta_2}}U_{1,3}\chi^2_{\beta_2}\Big)(x')\\ &=: \sum_{\beta_1=1}^{8} c^1_{\beta_1}\psi^1_{\beta_1}(x)\overline{\chi^1_{\beta_1}}(x').\end{aligned}$$

Therefore, J^1_1 can be represented by

$$J^1_1(x,x') = U^{(1)}_{0,1}\Theta_1(x,x') = \sum_{\beta_1=1}^{8} c^1_{\beta_1}U_{0,1}\psi^1_{\beta_1}(x)\overline{U_{0,1}\chi^1_{\beta_1}}(x'),$$

Similarly, we write the regular factor J^1_2 as

$$J^1_2(\sigma_2;t_2,t_4) = U^{(1)}_{0,1}\tilde{\Theta}_1(x,x') = \sum_{\tilde{\beta}_1=1}^{2} \tilde{c}^1_{\tilde{\beta}_1}U_{0,1}\tilde{\psi}^1_{\tilde{\beta}_1}(x)\overline{U_{0,1}\tilde{\chi}^1_{\tilde{\beta}_1}}(x'),$$

where

$$\begin{aligned}\tilde{\Theta}_1(x,x') &= (|U_{2,4}\phi|^2U_{2,4}\phi)(x)\overline{U_{2,4}\phi}(x') - U_{2,4}\phi(x)(|U_{2,4}\phi|^2\overline{U_{2,4}\phi})(x')\\ &=: \sum_{\tilde{\beta}_1=1}^{2} \tilde{c}^1_{\tilde{\beta}_1}\tilde{\psi}^1_{\tilde{\beta}_1}(x)\overline{\tilde{\chi}^1_{\tilde{\beta}_1}}(x').\end{aligned}$$

3.3. **Recursive Estimates.** Now, we estimate the example (3.1) using the structural properties obtained from the previous two subsections. The key tool is the trilinear estimates (Lemma 2.6).

Observe that in the example (3.1), the distinguished factor J^1_1 is independent of t_2, and the regular factor J^1_2 depends only on t_2 and t_4 (see (3.9) and (3.10)). Thus, (3.1) can be factored as

$$(3.1) = \Big(\int_{[0,T)^2} dt_1dt_3\mathrm{Tr}(|S^{(1,-d)}J^1_1|)\Big)\Big(\int_0^T dt_2\mathrm{Tr}(|S^{(1,-d)}J^1_2|)\Big). \tag{3.14}$$

We estimate these two factors separately.

3.3.1. *Distinguished factor.* By §3.1 and §3.2, we have

$$\int_{[0,T)^2} dt_1dt_3\mathrm{Tr}(|S^{(1,-d)}J^1_1|) \leqslant \sum_{\beta_1=1}^{8}\int_{[0,T)^2} dt_1dt_3\|\psi^1_{\beta_1}\|_{H^{-d}}\|\chi^1_{\beta_1}\|_{H^{-d}}, \tag{3.15}$$

where for each β_α, only one out of two terms $\psi^\alpha_{\beta_\alpha}$ and $\chi^\alpha_{\beta_\alpha}$ is cubic. Among the eight integrals on the right hand side of (3.15), we estimate the following two cases.

Case 1. Consider the integral whose $\psi^{\alpha}_{\beta_\alpha}$'s are all cubic, precisely

$$\begin{aligned} \psi^1_{\beta_1} &= U_{1,4}\phi U_{1,3}\psi^2_{\beta_2}\overline{U_{1,3}\chi^2_{\beta_2}}, & \chi^1_{\beta_1} &= U_{1,4}\phi, \\ \psi^2_{\beta_2} &= U_{3,4}\psi^3_{\beta_3}|U_{3,4}\phi|^2, & \chi^2_{\beta_2} &= U_{3,4}\chi^3_{\beta_3}, \\ \psi^3_{\beta_3} &= |\phi|^2\phi, & \chi^3_{\beta_3} &= \phi. \end{aligned} \tag{3.16}$$

We apply the trilinear estimates (2.19) recursively keeping the $W^{-s_c+\frac{\epsilon}{2},r_\epsilon}$ norm on $\psi^{\alpha}_{\beta_\alpha}$. Then, we obtain that

$$\begin{aligned} \int_{[0,T)^2} dt_1dt_3\|\psi^1_{\beta_1}\|_{H^{-d}}\|\chi^1_{\beta_1}\|_{H^{-d}} &\lesssim \int_{[0,T)^2} dt_1dt_3\|\psi^1_{\beta_1}\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}}\|\chi^1_{\beta_1}\|_{H^{s_\epsilon}} \quad \text{(by Sobolev ineq)} \\ &= \int_{[0,T)^2} dt_1dt_3\|U_{1,4}\phi U_{1,3}\psi^2_{\beta_2}\overline{U_{1,3}\chi^2_{\beta_2}}\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}}\|\phi\|_{H^{s_\epsilon}} \\ &\leqslant C_0T^\epsilon\int_0^T dt_3\|\psi^2_{\beta_2}\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}}\|\chi^2_{\beta_2}\|_{H^{s_\epsilon}}\|\phi\|^2_{H^{s_\epsilon}} \quad \text{(by (2.19))} \\ &= C_0T^\epsilon\int_0^T dt_3\|U_{3,4}\psi^3_{\beta_3}|U_{3,4}\phi|^2\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}}\|\phi\|^3_{H^{s_\epsilon}} \\ &\leqslant (C_0T^\epsilon)^2\|\psi^3_{\beta_3}\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}}\|\phi\|^5_{H^{s_\epsilon}} \quad \text{(by (2.19))} \\ &= (C_0T^\epsilon)^2\||\phi|^2\phi\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}}\|\phi\|^5_{H^{s_\epsilon}} \\ &\lesssim (C_0T^\epsilon)^2\|\phi\|^8_{H^{s_\epsilon}} \quad \text{(by Sobolev ineq).} \end{aligned}$$

Case 2. Consider the integral whose $\psi^{\alpha}_{\beta_\alpha}$'s are all linear except the last one, that is,

$$\begin{aligned} \psi^1_{\beta_1} &= U_{1,3}\psi^2_{\beta_2}, & \chi^1_{\beta_1} &= U_{1,3}\chi^2_{\beta_2}|U_{1,4}\phi|^2, \\ \psi^2_{\beta_2} &= U_{3,4}\psi^3_{\beta_3}, & \chi^2_{\beta_2} &= U_{3,4}\chi^3_{\beta_3}|U_{3,4}\phi|^2, \\ \psi^3_{\beta_3} &= |\phi|^2\phi, & \chi^3_{\beta_3} &= \phi. \end{aligned} \tag{3.17}$$

In this case, we first combine linear propagators acting on $\psi^3_{\beta_3}$ so that

$$\psi^1_{\beta_1} = U_{1,3}U_{3,4}(|\phi|^2\phi) = U_{1,4}(|\phi|^2\phi).$$

Then, applying the trilinear estimate (2.20) twice, we obtain

$$\begin{aligned} \int_{[0,T)^2} dt_1dt_3\|\psi^1_{\beta_1}\|_{H^{-d}}\|\chi^1_{\beta_1}\|_{H^{-d}} &\lesssim \int_{[0,T)^2} dt_1dt_3\|U_{1,4}(|\phi|^2\phi)\|_{H^{-d}}\|U_{1,3}\chi^2_{\beta_2}|U_{1,4}\phi|^2\|_{H^{s_\epsilon}} \\ &= \int_{[0,T)^2} dt_1dt_3\||\phi|^2\phi\|_{H^{-d}}\|U_{1,3}\chi^2_{\beta_2}|U_{1,4}\phi|^2\|_{H^{s_\epsilon}} \\ &\leqslant C_0T^\epsilon\int_0^T dt_3\||\phi|^2\phi\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}}\|\chi^2_{\beta_2}\|_{H^{s_\epsilon}}\|\phi\|^2_{H^{s_\epsilon}} \quad \text{(by (2.20))} \\ &\leqslant (C_0T^\epsilon)^2\||\phi|^2\phi\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}}\|\phi\|^5_{H^{s_\epsilon}} \quad \text{(by (2.20))} \\ &\lesssim (C_0T^\epsilon)^2\|\phi\|^8_{H^{s_\epsilon}} \quad \text{(by Sobolev ineq),} \end{aligned}$$

which is the same bound as in Example 1.

Similarly, one can show that the other six integrals satisfy the same bound. Then, it follows that

$$\int_{[0,T)^2} dt_1dt_3\mathrm{Tr}(|S^{(1,-d)}J^1_1|) \lesssim 8(C_0T^\epsilon)^2\|\phi\|^8_{H^{s_\epsilon}}.$$

3.3.2. *Regular factor.* For the regular factor, we have

$$\int_0^T dt_2 \mathrm{Tr}(|S^{(1,-d)} J_2^1|) \leqslant \sum_{\tilde{\beta}_1=1}^{2} \int_0^T dt_2 \|\tilde{\psi}^1_{\tilde{\beta}_1}\|_{H^{-d}} \|\tilde{\chi}^1_{\tilde{\beta}_1}\|_{H^{-d}}, \tag{3.18}$$

where for each $\tilde{\beta}_1$, only one out of two terms $\tilde{\psi}^1_{\tilde{\beta}_1}$ and $\tilde{\chi}^1_{\tilde{\beta}_1}$ is cubic. For instance, when $\tilde{\psi}^1_{\tilde{\beta}_1} = |U_{2,4}\phi|^2 U_{2,4}\phi$ and $\tilde{\chi}^1_{\tilde{\beta}_1} = U_{2,4}\phi$, it follows from the trilinear estimate (2.20) that

$$\int_0^T dt_2 \|\tilde{\psi}^1_{\tilde{\beta}_1}\|_{H^{-d}} \|\tilde{\chi}^1_{\tilde{\beta}_1}\|_{H^{-d}} \leqslant \int_0^T dt_2 \||U_{2,4}\phi|^2 U_{2,4}\phi\|_{H^{s_\epsilon}} \|U_{2,4}\phi\|_{H^{s_\epsilon}} \leqslant C_0 T^\epsilon \|\phi\|^4_{H^{s_\epsilon}}.$$

Similarly, one can also show that the other integral satisfies the same bound. Therefore, we get

$$\int_0^T dt_2 \mathrm{Tr}(|S^{(1,-d)} J_2^1|) \leqslant 2C_0 T^\epsilon \|\phi\|^4_{H^{s_\epsilon}}$$

3.3.3. *Conclusion.* Going back to (3.14)), we conclude that

$$(3.1) \lesssim 2^4 \cdot (C_0 T^\epsilon)^3 \|\phi\|^{12}_{H^{s_\epsilon}}.$$

4. Binary tree graphs for the general case

In order to prove Lemma 2.5 in the general case, we proceed as in [3], and use binary tree graphs. These graphs will help us keep track of the contraction operations applied iteratively in the Duhamel expansion (2.11).

4.1. **The binary tree graphs.**

We begin by recalling that, by (2.12), J^k is given by

$$J^k(\underline{t}_n;\sigma) = U_{0,1}^{(k)} B_{\sigma(k+1);k+1} U_{1,2}^{(k+1)} B_{\sigma(k+2);k+2} \cdots U_{n-1,n}^{(k+n-1)} B_{\sigma(k+n);k+n} (|\phi\rangle\langle\phi|)^{\otimes(k+n)},$$

where

$$(|\phi\rangle\langle\phi|)^{\otimes(k+n)}(\underline{x}_{k+n};\underline{x}'_{k+n}) = \prod_{i=1}^{k+n} (|\phi\rangle\langle\phi|)(x_i;x'_i)$$

is a product of one-particle kernels. Since the free evolution operators U and the contraction operators B preserve the product structure, it follows that we can also decompose

$$J^k(t,t_1,\ldots,t_r;\sigma;\underline{x}_k;\underline{x}'_k) = \prod_{j=1}^{k} J^1_j(t,t_{\ell_{j,1}},\ldots,t_{l_{j,m_j}};\sigma_j;x_j;x'_j) \tag{4.1}$$

into a product of one-particle kernels J^1_j. We associate to this decomposition k disjoint binary tree graphs $\tau_1,\tau_2,\ldots,\tau_k$. These graphs appear as *skeleton graphs* in [11, 12, 13, 14]. As in [3], we assign *root, internal,* and *leaf* vertices to for each tree τ_j.

- A *root* vertex labeled as W_j, $j=1,2,\cdots,k$, to represent $J^1_j(x_j,x'_j)$.
- An *internal* vertex labeled by v_l, $l=1,2,\cdots,n$, corresponding to $B_{\sigma(k+l),k+l}$ and attached to the time variable t_l.
- A *leaf* vertex u_i, $i=1,2,\cdots,k+n$, representing each factor $(|\phi\rangle\langle\phi|)(x_i,x'_i)$.

Next, we connect the vertices with *edges*, as described below.

- If v_l is the smallest value of l such that $\sigma(k+l)=j$, then we connect v_l to the root vertex W_j and write $W_j \sim v_l$ (or equivalently $W_j \sim B_{\sigma(k+l),k+l}$). If there is no internal vertex connected to a root vertex W_j, then we connect W_j to the leaf u_j, and write $W_j \sim u_j$.
- For any $1 < l \leqslant n$, if $\exists l' > l$ such that $\sigma(k+l) = \sigma(k+l')$ or $\sigma(k+l') = k+l$, then we connect v_l and $v_{l'}$ and write $v_l \sim v_{l'}$ (or equivalently $B_{\sigma(k+l),k+l} \sim B_{\sigma(k+l'),k+l'}$). In this case, we call v_l the *parent vertex* of $v_{l'}$, and $v_{l'}$ the *child vertex* of v_l. We denote the two child vertices of v_l by $v_{k_-(l)}$ and $v_{k_+(l)}$, with $k_-(l) < k_+(l)$.

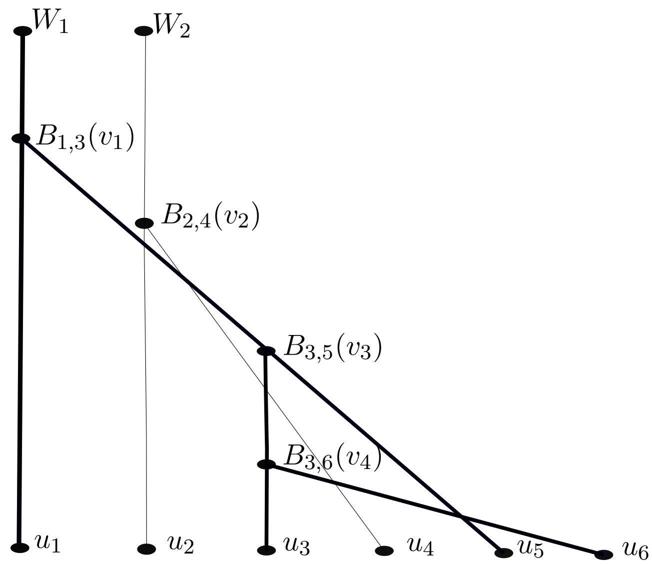

FIGURE 1. An example binary tree graphs of J^k. It is a disjoint union of two trees τ_1 and τ_2 with root vertices W_1 and W_2, respectively. Each tree corresponds to a one-particle kernel in the example in section 3, where $k = 2$ and $n = 4$.

- When there is no internal vertex with $r' > r$ and $k + l = \sigma(k + l')$, we connect v_l to the leaf vertex u_{k+l} and write $v_l \sim u_{k+l}$ (or equivalently $B_{\sigma(k+l),k+l} \sim u_{k+l}$). If there is no internal vertex with $l' > l$ and $\sigma(k + l) = \sigma(k + l')$, then we connect v_l to the leaf vertex $u_{\sigma(k+l)}$ and write $v_l \sim u_{\sigma(k+l)}$ (or equivalently $B_{\sigma(k+l),k+l} \sim u_{\sigma(k+l)}$).

We remark that it follows from the construction above that each root vertex has only one child vertex, and each internal vertex has exactly two child vertices (which can be internal and leaf). We call the tree τ_j *distinguished* if $v_n \in \tau_j$, and *regular* if $v_n \notin \tau_j$. The two leaves connected to v_n are called *distinguished leaf vertices*, and all other leaves are called *regular leaf vertices*. Clearly, there are $k - 1$ regular trees and one distinguished tree in each binary tree graph.

A sample binary tree graph is given in Figure 1, for J^k as in (3.3). Each tree τ_j has root vertex W_j, for $j = 1, 2$. The two leaf vertices u_3 and u_6 and the internal vertex v_4 (or $B_{3,6}$) are distinguished. τ_1 is the distinguished tree, and is drawn with thick edges.

4.2. **The distinguished one particle kernel J_j^1.** Let τ_j denote the distinguished tree graph. It has m_j internal vertices $(v_{\ell_j,\alpha})_{\alpha=1}^{m_j}$ and $m_j + 1$ leaf vertices $(u_{j,i})_{i=1}^{m_j+1}$. We enumerate the internal vertices with $\alpha \in \{1, \ldots, m_j\}$ and the leaf vertices with $\alpha \in \{m_j + 1, \ldots, 2m_j + 2\}$. To simplify notation, we refer to the vertex $v_{j,\alpha}$ by its label α. We observe that J_j^1 has the form

$$\begin{aligned} &J_j^1(t, t_{\ell_{j,1}}, \ldots, t_{\ell_j,m_j}; \sigma_j) \qquad (4.2)\\ &= U^{(1)}(t - t_{\ell_{j,1}}) \cdots U^{(1)}(t_{\ell_{j,1}-1} - t_{\ell_{j,1}}) B_{\sigma_j(2),2} \cdots \\ &\quad \cdots B_{\sigma_j(\alpha),\alpha} U^{(\alpha)}(t_{\ell_{j,\alpha-1}} - t_{\ell_{j,\alpha-1}+1}) \cdots U^{(\alpha)}(t_{\ell_{j,\alpha}-1} - t_{\ell_{j,\alpha}}) B_{\sigma_j(\alpha+1),\alpha+1} \cdots \\ &\quad \cdots U^{(m_j)}(t_{\ell_j,m_j-1} - t_{l_j,m_j}) B_{\sigma_j(m_j+1),m_j+1} (|\phi\rangle\langle\phi|)^{\otimes(m_j+1)}. \end{aligned}$$

By the group property

$$U^{(\alpha)}(t)U^{(\alpha)}(s) = U^{(\alpha)}(t + s),$$

and the fact that $\sigma_j(2) = 1$, (4.2) reduces to

$$J_j^1(t, t_{\ell_{j,1}}, \ldots, t_{\ell_j,m_j}; \sigma_j) \qquad (4.3)$$

$$
\begin{aligned}
&= U^{(1)}(t-t_{\ell_{j,1}})B_{1,2}\cdots \\
&\qquad \cdots B_{\sigma_j(\alpha),\alpha}U^{(\alpha)}(t_{\ell_{j,\alpha-1}}-t_{\ell_{j,\alpha}})B_{\sigma_j(\alpha+1),\alpha+1}\cdots \\
&\qquad \cdots U^{(m_j)}(t_{\ell_j,m_j-1}-t_{l_j,m_j})B_{\sigma_j(m_j+1),m_j+1}(|\phi\rangle\langle\phi|)^{\otimes(m_j+1)},
\end{aligned}
$$

where $\ell_{j,m_j} = r$.

4.3. **Definition of the kernels Θ_α at the vertices of the distinguished tree graph.** In this section, we proceed as in [3], and recursively assign a kernel Θ_α to each vertex α of the distinguished tree graph. The kernels at the vertices of the regular tree graph are defined similarly. We begin by assigning the kernel

$$\Theta_\alpha(x;x') := \phi(x)\overline{\phi(x')}$$

to the leave vertex with label $\alpha \in \{m_j+1,\dots,2m+j+2\}$ (corresponding to $u_{j,\alpha-m_j}$).

Next, we determine Θ_{m_j} at the distinguished vertex $\alpha = m_j$ from the term on the last line of (4.3), given by

$$
\begin{aligned}
B_{\sigma_j(m_j+1),m_j+1}(|\phi\rangle\langle\phi|)^{\otimes(m_j+1)} = (|\phi\rangle\langle\phi|)^{\otimes(\sigma_j(m_j+1)-1)} \otimes \Theta_{m_j} \\
\otimes (|\phi\rangle\langle\phi|)^{\otimes(m_j+1-\sigma_j(m_j+1)-1)}
\end{aligned}
$$

where

$$\Theta_{m_j}(x;x') := \tilde{\psi}(x)\overline{\phi(x')} - \phi(x)\overline{\tilde{\psi}(x')} \tag{4.4}$$

with $\tilde{\psi} := |\phi|^2\phi$. It is obtained from contracting two copies of $|\phi\rangle\langle\phi|$ at the two leaf vertices $\kappa_-(m_j),\kappa_+(m_j)$ which have m_j as their parent vertex.

Now we are ready to begin the induction. Let $\alpha \in \{1,\dots,m_j-1\}$. Suppose that the kernels $\Theta_{\alpha'}$ have been determined for all $\alpha' > \alpha$. We let $\kappa_-(\alpha),\kappa_+(\alpha)$ label the two child vertices (of internal or leaf type) of α,

$$\sigma_j(\alpha) = \sigma_j(\kappa_-(\alpha)) \quad , \quad \alpha = \sigma_j(\kappa_+(\alpha)).$$

Since $\Theta_{\kappa_-(\alpha)}$ and $\Theta_{\kappa_+(\alpha)}$ have already been determined, we can now define

$$
\begin{aligned}
&\Theta_\alpha(x;x') \\
&= B_{1,2}((U^{(1)}(t_\alpha - t_{\kappa_-(\alpha)}) \otimes (U^{(1)}(t_\alpha - t_{\kappa_+(\alpha)}\Theta_{\kappa_+(\alpha)}))(x;x') \\
&= (U^{(1)}(t_\alpha - t_{\kappa_-(\alpha)})\Theta_{k_-(\alpha)})(x;x')[(U^{(1)}(t_\alpha - t_{\kappa_+(\alpha)})\Theta_{\kappa_+(\alpha)})(x;x) \\
&\qquad - (U^{(1)}(t_\alpha - t_{\kappa_+(\alpha)})\Theta_{\kappa_+(\alpha)})(x';x')].
\end{aligned}
$$

The induction ends when we obtain the kernel Θ_1 at $\alpha = 1$.

4.4. **Key properties of the kernels Θ_α.** As in [3], we observe that the kernels Θ_α satisfy the following properties.

- Θ_α can be written as a sum of differences of factorized kernels
$$\Theta_\alpha(x;x') = \sum_{\beta_\alpha} c^\alpha_{\beta_\alpha}\chi^\alpha_{\beta_\alpha}(x)\overline{\psi^\alpha_{\beta_\alpha}(x')} \tag{4.5}$$
with at most $2^{m_j-\alpha}$ nonzero coefficients $c^\alpha_{\beta_\alpha} \in \{1,-1\}$.
- The product $\chi^\alpha_{\beta_\alpha}(x)\overline{\psi^\alpha_{\beta_\alpha}(x')}$ in (4.5) above is either of the form
$$
\begin{aligned}
\chi^\alpha_{\beta_\alpha}(x)\overline{\psi^\alpha_{\beta_\alpha}(x')} = (U_{\alpha;\kappa_-(\alpha)}\chi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}})(x)\overline{(U_{\alpha;\kappa_-(\alpha)}\psi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}})}(x') \\
(U_{\alpha;\kappa_+(\alpha)}\chi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}})(x)\overline{(U_{\alpha;\kappa_+(\alpha)}\psi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}})}(x)
\end{aligned} \tag{4.6}
$$

or

$$\chi^{\alpha}_{\beta_\alpha}(x)\overline{\psi^{\alpha}_{\beta_\alpha}(x')} = (U_{\alpha;\kappa_-(\alpha)}\chi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}})(x)\overline{(U_{\alpha;\kappa_-(\alpha)}\psi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}})(x')} \\ (U_{\alpha;\kappa_+(\alpha)}\chi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}})(x')\overline{(U_{\alpha;\kappa_+(\alpha)}\psi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}})(x')} \tag{4.7}$$

for some values of $\beta_{\kappa_-(\alpha)}, \beta_{\kappa_+(\alpha)}$ that depend on β_α. Observe that above, the function $\chi^{\alpha}_{\beta_\alpha}$ is either of the cubic form

$$\chi^{\alpha}_{\beta_\alpha}(x) = (U_{\alpha;\kappa_-(\alpha)}\chi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}})(x) \\ (U_{\alpha;\kappa_+(\alpha)}\chi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}})(x)\overline{(U_{\alpha;\kappa_+}(\alpha)\psi^{\kappa_+(\alpha)}_{\beta_{\kappa_+}})(x)} \tag{4.8}$$

or the linear form

$$\chi^{\alpha}_{\beta_\alpha}(x) = (U_{\alpha;\kappa_-(\alpha)}\chi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}})(x). \tag{4.9}$$

Accordingly, $\psi^{\alpha}_{\beta_\alpha}$ respectively is either of linear or cubic form, and the product $\chi^{\alpha}_{\beta_\alpha}(x)\overline{\psi^{\alpha}_{\beta_\alpha}(x')}$ always has quartic form (4.6) or (4.7).

- We call the functions $\chi^{\alpha}_{\beta_\alpha}, \psi^{\alpha}_{\beta_\alpha}$ in the sum (4.5) *distinguished* if they are a function of $|\phi|^2\phi$. In the product on the right hand side of (4.6), respectively (4.7), at most one of the four factors is distinguished. Indeed, this is true for all regular leaf vertices, and for the distinguished vertex (4.4). By induction along decreasing values of α, it is also true for the internal vertices.

5. Proof of Lemma 2.5

In this section, we prove Lemma 2.5. We begin by considering the contribution of each factor J^1_j on the right hand side of (4.1) separately. One of these factors is distinguished, and will be dealt with in Proposition 5.1 below. Proposition 5.4 will be for the regular factors.

We note that the analog of Proposition 5.1 in [3] has a shorter proof. This is because, where the authors of [3] work in L^2, we work in $W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}$ to achieve lower regularity. In $W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}$, the linear propagators $e^{it\Delta}$ are no longer isometries, and so we have to carefully rearrange them so that they do not interfere with our proof. This occurs in case 2 of our proof of Lemma 5.3.

We begin with Proposition 5.1, which addresses the contribution of the distinguished factor J^1_j. We prove Proposition 5.1 by induction. Lemma 5.2 will serve as our first induction step, and Lemma 5.3 will serve as the remainder of our proof by induction.

Proposition 5.1. *Let $d \geqslant 3$. Then, for the distinguished tree τ_j, we have the bound*

$$\int_{[0,T)^{m_j-1}} dt_1 \dots dt_{m_j-1} \mathrm{Tr}\left(\left|S^{(1,-d)}J^1_j(t,t_1,\cdots,t_{m_j};\sigma_j)\right|\right) \\ \leqslant 2^{m_j-1}C^{m_j-1}T^{\epsilon(m_j-1)}\|\phi\|^{2m_j-1}_{H^{s_\epsilon}}\||\phi|^2\phi\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}}. \tag{5.1}$$

Similarly, when $d = 2$, we have the bound

$$\int_{[0,T)^{m_j-1}} dt_1 \dots dt_{m_j-1} \mathrm{Tr}\left(\left|S^{(1,-d)}J^1_j(t,t_1,\cdots,t_{m_j};\sigma_j)\right|\right) \\ \leqslant 2^{m_j-1}C^{m_j-1}T^{\frac{1}{3}(m_j-1)}\|\phi\|^{2m_j-1}_{H^{1/3}}\||\phi|^2\phi\|_{W^{-(\frac{1}{3}-\frac{\epsilon}{2}),r_\epsilon}}, \tag{5.2}$$

and, when $d = 1$, we have the bound

$$\int_{[0,T)^{m_j-1}} dt_1 \dots dt_{m_j-1} \mathrm{Tr}\left(\left|S^{(1,-d)}J^1_j(t,t_1,\cdots,t_{m_j};\sigma_j)\right|\right)$$

$$\leqslant 2^{m_j-1}C^{m_j-1}T^{\frac{1}{2}(m_j-1)}\|\phi\|_{L^2}^{2m_j-1}\||\phi|^2\phi\|_{L^1}. \tag{5.3}$$

Proof. For $d \geqslant 3$, Proposition 5.1 follows immediately from Lemma 5.3 below. Indeed, in the statement of Lemma 5.3, there are at most 2^{m_j-1} terms in the sum over β_1.

Observe that in the proofs of Lemmas 5.2 and 5.3, we use the bounds for $d \geqslant 3$ presented in Lemma 2.6. The proof of Proposition 5.1 for $d = 1, 2$ is analogous (we use the corresponding bounds for $d = 1, 2$ presented in Lemma 2.6). □

We now prove Lemma 5.2, which will serve as the first induction step in our proof of Lemma 5.1.

Lemma 5.2. *Let $d \geqslant 3$. Then, the distinguished factor*

$$J_j^1(\underline{t}_n;\sigma_j;x,x') = U^{(1)}(t-t_1)\sum_{\beta_1} c_{\beta_1}^1\psi_{\beta_1}^1(x)\chi_{\beta_1}^1(x')$$

satisfies the following. For each value of β_1, either there exits a non-negative integer $\ell < m_j - 1$ such that

$$\int_{[0,T)^{m_j-1}} dt_1\dots dt_{m_j-1}\mathrm{Tr}\Big(\Big|S^{(1,-d)}U^{(1)}(t-t_1)c_{\beta_1}^1|\psi_{\beta_1}^1\rangle\langle\chi_{\beta_1}^1|\Big|\Big)$$
$$\leqslant (CT^\epsilon)^\ell\sum_{\beta_1}\int_{[0,T)^{m_j-\ell-1}} dt_{\ell+1}\cdots dt_{m_j-1}$$
$$\|(U_{\ell+2}f_{\ell+2}^1)(U_{\ell+2}f_{\ell+2}^2)(U_{\ell+2}f_{\ell+2}^3)\|_{W^{-s_c+\frac{\epsilon}{2},r_\epsilon}}\|U_{\ell+2}f_{\ell+2}^2\|_{H^{s_\epsilon}}\cdots\|U_{\ell+2}f_{\ell+2}^{2\ell+4}\|_{H^{s_\epsilon}}, \tag{5.4}$$

where the functions f are defined in terms of the functions $\psi_{\beta_\alpha}^\alpha$ and $\chi_{\beta_\alpha}^\alpha$ as described in the proof below, or

$$\int_{[0,T)^{m_j-1}} dt_1\dots dt_{m_j-1}\mathrm{Tr}\Big(\Big|S^{(1,-d)}U^{(1)}(t-t_1)c_{\beta_1}^1|\psi_{\beta_1}^1\rangle\langle\chi_{\beta_1}^1|\Big|\Big)$$
$$\leqslant C^{m_j-1}T^{\epsilon(m_j-1)}\|\phi\|_{H^{s_\epsilon}}^{2m_j-1}\||\phi|^2\phi\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}}. \tag{5.5}$$

Moreover, $f_{\ell+2}^1$ is the only distinguished fuction on the right hand side of (5.4).

Proof. We recall that $U_{i,j} := e^{i(t_i-t_j)\Delta}$, and let $U_j := U_{j,j+1}$. We have

$$\int_{[0,T)^{m_j-1}} dt_1\dots dt_{m_j-1}\mathrm{Tr}\Big(\Big|S^{(1,-d)}U^{(1)}(t-t_1)c_{\beta_1}^1|\psi_{\beta_1}^1\rangle\langle\chi_{\beta_1}^1|\Big|\Big)$$
$$\leqslant \int_{[0,T)^{m_j-1}} dt_1\cdots dt_{m_j-1}\|\psi_{\beta_1}^1\|_{H^{-d}}\|\chi_{\beta_1}^1\|_{H^{-d}}. \tag{5.6}$$

Now, we recall from subsection 4.4 that one of functions $\psi_{\beta_1}^1, \chi_{\beta_1}^1$ is distinguished. Moreover the distinguished function is either of the cubic form (4.8) or of the linear form (4.9). We will now label the distinguished function f_1^1 and the regular function f_1^2.

Case 1: f_1^1 is cubic. If f_1^1 is cubic, then, by (4.6) and (4.7), f_1^1 and f_1^2 are of the form

$$f_1^1 = (U_2f_2^1)(U_2f_2^2)(U_2f_2^3),$$
$$f_1^2 = U_2f_2^4.$$

As in Section 3, we apply the $W^{-s_c+\frac{\epsilon}{2},r_\epsilon}$ norm to the distinguished function f_1^1 and the H^{s_ϵ} norm to the regular function f_1^2 and find that

$$(5.6) = \int_{[0,T)^{m_j-1}} dt_1\cdots dt_{m_j-1}\|f_1^1\|_{H^{-d}}\|f_1^2\|_{H^{-d}}$$
$$= \int_{[0,T)^{m_j-1}} dt_1\cdots dt_{m_j-1}\|(U_2f_2^1)(U_2f_2^2)(U_2f_2^3)\|_{H^{-d}}\|U_2f_2^4\|_{H^{-d}}$$

$$\leqslant C \int_{[0,T)^{m_j-1}} dt_1 \cdots dt_{m_j-1} \|(U_2 f_2^1)(U_2 f_2^2)(U_2 f_2^3)\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \|U_2 f_2^4\|_{H^{s_\epsilon}},$$

which is of the form (5.4).

Case 2: f_1^2 is cubic. In this case, we have that f_1^1 and f_1^2 are of the form

$$\begin{aligned} f_1^1 &= U_2 f_2^1, \\ f_1^2 &= (U_2 f_2^2)(U_2 f_2^3)(U_2 f_2^4). \end{aligned}$$

Since f_1^1 is distinguished, there exists $\ell \geqslant 1$ such that

$$f_2^1 = U_3 f_3^1, \ f_3^1 = U_4 f_4^1, ..., f_\ell^1 = U_{\ell+1} f_{\ell+1}^1,$$

and

$$f_{\ell+1}^1 = (U_{\ell+2} f_{\ell+2}^1)(U_{\ell+2} f_{\ell+2}^2)(U_{\ell+2} f_{\ell+2}^3) \text{ or } f_{\ell+1}^1 = |\phi|^2\phi, \tag{5.7}$$

where $f_{\ell+2}^1$ (or $f_{\ell+2}^2$ or $f_{\ell+2}^3$) is a distinguished function. Thus, combining all propagators acting on $f_{\ell+1}^1$, we write

$$f_1^1 = U_{1,\ell+2} f_{\ell+1}^1.$$

Again, we apply the $W^{-s_c+\frac{\epsilon}{2},r_\epsilon}$ norm to the distinguished function f_1^1 and the H^{s_ϵ} norm to the regular function f_1^2 and find that

$$\begin{aligned} (5.6) &= \int_{[0,T)^{m_j-1}} dt_1 \cdots dt_{m_j-1} \|f_1^1\|_{H^{-d}} \|f_1^2\|_{H^{-d}} \\ &= \int_{[0,T)^{m_j-1}} dt_1 \cdots dt_{m_j-1} \|f_{\ell+1}^1\|_{H^{-d}} \|(U_2 f_2^2)(U_2 f_2^3)(U_2 f_2^4)\|_{H^{-d}} \\ &\lesssim \int_{[0,T)^{m_j-1}} dt_1 \cdots dt_{m_j-1} \|f_{\ell+1}^1\|_{W^{-s_c+\frac{\epsilon}{2},r_\epsilon}} \|(U_2 f_2^2)(U_2 f_2^3)(U_2 f_2^4)\|_{H^{s_\epsilon}}. \end{aligned} \tag{5.8}$$

Since $f_{\ell+1}$ doesn't depend on $t_1, \ldots, t_\ell$, we find that after ℓ applications of (2.20),

$$(5.8) \leqslant (CT^\epsilon)^\ell \int_{[0,T)^{m_j-\ell-1}} dt_{\ell+1} \cdots dt_{m_j-1} \|f_{\ell+1}^1\|_{W^{-s_c+\frac{\epsilon}{2},r_\epsilon}} \|f_{\ell+1}^2\|_{H^{s_\epsilon}} \cdots \|f_{\ell+1}^{2\ell+4}\|_{H^{s_\epsilon}}. \tag{5.9}$$

If $f_{\ell+1}^1 = |\phi|^2\phi$, then it follows from the binary tree graph structure presented in section 4 that $\ell = m_j - 1$ and $f_{\ell+1}^{\ell''} = \phi$ for $\ell'' \geqslant 2$, and so we have proven (5.5). Otherwise, if $f_{\ell+1}^1 = (U_{\ell+2} f_{\ell+2}^1)(U_{\ell+2} f_{\ell+2}^2)(U_{\ell+2} f_{\ell+2}^3)$, then we have that

$$\begin{aligned} (5.9) &\leqslant (CT^\epsilon)^\ell \int_{[0,T)^{m_j-\ell-1}} dt_{\ell+1} \cdots dt_{m_j-1} \\ &\qquad \|(U_{\ell+2} f_{\ell+2}^1)(U_{\ell+2} f_{\ell+2}^2)(U_{\ell+2} f_{\ell+2}^3)\|_{W^{-s_c+\frac{\epsilon}{2},r_\epsilon}} \|f_{\ell+1}^2\|_{H^{s_\epsilon}} \cdots \|f_{\ell+1}^{2\ell+4}\|_{H^{s_\epsilon}} \\ &= (CT^\epsilon)^\ell \int_{[0,T)^{m_j-\ell-1}} dt_{\ell+1} \cdots dt_{m_j-1} \\ &\qquad \|(U_{\ell+2} f_{\ell+2}^1)(U_{\ell+2} f_{\ell+2}^2)(U_{\ell+2} f_{\ell+2}^3)\|_{W^{-s_c+\frac{\epsilon}{2},r_\epsilon}} \|U_{\ell+2} f_{\ell+2}^2\|_{H^{s_\epsilon}} \cdots \|U_{\ell+2} f_{\ell+2}^{2\ell+4}\|_{H^{s_\epsilon}}, \end{aligned}$$

which is of the form (5.4). □

In Lemma 5.3, we complete the induction process. Observe that in the proof below, we proceed as in the proof of Lemma 5.2. In each induction step, we apply the $W^{s_c+\frac{\epsilon}{2},r_\epsilon}$ norm to the distinguished function, and the H^{s_ϵ} norm to the regular functions.

Lemma 5.3. *Let $d \geqslant 3$. Then, the distinguished factor*

$$J_j^1(\underline{t}_n; \sigma_j; x, x') = U^{(1)}(t - t_1) \sum_{\beta_1} c_{\beta_1}^1 \psi_{\beta_1}^1(x) \chi_{\beta_1}^1(x')$$

satisfies the following. For each value of β_1,

$$\begin{aligned} &\int_{[0,T)^{m_j-1}} dt_1 \ldots dt_{m_j-1} \mathrm{Tr}\left(\left| S^{(1,-d)} U^{(1)}(t-t_1) c_{\beta_1}^1 |\psi_{\beta_1}^1 \rangle\langle \chi_{\beta_1}^1| \right| \right) \\ &\leqslant C^{m_j-1} T^{\epsilon(m_j-1)} \|\phi\|_{H^{s_\epsilon}}^{2m_j-1} \||\phi|^2\phi\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}}. \end{aligned} \tag{5.10}$$

Proof. By Lemma 5.2, we have that for each β_1, either (5.10) holds, or there is a non-negative integer $\ell < m_j - 1$ such that

$$\begin{aligned} &\int_{[0,T)^{m_j-1}} dt_1 \ldots dt_{m_j-1} \mathrm{Tr}\left(\left| S^{(1,-d)} U^{(1)}(t-t_1) c_{\beta_1}^1 |\psi_{\beta_1}^1 \rangle\langle \chi_{\beta_1}^1| \right| \right) \\ &\leqslant (CT^\epsilon)^\ell 2^{m_j-1} \int_{[0,T)^{m_j-\ell-1}} dt_{\ell+1} \cdots dt_{m_j-1} \\ &\qquad \|(U_{\ell+2} f_{\ell+2}^1)(U_{\ell+2} f_{\ell+2}^2)(U_{\ell+2} f_{\ell+2}^3)\|_{W^{-s_c+\frac{\epsilon}{2},r_\epsilon}} \|U_{\ell+2} f_{\ell+2}^2\|_{H^{s_\epsilon}} \cdots \|U_{\ell+2} f_{\ell+2}^{2\ell+4}\|_{H^{s_\epsilon}}, \end{aligned} \tag{5.11}$$

where $f_{\ell+2}^1$ is the only distinguished function on the right hand side of (5.11). We recall from Section 4 that $f_{\ell+2}^1$ is either of the cubic form (4.8) or the linear for (4.9).

Now, we will proceed by induction, and show that in each induction step, we can bound 5.11 by an expression of the same form, but with a larger value of ℓ. In the last induction step, we find that (5.16) holds, which completes the proof of (5.10). Indeed, this follows from the binary tree graph structure presented in section 4.

Case 1: $f_{\ell+2}^1$ is cubic. If $f_{\ell+2}^1$ is cubic, then

$$\begin{aligned} &f_{\ell+2}^1 = (U_{\ell+3} f_{\ell+3}^1)(U_{\ell+3} f_{\ell+3}^2)(U_{\ell+3} f_{\ell+3}^3), \\ &f_{\ell+2}^2 = U_{\ell+3} f_{\ell+3}^4, \; f_{\ell+2}^3 = U_{\ell+3} f_{\ell+3}^5, ..., \; f_{\ell+2}^{2\ell+4} = U_{\ell+3} f_{\ell+3}^{2\ell+6}. \end{aligned}$$

Since $f_{\ell+2}^1$ is distinguished, one of $f_{\ell+3}^1, f_{\ell+3}^2, f_{\ell+3}^3$ is distinguished, say $f_{\ell+3}^1$. Then, applying (2.19), we get the integral of the form (5.11) back:

$$\begin{aligned} (5.11) &\lesssim (CT^\epsilon)^{\ell+1} 2^{m_j-1} \int_{[0,T)^{m_j-\ell-2}} dt_{\ell+2} \cdots dt_{m_j-1} \|f_{\ell+2}^1\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \|f_{\ell+2}^2\|_{H^{s_\epsilon}} \cdots \|f_{\ell+2}^{2\ell+4}\|_{H^{s_\epsilon}}. \\ &= (CT^\epsilon)^{\ell+1} 2^{m_j-1} \int_{[0,T)^{m_j-\ell-2}} dt_{\ell+2} \cdots dt_{m_j-1} \|(U_{\ell+3} f_{\ell+3}^1)(U_{\ell+3} f_{\ell+3}^2)(U_{\ell+3} f_{\ell+3}^3)\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \\ &\qquad \times \|f_{\ell+3}^4\|_{H^{s_\epsilon}} \cdots \|f_{\ell+3}^{2\ell+6}\|_{H^{s_\epsilon}}. \end{aligned}$$

Case 2: $f_{\ell+2}^2$ is cubic. If $f_{\ell+2}^1$ is cubic, then

$$\begin{aligned} &f_{\ell+2}^1 = U_{\ell+3} f_{\ell+3}^1, \\ &f_{\ell+2}^2 = (U_{\ell+3} f_{\ell+3}^2)(U_{\ell+3} f_{\ell+3}^3)(U_{\ell+3} f_{\ell+3}^4), \\ &f_{\ell+2}^3 = U_{\ell+3} f_{\ell+3}^5, ..., \; f_{\ell+2}^{2\ell+4} = U_{\ell+3} f_{\ell+3}^{2\ell+6}. \end{aligned}$$

Since $f_{\ell+2}^1$ is distinguished, there exists $\ell' \geqslant 1$ such that

$$f_{\ell+3}^1 = U_{\ell+4} f_{\ell+4}^1, f_{\ell+4}^1 = U_{\ell+5} f_{\ell+5}^1, \ldots, f_{\ell+1+\ell'}^1 = U_{\ell+2+\ell'} f_{\ell+2+\ell'}^1,$$

and

$$f_{\ell+2+\ell'}^1 = (U_{\ell+3+\ell'} f_{\ell+3+\ell'}^1)(U_{\ell+3+\ell'} f_{\ell+3+\ell'}^2)(U_{\ell+3+\ell'} f_{\ell+3+\ell'}^3) \text{ or } f_{\ell+2+\ell'}^1 = |\phi|^2\phi, \tag{5.12}$$

where $f^1_{\ell+3+\ell'}$ is a distinguished function. Thus, combining all linear propagators acting on $f^1_{\ell+2+\ell'}$, we write

$$f^1_{\ell+2} = U_{\ell+2,\ell+3+\ell'} f^1_{\ell+2+\ell'}.$$

Then, applying (2.19) and (2.20), we obtain

$$\begin{aligned}(5.11) &\leqslant (CT^\epsilon)^{\ell+1} 2^{m_j-1} \int_{[0,T)^{m_j-\ell-2}} dt_{\ell+2}\cdots dt_{m_j-1} \|f^1_{\ell+2+\ell'}\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \|f^2_{\ell+2}\|_{H^{s_\epsilon}} \cdots \|f^{2\ell+4}_{\ell+2}\|_{H^{s_\epsilon}} \\ &\leqslant (CT^\epsilon)^{\ell+2} 2^{m_j-1} \int_{[0,T)^{m_j-\ell-3}} dt_{\ell+3}\cdots dt_{m_j-1} \|f^1_{\ell+2+\ell'}\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \|f^2_{\ell+3}\|_{H^{s_\epsilon}} \cdots \|f^{2\ell+6}_{\ell+3}\|_{H^{s_\epsilon}},\end{aligned} \tag{5.13}$$

where, in the second inequality, we applied (2.20) to the cubic regular function $f^2_{\ell+2}$. After $\ell'-1$ applications of (2.20), we find that

$$(5.13) \leqslant (CT^\epsilon)^{\ell+1+\ell'} 2^{m_j-1} \int_{[0,T)^{m_j-\ell-2-\ell'}} dt_{\ell+2+\ell'}\cdots dt_{m_j-1} \tag{5.14}$$

$$\|f^1_{\ell+2+\ell'}\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \|f^2_{\ell+2+\ell'}\|_{H^{s_\epsilon}} \cdots \|f^{2\ell+2\ell'+4}_{\ell+2+\ell'}\|_{H^{s_\epsilon}}. \tag{5.15}$$

If

$$f^1_{\ell+2+\ell'} = |\phi|^2\phi, \tag{5.16}$$

then it follows from the binary tree graph structure presented in section 4 that $\ell+2+\ell' = m_j$ and $f^{\ell''}_{\ell+2+\ell'} = \phi$ for $\ell'' \geqslant 2$, and so we have completed the proof of (5.10). Otherwise, by (5.12),

$$\begin{aligned}(5.15) = (CT^\epsilon)^{\ell+1+\ell'} 2^{m_j-1} &\int_{[0,T)^{m_j-\ell-2-\ell'}} dt_{\ell+2+\ell'}\cdots dt_{m_j-1} \\ &\|(U_{\ell+3+\ell'} f^1_{\ell+3+\ell'})(U_{\ell+3+\ell'} f^2_{\ell+3+\ell'})(U_{\ell+3+\ell'} f^3_{\ell+3+\ell'})\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \\ &\times \|U_{\ell+3+\ell'} f^2_{\ell+3+\ell'}\|_{H^{s_\epsilon}} \cdots \|U_{\ell+3+\ell'} f^{2\ell+2\ell'+4}_{\ell+3+\ell'}\|_{H^{s_\epsilon}},\end{aligned}$$

which is of the form (5.11).

Case 3: $f^4_{\ell+2}$ is cubic. This case can be treated like Case 2. We choose $\ell' \geqslant 1$ satisfying (5.12), and combine linear propagators acting on $f^1_{\ell+2+\ell'}$. Then, we repeat the above procedure to bound (5.11) by (5.13). □

Next, we consider the contribution of the regular factors J^1_j.

Proposition 5.4. *Let $d \geqslant 3$. Then, for the regular tree τ_j, we have the bound*

$$\int_{[0,T)^{m_j}} dt_1 \dots dt_{m_j} \mathrm{Tr}\Big(\Big|S^{(1,-d)} J^1_j(t,t_1,\cdots,t_{m_j};\sigma_j)\Big|\Big) \leqslant 2^{m_j} C^{m_j} T^{\epsilon m_j} \|\phi\|^{2m_j+2}_{H^{s_\epsilon}}. \tag{5.17}$$

Similarly, when $d=2$, we have the bound

$$\int_{[0,T)^{m_j}} dt_1 \dots dt_{m_j} \mathrm{Tr}\Big(\Big|S^{(1,-d)} J^1_j(t,t_1,\cdots,t_{m_j};\sigma_j)\Big|\Big) \leqslant 2^{m_j} C^{m_j} T^{\frac{1}{3}m_j} \|\phi\|^{2m_j+2}_{H^{1/3}}, \tag{5.18}$$

and, when $d=1$, we have the bound

$$\int_{[0,T)^{m_j}} dt_1 \dots dt_{m_j} \mathrm{Tr}\Big(\Big|S^{(1,-d)} J^1_j(t,t_1,\cdots,t_{m_j};\sigma_j)\Big|\Big) \leqslant 2^{m_j} C^{m_j} T^{\frac{1}{2}m_j} \|\phi\|^{2m_j+2}_{L^2}. \tag{5.19}$$

Proof. Again, we consider the case $d \geqslant 3$, and note that the proof for $d = 1, 2$ is analogous (based on using the bounds for $d = 1, 2$ in Lemma 2.6).

We now proceed with the proof for $d \geqslant 3$.

$$\begin{aligned}
&\int_{[0,T)^{m_j}} dt_1 \dots dt_{m_j} \operatorname{Tr}\left(\left| S^{(1,-d)} J_j^1(t, t_1, \cdots, t_{m_j}; \sigma_j) \right| \right) \\
&= \int_{[0,T)^{m_j}} dt_1 \dots dt_{m_j} \operatorname{Tr}\left(\left| S^{(1,-d)} U^{(1)}(t - t_1) \Theta_1 \right| \right) \\
&\leqslant \sum_{\beta_1} \int_{[0,T)^{m_j}} dt_1 \cdots dt_{m_j} \|\psi^1_{\beta_1}\|_{H^{-d}} \|\chi^1_{\beta_1}\|_{H^{-d}} \\
&\leqslant \sum_{\beta_1} \int_{[0,T)^{m_j}} dt_1 \cdots dt_{m_j} \|\psi^1_{\beta_1}\|_{H^{s_\epsilon}} \|\chi^1_{\beta_1}\|_{H^{s_\epsilon}} \qquad (5.20)
\end{aligned}$$

By (4.6) and (4.7), one of $\psi^1_{\beta_1}, \chi^1_{\beta_1}$ is cubic, and the other is linear. We define f_1^1 to be the cubic function, and f_1^2 to be the linear one. Then, by (4.6) and (4.7), f_1^1 and f_1^2 are of the form

$$\begin{aligned}
f_1^1 &= (U_2 f_2^1)(U_2 f_2^2)(U_2 f_2^3). \\
f_1^2 &= U_2 f_2^4.
\end{aligned}$$

By (2.20), we have

$$\begin{aligned}
(5.20) &= \sum_{\beta_1} \int_{[0,T)^{m_j}} dt_1 \cdots dt_{m_j} \|(U_2 f_2^1)(U_2 f_2^2)(U_2 f_2^3)\|_{H^{s_\epsilon}} \|U_2 f_2^4\|_{H^{s_\epsilon}} \qquad (5.21) \\
&\leqslant (CT^\epsilon) \sum_{\beta_1} \int_{[0,T)^{m_j - 1}} dt_2 \cdots dt_{m_j} \|f_2^1\|_{H^{s_\epsilon}} \|f_2^2\|_{H^{s_\epsilon}} \|f_2^3\|_{H^{s_\epsilon}} \|f_2^4\|_{H^{s_\epsilon}}. \qquad (5.22)
\end{aligned}$$

By construction, only one of the factors f_2^ℓ is cubic. Without loss of generality, f_2^1 is cubic, and so we have

$$\begin{aligned}
f_2^1 &= (U_3 f_3^1)(U_3 f_3^2)(U_3 f_3^3), \\
f_2^\ell &= U_3 f_3^{\ell+2} \quad \text{for } \ell = 2, 3, 4.
\end{aligned}$$

Thus,

$$(5.22) = (CT^\epsilon) \sum_{\beta_1} \int_{[0,T)^{m_j - 1}} dt_2 \cdots dt_{m_j} \|(U_3 f_3^1)(U_3 f_3^2)(U_3 f_3^3)\|_{H^{s_\epsilon}} \|U_3 f_3^4\|_{H^{s_\epsilon}} \|U_3 f_3^5\|_{H^{s_\epsilon}} \|U_3 f_3^6\|_{H^{s_\epsilon}},$$

which is again of the form (5.21). Recall from subsection 4.4 that there are at most 2^{m_j} terms in the sum over β_1. Repeating this argument $m_j - 1$ more times yields the desired result (5.17). □

Before we proceed with the proof of Lemma 2.5, we present a short lemma that we use to bound the term $|\phi|^2\phi$ appearing on the right hand side of (5.1).

Lemma 5.5. *Let $\epsilon > 0$. Then, for $s_c = \frac{d}{2} - 1$, $r_\epsilon = \frac{2d}{d+2(1-\epsilon)}$, and $d \geqslant 3$, we have*

$$\||\phi|^2\phi\|_{W^{-(s_c + \frac{\epsilon}{2}), r_\epsilon}} \lesssim \|\phi\|^3_{H^{s_\epsilon}}. \qquad (5.23)$$

Similarly, when $d = 2$, we have

$$\||\phi|^2\phi\|_{W^{-(\frac{1}{3} - \frac{\epsilon}{2}), r_\epsilon}} \lesssim \|\phi\|^3_{H^{1/3}}. \qquad (5.24)$$

Proof. Let $d \geqslant 3$. By two applications of the Sobolev inequality, we have

$$\||\phi|^2\phi\|_{W^{-(s_c + \frac{\epsilon}{2}), r_\epsilon}} \lesssim \||\phi|^2\phi\|_{L^{\frac{2d}{2d-\epsilon}}} = \|\phi\|^3_{L^{\frac{6d}{2d-\epsilon}}} \lesssim \|\phi\|^3_{H^{\frac{d+\epsilon}{6}}} \leqslant \|\phi\|^3_{H^{s_\epsilon}}.$$

This establishes (5.23). The proof for the case $d = 2$ is similar.

∎

We are now ready to conclude the proof of Theorem 1.1 by proving Lemma 2.5.

Proof of Lemma 2.5. Recall from (4.1) that J^k can be decomposed into a product of k one particle kernels

$$J^k(t, t_1, \ldots, t_n; \sigma) = \prod_{j=1}^{k} J_j^1(t, t_{\ell_{j,1}}, \ldots, t_{\ell_{j,m_j}}; \sigma_j),$$

where only one of the factors J_j^1 distinguished. It now follows from Propositions 5.1 and 5.4 that

$$\begin{aligned}
&\int_{[0,T)^{n-1}} dt_1 \cdots dt_{n-1} \mathrm{Tr}\left(\left|S^{(k,-d)} J^k(t, t_1, \ldots, t_n; \sigma)\right|\right) \\
&= \int_{[0,T)^{n-1}} dt_1 \cdots dt_{n-1} \prod_{j=1}^{k} \mathrm{Tr}\left(\left|S^{(1,-d)} J_j^1(t, t_{\ell_{j,1}}, \ldots, t_{\ell_{j,m_j}}; \sigma_j)\right|\right) \\
&\leqslant \begin{cases} 2^n C^{n-1} T^{\epsilon(n-1)} \|\phi\|_{H^{s_\epsilon}}^{2(k+n)-3} \||\phi|^2\phi\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} & \text{if } d \geqslant 3 \\ 2^n C^{n-1} T^{\frac{1}{3}(n-1)} \|\phi\|_{H^{1/3}}^{2(k+n)-3} \||\phi|^2\phi\|_{W^{-(\frac{1}{3}-\frac{\epsilon}{2}),r_\epsilon}} & \text{if } d = 2 \\ 2^n C^{n-1} T^{\frac{1}{2}(n-1)} \|\phi\|_{L^2}^{2(k+n)-3} \||\phi|^2\phi\|_{L^1} & \text{if } d = 1. \end{cases}
\end{aligned}$$

Thus, for $t \in [0, T)$, it follows from Lemma 5.5 that

$$\begin{aligned}
&\int_{[0,T)^{n-1}} d\underline{t}_{n-1} \mathrm{Tr}(|S^{(k,-d)} J^k(\underline{t}_n; \sigma)|) \\
&\leqslant \begin{cases} (CT^\epsilon)^{n-1} \|\phi\|_{H^{s_\epsilon}}^{2(k+n)} & \text{if } d \geqslant 3 \\ (CT^{1/3})^{n-1} \|\phi\|_{H^{1/3}}^{2(k+n)} & \text{if } d = 2 \\ (CT^{1/2})^{n-1} \|\phi\|_{H^{1/6}}^{2(k+n)} & \text{if } d = 1, \end{cases}
\end{aligned}$$

which is precisely the statement of Lemma 2.5. ∎

Appendix A. Proof of Lemma 2.6

We prove Lemma 2.6 combining the dispersive estimate, the Strichartz estimates (see [24] for example) and negative order Sobolev norms.

Lemma A.1 (Dispersive estimates). *For $2 \leqslant r \leqslant \infty$, we have*

$$\|e^{it\Delta} f\|_{L_x^r} \lesssim |t|^{-d(\frac{1}{2}-\frac{1}{r})} \|f\|_{L_x^{r'}}. \tag{A.1}$$

Lemma A.2 (Homogeneous Strichartz estimates). *We call a pair of exponents (q, r) Schrödinger admissible if $2 \leqslant q, r \leqslant \infty$, $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$ and $(q, r, d) \neq (2, \infty, 2)$. Then for any admissible exponents (q, r) we have the homogeneous Strichartz estimate*

$$\|e^{it\Delta} f\|_{L_t^q L_x^r} \lesssim \|f\|_{L_x^2}. \tag{A.2}$$

Lemma A.3 (Negative order Sobolev norms). *Let $\epsilon > 0$ be a small number. Then, for $s \geqslant s_c + \frac{\epsilon}{2}$, we have*

$$\|fg\|_{W^{-s,r_\epsilon}} \lesssim \|f\|_{W^{-s,r'_\epsilon}} \|g\|_{W^{s,\frac{2d}{d+2-3\epsilon}}},$$

where $r_\epsilon = \frac{2d}{d+2(1-\epsilon)}$.

Proof. By Hölder's inequality, the fractional Leibniz rule and the Sobolev inequality, we have

$$\begin{aligned}\Big|\int f(x)g(x)\overline{h(x)}dx\Big| &\leqslant \|f\|_{W^{-s,r_\epsilon'}}\|g\bar{h}\|_{W^{s,r_\epsilon}}\\ &\lesssim \|f\|_{W^{-s,r_\epsilon'}}\Big(\|g\|_{W^{s,\frac{2d}{d+2-3\epsilon}}}\|h\|_{L^{\frac{2d}{\epsilon}}}+\|g\|_{L^{\frac{d}{2(1-\epsilon)}}}\|h\|_{W^{s,r_\epsilon'}}\Big)\\ &\lesssim \|f\|_{W^{-s,r_\epsilon'}}\|g\|_{W^{s,\frac{2d}{d+2-3\epsilon}}}\|h\|_{W^{s,r_\epsilon'}}.\end{aligned}$$

The lemma now follows from the standard duality argument. □

Proof of Lemma 2.6. (i). For notational convenience, we omit the time interval $[0, T)$ in the norms.
(2.19): By Lemma A.3, we get

$$\begin{aligned}\|T(f,g,h)\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} &\lesssim \|e^{i(t-t_1)\Delta}f\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon'}}\|(e^{i(t-t_2)\Delta}g)(e^{i(t-t_3)\Delta}h)\|_{W^{s_c+\frac{\epsilon}{2},\frac{2d}{d+2-3\epsilon}}}\\ &\lesssim \frac{1}{|t-t_1|^{1-\epsilon}}\|f\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}}\|g\|_{H^{s_\epsilon}}\|h\|_{H^{s_\epsilon}}.\end{aligned}\tag{A.3}$$

Here, in the second inequality, we use the dispersive estimate:

$$\|e^{i(t-t_1)\Delta}f\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon'}} \lesssim \frac{1}{|t-t_1|^{1-\epsilon}}\|f\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}}$$

and the fractional Leibniz rule and the Sobolev inequality:

$$\begin{aligned}&\|(e^{i(t-t_2)\Delta}g)(e^{i(t-t_3)\Delta}h)\|_{W^{s_c+\frac{\epsilon}{2},\frac{2d}{d+2-3\epsilon}}}\\ &\lesssim \|e^{i(t-t_2)\Delta}g\|_{W^{s_c+\frac{\epsilon}{2},\frac{2d}{d-\epsilon}}}\|e^{i(t-t_3)\Delta}h\|_{L^{\frac{d}{1-\epsilon}}}+\|e^{i(t-t_2)\Delta}g\|_{L^{\frac{d}{1-\epsilon}}}\|e^{i(t-t_3)\Delta}h\|_{W^{s_c+\frac{\epsilon}{2},\frac{2d}{d-\epsilon}}}\\ &\lesssim \|e^{i(t-t_2)\Delta}g\|_{H^{s_\epsilon}}\|e^{i(t-t_3)\Delta}h\|_{H^{s_\epsilon}} = \|g\|_{H^{s_\epsilon}}\|h\|_{H^{s_\epsilon}}.\end{aligned}\tag{A.4}$$

Integrating out the time variable t, we prove (2.19).

(2.20): By the fractional Leibniz rule, we have

$$\begin{aligned}\|T(f,g,h)\|_{L_t^1H_x^{s_\epsilon}} &\lesssim \|e^{i(t-t_1)\Delta}f\|_{L_t^3W_x^{s_\epsilon,\frac{6d}{3d-4}}}\|e^{i(t-t_2)\Delta}g\|_{L_t^3L_x^{3d}}\|e^{i(t-t_3)\Delta}h\|_{L_t^3L_x^{3d}}\\ &+\|e^{i(t-t_1)\Delta}f\|_{L_t^3L_x^{3d}}\|e^{i(t-t_2)\Delta}g\|_{L_t^3W_x^{s_\epsilon,\frac{6d}{3d-4}}}\|e^{i(t-t_3)\Delta}h\|_{L_t^3W_x^{3d}}\\ &+\|e^{i(t-t_1)\Delta}f\|_{L_t^3L_x^{3d}}\|e^{i(t-t_2)\Delta}g\|_{L_t^3L_x^{3d}}\|e^{i(t-t_3)\Delta}h\|_{L_t^3W_x^{s_\epsilon,\frac{6d}{3d-4}}}.\end{aligned}$$

Then, by the Sobolev inequality and the Strichartz estimates, we bound the first term by

$$\begin{aligned}&\lesssim \|e^{i(t-t_1)\Delta}f\|_{L_t^3W_x^{s_\epsilon,\frac{6d}{3d-4}}}\|e^{i(t-t_2)\Delta}g\|_{L_t^3W_x^{s_\epsilon,\frac{6d}{3d-4+6\epsilon}}}\|e^{i(t-t_3)\Delta}h\|_{L_t^3W_x^{s_\epsilon,\frac{6d}{3d-4+6\epsilon}}}\\ &\leqslant T^\epsilon\|e^{i(t-t_1)\Delta}f\|_{L_t^3W_x^{s_\epsilon,\frac{6d}{3d-4}}}\|e^{i(t-t_2)\Delta}g\|_{L_t^{\frac{6}{2-3\epsilon}}W_x^{s_\epsilon,\frac{6d}{3d-4+6\epsilon}}}\|e^{i(t-t_3)\Delta}h\|_{L_t^{\frac{6}{2-3\epsilon}}W_x^{s_\epsilon,\frac{6d}{3d-4+6\epsilon}}}\\ &\lesssim T^\epsilon\|f\|_{H^{s_\epsilon}}\|g\|_{H^{s_\epsilon}}\|h\|_{H^{s_\epsilon}}.\end{aligned}$$

Similarly, we bound the other two terms.

(ii). (2.21): The proof is similar to that of (2.19), but here we use Lemma A.3 with $s = (\frac{1}{3}-\frac{\epsilon}{2})$. Indeed, by the dispersive estimate and Lemma A.3,

$$\begin{aligned}\|T(f,g,h)\|_{W^{-(\frac{1}{3}-\frac{\epsilon}{2}),r_\epsilon}} &\lesssim \|e^{i(t-t_1)\Delta}f\|_{W^{-(\frac{1}{3}-\frac{\epsilon}{2}),r_\epsilon'}}\|(e^{i(t-t_2)\Delta}g)(e^{i(t-t_3)\Delta}h)\|_{W^{\frac{1}{3}-\frac{\epsilon}{2},\frac{2d}{d+2-3\epsilon}}}\\ &\lesssim \frac{1}{|t-t_1|^{1-\epsilon}}\|(e^{i(t-t_2)\Delta}g)(e^{i(t-t_3)\Delta}h)\|_{W^{\frac{1}{3}-\frac{\epsilon}{2},\frac{2d}{d+2-3\epsilon}}}.\end{aligned}$$

Then, modifying $(A.1)$, we obtain

$$\|(e^{i(t-t_2)\Delta}g)(e^{i(t-t_3)\Delta}h)\|_{W^{\frac{1}{3}-\frac{\epsilon}{2},\frac{2d}{d+2-3\epsilon}}}$$

$$\lesssim \|e^{i(t-t_2)\Delta}g\|_{W^{\frac{1}{3}-\frac{\epsilon}{2},\frac{2d}{d-\epsilon}}}\|e^{i(t-t_3)\Delta}h\|_{L^{\frac{d}{1-\epsilon}}} + \|e^{i(t-t_2)\Delta}g\|_{L^{\frac{d}{1-\epsilon}}}\|e^{i(t-t_3)\Delta}h\|_{W^{\frac{1}{3}-\frac{\epsilon}{2},\frac{2d}{d-\epsilon}}}$$
$$\lesssim \|e^{i(t-t_2)\Delta}g\|_{H^{1/3}}\|e^{i(t-t_3)\Delta}h\|_{H^{1/3}} = \|g\|_{H^{1/3}}\|h\|_{H^{1/3}},$$

Applying this to the above inequality and Integrating out t, we compete the proof.

(2.22): Although we set ϵ to be small and $d \geqslant 3$ in the proof of (2.20), it actually works for $\epsilon = \frac{1}{3}$ and $d = 2$ which is exactly (2.22).

(iii). For (2.23), by the Hölder inequality and the 1d dispersive estimates, we get

$$\|T(f,g,h)\|_{L^1} \leqslant \|e^{i(t-t_1)}f\|_{L^\infty}\|e^{i(t-t_2)}g\|_{L^2}\|e^{i(t-t_3)}h\|_{L^2} \lesssim \frac{1}{|t-t_1|^{1/2}}\|f\|_{L^1}\|g\|_{L^2}\|h\|_{L^2}.$$

Integrating out the time variable t, we prove (2.23).
For (2.24), by the Hölder inequality and the Strichartz estimate,

$$\|T(f,g,h)\|_{L^1_tL^2_x} \leqslant T^{1/2}\|e^{i(t-t_1)}f\|_{L^6_{t,x}}\|e^{i(t-t_2)}g\|_{L^6_{t,x}}\|e^{i(t-t_3)}h\|_{L^6_{t,x}} \lesssim T^{1/2}\|f\|_{L^2}\|g\|_{L^2}\|h\|_{L^2}.$$

□

Acknowledgements. *The authors would like to express their special appreciation and thanks to their mentors Thomas Chen and Nataša Pavlović for proposing the problem and for various useful discussions.*

References

[1] T. Cazenave. *Semilinear Schrödinger equations*, *Courant Lecture Notes in Mathematics* **10**. AMS/CIMS, 2003.

[2] T. Chen, C. Hainzl, N. Pavlović, and R. Seiringer. On the well-posedness and scattering for the Gross-Pitaevskii hierarchy via quantum de Finetti. *Preprint available at arXiv:1311.2136*, 2013.

[3] T. Chen, C. Hainzl, N. Pavlović, and R. Seiringer. Unconditional Uniqueness for the cubic Gross-Pitaevskii hierarchy via quantum de Finetti. *Commun. Pure Appl. Math., to appear. Preprint available at arXiv:1307.3168*, 2013.

[4] T. Chen and N. Pavlović. On the Cauchy problem for focusing and defocusing Gross-Pitaevskii hierarchies. *Discrete Contin. Dyn. Syst.*, 27(2):715–739, 2010.

[5] T. Chen and N. Pavlović. The quintic NLS as the mean field limit of a boson gas with three-body interactions. *J. Funct. Anal.*, 260(4):959–997, 2011.

[6] T. Chen and N. Pavlović. Derivation of the cubic NLS and GrossPitaevskii Hierarchy from manybody dynamics in d = 3 based on spacetime norms. *Annales Henri Poincaré*, pages 1–46, 2013.

[7] T. Chen and K. Taliaferro. Positive semidefiniteness and Global Well-Posedness of Solutions to the Gross-Pitaevskii Hierarchy. *Preprint available at arXiv:1305.1404*, 2013.

[8] X. Chen. Second order corrections to mean field evolution for weakly interacting bosons in the case of three-body interactions. *Arch. Ration. Mech. Anal.*, 203(2):455–497, 2012.

[9] X. Chen. On the Rigorous Derivation of the 3D Cubic Nonlinear Schrödinger Equation with a Quadratic Trap. *Arch. Ration. Mech. Anal.*, 210(2):365–408, 2013.

[10] X. Chen and J. Holmer. On the Klainerman-Machedon Conjecture of the Quantum BBGKY Hierarchy with self-interaction. *Preprint available at arXiv:1303.5385*, 2013.

[11] L. Erdős, B. Schlein, and H.-T. Yau. Derivation of the Gross-Pitaevskii hierarchy for the dynamics of Bose-Einstein condensate. *Comm. Pure Appl. Math.*, 59(12):1659–1741, 2006.

[12] L. Erdős, B. Schlein, and H.-T. Yau. Derivation of the cubic non-linear Schrödinger equation from quantum dynamics of many-body systems. *Invent. Math.*, 167(3):515–614, 2007.

[13] L. Erdős, B. Schlein, and H.-T. Yau. Rigorous derivation of the Gross-Pitaevskii equation with a large interaction potential. *J. Amer. Math. Soc.*, 22(4):1099–1156, 2009.

[14] L. Erdős, B. Schlein, and H.-T. Yau. Derivation of the Gross-Pitaevskii equation for the dynamics of Bose-Einstein condensate. *Ann. of Math. (2)*, 172(1):291–370, 2010.

[15] G. Furioli and E. Terraneo. Besov spaces and unconditional well-posedness for the nonlinear Schrödinger equation in $\dot{H}^s(\mathbb{R}^n)$. *Commun. Contemp. Math.*, 5(3):349–367, 2003.

[16] J. Ginibre and G. Velo. The classical field limit of scattering theory for nonrelativistic many-boson systems. I. *Comm. Math. Phys.*, 66(1):37–76, 1979.

[17] J. Ginibre and G. Velo. The classical field limit of scattering theory for nonrelativistic many-boson systems. II. *Comm. Math. Phys.*, 68(1):45–68, 1979.

[18] P. Gressman, V. Sohinger, and G. Staffilani. On the uniqueness of solutions to the periodic 3d gross-pitaevskii hierarchy. *Preprint available at arXiv:1212.2987*, 2013.

[19] M. Grillakis, M. Machedon, and D. Margetis. Second-order corrections to mean field evolution of weakly interacting bosons. II. *Adv. Math.*, 228(3):1788–1815, 2011.

[20] M. G. Grillakis, M. Machedon, and D. Margetis. Second-order corrections to mean field evolution of weakly interacting bosons. I. *Comm. Math. Phys.*, 294(1):273–301, 2010.

[21] Z. Han and D. Fang. On the unconditional uniqueness for NLS in $\dot{H}^s$. *SIAM J. Math. Anal.*, 45(3):1505–1526, 2013.

[22] K. Hepp. The classical limit for quantum mechanical correlation functions. *Comm. Math. Phys.*, 35:265–277, 1974.

[23] T. Kato. On nonlinear Schrödinger equations. II. H^s-solutions and unconditional well-posedness. *J. Anal. Math.*, 67:281–306, 1995.

[24] M. Keel and T. Tao. Endpoint Strichartz estimates. *Amer. J. Math.*, 120(5):955–980, 1998.

[25] K. Kirkpatrick, B. Schlein, and G. Staffilani. Derivation of the two-dimensional nonlinear Schrödinger equation from many body quantum dynamics. *Amer. J. Math.*, 133(1):91–130, 2011.

[26] S. Klainerman and M. Machedon. Space-time estimates for null forms and the local existence theorem. *Comm. Pure Appl. Math.*, 46(9):1221–1268, 1993.

[27] S. Klainerman and M. Machedon. On the uniqueness of solutions to the Gross-Pitaevskii hierarchy. *Comm. Math. Phys.*, 279(1):169–185, 2008.

[28] O. E. Lanford, III. The classical mechanics of one-dimensional systems of infinitely many particles. I. An existence theorem. *Comm. Math. Phys.*, 9:176–191, 1968.

[29] O. E. Lanford, III. The classical mechanics of one-dimensional systems of infinitely many particles. II. Kinetic theory. *Comm. Math. Phys.*, 11:257–292, 1968/1969.

[30] E. H. Lieb and R. Seiringer. Proof of Bose-Einstein Condensation for Dilute Trapped Gases. *Phys. Rev. Lett.*, 88(17):170409, Apr. 2002.

[31] E. H. Lieb, R. Seiringer, J. P. Solovej, and J. Yngvason. *The mathematics of the Bose gas and its condensation*, volume 34 of *Oberwolfach Seminars*. Birkhäuser Verlag, Basel, 2005.

[32] E. H. Lieb, R. Seiringer, and J. Yngvason. The ground state energy and density of interacting bosons in a trap. In *Quantum theory and symmetries (Goslar, 1999)*, pages 101–110. World Sci. Publ., River Edge, NJ, 2000.

[33] E. H. Lieb, R. Seiringer, and J. Yngvason. A rigorous derivation of the Gross-Pitaevskii energy functional for a two-dimensional Bose gas. *Comm. Math. Phys.*, 224(1):17–31, 2001.

[34] P. T. N. Mathieu Lewin and N. Rougerie. Derivation of Hartree's theory for generic mean-field Bose systems. *Preprint available at arXiv:1303.0981*, 2013.

[35] I. Rodnianski and B. Schlein. Quantum fluctuations and rate of convergence towards mean field dynamics. *Comm. Math. Phys.*, 291(1):31–61, 2009.

[36] K. M. Rogers. Unconditional well-posedness for subcritical NLS in H^s. *C. R. Math. Acad. Sci. Paris*, 345(7):395–398, 2007.

[37] H. Spohn. Kinetic equations from Hamiltonian dynamics: Markovian limits. *Rev. Modern Phys.*, 52(3):569–615, 1980.

[38] Y. Y. S. Win and Y. Tsutsumi. Unconditional uniqueness of solution for the Cauchy problem of the nonlinear Schrödinger equation. *Hokkaido Math. J.*, 37(4):839–859, 2008.

[39] Z. Xie. Derivation of a Nonlinear Schrödinger Equation with a General power-type nonlinearity. *Preprint available at arXiv:1305.7240*, 2013.

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