Maximizing the order of a regular graph of given valency and second eigenvalue

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Abstract

From Alon and Boppana, and Serre, we know that for any given integer $k \geq 3$ and real number $\lambda < 2\sqrt{k-1}$, there are only finitely many k-regular graphs whose second largest eigenvalue is at most λ. In this paper, we investigate the largest number of vertices of such graphs.

1 Introduction

For a k-regular graph G on n vertices, we denote by $\lambda_1(G) = k > \lambda_2(G) \geq \ldots \geq \lambda_n(G) = \lambda_{\min}(G)$ the eigenvalues of the adjacency matrix of G. For a general reference on the eigen-

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values of graphs, see [9, 18].

The second eigenvalue of a regular graph is a parameter of interest in the study of graph connectivity and expanders (see [1, 9, 24] for example). In this paper, we investigate the maximum order $v(k, \lambda)$ of a connected k-regular graph whose second largest eigenvalue is at most some given parameter λ. As a consequence of work of Alon and Boppana, and of Serre [1, 12, 16, 24, 25, 28, 31, 35, 36, 42], we know that $v(k, \lambda)$ is finite for $\lambda < 2\sqrt{k-1}$. The recent result of Marcus, Spielman and Srivastava [29] showing the existence of infinite families of Ramanujan graphs of any degree at least 3 implies that $v(k, \lambda)$ is infinite for $\lambda \geq 2\sqrt{k-1}$.

For any $\lambda < 0$, the parameter $v(k, \lambda)$ can be determined using the fact that a graph with only one nonnegative eigenvalue is a complete graph. Indeed, if a graph has only one nonnegative eigenvalue, then it must be connected. If a connected graph G is not a complete graph, then G contains an induced subgraph isomorphic to $K_{1,2}$, so Cauchy eigenvalue interlacing (see [9, Proposition 3.2.1]) implies $\lambda_2(G) \geq \lambda_2(K_{1,2}) = 0$, contradiction. Thus $v(k, \lambda) = k+1$ for any $\lambda < 0$ and the unique graph meeting this bound is K_{k+1}. The parameter $v(k, 0)$ can be determined using the fact that a graph with exactly one positive eigenvalue must be a complete multipartite graph (see [7, page 89]). The largest k-regular complete multipartite graph is the complete bipartite graph $K_{k,k}$, since a k-regular t-partite graph has $tk/(t-1)$ vertices. Thus $v(k, 0) = 2k$, and $K_{k,k}$ is the unique graph meeting this bound. The values of $v(k, -1)$ and $v(k, 0)$ also follow from Theorem 2.3 in Section 2 below.

Results from Bussemaker, Cvetković and Seidel [10] and Cameron, Goethals, Seidel, and Shult [11] give a characterization of the regular graphs with smallest eigenvalue $\lambda_{\min} \geq -2$. Since the second eigenvalue of the complement of a regular graph is $\lambda_2 = -1 - \lambda_{\min}$, the regular graphs with second eigenvalue $\lambda_2 \leq 1$ are also characterized. This characterization can be used to find $v(k, 1)$ (see Section 3).

The values remaining to be investigated are $v(k, \lambda)$ for $1 < \lambda < 2\sqrt{k-1}$. The parameter $v(k, \lambda)$ has been studied by Teranishi and Yasuno [44] and Høholdt and Justesen [22] for the class of bipartite graphs in connection with problems in design theory, finite geometry and coding theory. Some results involving $v(k, \lambda)$ were obtained by Koledin and Staníc [26, 27, 43] and Richey, Shutty and Stover [47] who implemented Serre's quantitative version of the Alon–Boppana Theorem [42] to obtain upper bounds for $v(k, \lambda)$ for several values of k and λ. For certain values of k and λ, Richey, Shutty and Stover [47] made some conjectures about $v(k, \lambda)$. We will prove some of their conjectures and disprove others in this paper. Reingold, Vadhan and Wigderson [38] used regular graphs with small second eigenvalue as the starting point of their iterative construction of infinite families of expander using the zig-zag product. Guo, Mohar, and Tayfeh-Rezaie [19, 32, 33] studied a similar problem involving the median eigenvalue. Nozaki [37] investigated a related, but different problem from the one studied in our paper, namely finding the regular graphs of given valency and order with smallest second eigenvalue. Amit, Hoory and Linial [2] studied a related problems of minimizing $\max(|\lambda_2|, |\lambda_n|)$ for regular graphs of given order n, valency k and girth g.

In this paper, we determine $v(k, \lambda)$ explicitly for several values of (k, λ), confirming or disproving several conjectures in [47], and we find the graphs (in many cases unique) which meet our bounds. In many cases these graphs are distance-regular. For definitions and notations related to distance-regular graphs, we refer the reader to [9, Chapter 12]. Table 1 contains a summary of the values of $v(k, \lambda)$ that we found for $k \leq 22$. Table 2 contains six infinite families of graphs and seven sporadic graphs meeting the bound $v(k, \lambda)$ for some values of k, λ due to Theorem 2.3. Table 3 illustrates that the graphs in Table 2 that meet the bound $v(k, \lambda)$ also meet the bound $v(k, \lambda')$ for certain $\lambda' > \lambda$ due to Proposition 2.9.

2 Linear programming method

In this section, we give a bound for $v(k, \lambda)$ using the linear programming method developed by Nozaki [37]. Let $F_i = F_i^{(k)}$ be orthogonal polynomials defined by the three-term recurrence relation:

$$F_0^{(k)}(x) = 1, \qquad F_1^{(k)}(x) = x, \qquad F_2^{(k)}(x) = x^2 - k,$$

and

$$F_i^{(k)}(x) = xF_{i-1}^{(k)}(x) - (k-1)F_{i-2}^{(k)}(x)$$

for $i \geq 3$. The following is called the linear programming bound for regular graphs.

Theorem 2.1 (Nozaki [37]). *Let G be a connected k-regular graph with v vertices. Let $\lambda_1 = k, \lambda_2, \ldots, \lambda_n$ be the distinct eigenvalues of G. Suppose there exists a polynomial $f(x) = \sum_{i\geq 0} f_i F_i^{(k)}(x)$ such that $f(k) > 0$, $f(\lambda_i) \leq 0$ for any $i \geq 2$, $f_0 > 0$, and $f_i \geq 0$ for any $i \geq 1$. Then we have*

$$v \leq \frac{f(k)}{f_0}.$$

Using Theorem 2.1, Nozaki [37] proved Theorem 2.2 below. Note that the paper [37] deals only with the problem of minimizing the second eigenvalue of a regular graph of given order and valency. While related to the problem of estimating $v(k, \lambda)$, the problem considered by Nozaki in [37] is quite different from the one we study in this paper.

Theorem 2.2 (Nozaki [37]). *Let G be a connected k-regular graph of girth g, with v vertices. Assume the number of distinct eigenvalues of G is $d + 1$. If $g \geq 2d$ holds, then G has the smallest second-largest eigenvalue in all k-regular graphs with v vertices.*

Table 1: Summary of our Results for $k \leq 22$

(k,λ)	$v(k,\lambda)$	(k,λ)	$v(k,\lambda)$	(k,λ)	$v(k,\lambda)$
$(2,-1)$	3	$(7,1)$	18	$(14,\sqrt{13})$	366
$(2,0)$	4	$(7,2)$	50	$(14,\sqrt{26})$	4760
$\left(2,\frac{1}{2}\left(\sqrt{5}-1\right)\right)$	5	$(8,-1)$	9	$(14,\sqrt{39})$	804468
$(2,1)$	6	$(8,0)$	16	$(15,-1)$	16
$(2,\sqrt{2})$	8	$(8,1)$	21	$(15,0)$	30
$\left(2,\frac{1}{2}\left(\sqrt{5}+1\right)\right)$	10	$(8,\sqrt{7})$	114	$(15,1)$	32
$(2,\sqrt{3})$	12	$(8,\sqrt{14})$	800	$(16,-1)$	17
$(3,-1)$	4	$(8,\sqrt{21})$	39216	$(16,0)$	32
$(3,0)$	6	$(9,-1)$	10	$(16,1)$	34
$(3,1)$	10	$(9,0)$	18	$(16,2)$	77
$(3,\sqrt{2})$	14	$(9,1)$	24	$(17,-1)$	18
$(3,\sqrt{3})$	18	$(9,2\sqrt{2})$	146	$(17,0)$	34
$(3,2)$	30	$(9,4)$	1170	$(17,1)$	36
$(3,\sqrt{6})$	126	$(9,2\sqrt{6})$	74898	$(18,-1)$	19
$(4,-1)$	5	$(10,-1)$	11	$(18,0)$	36
$(4,0)$	8	$(10,0)$	20	$(18,1)$	38
$(4,1)$	9	$(10,1)$	27	$(18,\sqrt{17})$	614
$(4,\sqrt{5}-1)$	10	$(10,2)$	56	$(18,\sqrt{34})$	10440
$(4,\sqrt{3})$	26	$(10,3)$	182	$(18,\sqrt{51})$	3017196
$(4,2)$	35	$(10,3\sqrt{2})$	1640	$(19,-1)$	20
$(4,\sqrt{6})$	80	$(10,3\sqrt{3})$	132860	$(19,0)$	38
$(4,3)$	728	$(11,-1)$	12	$(19,1)$	40
$(5,-1)$	6	$(11,0)$	22	$(20,-1)$	21
$(5,0)$	10	$(11,1)$	24	$(20,0)$	40
$(5,1)$	16	$(12,-1)$	13	$(20,1)$	42
$(5,2)$	42	$(12,0)$	24	$(20,\sqrt{19})$	762
$(5,2\sqrt{2})$	170	$(12,1)$	26	$(20,\sqrt{38})$	14480
$(5,2\sqrt{3})$	2730	$(12,\sqrt{11})$	266	$(20,\sqrt{57})$	5227320
$(6,-1)$	7	$(12,\sqrt{22})$	2928	$(21,-1)$	22
$(6,0)$	12	$(12,\sqrt{33})$	354312	$(21,0)$	42
$(6,1)$	15	$(13,-1)$	14	$(21,1)$	44
$(6,\sqrt{5})$	62	$(13,0)$	26	$(22,-1)$	23
$(6,\sqrt{10})$	312	$(13,1)$	28	$(22,0)$	44
$(6,\sqrt{15})$	7812	$(14,-1)$	15	$(22,1)$	46
$(7,-1)$	8	$(14,0)$	28	$(22,2)$	100
$(7,0)$	14	$(14,1)$	30		

Note also that while Table 2 is similar to [37, Table 2], the problems and tools in our paper are significantly different from the ones in [37].

Let $T(k,t,c)$ be the $t \times t$ tridiagonal matrix with lower diagonal $(1,1,\ldots,1,c)$, upper diagonal $(k,k-1,\ldots,k-1)$, and with constant row sum k, where c is a positive real number. Theorem 2.3 is the main theorem in this section and gives a new comprehension of the linear programming method and a general upper bound for $v(k,\lambda)$ without any assumption regarding the existence of some particular graphs.

Theorem 2.3. *If λ_2 is the second largest eigenvalue of $T(k,t,c)$, then*

$$v(k,\lambda_2) \le M(k,t,c) = 1 + \sum_{i=0}^{t-3} k(k-1)^i + \frac{k(k-1)^{t-2}}{c}. \tag{1}$$

Let G be a k-regular connected graph with second largest eigenvalue at most λ_2, valency k, and $v(k,\lambda_2)$ vertices. Then $v(k,\lambda_2) = M(k,t,c)$ if and only if G is distance-regular with quotient matrix $T(k,t,c)$ with respect to the distance-partition.

Proof. We first show that the eigenvalues of T that are not equal to k, coincide with the zeros of $\sum_{i=0}^{t-2} F_i(x) + F_{t-1}(x)/c$ (see also [7, Section 4.1 B]). Indeed,

$$[F_0, F_1, \ldots, F_{t-2}, F_{t-1}/c]T = [xF_0, xF_1, \ldots, xF_{t-2}, (k-1)F_{t-2} + (k-c)F_{t-1}/c],$$

and

$$\begin{aligned}
[F_0, F_1, \ldots, F_{t-2}, F_{t-1}/c](T - xI) &= [0, 0, \ldots, 0, (k-1)F_{t-2} + (-x+k-c)F_{t-1}/c] \\
&= [0, 0, \ldots, 0, (k-x)(\sum_{i=0}^{t-2} F_i + F_{t-1}/c)] \\
&= [0, 0, \ldots, 0, (k-x)((c-1)G_{t-2} + G_{t-1})/c]
\end{aligned}$$

by the three-term recurrence relation, where $G_i(x) = \sum_{j=0}^{i} F_j(x)$. This equation implies that the zeros of $(k-x)((c-1)G_{t-2} + G_{t-1})$ are eigenvalues of T. The monic polynomials G_i form a sequence of orthogonal polynomials with respect to some positive weight on the interval $[-2\sqrt{k-1}, 2\sqrt{k-1}]$ [37]. Since the zeros of G_{t-2} and G_{t-1} interlace on $[-2\sqrt{k-1}, 2\sqrt{k-1}]$, the zeros of $(k-x)((c-1)G_{t-2} + G_{t-1})$ are simple. Therefore all eigenvalues of T coincide with the zeros of $(k-x)((c-1)G_{t-2} + G_{t-1})$, and are simple.

Let $\lambda_1 = k > \lambda_2 > \ldots > \lambda_t$ be the eigenvalues of T. We prove that the polynomial

$$f(x) = \frac{1}{c} \cdot (x - \lambda_2) \prod_{i=3}^{t} (x - \lambda_i)^2 = \sum_{i=0}^{2t-3} f_i F_i(x) \tag{2}$$

satisfies $f_i > 0$ for $i = 0, 1, \ldots, 2t-3$. Note that it trivially holds that $f(k) > 0$, and $f(\lambda) \leq 0$ for any $\lambda \leq \lambda_2$. The polynomial $f(x)$ can be expressed as

$$f(x) = \frac{(c-1)G_{t-2} + G_{t-1}}{x - \lambda_2} \cdot \left(\sum_{i=0}^{t-2} F_i + F_{t-1}/c \right). \tag{3}$$

By [13, Proposition 3.2], $g(x) = ((c-1)G_{t-2} + G_{t-1})/(x - \lambda_2)$ has positive coefficients in terms of $G_0, G_1, \ldots, G_{t-2}$. This implies that $g(x)$ has positive coefficients in terms of $F_0, F_1, \ldots, F_{t-2}$. Therefore $f_i > 0$ for $i = 0, 1, \ldots, 2t-3$ by [37, Theorem 3].

The polynomial $g(x)$ can be expressed as $g(x) = \sum_{i=0}^{t-2} g_i F_i(x)$. By [37, Theorem 3], we get that $f_0 = \sum_{i=0}^{t-2} g_i F_i(k) = g(k)$. Using Theorem 2.1 for $f(x)$, we obtain that

$$\begin{aligned} v(k, \lambda_2) \leq \frac{f(k)}{f_0} &= \sum_{i=0}^{t-2} F_i(k) + F_{t-1}(k)/c \\ &= 1 + \sum_{i=0}^{t-3} k(k-1)^i + \frac{k(k-1)^{t-2}}{c}. \end{aligned}$$

By [37, Remark 2], the graph attaining the bound has girth at least $2t-2$, and at most t distinct eigenvalues. Therefore the graph is a distance-regular graph with quotient matrix $T(k, t, c)$ by [37, Theorem 6] and [14]. Conversely the distance-regular graph with quotient matrix $T(k, t, c)$ clearly attains the bound $M(k, t, c)$. $\square$

Remark 2.4. *The distance-regular graphs which have $T(k, t, c)$ as a quotient matrix of the distance partition are precisely the distance-regular graphs with intersection array $\{k, k-1, \ldots, k-1; 1, \ldots, 1, c\}$.*

Corollary 2.5. *Let H be a connected k-regular graph with at least $M(k, t, c)$ vertices. Let λ_2 be the second largest eigenvalue of $T(k, t, c)$. Then $\lambda_2 \leq \lambda_2(H)$ holds with equality only if H meets the bound $M(k, t, c)$.*

Proof. By Theorem 2.3, if $\lambda_2 > \lambda_2(H)$, then the order of H is at most $M(k, t, c)$. If the order of H is equal to $M(k, t, c)$, then H has at most $t-1$ distinct eigenvalues by [37, Remark 2]. However then the order of H is less than $M(k, t-1, 1)$ by the Moore bound, a contradiction. Therefore if $\lambda_2 > \lambda_2(H)$, then the order of H is less than $M(k, t, c)$. Namely if the order of H is at least $M(k, t, c)$, then $\lambda_2 \leq \lambda_2(H)$. If $\lambda_2 = \lambda_2(H)$ holds, then the order of H is bounded above by $M(k, t, c)$ in Theorem 2.3, and attains the bound. $\square$

We will discuss a possible second eigenvalue λ_2 of $T(k, t, c)$. Indeed for any $-1 \leq \lambda < 2\sqrt{k-1}$ there exist t, c such that λ is the second eigenvalue of $T(k, t, c)$. Let $\lambda^{(t)}$, $\mu^{(t)}$ be the largest zero of G_t, F_t, respectively. The zero $\lambda^{(t)}$ can be expressed by $\lambda^{(t)} = 2\sqrt{k-1}\cos\theta$, where $\pi/(t+1) < \theta < \pi/t$ [3, Section III.3].

Proposition 2.6. *The following hold:*

(1) $\lambda^{(t)} < \mu^{(t)}$ *for any* k, t.

(2) $\mu^{(t-1)} < \lambda^{(t)}$ *for* $k \geq 5$ *and any* t, $k = 4$ *and* $t \leq 5$, *or* $k = 3$ *and* $t \leq 3$.

(3) $\mu^{(t-1)} > \lambda^{(t)}$ *for* $k = 4$ *and* $t \geq 6$, *or* $k = 3$ *and* $t \geq 4$.

Proof. Since $F_t(\lambda^{(t)}) = G_t(\lambda^{(t)}) - G_{t-1}(\lambda^{(t)}) = -G_{t-1}(\lambda^{(t)}) < 0$, we have $\lambda^{(t)} < \mu^{(t)}$ for any k, t. Note that F_t has a unique zero greater than $\lambda^{(t)}$. By the equality $(k-1)F_{t-1} = (k-1-x)G_{t-1} + G_t$, we obtain that

$$\begin{aligned}(k-1)F_{t-1}(\lambda^{(t)}) &= (k-1-\lambda^{(t)})G_{t-1}(\lambda^{(t)}) + G_t(\lambda^{(t)}) \\ &= (k-1-\lambda^{(t)})G_{t-1}(\lambda^{(t)}) \\ &= (k-1-2\sqrt{k-1}\cos\theta)G_{t-1}(\lambda^{(t)}) \\ &\begin{cases} > (k-1-2\sqrt{k-1}\cos\frac{\pi}{t+1})G_{t-1}(\lambda^{(t)}) \geq 0 \text{ for } (k,t) \text{ in (2)}, \\ < (k-1-2\sqrt{k-1}\cos\frac{\pi}{t})G_{t-1}(\lambda^{(t)}) \leq 0 \text{ for } (k,t) \text{ in (3)}. \end{cases}\end{aligned}$$

This finishes the proof of the proposition. □

Remark 2.7. *The second largest eigenvalue* $\lambda_2(c)$ *of* $T(k,t,c)$ *is the largest zero of* $(c-1)G_{t-2}+G_{t-1}$. *Since the zeros of* G_{t-2} *and* G_{t-1} *interlace,* $\lambda_2(c)$ *is a monotonically decreasing function in* c. *In particular,* $\lim_{c\to\infty}\lambda_2(c) = \lambda^{(t-2)}$, $\lambda_2(1) = \lambda^{(t-1)}$, *and* $\lim_{c\to 0}\lambda_2(c) = \mu^{(t-1)}$.

Note that both F_i and G_i form a sequence of orthogonal polynomials with respect to some positive weight on the interval $[-2\sqrt{k-1}, 2\sqrt{k-1}]$. By Remark 2.7, the second eigenvalue $\lambda_2(t,c)$ of $T(k,t,c)$ may equal all possible values between $\lambda_2(2,1) = -1$ and $\lim_{t\to\infty}\lambda_2(t,c) = 2\sqrt{k-1}$. The following proposition shows that we may assume $c \geq 1$ in Theorem 2.3 to obtain better bounds.

Proposition 2.8. *For any* λ *such that* $\lambda^{(t-1)} < \lambda < \mu^{(t-1)}$, *there exist* $0 < c_1 < 1$, $c_2 > 0$ *such that both the second-largest eigenvalues of* $T(k,t,c_1)$ *and* $T(k,t+1,c_2)$ *are* λ. *Then we have* $M(k,t,c_1) > M(k,t+1,c_2)$.

Proof. Because $(c_1-1)G_{t-2}(\lambda)+G_{t-1}(\lambda) = 0$, we get $c_1 = -\frac{G_{t-1}(\lambda)-G_{t-2}(\lambda)}{G_{t-2}(\lambda)} = -F_{t-1}(\lambda)/G_{t-2}(\lambda)$. Similarly $c_2 = -F_t(\lambda)/G_{t-1}(\lambda)$. Note that $F_{t-1}(\lambda) = -c_1G_{t-2}(\lambda) < 0$ and $F_t(\lambda) =$

$-c_2 G_{t-1}(\lambda) < 0$. Therefore

$$\begin{aligned} M(k,t,c_1) - M(k,t+1,c_2) &= k(k-1)^{t-2}\big(\frac{1}{c_1} - 1 - \frac{1}{c_2}(k-1)\big) \\ &= k(k-1)^{t-2}\big(-\frac{G_{t-2}(\lambda)}{F_{t-1}(\lambda)} - 1 + (k-1)\frac{G_{t-1}(\lambda)}{F_t(\lambda)}\big) \\ &= k(k-1)^{t-2}\big(-\frac{G_{t-1}(\lambda)}{F_{t-1}(\lambda)} + (k-1)\frac{G_{t-1}(\lambda)}{F_t(\lambda)}\big) \\ &= \frac{k(k-1)^{t-2}G_{t-1}(\lambda)}{F_{t-1}(\lambda)F_t(\lambda)}\big(-F_t(\lambda) + (k-1)F_{t-1}(\lambda)\big) \\ &= \frac{k(k-1)^{t-2}(k-\lambda)G_{t-1}(\lambda)^2}{F_{t-1}(\lambda)F_t(\lambda)} > 0. \qquad \square \end{aligned}$$

Table 2 shows the known examples attaining the bound $M(k,t,c)$. The incidence graphs of $PG(2,q)$, $GQ(q,q)$, and $GH(q,q)$ are known to be unique for $q \le 8$, $q \le 4$, and $q \le 2$, respectively (see, for example, [7, Table 6.5 and the following comments]). The incidence graphs of $PG(2,2)$, $GQ(2,2)$, and $GH(2,2)$ are the Heawood graph, the Tutte-Coxeter graph (or Tutte 8-cage), and the Tutte 12-cage, respectively.

Table 2: Known graphs meeting the bound $M(k,t,c)$

(k,λ)	$v(k,\lambda)$	Graph meeting bound	Unique?	Ref.
$(2, 2\cos(2\pi/n))$	n	n-cycle C_n	yes	
$(k,-1)$	$k+1$	Complete graph K_{k+1}	yes	
$(k,0)$	$2k$	Complete bipartite graph $K_{k,k}$	yes	
$(q+1,\sqrt{q})$	$2(q^2+q+1)$	incidence graph of $PG(2,q)$	?	[7, 41]
$(q+1,\sqrt{2q})$	$2(q+1)(q^2+1)$	incidence graph of $GQ(q,q)$	?	[4, 7]
$(q+1,\sqrt{3q})$	$2(q+1)(q^4+q^2+1)$	incidence graph of $GH(q,q)$	?	[4, 7]
$(3,1)$	10	Petersen graph	yes	[21]
$(4,2)$	35	Odd graph O_4	yes	[34]
$(7,2)$	50	Hoffman–Singleton graph	yes	[21]
$(5,1)$	16	Clebsch graph	yes	[18, 40]
$(10,2)$	56	Gewirtz graph	yes	[8, 17]
$(16,2)$	77	M_{22} graph	yes	[6, 20]
$(22,2)$	100	Higman–Sims graph	yes	[17, 20]

$PG(2,q)$: projective plane, $GQ(q,q)$: generalized quadrangle,
$GH(q,q)$: generalized hexagon, q: prime power

The bounds in Table 2 solve several conjectures of Richey, Shutty, and Stover [47]. Richey, Shutty, and Stover prove that $v(3,2) \le 105$, but they note that the largest 3-regular graph with $\lambda_2 \le 2$ they are aware of is the Tutte-Coxeter graph on 30 vertices. They conjectured that $v(3,2) = 30$. They show that $v(4,2) \leq 77$ and conjecture that the largest 4-regular graph with $\lambda_2 \leq 2$ is the so-called rolling cube graph on 24 vertices (that is, the bipartite double of the cuboctahedral graph which is the line graph of the 3-cube). They also conjectured that $v(4,3) = 27$ and the largest 4-regular graph with $\lambda_2 \leq 3$ is the Doyle graph on 27 vertices (see [15, 23] for a description of this graph). In Table 2 we confirm that $v(3,2) = 30$ and the Tutte-Coxeter graph (the incidence graph of $GQ(2,2)$) is, in fact, the unique graph which meets this bound (see [7, Theorem 7.5.1] for uniqueness). However, Table 2 shows that $v(4,2) = 35$ (the Odd graph O_4) and that $v(4,3) = 728$ (the incidence graph of $GH(3,3)$), disproving the latter two conjectures.

Since the order of a graph is an integer, $v(k,\lambda)$ can be bounded above by $\lfloor M(k,t,c) \rfloor$. The graphs meeting the bound $M(k,t,c)$ can be maximal under the assumption of a larger second eigenvalue.

Proposition 2.9. *Let λ_1, λ_2 be the second largest eigenvalues of $T(k,t+1,c_1)$ and $T(k,t,c_2)$, respectively. Suppose there exists a graph which attains the bound $M(k,t,c)$ of Theorem 2.3. Then*

(1) *If $c = 1$, then $v(k,\lambda_1) = v(k,\lambda)$ for $c_1 > k(k-1)^{t-1}$. Moreover if $M(k,t,c)$ is even, and k is odd, then $v(k,\lambda_1) = v(k,\lambda)$ for $c_1 > k(k-1)^{t-1}/2$.*

(2) *If $c > 1$, $v(k,\lambda_2) = v(k,\lambda)$ for $c_2 > c - c^2/(k(k-1)^{t-2} + c)$. Moreover if $M(k,t,c)$ is even, and k is odd, then $v(k,\lambda_2) = v(k,\lambda)$ for $c_2 > c - 2c^2/(k(k-1)^{t-2} + 2c)$.*

Proof. We show only (1) because (2) can be proved similarly. For $c_1 > k(k-1)^{t-1}$, we have

$$M(k,t,c) = v(k,\lambda) \leq v(k,\lambda_1) \leq \lfloor M(k,t,c_1) \rfloor = M(k,t,c).$$

Therefore $v(k,\lambda) = v(k,\lambda_1)$. If k is odd, $v(k,\lambda_1)$ must be even. For $c_1 > k(k-1)^{t-1}/2$, we have

$$M(k,t,c) = v(k,\lambda) \leq v(k,\lambda_1) \leq \lfloor M(k,t,c_1) \rfloor = M(k,t,c) + 1.$$

Thus if $M(k,t,c)$ is even, then $v(k,\lambda) = v(k,\lambda_1)$. □

The larger second eigenvalues in Proposition 2.9 are calculated in Table 3. The graphs in Table 3 meet $v(k,\lambda)$ for any $\lambda_2 \leq \lambda < \lambda'$, where λ' is the largest zero of $f(x)$ in the table.

By Theorem 2.3, we can obtain an alternative proof of the theorem due to Alon and Boppana, and Serre (see [1, 12, 16, 24, 25, 28, 31, 35, 36, 42] for more details).

Corollary 2.10 (Alon–Boppana, Serre)**.** *For given k, $\lambda < 2\sqrt{k-1}$, there exist finitely many k-regular graphs whose second largest eigenvalue is at most λ.*

Proof. The second largest eigenvalue $\lambda_2(t)$ of $T(k,t,1)$ is equal to the largest zero of G_{t-1}. The zero is expressed by $\lambda_2(t) = 2\sqrt{k-1}\cos\theta$, where θ is less than $\pi/(t-1)$ [3, Section III.3]. This implies that there exists a sufficiently large t' such that $\lambda_2(t') > \lambda$. Therefore we

Table 3: Graphs meeting $v(k,\lambda)$ for $\lambda_2 \le \lambda < \lambda'$

Graph	t	c	$f(x)$	λ'
K_{k+1} (k: even)	2	1	$x^2-(k-k^2)\,x+k^2-2k$	
K_{k+1} (k: odd)	2	1	$2x^2-(k-k^2)\,x+k^2-3k$	
$K_{k,k}$ (k: even)	3	k	$x^2-(1-k)x-1$	
$K_{k,k}$ (k: odd)	3	k	$(k+1)x^2+(k^2-k)\,x-2k$	
$PG(2,q)$ ($q+1$: even)	4	$q+1$	$(q^2+1)\,x^3+(q^3+q^2)\,x^2$ $+(-q^3-2q-1)\,x-q^4-q^3$	
$PG(2,q)$ ($q+1$: odd)	4	$q+1$	$+(q^2+2)\,x^3+(q^3+q^2)\,x^2$ $+(-q^3-4q-2)\,x-q^4-q^3$	
$GQ(q,q)$ ($q+1$: even)	5	$q+1$	$(-q^2+q-1)\,x^4-q^3x^3$ $+(2q^3-2q^2+2q+1)\,x^2$ $+2q^4x-q$	
$GQ(q,q)$ ($q+1$: odd)	5	$q+1$	$(-q^3-2)\,x^4+(-q^4-q^3)\,x^3$ $+(2q^4+6q+2)\,x^2+(2q^5+2q^4)\,x$ $-2q^2-2q$	
$GH(q,q)$ ($q+1$: even)	7	$q+1$	$(-q^4+q^3-q^2+q-1)\,x^6$ $+(4q^5-4q^4+4q^3-4q^2+4q+1)\,x^4$ $+(-3q^6+3q^5-3q^4+3q^3-3q^2-3q)\,x^2$ $-q^5x^5+4q^6x^3-3q^7x+q^2$	
$GH(q,q)$ ($q+1$: odd)	7	$q+1$	$(-q^5-2)\,x^6+(-q^6-q^5)\,x^5$ $+(4q^6+10q+2)\,x^4+(4q^7+4q^6)\,x^3$ $+(-3q^7-12q^2-6q)\,x^2$ $+(-3q^8-3q^7)\,x+2q^3+2q^2$	
Petersen	3	1	$x^3+12x^2+7x-24$	1.11207
Odd graph O_4	4	2	$19x^3+36x^2-97x-108$	2.02156
Hoffman–Singleton	3	1	$x^3+126x^2+113x-756$	2.02845
Clebsch	3	2	$3x^2+5x-10$	1.1736
Gewirtz	3	2	$23x^2+45x-185$	2.02182
M_{22}	3	4	$61x^2+240x-736$	2.02472
Higman–Sims	3	6	$13x^2+77x-209$	2.0232

λ' is the largest zero of $f(x)$

have

$$v(k,\lambda) \le v(k,\lambda_2(t')) \le 1 + \sum_{i=0}^{t'-2} k(k-1)^i. \qquad \square$$

3 Second largest eigenvalue 1

In this section, we classify the graphs meeting $v(k,1)$. The complement of a regular graph with second eigenvalue at most 1 has smallest eigenvalue at least -2. The structure of such graph is obtained from a subset of a root system, and it is characterized as a line graph except for sporadic examples [7, Theorem 3.12.2]. The following theorem is immediate by [7, Theorem 3.12.2].

Theorem 3.1. *Let G be a connected regular graph with v vertices, valency k, and second largest eigenvalue at most 1. Then one of the following holds:*

(1) *G is the complement of the line graph of a regular or a bipartite semiregular connected graph.*

(2) *$v = 2(k-1) \le 28$, and G is a subgraph of the complement of $E_7(1)$, switching-equivalent to the line graph of a graph Δ on eight vertices, where all valencies of Δ have the same parity (graphs nos.* 1–163 *in Table* 9.1 *in* [10]*).*

(3) *$v = 3(k-1) \le 27$, and G is a subgraph of the complement of the Schläfli graph (graphs nos.* 164–184 *in Table* 9.1 *in* [10]*).*

(4) *$v = 4(k-1) \le 16$, and G is a subgraph of the complement of the Clebsch graph (graphs nos.* 185–187 *in Table* 9.1 *in* [10]*).*

The following theorem shows the classification of graphs meeting $v(k,1)$. Note that this result will show that $v(k,1) = 2k+2$ for k large whereas Theorem 2.3 would give a larger upper bound for $v(k,1)$.

Theorem 3.2. *Let G be a connected k-regular graph with second largest eigenvalue at most 1, with $v(k,1)$ vertices. Then the following hold:*

(1) *$v(2,1) = 6$, and G is the hexagon.*

(2) *$v(3,1) = 10$, and G is the Petersen graph.*

(3) *$v(4,1) = 12$, and G is the complement of the graph no.* 186 *in Table* 9.1 *in* [10].

(4) *$v(5,1) = 16$, and G is the Clebsch graph.*

(5) $v(6,1) = 15$, *and* G *is the complement of the line graph of the complete graph with* 6 *vertices, or the complement of one of the graphs nos.* 171–176 *in Table* 9.1 *in* [10].

(6) $v(7,1) = 18$, *and* G *is the complement of one of the graphs nos.* 177–180 *in Table* 9.1 *in* [10].

(7) $v(8,1) = 21$, *and* G *is the complement of one of the graphs nos.* 181, 182 *in Table* 9.1 *in* [10].

(8) $v(9,1) = 24$, *and* G *is the complement of the graph no.* 183 *in Table* 9.1 *in* [10].

(9) $v(10,1) = 27$, *and* G *is the complement of the Schläfli graph.*

(10) $v(k,1) = 2k+2$ *for* $k \geq 11$, *and* G *is the complement of the line graph of* $K_{2,k+1}$.

Proof. (1): A connected 2-regular graph is an n-cycle, whose eigenvalues are $2\cos(2\pi j/n)$ ($j = 0, 1, \ldots, n-1$). This implies (1).

(2), (4): By Theorem 2.3 for $T(k, 3, (k-1)/2)$, we have $v(k,1) \leq 3k+1$. The two graphs are unique graphs attaining this bound (see [18, Theorem 10.6.4] and [21, 37]).

(10): The complement of the line graph of $K_{2,k+1}$ is of degree k and has $2k+2$ vertices for any k. We will prove that there exists no graph with at least $2k+2$ vertices except for these graphs for $k \geq 11$. In the case of Theorem 3.1 (3) (4), we have no graph for $k \geq 11$. In the case of Theorem 3.1 (2), trivially $v = 2(k-1) < 2k+2$. We consider the case of Theorem 3.1 (1). Let G be the complement of the line graph of a t-regular graph with u vertices. Then G is of degree $k = (u/2-2)t+1$, and has $v = ut/2$ vertices. Therefore $v = ut/2 = u(k-1)/(u-4) \leq 2(k-1) < 2k+2$ because $u \geq 8$ for $k \geq 11$. Let G be the complement of the line graph of a bipartite semiregular connected graph (V_1, V_2, E). Let $|V_i| = u_i$ and the degree of $x \in V_i$ be t_i, where we suppose $t_1 \geq t_2$. Then G is of degree $k = (u_1-1)t_1 - t_2 + 1 \geq (u_1-2)t_1 + 1$, and has $v = u_1 t_1$ vertices. If $u_1 = 1$ holds, then G has no edge. For $u_1 > 3$, it is satisfied that

$$v \leq \left(1 + \frac{2}{u_1 - 2}\right)(k-1) \leq 2(k-1) < 2k+2 \tag{4}$$

for any k. For $u_1 = 3$, we have $t_2 \leq u_1 = 3$ and

$$v = 3t_1 = \frac{3}{2}(k + t_2 - 1) \leq \frac{3}{2}(k+2) < 2k+2 \tag{5}$$

for $k > 2$. For $u_1 = 2$, similarly $t_2 \leq u_1 = 2$ and

$$v = 2t_1 = 2(k + t_2 - 1) \leq 2k+2 \tag{6}$$

for any k, with equality only if $t_1 = k+1$, $t_2 = 2$, $u_1 = 2$ and $u_2 = k+1$. Thus (10) holds.

(3), (5)–(9): Every candidate of maximal graphs comes from Theorem 3.1 (3) or (4) except for the case of the complete graph in (5). We prove that there does not exist a larger graph which comes from Theorem 3.1 (1). By inequalities (4)–(6), the complement of the line graph of a bipartite semiregular graph is not maximal for $k > 2$. We consider the case of the complements of the line graphs of t-regular graphs with u vertices. Since $v = k - 1 + 2t$ is at least $12, 15, 18, 21, 24, 27$, we have $u-1 \geq t \geq 5, 5, 6, 7, 8, 9$ for $k = 4, 6, 7, 8, 9, 10$, respectively. Therefore $k = (u/2 - 2)t + 1 \geq (t-2)(t-1)/2 \geq 6, 6, 10, 15, 21, 28$ for $k = 4, 6, 7, 8, 9, 10$, respectively. The only parameter $(v, k, u, t) = (15, 6, 6, 5)$ satisfies the conditions and it corresponds to the case of the complete graph in (5). □

4 Other Values of $v(k, \lambda)$

When no graph meets the bound given by Theorem 2.3, other techniques may be necessary to find $v(k, \lambda)$. However, the bound is still useful in reducing the size of graphs which must be checked. In this section we describe several tools which we will use (Lemma 4.3 and Lemma 4.4), and then find $v(k, \lambda)$ in a few more cases (Proposition 4.5, Proposition 4.6, Proposition 4.7).

Let $n(k, g)$ denote the minimum possible number of vertices of a k-regular graph with girth g. A (k, g)-cage is a graph which attains this minimum. The following lower bound on $n(k, g)$ due to Tutte [46] will be useful.

Lemma 4.1. *Define $n_l(k, g)$ by*

$$n_l(k,g) = \begin{cases} \frac{k(k-1)^{(g-1)/2}-2}{k-2} & \text{if } g \text{ is odd,} \\ \frac{2(k-1)^{g/2}-2}{k-2} & \text{if } g \text{ is even.} \end{cases}$$

Then $n(k, g) \geq n_l(k, g)$.

The following lemma is easily verified.

Lemma 4.2. *' Each of the graphs in Figure 1 has spectral radius greater than 2.*

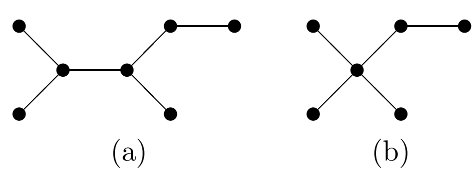


Figure 1: Graphs with spectral radius greater than 2.

For a graph G, a vertex $v \in V(G)$, and a subset $U \subset V(G)$, define the distance $\mathrm{dist}(v, U) = \min_{u \in U} \mathrm{dist}(u, v)$. For an induced subgraph H of G, let $\Gamma_i(H)$ and $\Gamma_{\geq i}(H)$

be the sets of vertices in G at distance exactly i and at least i from $V(H)$ in G, respectively. Let $\rho(G)$ and $d(G)$ denote the spectral radius and average degree of G, respectively. Note that $d(G) \le \rho(G)$.

Lemma 4.3. *Suppose G is a connected, k-regular graph with second largest eigenvalue $\lambda_2(G) \le \lambda < k$, and H is an induced subgraph of G with $d(H) \ge \lambda$. Then for the subgraph K induced by $\Gamma_{\ge 2}(H)$ we have $d(K) \le \lambda$, with equality only if $d(H) = \lambda_2(G) = \lambda$.*

Proof. Consider the quotient matrix Q of the partition $\{V(H), \Gamma_1(H), \Gamma_{\ge 2}(H)\}$ of $V(G)$. We have

$$Q = \begin{pmatrix} \alpha & k-\alpha & 0 \\ \gamma & k-(\gamma+\epsilon) & \epsilon \\ 0 & k-\beta & \beta \end{pmatrix},$$

where $\alpha = d(H)$, $\beta = d(K)$, and γ and ϵ are the average numbers of neighbors in H and K, respectively, of the vertices in $\Gamma_1(H)$. The eigenvalues of Q interlace those of G (see [9, Corollary 2.5.4]), so we must have $\lambda_2(Q) \le \lambda_2(G) \le \lambda$. It is straightforward to verify that $\lambda_1(Q) = k$ and

$$\lambda_2(Q) = \frac{1}{2}\left(\alpha + \beta - (\gamma+\epsilon) + \sqrt{\Delta}\right), \tag{7}$$

where $\Delta = (\alpha+\beta-(\gamma+\epsilon))^2 - 4(\alpha\beta - \beta\gamma - \alpha\epsilon)$. By hypothesis we have $\alpha \ge \lambda$. If also $\beta \ge \lambda$, then we find that $\alpha = \beta = \lambda_2(Q) = \lambda$, as we will prove below.

Indeed, if both $\alpha > \lambda$ and $\beta > \lambda$, then by Cauchy interlacing [9, Proposition 3.2.1] $\lambda_2(G) \ge \lambda_2(H+K) > \lambda$, where $H+K$ is the disjoint union of H and K, a contradiction. Suppose $\alpha \ge \lambda$ and $\beta \ge \lambda$. If $\alpha = \beta = \lambda$, then (7) becomes $\lambda_2(Q) = \lambda$. Otherwise we must have $\alpha > \beta = \lambda$ or $\beta > \alpha = \lambda$. If $\sqrt{\Delta} \ge \gamma + \epsilon$, then clearly $\lambda_2(Q) > \lambda$, a contradiction. If $\sqrt{\Delta} < \gamma+\epsilon$, then $\Delta < (\gamma+\epsilon)^2$, which implies $(\alpha-\beta)(\alpha-\beta+2(\epsilon-\gamma)) < 0$. Thus we have either $\alpha > \beta$ and $\epsilon < \gamma - \frac{1}{2}(\alpha-\beta)$, or $\beta > \alpha$ and $\gamma < \epsilon - \frac{1}{2}(\beta-\alpha)$. Suppose the former is true. Then $\beta = \lambda$ and we can write $\alpha = \beta + s = \lambda + s$ and $\epsilon = \gamma - \frac{s}{2} - t$ for some $s, t > 0$. Then (7) becomes

$$\lambda_2(Q) = \frac{1}{4}\left(4\lambda - 4\gamma + 3s + 2t + \sqrt{\Delta'}\right),$$

where $\Delta' = 16\gamma^2 + (s-2t)^2 - 8\gamma(s+2t)$. If $\sqrt{\Delta'} > 4\gamma - 3s - 2t$, then clearly $\lambda_2(Q) > \lambda$, a contradiction. If $\sqrt{\Delta'} \le 4\gamma - 3s - 2t$, then $\Delta' \le (4\gamma-3s-2t)^2$, which implies $\gamma \le \frac{s}{2} + t$. However, this implies $\epsilon = \gamma - \frac{s}{2} - t \le 0$, a contradiction. If $\beta > \alpha$ and $\gamma < \epsilon - \frac{1}{2}(\beta - \alpha)$, the same argument holds (simply swap the roles of α and β and of γ and ϵ in the above argument). Thus we cannot have $\alpha \ge \lambda$ and $\beta \ge \lambda$ unless $\alpha = \beta = \lambda$, so we must have $\beta < \lambda$ or $\alpha = \beta = \lambda_2(Q) = \lambda$. □

Lemma 4.4. *Suppose G is a connected, k-regular graph with second largest eigenvalue $\lambda_2(G) \leq \lambda < k$. If G contains an induced subgraph H on s vertices with t edges and either $d(H) \geq \lambda$ or $\rho(H) > \lambda$, then*

$$|V(G)| \leq s + \frac{2k-\lambda-1}{k-\lambda}(ks-2t). \tag{8}$$

Proof. Since G is k-regular, there are $ks-2t$ edges from H to $\Gamma_1(H)$, which implies $|\Gamma_1(H)| \leq ks-2t$. We will show that $|\Gamma_{\geq 2}(H)| \leq \frac{k-1}{k-\lambda} |\Gamma_1(H)|$, which completes the proof that (8) holds.

First, note that each vertex in $\Gamma_1(H)$ has a neighbor in H, so each such vertex has at most $k-1$ neighbors in $\Gamma_{\geq 2}(H)$. Then there are at most $(k-1)|\Gamma_1(H)|$ edges from $\Gamma_1(H)$ to $\Gamma_{\geq 2}(H)$. If $d(H) \geq \lambda$ then by Lemma 4.3 we have $d(K) \leq \lambda$, where K is the subgraph induced by $\Gamma_{\geq 2}(H)$. If not, then $\rho(H) > \lambda$, so $\rho(K) \leq \lambda$ (and so also $d(K) \leq \lambda$) by eigenvalue interlacing. Since G is k-regular, this implies that the average number of neighbors in $\Gamma_1(H)$ of the vertices in $\Gamma_{\geq 2}(H)$ is at least $k-\lambda$, so there are at least $(k-\lambda)|\Gamma_{\geq 2}(H)|$ edges from $\Gamma_{\geq 2}(H)$ to $\Gamma_1(H)$. This completes the proof. □

Proposition 4.5. *If G is a connected, 3-regular graph with $\lambda_2(G) > 1$, then $\lambda_2(G) \geq \sqrt{2}$, with equality if and only if G is the Heawood graph.*

Proof. We have already seen in Table 2 that $v(3, \sqrt{2}) = 14$ and the Heawood graph (the incidence graph of $PG(2,2)$) is the unique graph meeting this bound. Thus we only need to show that no 3-regular graph has second eigenvalue between 1 and $\sqrt{2}$. Suppose G is a 3-regular graph with $1 < \lambda_2(G) < \sqrt{2}$. We will show that this yields a contradiction. We have immediately that $|V(G)| < 14$. Since G is 3-regular, this implies $|V(G)| \leq 12$.

We note that the average degree of any cycle is $2 > \sqrt{2} > \lambda_2(G)$. If G has girth 3, then Lemma 4.4 implies $|V(G)| \leq \frac{6}{7}(\sqrt{2}+10) \approx 9.78$. Since G is 3-regular, this implies $|V(G)| \leq 8$. Lemma 4.1 implies that a graph with girth more than 5 has at least 14 vertices, so G has girth at most 5.

We partition the vertices of G by $P_1 = \{V(H), \Gamma_1(H), \Gamma_{\geq 2}(H)\}$, where H is a subgraph of G isomorphic to C_m, where $m \in \{3,4,5\}$ is the girth of G. This partition has quotient matrix Q given by

$$Q = \begin{pmatrix} 2 & 1 & 0 \\ \gamma & 3-(\alpha+\gamma) & \alpha \\ 0 & \beta & 3-\beta \end{pmatrix},$$

where $\gamma |\Gamma_1(H)| = m$ (by counting edges from H to $\Gamma_1(H)$) and $\alpha |\Gamma_1(H)| = \beta |\Gamma_{\geq 2}(H)|$ (by counting edges from $\Gamma_1(H)$ to $\Gamma_{\geq 2}(H)$).

We first suppose G has girth 3. Then $4 \leq |V(G)| \leq 8$. If $|V(G)| = 4$, then $G \cong K_4$, and we have $\lambda_2(G) = -1$. If $|V(G)| = 6$, it is straightforward to show that $G \cong C_3 \square K_2$, where $\square$ denotes the graph Cartesian product, and we have $\lambda_2(G) = 1$. Either case is a contradiction.

If $|V(G)| = 8$ then $\Gamma_1(H)$ has 2 or 3 vertices. If $|\Gamma_1(H)| = 2$, then we have $|\Gamma_{\geq 2}(H)| = 3$, $\gamma = 3/2$, and depending on whether there is an edge in $\Gamma_1(H)$ or not we have $\alpha = 1/2$ or $3/2$, $\beta = 1/3$ or 1, and $\lambda_2(Q) = \frac{1}{3}(\sqrt{13} + 4) \approx 2.54$ or 2, respectively. Either case is a contradiction. If $|\Gamma_1(H)| = 3$, then $|\Gamma_{\geq 2}(H)| = 2$, $\gamma = 1$, and depending on whether there is an edge in $\Gamma_{\geq 2}(H)$ or not we have $\beta = 2$ or 3, $\alpha = 4/3$ or 2, and $\lambda_2(Q) = 5/3$ or $\frac{1}{2}(\sqrt{17} - 1) \approx 1.56$, respectively. Either case is a contradiction. Thus G cannot have girth 3.

Suppose G has girth 4. Then we have $6 \leq |V(G)| \leq 12$. If $|V(G)| = 6$, then $G \cong K_{3,3}$ and we have $\lambda_2(G) = 0$. If $|V(G)| = 8$, then it is straightforward to verify that G must either be the 3-cube Q_3 or the graph in Figure 2. In either case we have $\lambda_2(G) = 1$, a contradiction.

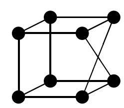


Figure 2: A 3-regular graph on 8 vertices with girth 4.

If $|V(G)| = 10$, then $\Gamma_1(H)$ has 2, 3, or 4 vertices. If $|\Gamma_1(H)| = 2$, then $|\Gamma_{\geq 2}(H)| = 4$, $\gamma = 2$, $\alpha = 1$, $\beta = 1/2$, and $\lambda_2(Q) = \frac{1}{4}(\sqrt{41} + 3) \approx 2.35$, a contradiction. If $|\Gamma_1(H)| = 3$, then $|\Gamma_{\geq 2}(H)| = 3$, $\gamma = 4/3$, and $\alpha = \beta$. Then $\alpha \leq 5/3$ (since $3 - (\alpha + \gamma) \geq 0$) implies $\beta \leq 5/3$, which implies $\Gamma_{\geq 2}(H)$ has at least 2 edges. Since G has girth 4, $\Gamma_{\geq 2}(H)$ cannot have 3 edges, so $\Gamma_{\geq 2}(H)$ has exactly 2 edges, $\alpha = \beta = 5/3$, and $\lambda_2(Q) = \frac{1}{2}(\sqrt{241} + 7) \approx 1.88$, a contradiction. If $|\Gamma_1(H)| = 4$, then $|\Gamma_{\geq 2}(H)| = 2$, $\gamma = 2$, and depending on whether there is an edge in $\Gamma_{\geq 2}(H)$ or not we have $\beta = 2$ or 3, $\alpha = 1$ or $3/2$, and $\lambda_2(Q) = \frac{1}{2}(\sqrt{5} + 1) \approx 1.62$ or $3/2$, respectively. Either case is a contradiction. If $|V(G)| = 12$, then $\Gamma_1(H)$ must be a coclique on 4 vertices (otherwise there are at most 6 edges from $\Gamma_1(H)$ to $\Gamma_{\geq 2}(H)$, so Lemma 4.3 implies $|\Gamma_{\geq 2}(H)| < 6/(3 - \sqrt{2}) \approx 3.78$, which implies $|V(G)| < 11.78$, a contradiction). Then we have $|\Gamma_1(H)| = |\Gamma_{\geq 2}(H)| = 4$, $\gamma = 1$, $\alpha = \beta = 2$, and $\lambda_2(Q) = \sqrt{3}$. This is a contradiction, so G cannot have girth 4.

Suppose G has girth 5. Then $10 \leq |V(G)| \leq 12$. The Petersen graph with 10 vertices and $\lambda_2 = 1$ is the unique $(3, 5)$-cage (see [21]), so G must have 12 vertices. Note we must have $|\Gamma_1(H)| = 5$ and $\gamma = 1$, since vertices in H cannot have common neighbors outside of H. Since $|V(G)| = 12$, we have $|\Gamma_{\geq 2}(H)| = 2$, and depending on whether there is an edge in $\Gamma_{\geq 2}(H)$ or not we have $\beta = 2$ or 3, $\alpha = 4/5$ or $6/5$, and $\lambda_2(Q) = \frac{1}{5}(2\sqrt{6} + 3) \approx 1.58$ or $\frac{1}{10}(\sqrt{241} - 1) \approx 1.45$, respectively. Either case is a contradiction.

Thus G cannot exist as described, which completes the proof. $\square$

Proposition 4.6. *If G is a connected, 4-regular graph with $\lambda_2(G) > 1$, then $\lambda_2(G) \geq \sqrt{5} - 1$, with equality if and only if G is either the graph in Figure 3 or the circulant graph $Ci_{10}(1, 4)$ (the Cayley graph of $(\mathbb{Z}_{10}, +)$ with generating set $\{\pm 1, \pm 4\}$).*

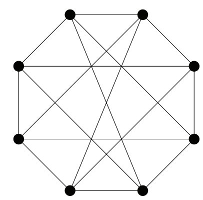


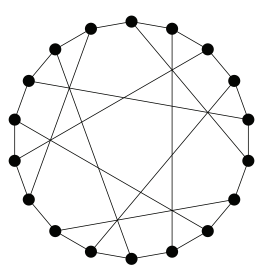
Figure 3: The 4-regular graph G on 8 vertices with $\lambda_2(G) = \sqrt{5} - 1$.

Proof. It is straightforward to verify that the second eigenvalue of $T(4, 3, (4-(\sqrt{5}-1)^2)/\sqrt{5}) = \sqrt{5}-1$ and $M(4, 3, (4-(\sqrt{5}-1)^2)/\sqrt{5}) = 5+12\sqrt{5}/(4-(\sqrt{5}-1)^2) \approx 15.85$, so by Theorem 2.3 we have $v(4, \sqrt{5} - 1) \leq 15$. We checked by computer all 4-regular graphs on at most 15 vertices and found that, in each case where $\lambda_2(G) > 1$, we have $\lambda_2(G) \geq \sqrt{5} - 1$, with equality if and only if G is either the graph in Figure 3 or the circulant graph $\mathrm{Ci}_{10}(1, 4)$. $\square$

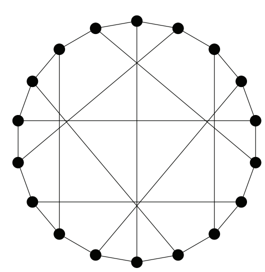
The previous result and Theorem 3.2 part (iii) imply that $v(4, \sqrt{5} - 1) = 12$. It would be interesting to find a proof of Proposition 4.6 which does not require a computer search. For the proof above the computer must check 906,331 graphs.

Richey, Shutty, and Stover [47] conjectured that $v(3, 1.9) = 18$. We confirm this conjecture, and show that there are exactly two graphs meeting this bound.

Proposition 4.7. *If G is a connected, 3-regular graph with second largest eigenvalue $\lambda_2(G) \leq 1.9$, then $|V(G)| \leq 18$, with equality if and only if G is the Pappus graph (see Figure 4(a)) or the graph in Figure 4(b).*



(a) The Pappus graph with second eigenvalue $\sqrt{3}$.



(b) A graph with $\lambda_2 = \gamma \approx 1.8662$, the largest root of $f(x) = x^3 + 2x^2 - 4x - 6$.

Figure 4: The 3-regular graphs on 18 vertices with $\lambda_2 < 1.9$.

Proof. It is straightforward to verify that the second eigenvalue of $T(3, 4, 2641/3510) = 19/10 = 1.9$ and $M(3, 4, 2641/3510) = 68530/2641 \approx 25.95$, so by Theorem 2.3 we have $v(3, 1.9) \leq 25$. Since G is 3-regular, this implies $v(3, 1.9) \leq 24$. We note again that any cycle has spectral radius 2. Then, by Lemma 4.4, if G has girth 3, 4, 5, or 6, then G has at most

11.45, 15.27, 19.09, or 22.91 vertices, respectively. Since G is 3-regular, this implies G has at most 10, 14, 18, or 22 vertices, respectively. A 3-regular graph of girth 8 has at least 30 vertices by Lemma 4.1 (or note that the Tutte-Coxeter graph is the unique (3,8)-cage, see [45, 46]). Thus, we have shown that a 3-regular graph G with $\lambda_2(G) \leq 1.9$ and more than 18 vertices must have girth 6 or 7.

If G has girth 7, we note that the McGee graph on 24 vertices is the unique (3,7)-cage (see [7, p.209] or [30, 46]), so G must be the McGee graph. Since the McGee graph has second eigenvalue 2, we have proved that G does not have girth 7.

Now, if G has more than 18 vertices then G must have girth 6 and at most 22 vertices. Among 3-regular graphs, we checked by computer the 32 graphs with girth 6 on 20 vertices and the 385 graphs with girth 6 on 22 vertices and found that each has second eigenvalue more than 1.9. Thus G has at most 18 vertices. If G has 18 vertices, then G must have girth 5 or 6. Among 3-regular graphs, we checked by computer the 450 graphs with girth 5 on 18 vertices and found that each has second eigenvalue more than 1.9. We checked the 5 graphs with girth 6 on 18 vertices and found that all but two of them have second eigenvalue more than 1.9. The exceptions were the Pappus graph with second eigenvalue $\sqrt{3}$ and the graph in Figure 4(b) with second eigenvalue γ, where $\gamma \approx 1.8662$ is the largest root of $f(x) = x^3 + 2x^2 - 4x - 6$. $\square$

Note that this implies $v(3, \sqrt{3}) = 18$ and $v(3, \gamma \approx 1.8662) = 18$ (and, of course, $v(3, 1.9) = 18$). It would be nice to find a proof of Proposition 4.7 that does not require a computer search.

5 Final Remarks

We conclude the paper with some questions and problems for future research.

Problem 5.1. *Determine $v(k, \sqrt{k})$ for $k \geq 3$.*

We have $\lambda_2(T(k, 4, k - \sqrt{k})) = \sqrt{k}$ and $M(k, 4, k - \sqrt{k}) = 2k^2 + k^{3/2} - k - \sqrt{k} + 1$, which yields

$$v(k, \sqrt{k}) \leq 2k^2 + k^{3/2} - k - \sqrt{k} + 1.$$

The Odd graph O_4 meets this bound (see Table 2). We do not know what other graphs, if any, meet this bound. Odd graphs, in general, do not have $T(k, t, c)$ as a quotient matrix.

Problem 5.2. *Determine $v(k, \sqrt{2})$ for $k \geq 3$.*

Recall that for $k = 3$ we have $v(3, \sqrt{2}) = 14$ and the Heawood is the unique graph meeting this bound. For $k > 3$ we note that Lemma 4.4 with $H = K_3$ implies that a graph G with $\lambda_2(G) \le \sqrt{2}$ and girth 3 satisfies $|V(G)| \le 3(k-1)\left(1+\frac{k-2}{k-\sqrt{2}}\right)$, and Lemma 4.4 with $H = K_{1,3}$ implies that such a graph with girth more than 3 satisfies $|V(G)| \le 4 + 2(2k-3)\left(1+\frac{k-1}{k-\sqrt{2}}\right)$ (note that in both cases we have $\rho(H) > \lambda_2(G)$). Combining this with Lemma 4.1 allows one to restrict the search to graphs with certain girth. For $k \ge 7$, $n_l(k,g)$ is larger than these bounds unless the girth is at most 4, and for $k = 4$, 5, or 6 $n_l(k,g)$ is larger than these bounds unless the girth is at most 5. Thus the graphs sought in Problem 5.2 must have girth at most 5 for $k = 4, 5, 6$ and girth at most 4 for $k \ge 7$.

Problem 5.3. *Among regular graphs, what is the smallest second eigenvalue larger than 1?*

Yu [48] found a 3-regular graph G on 16 vertices (see Figure 5) with smallest eigenvalue

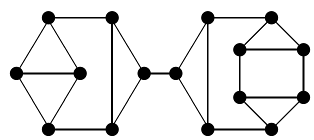


Figure 5: The unique 3-regular graph with largest least eigenvalue less than -2.

$\lambda_{\min} = \gamma \approx -2.0391$, where γ is the smallest root of $f(x) = x^6 - 3x^5 - 7x^4 + 21x^3 + 13x^2 - 35x - 4$, and moreover proved that there is no connected, 3-regular graph with smallest eigenvalue in the interval $(\gamma, -2)$ (that is, among all connected, 3-regular graphs G has the largest least eigenvalue less than -2). Since the second eigenvalue of the complement of a regular graph is $\lambda_2 = -1 - \lambda_{\min}$, the complement $\overline{G}$ of G, a 12-regular graph on 16 vertices, has second eigenvalue $\lambda_2(\overline{G}) = -1 - \gamma \approx 1.0391$. We do not know if $\overline{G}$ has smallest second eigenvalue larger than 1 among regular graphs, but it is not unique. Indeed, the complement of the disjoint union $G + kK_4$ of G and k copies of K_4 is a connected, $(12+4k)$-regular graph on $16 + 4k$ vertices with second eigenvalue $\lambda_2(\overline{G + kK_4}) = -1 - \gamma$, so we have found an infinite family of regular graphs with second eigenvalue $-1 - \gamma$.

Problem 5.4. *For any integer $k \ge 2$, let $\lambda(k) := (-1 + \sqrt{4k-3})/2$. Then we find that $v(k, \lambda(k)) \le k^2 + 1$ with equality if and only if the associated graph is a Moore graph of diameter* 2. *Moore graphs of diameter* 2 *only exists for $k = 2, 3, 7$, and possibly* 57. *If k is not* 2, 3, 7, 57, *then $v(k, \lambda(k)) \le k^2$. Determine the exact value of $v(k, \lambda(k))$ in these cases.*

An (n, k, λ)-graph is a k-regular graph with n vertices such that $|\lambda_i| \le \lambda$ for $i \ge 2$. This notion was introduced by Alon (see [1, 25]) motivated by the study of pseudo-random graphs and expanders among other things. The following question seems natural and interesting.

Problem 5.5. *Given $k \ge 3$ and $1 < \lambda < 2\sqrt{k-1}$, what is the maximum order n of an (n, k, λ)-graph ?*

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