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SPARSE SPANNING k-CONNECTED SUBGRAPHS IN TOURNAMENTS

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Abstract. In 2009, Bang-Jensen asked whether there exists a function $g(k)$ such that every strongly k-connected n-vertex tournament contains a strongly k-connected spanning subgraph with at most $kn + g(k)$ arcs. In this paper, we answer the question by showing that every strongly k-connected n-vertex tournament contains a strongly k-connected spanning subgraph with at most $kn + 750k^2 \log_2(k + 1)$ arcs, and there is a polynomial-time algorithm to find the spanning subgraph.

1. Introduction

Search of certain subgraphs which inherit the properties of the original graph has a long history. For example, Hajnal [7] and Thomassen [15] proved that a graph G with high enough connectivity has two vertex disjoint k-connected subgraphs which together cover all vertices. Thomassen [14] also made a conjecture that a graph G with high enough connectivity has a k-connected spanning bipartite subgraph.

For directed graphs, such problems become more difficult. One of most important problems in this direction is the following *$MSSS_k$ problem*, where MSSS_k stands for Minimum Spanning Strongly k-connected Subgraph: for a given strongly k-connected digraph D, find a spanning strongly k-connected subgraph of D with as few arcs as possible. For $k = 1$, we call it *MSSS problem* by omitting k. It is known that the Hamilton cycle problem can be solved if one can solve the MSSS problem. Thus MSSS problem is a generalization of Hamilton cycle problem, so it has been studied extensively (see e.g [2, 3] for a survey). Since the Hamilton cycle problem is NP-hard for general directed graphs, MSSS problem is also NP-hard for general directed graphs. Thus it makes sense to consider subclasses of directed graphs for this problem, and this problem is solvable in polynomial-time for several classes of graphs (see [4, 5]). In particular, MSSS problem for tournaments is trivial as any strongly-connected tournament contains a Hamilton cycle (see [3, Corollary 1.5.2]). However, it is not known whether MSSS_k problem is solvable in polynomial-time for tournaments for $k \geq 2$.

Naturally, one can ask about the size (the number of arcs) of minimum spanning strongly k-connected subgraphs for strongly k-connected tournaments. If we consider the same question for arc-connectivity, the following theorem was proved by Bang-Jensen, Huang and Yeo in 2004.

Theorem 1.1. [6] *For $k \geq 1$, every strongly k-arc-connected n-vertex tournament contains a strongly k-arc-connected spanning subgraph D with $|E(D)| \leq nk + 136k^2$.*

This gives us an upper bound of the number of arcs in minimum spanning strongly k-arc-connected subgraphs for strongly k-arc-connected tournaments. However, for vertex-connectivity, no good upper bound was known. Indeed, Bang-Jensen [2] asked the following question in 2009.

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Question 1.2. [2] *For $k \geq 1$, does there exist a function $g = g(k)$ such that every strongly k-connected n-vertex tournament has a strongly k-connected spanning subgraph with at most $kn + g(k)$ arcs?*

In this paper, we answer this question by proving the following theorem.

Theorem 1.3. *For $k \geq 1$, every strongly k-connected tournament T with n vertices has a strongly k-connected spanning subgraph D with at most $kn + 750k^2 \log_2(k+1)$ arcs.*

Thus $g(k) = 750k^2 \log_2(k+1)$ is sufficient for answering Question 1.2, and this is asymptotically best possible up to logarithmic factor. Indeed, Bang-Jensen, Huang and Yeo [6] introduced an n-vertex tournament $\mathcal{T}_{n,k}$ for $n \geq k$ such that every strongly k-arc-connected spanning subgraph of $\mathcal{T}_{n,k}$ contains at least $nk + \frac{k(k-1)}{2}$ arcs. Since every strongly k-connected digraphs are also strongly k-arc-connected, this example shows that Theorem 1.3 is asymptotically best possible up to logarithmic factor. We conjecture that we can reduce $g(k)$ to $O(k^2)$.

Conjecture 1.4. *There is $C > 0$ such that for any positive integer k, every strongly k-connected n-vertex tournament T contains a strongly k-connected spanning subgraph D with at most $kn + Ck^2$ arcs.*

One of two main ingredients for the proof of Theorem 1.3 is Lemma 3.4 which is, roughly speaking, a tool guaranteeing a sparse linkage structure from/to certain vertex-sets for any tournament. The other main ingredient is "robust linkage structures" introduced by Kühn, Lapinskas, Osthus and Patel in [9] to prove a conjecture of Thomassen on edge-disjoint Hamilton cycles in highly connected tournaments. Robust linkage structure is a very powerful tool for studying highly connected tournament. Further results were obtained by this method [8, 10, 12, 13]. The novelty of the proof of Theorem 1.3 is that it produces a highly connected 'sparse' subgraph in the tournament, whereas previous applications of the method only produced highly connected relatively dense subgraphs.

2. Basic terminology and tools

For any positive integer $N \geq 1$, $[N]$ denotes the set $\{1, \dots, N\}$. Let $\log := \log_2$, where we omit the base 2. A *graph* or *simple graph* is an undirected graph without multiple edges between two vertices and loops. A *directed graph* or *digraph* $D = (V, E)$ is a pair of a vertex set $V(D) = V$ and an arc set $E(D) = E$, where E is a collection of ordered pairs in $V \times V$. We let $\overrightarrow{uv}$ denote $(u, v) \in V \times V$ an *arc from u to v*. An *oriented graph* is a digraph obtained by orienting each edge $e \in E(G)$ for a simple graph G. An n-vertex *tournament* is an oriented graph obtained by orienting each edge $e \in E(K_n)$, where K_n is a simple complete graph of order n. For a set S of vertices, $D - S$ denotes the induced digraph $D[V(D) \setminus S]$. For a set E' of arcs, $D - E'$ denotes the digraph $(V(D), E(D) \setminus E')$. We say a digraph $D' = (V', E')$ is a *subgraph of* $D = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. We denote $D' \subseteq D$ if D' is a subgraph of D.

For a collection of arcs E, we let $V(E) := \{u : \exists\ v \text{ such that } \overrightarrow{uv} \in E \text{ or } \overrightarrow{vu} \in E\}$. A *path* always denotes a directed path. A path $P = (v_1, v_2, \dots, v_n)$ is called a *path from v_1 to v_n*, and we say v_i is the *ith vertex* of P. Sometimes, we consider the path P as a collection of arcs and $V(P)$ denotes $\{v_1, \dots, v_n\}$. A directed graph $D = (V, E)$ is *strongly connected* if for any $u, v \in V$, there is a path from u to v. We say that digraph D is *strongly k-connected*, if $|V| \geq k + 1$ and for $S \subseteq V$ with $|S| \leq k - 1$, the digraph $D - S$ remains strongly connected. Similarly, D is *strongly k-arc-connected*, if for $W \subseteq E$ with $|W| \leq k - 1$, the digraph $D - W$ remains strongly connected. It is easy to see that every strongly k-connected digraph is strongly k-arc-connected. For a directed graph $D = (V, E)$ and $v \in V$, let

$$N_D^+(v) := \{u \in V(D) : \overrightarrow{vu} \in E(D)\} \text{ and } N_D^-(v) := \{u \in V(D) : \overrightarrow{uv} \in E(D)\}.$$

We call u an *out-neighbor* of v if $\overrightarrow{vu} \in E(D)$ and u an *in-neighbor* of v if $\overrightarrow{uv} \in E(D)$. We define

$$d_D^+(v) := |N_D^+(v)|, \ \ d_D^-(v) := |N_D^-(v)|, \ \ d_D(v) := d_D^+(v) + d_D^-(v),$$
$$\delta^+(D) = \min_{v \in V(D)} d_D^+(v), \ \ \delta^-(D) = \min_{v \in V(D)} d_D^-(v) \ \text{ and } \ \delta(D) = \min_{v \in V(D)} d_D(v).$$

For a digraph D, $B \subseteq V(D)$ *out/in-dominates* $C \subseteq V(D)$ if every vertex in C is an out/in-neighbor of a vertex in B, respectively. A tournament T is *transitive* if $V(T)$ can be ordered into $v_1, \dots, v_n$ such that $\overrightarrow{v_i v_j} \in E(T)$ if and only if $i < j$. We say that T is a transitive tournament *with respect to* the ordering $\sigma = (v_1, \dots, v_n)$ with the *source vertex* v_1 and the *sink vertex* v_n.

We say a directed path $P = (v_1, \dots, v_p)$ in T is *backwards-transitive* if $\overrightarrow{v_i v_j} \in E(T)$ whenever $i \geq j+2$. For a vertex v and a vertex-set $U = \{u_1, \dots, u_k\}$, a collection $\{P_1, \dots, P_k\}$ of k paths is a *k-fan from v to U* if P_i is a path from v to $u_i \in U$, $U \cap V(P_i) = \{u_i\}$ for each $i \in [k]$, and $V(P_i) \cap V(P_j) = \{v\}$ for distinct $i, j \in [k]$. Similarly, a collection $\{P_1, \dots, P_k\}$ of k paths is a *k-fan from U to v* if P_i is a path from $u_i \in U$ to v, $U \cap V(P_i) = \{u_i\}$ for each $i \in [k]$, and $V(P_i) \cap V(P_j) = \{v\}$ for distinct $i, j \in [k]$.

We will use the following well-known fact deduced from Menger's theorem later. We omit the proof.

Fact 1. *For any strongly k-connected digraph D, a vertex $v \in V(D)$ and $U \subseteq V(D)$ with $|U| \geq k$, there exists a k-fan from v to U and a k-fan from U to v.*

Note that if $v \in U$, then one of the paths in the k-fan is a trivial path from v to v.

Lemma 2.1. *For positive integers n, k with $n \geq 2$ and $k \leq n$, an n-vertex tournament T has at least k vertices of out-degree at least $(n-k)/2$ and k vertices of in-degree at least $(n-k)/2$. Moreover, T has a vertex v with $n/4 \leq d^+_T(v) \leq 3n/4$ and a vertex u with $n/4 \leq d^-_T(u) \leq 3n/4$.*

Proof. Note that any n-vertex tournament contains a vertex with out-degree at least $(n-1)/2$. Let $v_1, \dots, v_n$ be an ordering of $V(T)$ such that $d^+_T(v_1) \geq \dots \geq d^+_T(v_n)$. Then $T[\{v_k, \dots, v_n\}]$ contains a vertex with out-degree at least $(n-k)/2$, thus $d^+_T(v_k) \geq (n-k)/2$. Hence T contains k vertices of out-degree at least $(n-k)/2$. It follows that T also contains k vertices of in-degree at least $(n-k)/2$ by reversing every arc of T and applying the same argument.

This also gives us at least $\lfloor n/2 \rfloor$ vertices with out-degree at least $\frac{n - \lfloor n/2 \rfloor}{2} \geq n/4$, and at least $\lceil n/2 \rceil + 1$ vertices with in-degree at least $\frac{n - \lceil n/2 \rceil - 1}{2} \geq \frac{n}{4} - 1$. Hence there exists a vertex v with $n/4 \leq d^+_T(v) \leq (n-1) - (n/4 - 1) = 3n/4$. By reversing every arc of T and applying the same argument, it follows that there is a vertex u with $n/4 \leq d^-_T(u) \leq 3n/4$. □

We introduce the following useful lemmas regarding in-dominating sets and out-dominating sets of tournaments.

Lemma 2.2. *Let v be a vertex in an n-vertex tournament T with $d^+_T(v) = d$. Then there exist $A \subseteq V(T)$ and a vertex $a \in A$ such that the following properties hold:*

(a1) *We have $\frac{1}{2}\log(d+1) + 1 \leq s \leq \frac{5}{2}\log(d+1) + 2$ where $s = |A|$.*

(a2) *$T[A]$ is a transitive tournament with respect to the ordering $(v_1, \dots, v_s)$ with source v and sink a.*

(a3) *A in-dominates $V(T) \setminus A$.*

(a4) *For $1 \leq i \leq s/5 - 13$, we have*

$$|N^+_T(v_i) \setminus A|, |N^-_T(v_i) \setminus A| \geq 8d^{1/7} - 1.$$

(a5) *For any positive integers i, k with $1 \leq i \leq s - 5\log(k) - 30$, we have*

$$|N^+_T(v_i) \setminus A|, |N^-_T(v_i) \setminus A| \geq 1000k^2.$$

Proof. Let $L_0 = V(T)$. If $d = 0$, then let $L_1 = \emptyset$ and $A := \{v_1\}$. Then it is obvious that A with an ordering (v_1) satisfies all (a1)–(a5). Now suppose $d \geq 1$. Let $v_1 := v$, $A_1 := \{v_1\}$ and $L_1 := N^+_T(v_1)$. Suppose $L_1, \dots, L_i$ has already been defined with $|L_i| \geq 1$. If L_i contains only one vertex u, let $v_{i+1} := u$ and $A_{i+1} := A_i \cup \{v_{i+1}\}$. If $|L_i| \geq 2$, Lemma 2.1 implies that there exists a vertex $u \in L_i$ with $|L_i|/4 \leq d^+_{T[L_i]}(u) \leq 3|L_i|/4$. Let $v_{i+1} := u$ and $L_{i+1} := L_i \cap N^+_T(v_{i+1})$. This procedure gives vertices $v_1, \dots, v_s$ and sets $L_1, \dots, L_s$ with $L_s = \emptyset$. We let $A := A_s$ with ordering $(v_1, \dots, v_s)$ and let $a := v_s$. From the construction, (a2) and (a3) are obvious.

The construction also implies that

$$\frac{|L_i|}{4} \leq |L_{i+1}| \leq \frac{3|L_i|}{4} \text{ for } i \in [s-2] \quad \text{and} \quad |L_{s-1}| = 1. \tag{2.1}$$

Note that we have $s \geq 2$ because $d \geq 1$. This implies

$$(\frac{4}{3})^{s-i-1} \leq |L_i| \leq 4^{s-i-1} \text{ for } i \in [s-1]. \tag{2.2}$$

In particular, (2.2) with $i = 1$ and the fact that $d = |L_1|$ together imply

$$\frac{1}{2}\log(d) + 2 \leq s \leq \frac{\log(d)}{2 - \log(3)} + 2 \leq \frac{5}{2}\log(d) + 2.$$

Thus we get (a1).

Note that $L_i \setminus (L_{i+1} \cup \{v_{i+1}\}) \subseteq N^+_T(v_i) \setminus A$ and $L_{i-1} \setminus (L_i \cup \{v_i\}) \subseteq N^-_T(v_i)$. Thus, for $1 \leq i \leq s/5 - 13$ we have

$$\begin{aligned} |N^+_T(v_i) \setminus A| &\geq |L_i \setminus L_{i+1}| - 1 \overset{(2.1)}{\geq} \frac{1}{4}|L_i| - 1 \overset{(2.2)}{\geq} \frac{1}{4}(\frac{4}{3})^{s-i-1} - 1 \geq \frac{1}{4}(\frac{4}{3})^{4s/5+12} - 1 \\ &\overset{(a1)}{\geq} \frac{1}{4}(\frac{4}{3})^{\frac{2}{5}\log(d+1)+64/5} - 1 \geq 8d^{1/7} - 1 \end{aligned}$$

Similarly we also get $|N^-_T(v_i) \setminus A| \geq |L_{i-1} \setminus L_i| - 1 \geq 8d^{1/7} - 1$. Thus (a4) holds.

For $i \leq s - 5\log(k) - 30$, (2.2) implies that

$$|L_i| \geq \left(\frac{4}{3}\right)^{s-i-1} \geq \left(\frac{4}{3}\right)^{5\log(k)+29} > 4100k^2.$$

Therefore, (a5) follows from

$$|N^+_T(v_i) \setminus A| \geq |L_i \setminus L_{i+1}| - 1 \overset{(2.1)}{\geq} \frac{1}{4}|L_i| - 1 \geq 1000k^2, \ \ |N^+_T(v_i) \setminus A| \geq |L_{i-1} \setminus L_i| - 1 \geq 1000k^2.$$

□

By reversing arcs of a tournament T in Lemma 2.2, we have the following analogue.

Lemma 2.3. *Let v be a vertex in an n-vertex tournament T with $d = d^-_T(v)$. Then there exist $B \subseteq V(T)$ and a vertex $b \in B$ such that the following properties hold:*

- (b1) *We have $\frac{1}{2}\log(d+1) + 1 \leq s \leq \frac{5}{2}\log(d+1) + 2$ where $s = |B|$*
- (b2) *$T[B]$ is a transitive tournament with respect to the ordering $(v_1, \dots, v_s)$ with source b and sink v.*
- (b3) *B out-dominates $V(T) \setminus B$.*
- (b4) *For $i \geq 4s/5 + 14$, we have*
$$|N^+_T(v_i) \setminus B|, |N^-_T(v_i) \setminus B| \geq 8d^{1/7} - 1.$$
- (b5) *For any positive integers i, k with $5\log(k) + 31 \leq i \leq s$, we have*
$$|N^+_T(v_i) \setminus B|, |N^-_T(v_i) \setminus B| \geq 1000k^2.$$

3. Sparse linkage structure

In this section, we will prove Lemma 3.4. For an ordering $\sigma = (v_1, \cdots, v_n)$ of vertices, we say that an arc $\overrightarrow{v_iv_j}$ is *σ-forward* if $i < j$, and *σ-backward* if $j < i$. For two integers a, b, we let $\sigma(a, b) := \{v_\ell : a \leq \ell \leq b, \ell \in [n]\}$. For positive integers n, k, t, an n-vertex digraph D and an ordering σ of $V(D)$, we say an D is *(σ, k, t)-good* if it satisfies the following.

- (D1) Every arc in D is a σ-forward arc.
- (D2) Every vertex in $\sigma(1, n-t)$ has out-degree at least k in D.
- (D3) Every vertex in $\sigma(t+1, n)$ has in-degree at least k in D.

Note that if $n \leq t$, then $\sigma(1, n-t) = \sigma(t+1, n) = \emptyset$, so (D2) and (D3) are vacuous. Also note that (D2) or (D3) never holds together with (D1) if $t < k$. In Lemma 3.4, we will show that every almost complete oriented graph has a spanning subgraph D' and an ordering σ such that D' is a sparse (σ, k, t)-good digraph for appropriate k, t. The following shows that (σ, k, t)-good digraph D' provides a sparse linkage structure from/to certain vertex sets.

Claim 3.1. *Let k, t be two positive integers with $t \geq k$. Let D' be a (σ, k, t)-good digraph for an ordering σ of $V(D')$. Then for a set $S \subseteq V(D')$ of $k-1$ vertices and $v \in V(D') \setminus S$, there exists a path P in $D' - S$ from v to $\sigma(n-t+1, n)$ and a path P' in $D' - S$ from $\sigma(1, t)$ to v.*

Proof. If $n \leq t$, then the claim is trivial as $\sigma(n-t+1, n) = \sigma(1, t) = V(D')$. Assume $n \geq t+1$. Let $\sigma = (v_1, \dots, v_n)$. Take a path P starting at v and ending at v_j with the largest possible j. If $j \leq n-t$, then (D1) and (D2) imply that v_j has at least k out-neighbors with larger indices. Thus $N^+_{D'}(v_j) \setminus S$ contains a vertex $v_{j'}$ with $j' > j$. However, $P \cup \{\overrightarrow{v_j v_{j'}}\}$ contradicts the maximality of j. Thus we have $j > n-t$. Therefore there exists a path P in $T - S$ from v to $v_j \in \sigma(n-t+1, n)$. We can find P' in a similar way. □

The following two claims are useful to prove Lemma 3.4.

Claim 3.2. *For an integer $s \geq 0$, let G be a bipartite graph with bipartition $A \cup B$ with $A = \{a_1, \dots, a_n\}, B = \{b_1 \dots, b_n\}$ satisfying the following.*

(P1$_s$) *For all $i, j \in [n]$ with $i < j$, we have $|N_G(a_i) \cap \{b_{i+1}, \dots, b_j\}| \geq \frac{j-i-s}{2}$,*

(P2$_s$) *for all $i, j \in [n]$ with $i < j$, we have $|N_G(b_j) \cap \{a_i, \dots, a_{j-1}\}| \geq \frac{j-i-s}{2}$.*

Then G contains a matching of size at least $n - s - 1$.

Proof. We may assume that $n - s - 1 > 0$, otherwise the claim is obvious. By König's theorem, it is enough to show that minimum vertex cover has size at least $n - s - 1$. Assume we have a minimum vertex cover W of G. If $A \subseteq W$ or $B \subseteq W$, then $|W| \geq n \geq n - s - 1$. So we may assume that each of $A \setminus W$ and $B \setminus W$ contains an element. Consider the smallest index i such that $a_i \in A \setminus W$, and the largest index j such that $b_j \in B \setminus W$. We have $i < j$, otherwise W contains at least $n-1$ vertices. Then we have

$$\{a_1, \dots, a_{i-1}\} \cup \{b_{j+1}, \dots, b_n\} \cup (N_G(b_j) \cap \{a_i, \dots, a_{j-1}\}) \cup (N_G(a_i) \cap \{b_f i + 1, \dots, b_j\}) \subseteq W.$$

By (P1$_s$) and (P2$_s$), we have

$$|W| \geq i - 1 + (n-j) + \frac{j-i-s}{2} + \frac{j-i-s}{2} \geq n - s - 1$$

as desired. □

Claim 3.3. *For $s \geq 0$, let D be an n-vertex oriented graph with $\delta(D) \geq n - s - 1$. Then there exists an ordering $\sigma = (v_1, \dots, v_n)$ of $V(D)$ that satisfies the following.*

(Q1$_s$) *For any $i, j \in [n]$ with $i < j$, v_i has at least $\frac{j-i-s}{2}$ out-neighbours in $\{v_{i+1}, \dots, v_j\}$,*

(Q2$_s$) *For any $i, j \in [n]$ with $i < j$, v_j has at least $\frac{j-i-s}{2}$ in-neighbours in $\{v_i, \dots, v_{j-1}\}$.*

Moreover, we can find such an ordering in polynomial-time on n.

Proof. We start with an arbitrary ordering $\sigma_1 = (v_1, \dots, v_n)$ of $V(D)$. Assume we have an ordering σ_ℓ of $V(D)$ for some $\ell \geq 1$. If σ_ℓ satisfies (Q1$_s$) and (Q2$_s$), then we are done. Otherwise consider $1 \leq i < j \leq n$ that does not satisfy (Q1$_s$) or (Q2$_s$). Let us define

$$\sigma_{\ell+1} := \begin{cases} (v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_j, v_i, v_{j+1}, \dots, v_n) & \text{if } i < j \text{ does not satisfy (Q1}_s\text{)}, \\ (v_1, \dots, v_{i-1}, v_j, v_i, \dots, v_{j-1}, v_{j+1}, \dots, v_n) & \text{if } i < j \text{ does not satisfy (Q2}_s\text{)}. \end{cases}$$

Note that $\sigma_{\ell+1}$ has at least one more σ-forward arc than σ_ℓ. The number of σ-forward arcs in D is at most $\binom{n}{2}$, so the procedure must end before we have $\sigma_{\binom{n}{2}}$. Thus we obtain a desired ordering in polynomial-time in n. □

Now we prove Lemma 3.4. It will be frequently used in the proof of Theorem 1.3.

Lemma 3.4. *For integers $s \geq 0$ and $k \geq 1$, let D be an n-vertex oriented graph with $\delta(D) \geq n-1-s$. Then there exist an ordering σ of $V(D)$ and a $(\sigma, k, 2k+s-1)$-good spanning subgraph D' of D with $|E(D')| \leq kn - k + sk$.*

Proof. If $n < 2k+s$, then an arbitrary ordering σ of $V(D)$ with a digraph D' with no arcs is $(\sigma, k, 2k+s-1)$-good. Thus we may assume that $n \geq 2k+s$. By Claim 3.3, we can find an ordering $\sigma = (v_1, \dots, v_n)$ which satisfies condition (Q1$_s$) and (Q2$_s$) in Claim 3.3. We consider an auxiliary bipartite graph H_0 with a bipartition $A \cup B$, where $A = \{v_1, \dots, v_n\}$ and $B = \{v'_1, \dots, v'_n\}$, such that $v_i v'_j \in H_0$ if and only if $\overrightarrow{v_i v_j}$ is a σ-forward arc of D. (i.e. $i < j$ and $\overrightarrow{v_i v_j} \in E(D)$.)

Note that the conditions (Q1$_s$) and (Q2$_s$) imply that the graph H_0 satisfies the condition (P1$_s$) and (P2$_s$). Assume we have a graph H_ℓ satisfying the condition (P1$_{s+2\ell}$) and (P2$_{s+2\ell}$). By Claim 3.2, H_ℓ contains a matching M_ℓ of size at least $n-s-2\ell-1$. Let $H_{\ell+1} := H_\ell \setminus M_\ell$. Then for any $i, j \in [n]$, we have $|N_{H_\ell}(a_i) \setminus N_{H_{\ell+1}}(a_i)| \leq 1$ and $|N_{H_\ell}(b_j) \setminus N_{H_{\ell+1}}(b_j)| \leq 1$. Thus the graph $H_{\ell+1}$ satisfies the condition (P1$_{s+2\ell+2}$) and (P2$_{s+2\ell+2}$). Repeating this for $0 \leq \ell \leq k-1$ provides arc-disjoint matchings $M_0, M_1, \dots, M_{k-1}$ of H_0 where the size of M_ℓ is at least $n-s-2\ell-1$ for $0 \leq \ell \leq k-1$. By deleting some arcs, we may assume that for $0 \leq \ell \leq k-1$ we have

$$|E(M_\ell)| = n - s - 2\ell - 1. \tag{3.1}$$

Let M be a subgraph of H_0 such that $E(M) := \bigcup_{\ell=0}^{k-1} E(M_\ell)$ and let D_1 be a subgraph of D such that

$$V(D_1) := V(D), \;\; E(D_1) := \{\overrightarrow{v_i v_j} : v_i v'_j \in E(M)\}.$$

Then by construction of H_0, every arc of D_1 is a σ-forward arc and

$$\Delta(M) \leq k \quad \text{and} \quad |E(M)| = \sum_{\ell=0}^{k-1} |E(M_\ell)| \overset{(3.1)}{=} kn - k^2 - sk. \tag{3.2}$$

Also this implies that

$$\begin{aligned} &\Delta^+(D_1) \leq k, \;\; \Delta^-(D_1) \leq k, \;\; |E(D_1)| = kn - k^2 - sk, \\ &d^-_{D_1}(v_i) \leq \min\{k, i-1\} \quad \text{and} \quad d^+_{D_1}(v_i) \leq \min\{k, n-i\}. \end{aligned} \tag{3.3}$$

For each vertex $2k+s \leq i \leq n$, the number of σ-forward arcs towards v_i in D is at least $\lceil \frac{i-1-s}{2} \rceil \geq \lceil \frac{2k+s-1-s}{2} \rceil \geq k$ by (Q2$_s$). Thus for each $2k+s \leq i \leq n$, we can choose a set N_i^- of σ-forward arcs towards v_i such that $N_i^- \subseteq E(D) \setminus E(D_1)$ and $|N_i^-| = k - d^-_{D_1}(v_i)$. Similarly, for each $1 \leq i \leq n-2k-s+1$, we can choose a set N_i^+ of σ-forward arcs from v_i such that $N_i^+ \cap E(D_1) = \emptyset$ and $|N_i^+| = k - d^+_{D_1}(v_i)$. Define a digraph $D' \subseteq D$ with

$$V(D') := V(D), \;\; E(D') := E(D_1) \cup \bigcup_{i=2k+s}^{n} N_i^- \cup \bigcup_{i=1}^{n-2k-s+1} N_i^+.$$

Then D' satisfies (D1) by construction, and satisfies (D2) since $|d^+_{D'}(v_i)| \geq d^+_{D_1}(v_i) + |N_i^+| \geq k$ for $i \in [n-2k-s+1]$. Similarly, D' also satisfies (D3), thus D' is $(\sigma, k, 2k+s-1)$-good. Note that

$$\begin{aligned} \left| \bigcup_{i=2k+s}^{n} N_i^- \right| \;\; &\leq \;\; \sum_{i=2k+s}^{n} (k - d^-_{D_1}(v_i)) \;\; = \;\; k(n-2k-s+1) - \sum_{i=1}^{n} d^-_{D_1}(v_i) + \sum_{i=1}^{2k+s-1} d^-_{D_1}(v_i) \\ &\overset{(3.3)}{\leq} \;\; k(n-2k-s+1) - |E(D_1)| + \sum_{i=1}^{2k+s-1} \min\{k, i-1\} \;\; \overset{(3.3)}{=} \;\; \binom{k}{2} + sk. \end{aligned}$$

Here, we get the second inequality because $E(D_1) = \sum_{i=1}^{n} d^-_{D_1}(v_i)$. Similarly, we also have $|\bigcup_{i=1}^{n-2k-s+1} N_i^+| \leq \binom{k}{2} + sk$. Thus we have

$$\begin{aligned} |E(D')| &\leq |E(D_1)| + \left|\bigcup_{i=2k+s}^{n} N_i^-\right| + \left|\bigcup_{i=1}^{n-2k-s+1} N_i^+\right| \\ &\overset{(3.3)}{\leq} kn - k^2 - sk + 2\binom{k}{2} + 2sk = kn - k + sk. \end{aligned}$$

□

4. Small tournaments

In this section, we show that Theorem 1.3 holds for any strongly k-connected tournament T with at most $100k\log(k+1)$ vertices. Note that Theorem 4.2 is sufficient for our purpose. To prove Theorem 4.2, we use the following lemma, which is a modification of Lemma 2.1 in [12], and the proof is almost identical except a few changes.

Lemma 4.1. [12] *Let $k \geq 1$ and $n \geq 5k$ be integers. Every n-vertex tournament T contains two disjoint sets of vertices X and Y of size k such that for any set S of $k-1$ vertices and any $x \in X \setminus S, y \in Y \setminus S$ there is a path P in $T - S$ from x to y.*

Proof. Let $\overrightarrow{K_{k,k}}$ be a bipartite digraph with partition A, B such that $|A| = |B| = k$ and for every $u \in A, v \in B$, we have $\overrightarrow{uv} \in E(\overrightarrow{K_{k,k}})$. If T contains $\overrightarrow{K_{k,k}}$ with bipartition A and B as a subgraph, then $X := A, Y := B$ are sufficient for our purpose. Thus we may assume that T does not contain $\overrightarrow{K_{k,k}}$ as a subgraph.

Let $X = \{x_1, \dots, x_k\}$ be a set of k vertices in T of largest out-degree and $\{y_1, \dots, y_k\}$ be a set of k vertices in T of largest in-degree. Since $n \geq 5k$, we may assume $X \cap Y = \emptyset$. From Lemma 2.1, we have $d^+_T(x_i) \geq (n-k)/2 \geq 2k$ and $d^-_T(y_i) \geq (n-k)/2 \geq 2k$ for all $i \in [k]$. Consider a set $S \subseteq V(T)$ of size $k-1$. For each $i, j \in [k]$ let $X_{i,j} := N^+(x_i) \setminus N^-(y_j)$, $Y_{i,j} := N^-(y_j) \setminus N^+(x_i)$, $I_{i,j} = N^+(x_i) \cap N^-(y_j)$. Let $M_{i,j}$ be a maximum matching between $X_{i,j}$ and $Y_{i,j}$ such that every arc is directed from $X_{i,j}$ to $Y_{i,j}$. For each $z \in I_{i,j}$, T contains a path (x_i, z, y_j) and for each $\overrightarrow{ww'} \in M_{i,j}$, T contains a path (x_i, w, w', y_j). Moreover, those paths are all pairwise internally vertex disjoint. Thus if $|M_{i,j}| + |I_{i,j}| \geq k$ for all $i, j \in [k]$, then for any x_i and y_j, there are at least k internally vertex disjoint paths from x_i to y_j. So we are done since for each $i, j \in [k]$ at least one path from x_i to y_j does not intersect with S. If there exist $i, j \in [k]$ such that $|M_{i,j}| + |I_{i,j}| < k$, then we have

$$|X_{i,j} \setminus V(M_{i,j})| \geq |N^+_T(x_i) - I_{i,j} - V(M_{i,j})| \geq d^+_T(x_i) - k \geq k.$$

Similarly we get $|Y_{i,j} \setminus V(M_{i,j})| \geq k$. Since $M_{i,j}$ is a maximal matching from $X_{i,j}$ to $Y_{i,j}$, for any $x' \in X_{i,j} \setminus V(M_{i,j})$ and $y' \in Y_{i,j} \setminus V(M_{i,j})$ we have $\overrightarrow{y'x'} \in E(T)$. This contradicts the fact that T does not contain $\overrightarrow{K_{k,k}}$. □

Now we prove the theorem, which has worse upper bound than the upper bound in Theorem 1.3 for sufficiently large n. However, if n is small enough, for example, $n \leq 100k\log(k+1)$, then the following theorem implies Theorem 1.3.

Theorem 4.2. *For any integer $k \geq 1$, every strongly k-connected tournament T contains a strongly k-connected spanning subgraph D with $|E(D)| \leq (5k-2)n + \binom{5k}{2}$.*

Proof. If T has less than $5k$ vertices, then T itself is sufficient to be D. Otherwise, let $V' \subseteq V$ be a set of $5k$ vertices. By applying Lemma 4.1, we can find two disjoint sets $X = \{x_1, \dots, x_k\}, Y = \{y_1, \dots, y_k\}$ of size k such that for any set $S \subseteq V'$ of size $k-1$ and vertices $x \in X, y \in Y$, there exists a path from x to y in $T[V'] - S$. We apply Lemma 3.4 to T with parameters $0, k$ corresponding to s, k, and we obtain an ordering $\sigma = (v_1, \dots, v_n)$ of $V(T)$ and a $(\sigma, k, 2k-1)$-good spanning subgraph $D' \subseteq T$ with $|E(D')| \leq kn - k$.

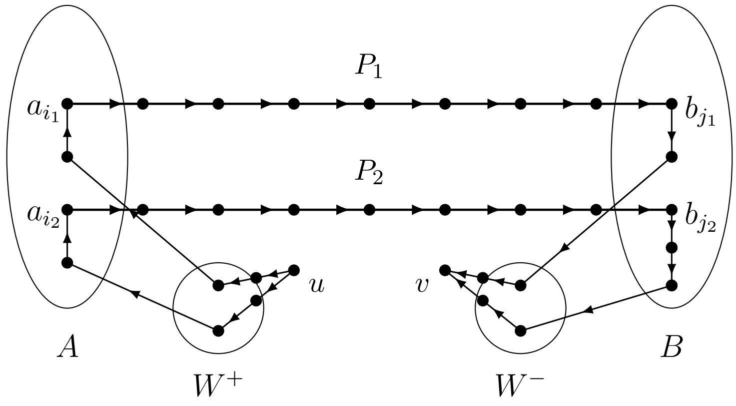

FIGURE 1. Two paths from u to v in the outline of the idea when $k = 2$.

For each $n-2k+2 \leq i \leq n$, let $\{P(v_i, j) : j \in [k]\}$ be a k-fan from v_i to X (which exists since T is strongly k-connected) such that $P(v_i, j)$ is a path from v_i to x_j. Note that if $v_i = x_j$, then $P(v_i, j)$ is a path of one vertex. Similarly, for each $1 \leq i \leq 2k-1$, let $\{Q(v_i, j) : j \in [k]\}$ be a k-fan from Y to v_i such that $Q(v_i, j)$ is a path from y_j to v_i. Note that if $v_i = y_j$, then $Q(v_i, j)$ is a path of one vertex.

For each $n-2k+2 \leq i \leq n$ and $1 \leq i' \leq 2k-1$, it follows that

$$\sum_{j=1}^{k} |E(P(v_i, j))| \leq n-1, \quad \sum_{j=1}^{k} |E(Q(v_{i'}, j))| \leq n-1,$$

because no vertex other than v_i is covered by two distinct paths in a k-fan from v_i to X or by two distinct paths in a k-fan from Y to v_i. Let D be the subgraph of T such that

$$V(D) := V(T), \quad E(D) := E(T(V')) \cup E(D') \cup \bigcup_{i=1}^{2k-1} \bigcup_{j=1}^{k} Q(v_i, j) \cup \bigcup_{i=n-2k+2}^{n} \bigcup_{j=1}^{k} P(v_i, j).$$

Then

$$\begin{aligned} |E(D)| &\leq |E(T(V'))| + |E(D')| + (2k-1)(n-1) + (2k-1)(n-1) \\ &\leq \binom{5k}{2} + kn - k + (4k-2)n \leq (5k-2)n + \binom{5k}{2}. \end{aligned}$$

Moreover, for any set $S \subseteq V(D)$ of $k-1$ vertices and any vertices $u, v \in V(T) \setminus S$, there is a path P from v to v_i and a path P' from $v_{i'}$ to u in $D' - S$ for some $i \geq n-2k+2$ and $i' \leq 2k+1$, by Claim 3.1. Since $\{P(v_i, j) : j \in [k]\}$ and $\{Q(v_{i'}, j) : j \in [k]\}$ are k-fans, there are $s, s' \in [k]$ such that both $P(v_i, s)$ and $Q(v_{i'}, s')$ do not intersect S. Let $x_s^* \in X$ and $y_{s'}^* \in Y$ be the endpoints of $P(v_i, s)$ and $Q(v_{i'}, s')$, respectively. (note that if $v_i \in X$ ($v_{i'} \in Y$), then $x_s^* = v_i$ ($y_{s'}^* = v_{i'}$).) By Claim 4.1, there is a path P'' in $T[V'] - S$ from x_s^* to $y_{s'}^*$. Hence $E(P) \cup E(P(v_i, s)) \cup E(P'') \cup E(Q(v_{i'}, s')) \cup E(P')$ contains a path in $D - S$ from u to v. Thus D is strongly k-connected. □

5. Proof of Theorem 1.3

Outline of the idea. For a strongly k-connected tournament T, we construct a set A which is the union of many in-dominating sets, a set B which is the union of many out-dominating sets and k pairwise vertex disjoint paths $P_1, \ldots, P_k$ from A to B such that the path P_t is from a_{i_t} to b_{j_t} for each $t \in [k]$. We choose the size of in-dominating sets and out-dominating sets in A and B to be sufficiently small (Lemmas 2.2 and 2.3) so that there are few vertices in both A and B.

To find a sparse subgraph D, we divide the vertex set $V(T)$ into V_1, V_1', V_2, V_3, V_4 and apply Lemma 3.4 to each set and find two small sets W^+ and W^- such that D contains k internally vertex-disjoint paths from any vertex u to W^+ and k internally vertex-disjoint paths from W^-

to any vertex v. We also add some arcs to the subgraph D so that there are k arcs in D from each vertex in W^+ to A, and k arcs in D from B to each vertex in W^-. Note that this is possible since A is a union of many in-dominating sets and B is a union of many out-dominating sets. By adding some arcs inside A and B, we can also ensure that there are k internally vertex-disjoint paths from any vertex in A to the vertices $a_{i_1}, \dots, a_{i_k}$ and k internally vertex-disjoint paths from $b_{j_1}, \dots b_{j_k}$ to any vertex in B. Then for each distinct vertices $u, v \in V(T)$, the paths from u to W^+, the arcs from W^+ to A, the paths inside A to $a_{i_1}, \dots, a_{i_k}$, the paths $P_1, \dots, P_k$, the paths inside B from $b_{j_1}, \dots, b_{j_k}$, the arcs from B to W^-, and the paths from W^- to v all together form k internally vertex-disjoint paths from u to v as in Figure 1. Since u and v are arbitrarily chosen, D is strongly k-connected while D is sparse enough.

Proof of Theorem 1.3. Let T be a strongly k-connected n-vertex tournament with a vertex-set V. Note that Theorem 1.3 is trivial for $k = 1$ since every strongly connected n-vertex tournament contains a Hamilton cycle (see [3, Theorem 1.5.1]). There is an algorithm that finds a Hamilton cycle in an n-vertex tournament and runs in $O(n^2)$ (see [11]). If $k \geq 2$ and $n \leq 100k \log(k+1)$, then Theorem 4.2 implies Theorem 1.3. Thus we may assume that

$$k \geq 2, \;\; n > 100k \log(k+1).$$

Now we construct an appropriate in-dominating set A and out-dominating set B as we sketched before. Let X and Y be two disjoint sets such that X is a set of $3k-1$ vertices with smallest out-degrees, and let Y is a set of $3k-1$ vertices with smallest in-degrees. Let $\delta^- := \max_{y \in Y} d^-_T(y)$ and $\delta^+ := \max_{x \in X} d^+_T(x)$. Without loss of generality, we assume

$$\delta^- \geq \delta^+. \tag{5.1}$$

Choose $x_1 \in X$ having the largest number of out-neighbors in $V \setminus (X \cup Y)$ among all vertices in X, and let

$$d^+_1 := |(V \setminus (X \cup Y)) \cap N^+_T(x_1)|.$$

We apply Lemma 2.2 with $T - ((X - \{x_1\}) \cup Y), x_1, d^+_1$ corresponding to T, v, d to find a set A_1 and a sink vertex $a_1 \in A_1$ satisfying (a1)–(a5). Note that (a1) implies that A_1 is nonempty and $a_1 = x_1$ could happen when $d^+_1 = 0$. For given $x_1, \dots, x_i$ and $A_1, \dots, A_i$, let us choose $x_{i+1} \in X \setminus \{x_1, \dots, x_i\}$ having the largest number of out-neighbours in $V \setminus (X \cup Y \cup \bigcup_{j=1}^{i} A_j)$ among all the vertices in $X \setminus \{x_1, \dots, x_i\}$ and let

$$d^+_{i+1} := |(V \setminus (X \cup Y \cup \bigcup_{j=1}^{i} A_j)) \cap N^+_T(x_{i+1})|.$$

We apply Lemma 2.2 with $T - ((X - \{x_{i+1}\}) \cup Y \cup \bigcup_{j=1}^{i} A_j), x_{i+1}, d^+_{i+1}$ corresponding to T, v, d to find a set A_{i+1} and a sink vertex $a_{i+1} \in A_{i+1}$ satisfying (a1)–(a5). By repeating this $3k-1$ times, we get $A_1, \dots, A_{3k-1}$ and $a_1, \dots, a_{3k-1}$. We let $A := \bigcup_{i=1}^{3k-1} A_i$.

Next, we choose $y_1 \in Y$ having the largest number of in-neighbours in $V \setminus (X \cup Y \cup A)$. Let

$$d^-_1 := |(V \setminus (X \cup Y \cup A)) \cap N^-_T(y_1)|.$$

Then we apply Lemma 2.3 with $T - (X \cup (Y - \{y_1\}) \cup A), y_1, d^-_1$ corresponding to T, v, d to find a set B_1 and a source vertex $b_1 \in B_1$ satisfying (b1)–(b5). Note that (b1) implies that B_1 is nonempty and $b_1 = y_1$ could happen when $d^-_1 = 0$. For given $A, y_1, \dots, y_i$ and $B_1, \dots, B_i$, let us choose $y_{i+1} \in Y \setminus \{y_1, \dots, y_i\}$ having the largest number of in-neighbours in $V \setminus (X \cup Y \cup A \cup \bigcup_{j=1}^{i} B_j)$ among all the vertices in $Y \setminus \{y_1, \dots, y_i\}$ and let

$$d^-_{i+1} := |(V \setminus (X \cup Y \cup A \cup \bigcup_{j=1}^{i} B_j)) \cap N^-_T(y_{i+1})|.$$

We apply Lemma 2.3 with $T - (X \cup (Y - \{y_{i+1}\}) \cup A \cup \bigcup_{j=1}^{i} B_j), y_{i+1}, d^-_{i+1}$ corresponding to T, v, d to find a set B_{i+1} and a source vertex $b_{i+1} \in B_{i+1}$ satisfying (b1)–(b5). By repeating this $3k-1$ times, we get $B_1, \dots, B_{3k-1}$ and $b_1, \dots, b_{3k-1}$. We let $B := \bigcup_{i=1}^{3k-1} B_i$. Note that $T[B_i]$ is a transitive tournament for each $i \in [3k-1]$. For each i, we let B'_i be the set of the last $\max(\lceil |B_i|/5 - 13\rceil, 0)$ vertices, and let B_i'' be the set of the first $\min(\lceil 5\log(k)+30\rceil, |B_i|)$ vertices in the transitive ordering of $T[B_i]$, respectively. Note that B_i' and B_i'' are not necessarily disjoint.

We define

$$A_{\text{sink}} := \{a_1, \dots, a_{3k-1}\}, B_{\text{source}} := \{b_1, \dots, b_{3k-1}\}, B' := \bigcup_{i=1}^{3k-1} B_i', \text{ and } B'' := \bigcup_{i=1}^{3k-1} B_i''.$$

From this construction, we get numbers $d_1^+, \dots, d_{3k-1}^+, d_1^-, \dots, d_{3k-1}^-$ satisfying

$$\delta^+ \geq d_1^+ \geq d_2^+ \geq \cdots \geq d_{3k-1}^+ \quad \text{and} \quad \delta^- \geq d_1^- \geq d_2^- \geq \cdots \geq d_{3k-1}^-, \tag{5.2}$$

and sets $A_1, \dots, A_{3k-1}$, $B_1, \dots, B_{3k-1}$, $B_1', \dots, B_{3k-1}'$, $B_1'', \dots, B_{3k-1}''$ and vertices $a_1, \dots, a_{3k-1}$, $b_1, \dots, b_{3k-1}$ satisfying the following (A1)–(A3) and (B1)–(B6) for all $i \in [3k-1]$.

(A1) $\frac{1}{2}\log(d_i^+ + 1) + 1 \leq |A_i| \leq \frac{5}{2}\log(d_i^+ + 1) + 2$,
(A2) $T[A_i]$ is a transitive tournament with source x_i and sink a_i,
(A3) A_i in-dominates $V \setminus (A \cup B)$,
(B1) $\frac{1}{2}\log(d_i^- + 1) + 1 \leq |B_i| \leq \frac{5}{2}\log(d_i^- + 1) + 2$,
(B2) $T[B_i]$ is a transitive tournament with sink y_i and source b_i,
(B3) B_i out-dominates $V \setminus (A \cup B)$,
(B4) $|B_i'| \geq |B_i|/5 - 13$ and for $v \in B_i'$ we have

$$|N_T^+(v) \setminus (A \cup \bigcup_{j=1}^{i} B_j)| \geq 8(d_i^-)^{1/7} - 1, \;\; |N_T^-(v) \setminus (A \cup \bigcup_{j=1}^{i} B_j)| \geq 8(d_i^-)^{1/7} - 1.$$

(B5) $|B_i''| < 5\log(k) + 31$ and for $v \in B_i \setminus B_i''$ we have

$$|N_T^+(v) \setminus (A \cup \bigcup_{j=1}^{i} B_j)| \geq 1000k^2, \;\; |N_T^-(v) \setminus (A \cup \bigcup_{j=1}^{i} B_j)| \geq 1000k^2.$$

(B6) For any vertex $v \in B_i \setminus B_i'$, we have $B_i' \subseteq N_T^+(v)$.

By Lemma 2.1, each of $T[A_{\text{sink}}]$ and $T[B_{\text{source}}]$ contains k vertices of in-degree at least k and k vertices of out-degree at least k. Let $a_{i_1}, \dots, a_{i_k} \in A_{\text{sink}}$ be k distinct vertices having in-degree at least k in $T[A_{\text{sink}}]$ and let $b_{j_1}, \dots, b_{j_k} \in B_{\text{source}}$ be distinct k vertices having out-degree at least k in $T[B_{\text{source}}]$. By (A1), (B1) and the fact that $\delta^- \leq n-1$, we have $|A \cup B| \leq (6k-2)(\frac{5}{2}\log(n) + 2) < n - k$ since $n \geq 100k\log(k+1)$ and $k \geq 2$. Thus we have

$$|V \setminus (A \cup B)| \geq k. \tag{5.3}$$

Our aim is to find collections of arcs E_0, E_1, E_2, E_3, E_4 and E_5 which together form a desired digraph D. Since the tournament T is strongly k-connected, by Menger's theorem, let $P_1, \dots, P_k$ be k vertex-disjoint paths from $\{a_{i_1}, \dots, a_{i_k}\}$ to $\{b_{j_1}, \dots, b_{j_k}\}$. We choose those k vertex-disjoint paths with the minimum length $\sum_{i=1}^k |E(P_i)|$, and thus each path P_i is backwards-transitive for $1 \leq i \leq k$. Note that $V(P_i)$ is not necessarily disjoint from $A \cup B \setminus \{a_{i_1}, \dots, a_{i_k}, b_1, \dots, b_{j_k}\}$. By permuting indices, we may assume that P_s is a backwards-transitive path from a_{i_s} to b_{j_s}. See Figure 2 for the picture which we currently have. Let $V^{\text{int}}(P_s)$ be the set of internal vertices of P_s. We define

$$V_1 := (A \cup B) \setminus (\bigcup_{i=1}^{k} V^{\text{int}}(P_i)), \;\; V_1' := (A \cup B) \cap (\bigcup_{i=1}^{k} V^{\text{int}}(P_i)) \;\text{ and }\; E_0 := \bigcup_{s=1}^{k} E(P_s). \tag{5.4}$$

Before starting the construction of E_1, E_2, E_3, E_4 and E_5, we prove Claim 5.1 and Claim 5.3 showing that for any $v \in A \cup B$ there exists a k-fan from v to $V \setminus (A \cup B)$ and a k-fan from $V \setminus (A \cup B)$ to v consisting of short paths.

Claim 5.1. *For any vertex $v \in A \cup B$, we can find a k-fan $\{P^-(v,1), \dots, P^-(v,k)\}$ from $V \setminus (A \cup B)$ to v such that $\sum_{i=1}^k |E(P^-(v,i))| \leq 70k\log(k+1)$.*

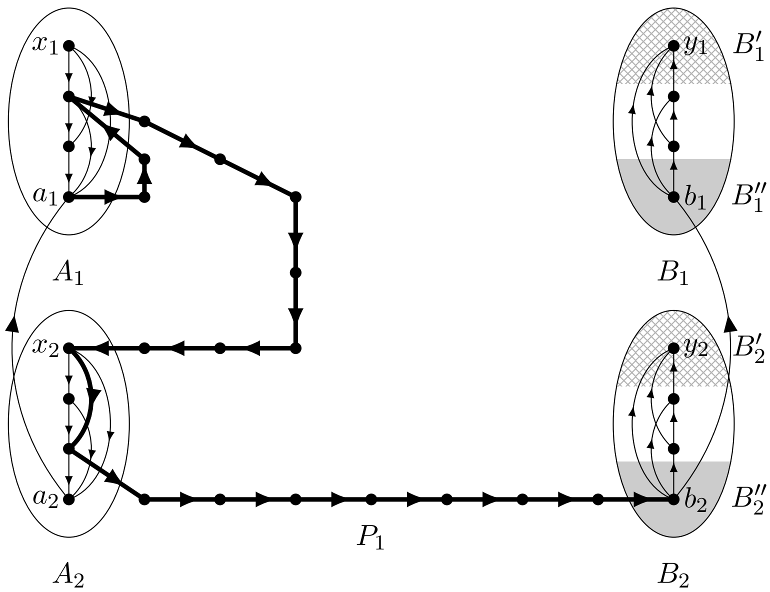

FIGURE 2. A picture when $k = 1, i_1 = 1$ and $j_1 = 2$.

Proof of Claim 5.1. Note that (5.1), (5.2), (A1) and (B1) together imply that

$$|A \cup B| \leq (6k-2)(\frac{5}{2}\log(\delta^- + 1) + 2). \tag{5.5}$$

We consider the following two cases.

Case 1. $\delta^- \leq 60k^2$.

In this case, consider $\{P^-(v,1), \dots, P^-(v,k)\}$, a k-fan from $V \setminus (A \cup B)$ to v. Such a k-fan exists because of Fact 1 and (5.3). By (5.5), we have $|A \cup B| \leq (6k-2)(\frac{5}{2}\log(60k^2+1)+2) \leq 69k\log(k+1)$. Since every vertex in each $P^-(v,i)$ is in $A \cup B$ except for one vertex, we have $\sum_{i=1}^{k} |E(P^-(v,i))| \leq |A \cup B| + k \leq 70k\log(k+1)$.

Case 2. $\delta^- > 60k^2$.

Since $k \geq 2$, we have

$$\delta^- \geq (6k-2)(\frac{5}{2}\log(\delta^- + 1) + 2) + 2k \overset{(5.5)}{\geq} |A \cup B| + 2k.$$

Thus for any vertex $u \notin Y$, we have $d^-(u) \geq \delta^- \geq |A \cup B| + 2k$.

If $v \notin Y$, take k distinct paths of length 1 from $V \setminus (A \cup B)$ to v, and let $P^-(v,1), \dots, P^-(v,k)$ be those paths of length 1. Then we have $\sum_{i=1}^{k} |E(P^-(v,i))| \leq k \leq 70k\log(k+1)$. If $v \in Y$, then take $\{Q_1, \dots, Q_k\}$, a k-fan from $V \setminus Y$ to v given by Fact 1 and (5.3). Let v_i be the starting vertex of Q_i for $1 \leq i \leq k$. Then we have

$$\sum_{i=1}^{k} |E(Q_i)| \leq |Y| + k \leq 4k - 1.$$

Consider $i \in [k]$ with $v_i \in A \cup B$. Since each v_i is not in Y, $d_T^-(v_i) \geq \delta^- \geq |A \cup B| + 2k$ and v_i has at least $2k$ in-neighbors outside $A \cup B$. For each $i \in [k]$ with $v_i \in A \cup B$, we choose v_i' in $N_T^-(v_i) \setminus (A \cup B \cup \{v_1, \dots, v_k\})$ in the way that v_i's are all distinct. Let

$$P^-(v,i) := \begin{cases} Q_i \cup \{\overrightarrow{v_i'v_i}\} & \text{if } v_i \in A \cup B, \\ Q_i & \text{if } v_i \notin A \cup B. \end{cases}$$

Then the paths $P^-(v,1),\dots,P^-(v,k)$ form a k-fan from $V\setminus(A\cup B)$ to v such that

$$\sum_{i=1}^{k}|E(P^-(v,i))| \le k+\sum_{i=1}^{k}|E(Q_i)| \le |Y|+2k = 5k-1 \le 70k\log(k+1).$$

This proves Claim 5.1. □

Claim 5.2. *For each $v\in A\cup B''$, there exists a k-fan $\{P_*^+(v,1),\dots,P_*^+(v,k)\}$ from v to $V\setminus(A\cup B'')$ such that $\sum_{i=1}^{k}|E(P_*^+(v,i))|\le 98k\log(k+1)$.*

Proof of Claim 5.2. Note that we have

$$\begin{aligned}|A\cup B''| &\overset{(A1)}{\le} \sum_{i=1}^{3k-1}\left(\frac{5}{2}\log(d_i^+ +1)+2\right)+|B''| \\ &\overset{(5.2),(B5)}{<} (3k-1)\left(\frac{5}{2}\log(\delta^+ +1)+2\right)+(3k-1)(5\log(k)+31) \qquad (5.6)\end{aligned}$$

To prove Claim 5.2, we consider the following two cases.

Case 1. $\delta^+\le 100k^2$.

Since T is strongly k-connected, there exists $\{P_*^+(v,1),\dots,P_*^+(v,k)\}$, a k-fan from v to $V\setminus(A\cup B'')$ by Fact 1 and (5.3). Since $P_*^+(v,1),\dots,P_*^+(v,k)$ contains at most k vertices outside $A\cup B''$ and $\delta^+\le 100k^2$, we have

$$\sum_{i=1}^{3k-1}|E(P_*^+(v,i))| \le |A\cup B''|+k \overset{(5.6)}{\le} 98k\log(k+1).$$

Case 2. $\delta^+\ge 100k^2$.

In this case, we have

$$|A\cup B''|+2k \overset{(5.6)}{<} (3k-1)\left(\frac{5}{2}\log(\delta^+ +1)+2\right)+(3k-1)(5\log(k)+31)+2k \le \delta^+$$

If $v\notin X$, then $d_T^+(v)\ge\delta^+\ge|A\cup B''|+2k$. So we can find k paths $Q_1',\dots,Q_k'$ of length 1 from v to $V\setminus(A\cup B'')$. Let $P_*^+(v,1),\dots,P_*^+(v,k)$ be those paths of length 1. Then $\sum_{i=1}^{k}|E(P_*^+(v,i))|\le k\le 98k\log(k+1)$.

If $v\in X$, then we find a k-fan $\{Q_1',\dots,Q_k'\}$ from v to $V\setminus X$ by Fact 1 and (5.3). Then because all vertices of Q_i' except the last vertex belong to X, we have $\sum_{i=1}^{k}|E(Q_i')|\le|X|+k$. Let u_i' be the end vertex of Q_i', for $1\le i\le k$. Consider $i\in[k]$ with $u_i'\in A\cup B''$. Since $u_i'\notin X$ and $d_T^+(u_i')\ge\delta^+\ge|A\cup B''|+2k$, u_i' has at least $2k$ out-neighbors in $V\setminus(A\cup B'')$, we can choose $u_i''\in N_T^+(u_i')\setminus(A\cup B''\cup\{u_1',\dots,u_k'\})$ such that u_i''s are distinct. We let

$$P_*^+(v,i) := \begin{cases} Q_i'\cup\{\overrightarrow{u_i'u_i''}\} & \text{if } u_i'\in A\cup B'', \\ Q_i' & \text{if } u_i'\notin A\cup B''. \end{cases}$$

Then we have a k-fan $\{P_*^+(v,1),\dots,P_*^+(v,k)\}$ from v to $V\setminus(A\cup B'')$ such that

$$\sum_{i=1}^{k}|E(P_*^+(v,i))| \le \sum_{i=1}^{k}|E(Q_i')|+k \le |X|+2k = 5k-1 \le 98k\log(k+1).$$

This proves Claim 5.2. □

Now we prove Claim 5.3 by using Claim 5.2.

Claim 5.3. *For any vertex $v \in A \cup B$, there exists a k-fan $\{P^+(v,1),\dots,P^+(v,k)\}$ from v to $V \setminus (A \cup B)$ with $\sum_{i=1}^k |E(P^+(v,i))| \le 100k\log(k+1)$.*

Proof of Claim 5.3. We first use Claim 5.2 to find a k-fan from v to $V \setminus (A \cup B'')$ such that $\sum_{i=1}^k |E(P_*^+(v,i))| \le 98k\log(k+1)$. Let u_i be the last vertex in $P_*^+(v,i)$ and let $U := \{u_1,\dots,u_k\}$. Then for each $i \in [k]$ all vertices in $P_*^+(v,i)$ except u_i belong to $A \cup B''$, and u_i is either in $V \setminus (A \cup B)$ or in $B \setminus B''$. For each i with $u_i \in B \setminus B''$, let ℓ_i be the index such that $u_i \in B_{\ell_i}$. Then we can partition $[k]$ into four sets I_1, I_2, I_3 and I_4 as follows.

$$\begin{aligned}
&\text{For } i \in I_1, \text{ we have } |B_{\ell_i}| \ge 18k+80, u_i \in B \setminus B'' \text{ and } u_i \notin B'_{\ell_i},\\
&\text{for } i \in I_2, \text{ we have } |B_{\ell_i}| \ge 18k+80, u_i \in B \setminus B'' \text{ and } u_i \in B'_{\ell_i},\\
&\text{for } i \in I_3, \text{ we have } |B_{\ell_i}| < 18k+80 \text{ and } u_i \in B \setminus B'',\\
&\text{for } i \in I_4, \text{ we have } u_i \notin A \cup B.
\end{aligned}$$

First, consider $i \in I_1 \cup I_2$. Since $|B_{\ell_i}| \ge 18k+80$, (B1) implies that

$$d_{\ell_i}^- \ge 2^{\frac{2}{5}(|B_{\ell_i}|-2)} - 1 \ge 2^{7k+30}. \tag{5.7}$$

For any $u \in B'_{\ell_i}$ we have

$$\begin{aligned}
|N_T^+(u) \setminus (A \cup B)| &\ge \left| N_T^+(u) \setminus (A \cup \bigcup_{p=1}^{\ell_i} B_p) \right| - \left| \bigcup_{p=\ell_i+1}^{3k-1} B_p \right| \\
&\overset{(B4)}{\ge} 8(d_{\ell_i}^-)^{1/7} - 1 - \left| \bigcup_{p=\ell_i+1}^{3k-1} B_p \right| \\
&\overset{(5.7)}{\ge} (3k-1)(\frac{5}{2}\log(d_{\ell_i}^- + 1) + 2) + 3k - \left| \bigcup_{p=\ell_i+1}^{3k-1} B_p \right| \\
&\overset{(B1),(5.2)}{\ge} 3k.
\end{aligned} \tag{5.8}$$

Here, we get the third inequality since $8x^{1/7} - 1 \ge (3k-1)(\frac{5}{2}\log(x+1)+2) + 3k$ holds for $x \ge 2^{7k+30}$ and $k \ge 2$. Thus any vertex $u \in B'_{\ell_i}$ has at least $3k$ out-neighbors in $V \setminus (A \cup B)$.

For $i \in I_1$, (B4) implies that $|B'_{\ell_i}| \ge |B_{\ell_i}|/5 - 13 \ge 3k$ and (B6) implies that $B'_{\ell_i} \subseteq N_T^+(u_i)$. From this we obtain $|(N_T^+(u_i) \cap B'_{\ell_i}) \setminus U| = |B'_{\ell_i} \setminus U| \ge 3k - k \ge 2k$. Thus we can choose a set $W = \{w_i : i \in I_1\}$ of $|I_1|$ distinct vertices such that $w_i \in N_T^+(u_i) \cap (B'_{\ell_i} \setminus U)$. Again, (5.8) implies that

$$|N_T^+(w_i) \setminus (A \cup B \cup U \cup W)| \ge k,$$

so we can further choose a set $W' = \{w'_i : i \in I_1\}$ of $|I_1|$ distinct vertices such that $w'_i \in N_T^+(w_i) \setminus (A \cup B \cup U \cup W)$.

Now we consider $i \in I_2$. In this case $u_i \in B'_{\ell_i}$ and (5.8) imply that

$$|N_T^+(u_i) \setminus (A \cup B \cup U \cup W \cup W')| \ge 2k - 2|I_1| \ge |I_2|,$$

so we can further choose a set $W^* = \{w^*_i : i \in I_2\}$ of $|I_2|$ distinct vertices such that $w^*_i \in N_T^+(u_i) \setminus (A \cup B \cup U \cup W \cup W')$.

Now we consider $i \in I_3$. In this case, u_i belongs to $B_{\ell_i} \setminus B''_{\ell_i}$. Thus

$$\begin{aligned}
\left|N^+_T(u'_i) \setminus (A \cup B)\right| &\geq \left|N^+_T(u'_i) \setminus (A \cup \bigcup_{p=1}^{\ell_i} B_p)\right| - \left|\bigcup_{p=\ell_i+1}^{3k-1} B_p\right| \\
&\overset{(B1),(B5)}{\geq} 1000k^2 - \sum_{p=\ell_i+1}^{3k-1} (\frac{5}{2}\log(d^-_p + 1) + 2) \\
&\overset{(5.2)}{\geq} 1000k^2 - (3k-1)(\frac{5}{2}\log(d^-_{\ell_i} + 1) + 2) \\
&\overset{(B1)}{\geq} 1000k^2 - 5(3k-1)|B_{\ell_i}| \\
&\geq 1000k^2 - 5(3k-1)(18k + 80) \geq 5k \geq |I_3| + 4k.
\end{aligned}$$

Thus we can choose a set $W^{**} := \{w^{**}_i : i \in I_3\}$ of $|I_3|$ distinct vertices such that $w^{**}_i \in N^+_T(u_i) \setminus (A \cup B \cup U \cup W \cup W' \cup W^*)$. Note that U, W, W', W^*, W^{**} are pairwise disjoint sets by construction. For $i \in [k]$, let $P^+(v, i)$ be a path from v to $V \setminus (A \cup B)$ as follows.

$$E(P^+(v,i)) := \begin{cases} E(P^+_*(v,i)) \cup \{\overrightarrow{u_i w_i}, \overrightarrow{w_i w'_i}\} & \text{if } i \in I_1, \\ E(P^+_*(v,i)) \cup \{\overrightarrow{u_i w^*_i}\} & \text{if } i \in I_2, \\ E(P^+_*(v,i)) \cup \{\overrightarrow{u_i w^{**}_i}\} & \text{if } i \in I_3, \\ E(P^+_*(v,i)) & \text{if } i \in I_4. \end{cases}$$

We claim that $\{P^+(v,i)\}_{i=1}^k$ is a k-fan from v to $V \setminus (A \cup B)$, and the sum of lengths is small. Indeed, for any $i \in [k]$, $P^+(v,i)$ is a path from v to $V \setminus (A \cup B)$. Note that paths $\{V(P^+(v,i))\}_{i=1}^k$ form a k-fan since the paths $\{V(P^+_*(v,i)) \setminus \{v\}\}_{i=1}^k$ are pairwise-disjoint, and U, W, W', W^*, W^{**} are pairwise disjoint. Moreover,

$$\sum_{i=1}^k |E(P^+(v,i))| = \sum_{i=1}^k |E(P^+_*(v,i))| + 2|I_1| + |I_2| + |I_3| \leq 98k\log(k+1) + 2k \leq 100k\log(k+1).$$

This proves Claim 5.3. □

Recall that V_1, V'_1 and E_0 are defined in (5.4) and note that we have $\{a_{i_1}, \ldots, a_{i_k}, b_{j_1}, \ldots, b_{j_k}\} \subseteq V_1$. Now we will find a set of arcs E_1 as in the following claim.

Claim 5.4. *There exist a set of arcs $E_1 \subseteq E(T)$ and a set of vertices $V_2 \subseteq V \setminus (A \cup B)$ satisfying the following.*

- (E1)$_1$ $|E_1| \leq k|V_1| + (k-1)|V'_1| + 680k^2\log(k+1)$ *and* $|V_2| \leq 8k^2$.
- (E1)$_2$ *For any set $S \subseteq V(T)$ of size $k-1$ and a vertex $v \in (V_1 \cup V'_1) \setminus S$, we can find a path P in $T - S$ from v to V_2 such that $E(P) \subseteq E_0 \cup E_1$.*
- (E1)$_3$ *For any set $S \subseteq V(T)$ of size $k-1$ and a vertex $v \in (V_1 \cup V'_1) \setminus S$, we can find a path P in $T - S$ from V_2 to v such that $E(P) \subseteq E_0 \cup E_1$.*

Proof of Claim 5.4. We apply Lemma 3.4 to $T[V_1]$ with parameters $0, k$ corresponding to s, k, respectively. Then we obtain an ordering σ_1 of V_1 with a $(\sigma_1, k, 2k-1)$-good digraph $D_1 \subseteq T[V_1]$ such that $|E(D_1)| \leq k|V_1| - k$. We also consider a digraph $T[V'_1] - E_0$. Since $\delta(T[V'_1] - E_0) \geq |V'_1| - 3$, we can apply Lemma 3.4 to $T[V'_1] - E_0$ with parameters $2, (k-1)$ corresponding to s, k, respectively. Then we obtain an ordering σ'_1 of V'_1 and a $(\sigma'_1, k-1, 2k-1)$-good digraph $D'_1 \subseteq T[V'_1] - E_0$ with $|E(D'_1)| \leq (k-1)|V'_1| + (k-1)$. Here, it is important to take $(\sigma'_1, k-1, 2k-1)$-good subgraph of $T[V'_1] - E_0$ instead of $(\sigma'_1, k, 2k-1)$-good subgraph of $T[V'_1]$, otherwise we would get $|E(D'_1)| \leq k|V'_1| + k$ which is too much for our purpose.

Now we define W^-_1 and W^+_1 as follows.

$$W^-_1 := \sigma_1(1, 2k-1) \cup \sigma'_1(1, 2k-1) \text{ and } W^+_1 := \sigma_1(|V_1| - 2k + 1, |V_1|) \cup \sigma'_1(|V'_1| - 2k + 1, |V'_1|)$$

This gives

$$|W_1^-|, |W_1^+| \leq 4k - 2. \tag{5.9}$$

For each vertex $u \in W_1^-$ we use Claim 5.1 to obtain a k-fan $\{P^-(u,1), \dots, P^-(u,k)\}$ in T from $V \setminus (A \cup B)$ to u with

$$\sum_{i=1}^{k} |E(P^-(u,i))| \leq 70k \log(k+1). \tag{5.10}$$

For each vertex $u \in W_1^+$, we use Claim 5.3 to obtain a k-fan $\{P^+(u,1), \dots, P^+(u,k)\}$ in T from u to $V \setminus (A \cup B)$ with

$$\sum_{i=1}^{k} |E(P^+(u,i))| \leq 100k \log(k+1). \tag{5.11}$$

Let

$$E_1 := E(D_1) \cup E(D_1') \cup \bigcup_{u \in W_1^-, i \in [k]} E(P^-(u,i)) \cup \bigcup_{u \in W_1^+, i \in [k]} E(P^+(u,i)), \tag{5.12}$$

$$V_2 := V(E_1) \setminus (V_1 \cup V_1').$$

Since $V_1 \cup V_1' = A \cup B$, every vertex in V_2 is either one of the last vertices of $P^+(u,i)$ for some $i \in [k]$ and $u \in W_1^+$ or one of the first vertex of $P^-(u,i)$ for some $i \in [k]$ and $u \in W_1^-$. Thus we have $|V_2| \leq k(|W_1^+| + |W_1^-|) \overset{(5.9)}{\leq} 8k^2$. Moreover,

$$\begin{aligned} |E_1| &\overset{(5.10),(5.11)}{\leq} |E(D_1)| + |E(D_2)| + 70k \log(k+1)|W_1^-| + 100k \log(k+1)|W_1^+| \\ &\overset{(5.9)}{\leq} k|V_1| + (k-1)|V_1'| + 680k^2 \log(k+1). \end{aligned}$$

This proves $(\mathrm{E1})_1$. To prove $(\mathrm{E1})_2$, let S be a set of $k-1$ vertices in V and let v be a vertex with $v \in (V_1 \cup V_1') \setminus S$. We consider the following two cases.

Case 1. $v \in V_1$.

By Claim 3.1 and the fact that D_1 is $(\sigma_1, k, 2k-1)$-good, we can find a path P' from v to a vertex $u \in W_1^+$ in $T - S$ such that $E(P') \subseteq E_1$. Also $P^+(u,1), \dots, P^+(u,k)$ are disjoint paths except the common starting vertex $u \notin S$, thus there exists $j \in [k]$ such that $P^+(u,j)$ does not intersect with S. Then $E(P') \cup E(P^+(u,j))$ contains a path P in $T - S$ from v to V_2 with $E(P) \subseteq E_1$.

Case 2. $v \in V_1'$.

Assume $\sigma_1' = (v_1', \dots, v_{|V_1'|}')$. We consider the maximum index i such that there is a path P' from v to v_i' in $D_1' - S$. If $i \geq |V_1'| - 2k + 2$, then we have $v_i' \in W_1^+$ and we can choose $j \in [k]$ such that $P^+(v_i', j)$ does not intersect with S. Then $E(P') \cup E(P^+(v_i', j))$ contains a path P in $T - S$ from v to V_2 with $E(P) \subseteq E_1$. If $i < |V_1'| - 2k + 2$, then the maximality of i implies $N^+_{D_1'}(v_i') \subseteq S$ by (D1) and the fact that D_1' is $(\sigma_1', k-1, 2k-1)$-good. Since

$$k - 1 \overset{(\mathrm{D2})}{\leq} |N^+_{D_1'}(v_i')| \leq |S| = k - 1,$$

we have

$$S = N^+_{D_1'}(v_i'). \tag{5.13}$$

By (5.4) and the fact that $v_i' \in V_1'$, there exists $s \in [k]$ such that $v_i' \in V^{\mathrm{int}}(P_s)$. We let P'' be the sub-path of P_s from v_i' to b_{j_s}. Since P_s is backwards-transitive, every vertex in $V(P'')$ belongs to $N^-_T(v_i')$ except the first vertex v_i' and the second vertex, say u', of P''. Since $\overrightarrow{v_i' u'} \in E(P_s) \subseteq E_0$

and $D'_1 \subseteq T[V'_1] - E_0$, we obtain $\overrightarrow{v'_i u'} \notin E(D'_1)$. Thus $u' \notin N^+_{D'_1}(v'_i)$. This with the fact that $V(P'') \subseteq N^-_T(v'_i) \cup \{v'_i, u'\}$ implies that

$$V(P'') \cap S \subseteq (N^-_T(v'_i) \cup \{v'_i, u'\}) \cap S \overset{(5.13)}{=} (N^-_T(v'_i) \cup \{v'_i, u'\}) \cap N^+_{D'_1}(v'_i) = \emptyset.$$

Thus P'' does not intersect with S. Since $b_{j_s} \in V_1$, Case 1 implies that there exists a path P^* from b_{j_s} to V_2 in $T[V \setminus S]$ with $E(P^*) \subseteq E_1$. Then $E(P') \cup E(P'') \cup E(P^*)$ contains a path P in $T - S$ from v to V_2 with $E(P) \subseteq E_0 \cup E_1$. Thus we have (E1)$_2$. We can prove (E1)$_3$ in a similar way. This proves Claim 5.4. □

Claim 5.5. *There exist a set of arcs $E_2 \subseteq E(T)$ and two sets $W^+_2, W^-_2 \subseteq V_2$ satisfying the following.*

(E2)$_1$ $|E_2| \leq k|V_2| - k$ *and* $|W^+_2|, |W^-_2| \leq 2k - 1$.
(E2)$_2$ *For a set $S \subseteq V(T)$ of size $k-1$ and a vertex $v \in V_2 \setminus S$, there exists a path P in $T - S$ from v to W^+_2 with $E(P) \subseteq E_2$.*
(E2)$_3$ *For a set $S \subseteq V(T)$ of size $k-1$ and a vertex $v \in V_2 \setminus S$, there exists a path P in $T - S$ from W^-_2 to v with $E(P) \subseteq E_2$.*

Proof of Claim 5.5. We apply Lemma 3.4 to $T[V_2]$ with parameters $0, k$ corresponding to s, k, respectively. Then we obtain an ordering σ_2 of V_2 and a $(\sigma_2, k, 2k-1)$-good digraph $D_2 \subseteq T[V_2]$ such that $|E(D_2)| \leq k|V_2| - k$. Let

$$E_2 := E(D_2),\ \ W^-_2 := \sigma_1(1, 2k-1)\ \text{ and }\ W^+_2 := \sigma_1(|V_2| - 2k + 2, |V_2|),$$

then we have $|E_2| = |E(D_2)| \leq k|V_2| - k$ and $|W^-_2|, |W^+_2| \leq 2k-1$. Hence we have (E2)$_1$. Since D_2 is $(\sigma_2, k, 2k-1)$-good, Claim 3.1 implies that for any set S of $k-1$ vertices in V and a vertex $v \in V_2 \setminus S$, we can find a path P in $T - S$ from v to W^+_2 and a path P' in $T - S$ from W^-_2 to v such that $E(P), E(P') \subseteq E_2$, proving (E2)$_2$ and (E2)$_3$. □

Now we define V_3, V_4 as follows.

$$V_3 := \bigcup_{i=1}^{k} V^{\text{int}}(P_i) \setminus (V'_1 \cup V_2) \quad \text{and} \quad V_4 := V \setminus (V_1 \cup V'_1 \cup V_2 \cup V_3). \tag{5.14}$$

Claim 5.6. *There exist a set of arcs $E_3 \subseteq E(T)$ and two sets $W^+_3, W^-_3 \subseteq V_3$ satisfying the following.*

(E3)$_1$ $|E_3| \leq (k-1)|V_3| + (k-1)$ *and* $|W^+_3|, |W^-_3| \leq 2k - 1$.
(E3)$_2$ *For a set $S \subseteq V(T)$ of size $k-1$ and a vertex $v \in V_3 \setminus S$, there exists a path P in $T - S$ from v to $W^+_3 \cup V_1$ with $E(P) \subseteq E_0 \cup E_3$.*
(E3)$_3$ *For a set $S \subseteq V(T)$ of size $k-1$ and a vertex $v \in V_3 \setminus S$, there exists a path P in $T - S$ from $W^-_3 \cup V_1$ to v with $E(P) \subseteq E_0 \cup E_3$.*

Proof of Claim 5.6. Consider a digraph $T[V_3] - E_0$. Note that $\delta(T[V_3] - E_0) \geq |V_3| - 3$. Apply Lemma 3.4 to $T[V_3] - E_0$ with parameters $2, k-1$ corresponding to s, k, respectively. Then we obtain an ordering $\sigma_3 = (v_1, \dots, v_{|V_3|})$ and a $(\sigma_3, k-1, 2k-1)$-good digraph $D_3 \subseteq T[V_3] - E_0$ with $|E(D_3)| \leq (k-1)|V_3| + (k-1)$. Here, it is important to take $(\sigma_3, k-1, 2k-1)$-good subgraph of $T[V_3] - E_0$ instead of $(\sigma_3, k, 2k-1)$-good subgraph of $T[V_3]$, otherwise we would get $|E(D_3)| \leq k|V_3| - k$ instead of (E3)$_1$.

Let

$$E_3 := E(D_3),\ \ W^-_3 := \sigma_3(1, 2k-1)\ \text{ and }\ W^+_3 := \sigma_3(|V_3| - 2k + 2, |V_3|).$$

This verifies (E3)$_1$. To verify (E3)$_2$, we consider a set $S \subseteq V(T)$ with $k-1$ vertices and a vertex $v \in V_3 \setminus S$. Then we consider a path P' in $D_3 - S$ with $E(P') \subseteq E(D_3)$ from v to v_i which maximizes i. If $i \geq |V_3| - 2k + 2$, then $v_i \in W^+_3$ and we are done. If $i < |V_3| - 2k + 2$, the maximality of i implies $N^+_{D_3}(v_i) \subseteq S$ by (D1) and the fact that D_3 is $(\sigma, k-1, 2k-1)$-good. Since

$$k - 1 \overset{\text{(D2)}}{\leq} |N^+_{D_3}(v_i)| \leq |S| = k - 1,$$

we have $S = N^+_{D_3}(v_i)$. Because $v_i \in V_3$, by (5.14) there exists $s \in [3k-1]$ such that $v_i \in V^{\rm int}(P_s)$. We let P'' be the sub-path of P_s from v_i to b_{j_s}. Since P_s is backwards-transitive, every vertex in $V(P'')$ should be in $N^-_T(v_i)$ except v_i and the second vertex, say u', of P''. Since $\overrightarrow{v_iu'} \in E_0$ and $E(D_3) \subseteq T[V_3] - E_0$, $u' \notin N^+_{D_3}(v_i)$. Thus

$$V(P'') \cap S \subseteq (N^-_T(v_i) \cup \{v_i, u'\}) \cap N^+_{D_3}(v_i) = \emptyset.$$

Thus P'' does not intersect with S. So $E(P') \cup E(P'')$ contains a path P in $T - S$ from v to V_1 with $E(P) \subseteq E_0 \cup E_3$. This proves (E3)$_2$. We can prove (E3)$_3$ in a similar way. This proves Claim 5.6. □

Claim 5.7. *There exist a set of arcs $E_4 \subseteq A(T)$ and two sets $W^+_4, W^-_4 \subseteq V_4$ satisfying the following.*

(E4)$_1$ $|E_4| \leq k|V_4| - k$ *and* $|W^+_4|, |W^-_4| \leq 2k - 1$.
(E4)$_2$ *For a set $S \subseteq V(T)$ of size $k-1$ and a vertex $v \in V_4 \setminus S$, there exists a path P in $T - S$ from v to W^+_4 with $E(P) \subseteq E_4$.*
(E4)$_3$ *For a set $S \subseteq V(T)$ of size $k-1$ and a vertex $v \in V_4 \setminus S$, there exists a path P in $T - S$ from W^-_4 to v with $E(P) \subseteq E_4$.*

Proof of Claim 5.7. We apply Lemma 3.4 to $T[V_4]$ with parameters $0, k$ corresponding to s, k, respectively. Then we obtain an ordering σ_4 and a $(\sigma_4, k, 2k-1)$-good digraph $D_4 \subseteq T[V_4]$ with $|E(D_4)| \leq k|V_4| - k$. Let

$$E_4 := E(D_4),\ \ W^+_4 := \sigma_4(|V_4| - 2k + 2, |V_4|) \text{ and } W^-_4 := \sigma_4(1, 2k-1),$$

then we have $|E_4| = |E(D_4)| \leq k|V_4| - k$, $|W^-_4| \leq 2k - 1$ and $|W^+_4| \leq 2k - 1$. Hence (E4)$_1$ holds. By Claim 3.1, for any $S \subseteq V(T)$ of $k-1$ vertices and $v \in V_4 \setminus S$, we can find a path P in $T[V_4] \setminus S$ from v to W^+_4 and a path P' in $T[V_4] \setminus S$ from W^-_4 to v. This proves (E4)$_2$ and (E4)$_3$. This proves Claim 5.7. □

We define W^+ and W^- as follows.

$$W^+ := W^+_2 \cup W^+_3 \cup W^+_4 \quad \text{and} \quad W^- := W^-_2 \cup W^-_3 \cup W^-_4.$$

Note that $W^+, W^- \subseteq V \setminus (A \cup B)$. Thus A in-dominates W^+ and B out-dominates W^-. Now we take E_5 as follows to make connections from W^+ to $\{a_{i_1}, \dots, a_{i_k}\}$ and from $\{b_{j_1}, \dots, b_{j_k}\}$ to W^-.

Claim 5.8. *There exists a set of arcs $E_5 \subseteq E(T)$ satisfying the following.*

(E5)$_1$ $|E_5| \leq 81k^2$
(E5)$_2$ *For $t \in [k]$, a vertex $v \in W^+$ and a set $S \subseteq V(T) \setminus \{a_{i_t}, v\}$ of at most $k-1$ vertices, there exists a path $P(v,t)$ in $T - S$ from v to a_{i_t} such that $E(P(v,t)) \subseteq E_5$.*
(E5)$_3$ *For $t \in [k]$, a vertex $v \in W^-$ and a set $S \subseteq V(T) \setminus \{b_{j_t}, v\}$ of at most $k-1$ vertices, there exists a path $Q(v,t)$ in $T - S$ from b_{j_t} to v such that $E(Q(v,t)) \subseteq E_5$.*

Proof of Claim 5.8. By (A2) and (A3), for each $u \in W^+$ and $s \in [3k-1]$ there exists $c_{u,s} \in N^+_T(u) \cap A_s$ such that $c_{u,s} = a_s$ or $a_s \in N^+_T(c_{u,s})$. Let

$$P(u,s) := \begin{cases} (u, c_{u,s}, a_s) & \text{if } c_{u,s} \neq a_s, \\ (u, a_s) & \text{otherwise.} \end{cases}$$

Similarly, for $u \in W^-$ and $s \in [3k-1]$, there is a path $Q(u,s)$ from b_s to u with length at most 2 lying entirely in $B_s \cup \{u\}$. Let

$$E_5 := E(T[A_{\rm sink}]) \cup E(T[B_{\rm source}]) \cup \bigcup_{u \in W^+} \bigcup_{s=1}^{3k-1} E(P(u,s)) \cup \bigcup_{u \in W^-} \bigcup_{s=1}^{3k-1} E(Q(u,s)).$$

Then we have

$$|E_5| \le |E(T[A_{\text{sink}}])| + |E(T[B_{\text{source}}])| + \sum_{u\in W^+}\sum_{s=1}^{3k-1}|E(P(u,s))| + \sum_{u\in W^-}\sum_{s=1}^{3k-1}|E(Q(u,s))|$$
$$\le \binom{3k-1}{2} + \binom{3k-1}{2} + (6k-2)|W^+| + (6k-2)|W^-| \le 81k^2.$$

We get the final inequality from $(\text{E2})_1$, $(\text{E3})_1$ and $(\text{E4})_1$. To verify $(\text{E5})_2$, consider a set S of $k-1$ vertices and an index $t \in [k]$ such that $a_{i_t} \notin S$ and a vertex $v \in W^+ \setminus S$. Recall that a_{i_t} has at least k in-neighbors in A_{sink} as defined before Claim 5.1. This together with the fact that $A_1, \dots, A_{3k-1}$ are pairwise disjoint implies that there exists an index $s \in [3k-1]$ such that $a_s \in N_T^-(a_{i_t})$ and $A_s \cap S = \emptyset$. Then $P(v,s) \cup \overrightarrow{a_s a_{i_t}}$ contains a path P from v to a_{i_t}, where P does not intersect with S because P is contained in $A_s \cup \{v\} \cup \{a_{i_t}\}$. Also $E(P) \subseteq E_5$, this proves $(\text{E5})_2$. We can also prove $(\text{E5})_3$ similarly. This proves Claim 5.8. $\square$

Now we define the desired spanning strongly k-connected digraph $D \subseteq T$. Let

$$V(D) := V(T) \quad \text{and} \quad E(D) := E_0 \cup E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5.$$

Because $\bigcup_{s=1}^{k} V^{\text{int}}(P_s) \subseteq V_1' \cup V_2 \cup V_3$, we have $|E_0| \le |V_1'| + |V_2| + |V_3| - k$. By $(\text{E1})_1$, $(\text{E2})_1$, $(\text{E3})_1$, $(\text{E4})_1$ and $(\text{E5})_1$ we have

$$\begin{aligned}
|E(D)| &\le |E_0| + |E_1| + |E_2| + |E_3| + |E_4| + |E_5| \\
&\le (|V_1'| + |V_2| + |V_3| - k) + (k|V_1| + (k-1)|V_1'| + 680k^2\log(k+1)) + (k|V_2| - k) \\
&\quad + ((k-1)|V_3| + (k-1)) + (k|V_4| - k) + 81k^2 \\
&\le k(|V_1| + |V_1'| + |V_2| + |V_3| + |V_4|) + |V_2| + 740k^2\log(k+1) \\
&\overset{(\text{E1})_1}{\le} k|V| + 750k^2\log(k+1)
\end{aligned}$$

since $680k^2\log(k+1) + 81k^2 \le 740k^2\log(k+1)$ for $k \ge 2$.

Now it suffices to show that D is strongly k-connected. For any set $S \subseteq V(T)$ of $k-1$ vertices and any two distinct vertices $u, v \in V(T) \setminus S$, we claim that there is a path from u to v in $D-S$. First of all, since $P_1, \dots, P_k$ are vertex-disjoint there exists $t \in [k]$ such that $V(P_t) \cap S = \emptyset$. We find a path P in $D-S$ from u to $u' \in W^+$ as follows.

Case 1. $u \in V_2 \cup V_4$.
There exists a path P in $D-S$ from u to $u' \in W^+$ by $(\text{E2})_2$ and $(\text{E4})_2$.

Case 2. $u \in V_1 \cup V_1'$.
By $(\text{E1})_2$, there is a path Q in $D-S$ from u to a vertex $u_0 \in V_2$. Also $(\text{E2})_2$ implies that there is a path Q' in $D-S$ from u_0 to $u' \in W^+$. Thus $E(Q) \cup E(Q')$ contains a path P in $D-S$ from u to $u' \in W^+$.

Case 3. $u \in V_3$.
By $(\text{E3})_2$, there is a path R in $D-S$ from u to a vertex $u_0 \in W^+ \cup V_1$. If $u_0 \in W^+$, then let $u' = u_0$ and $P := R$. Otherwise, there is a path R' in $D-S$ from u_0 to $u' \in W^+$ by Case 2. Thus $E(R) \cup E(R')$ contains a path P in $D-S$ from u to $u' \in W^+$.

Similarly, there is a path Q in $D-S$ from a vertex $v' \in W^-$ to v. By Claim 5.8, there is a path $P(u',t)$ in $D-S$ from u' to a_{i_t}, and a path $Q(v',t)$ in $D-S$ from b_{j_t} to v'. Thus $E(P) \cup E(P(u',t)) \cup E(P_t) \cup E(Q(v',t)) \cup E(Q)$ contains a path in $D-S$ from u to v. This proves that D is strongly k-connected. $\square$

Algorithmic aspect of Theorem 1.3. The proof of Theorem 1.3 is trivially algorithmic up to the following three optimization problems: finding a k-fan from a fixed vertex to a set with minimum total length, finding a maximum matching in a bipartite graph, and finding k vertex-disjoint paths between two sets with minimum total length. These optimization problems can be solved in polynomial-time on $n = |V(T)|$ by standard application of algorithms finding maximum-flows and minimum cost flows of digraphs (see [1, Chapter 7,8 and 9]). Note that when we apply Lemma 3.4, we use Claim 3.3 to find the ordering σ and a subgraph D in polynomial time on n. With these tools, the proof itself immediately gives a polynomial-time algorithm to find the desired digraph D as in Theorem 1.3.

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