Almost Sure and Moment Exponential Stability of Regime-Switching Jump Diffusions

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October 4, 2018

Abstract

This work is devoted to almost sure and moment exponential stability of regime-switching jump diffusions. The Lyapunov function method is used to derive sufficient conditions for stabilities for general nonlinear systems; which further helps to derive easily verifiable conditions for linear systems. For one-dimensional linear regime-switching jump diffusions, necessary and sufficient conditions for almost sure and pth moment exponential stabilities are presented. Several examples are provided for illustration.

Keywords. Regime-switching jump diffusion, almost sure exponential stability, pth moment exponential stability, Lyapunov exponent, Poisson random measure.

Mathematics Subject Classification. 60J60, 60J75, 47D08.

1 Introduction

Applications of stochastic analysis have emerged in various areas such as financial engineering, wireless communications, mathematical biology, and risk management. One of the salient features of such systems is the coexistence of and correlation between continuous dynamics and discrete events. Often, the trajectories of these systems are not continuous: there is day-to-day jitter that causes minor fluctuations as well as big jumps caused by rare events arising from, e.g., epidemics, earthquakes, tsunamis, or terrorist atrocities. On the other hand, the systems often display qualitative changes. For example, as demonstrated in

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Barone-Adesi and Whaley (1987), the volatility and the expected rate of return of an asset are markedly different in the bull and bear markets. Regime-switching diffusion with Lévy type jumps naturally captures these inherent features of these systems: the Lévy jumps are well-known to incorporate both small and big jumps (Applebaum (2009), Cont and Tankov (2004)) while the regime switching mechanisms provide the qualitative changes of the environment (Mao and Yuan (2006), Yin and Zhu (2010)). In other words, regime-switching diffusion with Lévy jumps provides a uniform and realistic yet mathematically tractable platform in modeling a wide range of applications. Consequently increasing attention has been drawn to the study of regime-switching jump diffusions in recent years. Some recent work in this vein can be found in Shao and Xi (2014), Xi (2009), Yin and Xi (2010), Zhu et al. (2015), Zong et al. (2014) and the references therein.

Regime-switching jump diffusion processes can be viewed as jump diffusion processes in random environments, in which the evolution of the random environments is modeled by a continuous-time Markov chain or more generally, a continuous-state-dependent switching process with a discrete state space. Seemingly similar to the usual jump diffusion processes, the behaviors of regime-switching jump diffusion processes can be markedly different. For example, (Yin and Zhang, 2013, Section 5.6) illustrates that two stable diffusion processes can be combined via a continuous-time Markov chain to produce an unstable regime-switching diffusion process. See also Costa et al. (2013) for similar observations.

This paper aims to investigate almost sure and moment exponential stability for regime-switching diffusions with Lévy type jumps. This is motivated by the recent advances in the investigations of stability of regime-switching jump diffusions in Yin and Xi (2010), Zong et al. (2014) and the references therein. In Yin and Xi (2010), Zong et al. (2014), the Lévy measure ν on some measure space $(U, \mathfrak{U})$ is assumed to be a finite measure with $\nu(U) < \infty$. Consequently, in these models, the jump mechanism is modeled by compound Poisson processes and there are finitely many jumps in any finite time interval. In contrast, in our formulation, the Lévy measure ν on $(\mathbb{R}^n - \{0\}, \mathcal{B}(\mathbb{R}^n - \{0\}))$ merely satisfies $\int_{\mathbb{R}^n - \{0\}} (1 \wedge |z|^2) \nu(\mathrm{d}z) < \infty$ and hence it is not necessarily finite. This formulation allows the possibility of infinite number of "small jumps" in a finite time interval. Indeed, such "infinite activity models" are studied in the finance literature, such as the variance gamma model in Seneta (2004) and the normal inverse Gaussian model in Barndorff-Nielsen (1998). See also the recent paper Barndorff-Nielsen et al. (2013) for energy spot price modeling using Lévy processes.

Our focus of this paper is to study almost sure and moment exponential stabilities of the equilibrium point $x = 0$ of regime-switching jump diffusion processes. To this end, we first observe the "nonzero" property, which asserts that almost all sample paths of all solutions to (2.3) starting from a nonzero initial condition will never reach the origin with probability one. This phenomenon was first established for diffusion processes in Khasminskii (2012) and later extended to regime-switching diffusions in Mao and Yuan (2006), Yin and Zhu (2010) under the usual Lipschitz and linear growth conditions. For processes with Lévy type jumps, additional assumptions are needed to handle the jumps to obtain the "nonzero" property. For instance, Applebaum and Siakalli (2009) and Wee (1999) contain different sufficient conditions. The differences are essentially on the assumptions concerning the jumps. Here we propose a different sufficient condition than those in Applebaum and Siakalli (2009), Wee

(1999) for the "nonzero" property for regime-switching jump diffusion. We show in Lemma 2.6 that the "nonzero" property holds under the usual Lipschitz and linear growth conditions on the coefficients of (2.3) together with Assumption 2.4. Note that it is quite easy to verify Assumption 2.4 in many practical situations; see, for example, the discussions in Remark 2.5.

With the "nonzero" property at our hands, we proceed to obtain sufficient conditions for almost sure and pth moment exponential stabilities of the equilibrium point of nonlinear regime-switching jump diffusions. Similar to the related results in Applebaum and Siakalli (2009) for jump diffusions, these sufficient conditions for stability are expressed in terms of the existence of appropriate Lyapunov functions. The details are spelled out in Theorems 3.1 and 3.4, and Corollary 3.3. Also, as observed in Costa et al. (2013), Yin and Zhang (2013), Yin and Zhu (2010) for regime-switching diffusions, our results demonstrate that the switching mechanism can contribute to the stabilization or destabilization of jump diffusion processes. Next we show in Theorem 3.5 that pth ($p \geq 2$) moment exponential stability implies almost sure exponential stability for regime-switching jump diffusions under a certain integrability condition on the jump term. Such a result has been established for diffusions in Khasminskii (2012), jump diffusions in Applebaum and Siakalli (2009), and regime-switching diffusions in Mao and Yuan (2006). In addition, we derive a sufficient condition for pth moment exponential stability using M-matrices in Theorem 3.7.

The aforementioned general results are then applied to treat linear regime-switching jump diffusions. For one-dimensional systems, we obtain necessary and sufficient conditions for almost sure and pth moment exponential stabilities in Propositions 4.1 and 4.5, respectively. For the multidimensional system, we present verifiable sufficient conditions for almost sure and moment exponential stability in Propositions 4.2, 4.6, and 4.7. To illustrate the results, we also study several examples in Section 4.3.

The remainder of the paper is organized as follows. After a brief introduction to regime-switching jump diffusion processes in Section 2, we proceed to deriving sufficient conditions for almost sure and pth moment exponential stabilities of the equilibrium point of the nonlinear system (2.3) in Section 3. Section 4 treats stability of the equilibrium point of linear systems. Finally we conclude the paper with conclusions and remarks in Section 5.

To facilitate the presentation, we introduce some notation that will be used often in later sections. Throughout the paper, we use x' to denote the transpose of x, and $x'y$ or $x \cdot y$ interchangeably to denote the inner product of the vectors x and y. If A is a vector or matrix, then $|A| := \sqrt{\operatorname{tr}(AA')}$, $\|A\| := \sup\{|Ax| : x \in \mathbb{R}^n, |x| = 1\}$, and $A \gg 0$ means that every element of A is positive. For a square matrix A, $\rho(A)$ is the spectral radius of A. Moreover if A is a symmetric square matrix, then $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the maximum and minimum eigenvalues of A, respectively. For sufficiently smooth function $\phi : \mathbb{R}^n \to \mathbb{R}$, $D_{x_i}\phi = \frac{\partial \phi}{\partial x_i}$, $D_{x_i x_j}\phi = \frac{\partial^2 \phi}{\partial x_i \partial x_j}$, and we denote by $D\phi = (D_{x_1}\phi, \dots, D_{x_n}\phi)' \in \mathbb{R}^n$ and $D^2\phi = (D_{x_i x_j}\phi) \in \mathbb{R}^{n\times n}$ the gradient and Hessian of ϕ, respectively. For $k \in \mathbb{N}$, $C^k(\mathbb{R}^n)$ is the collection of functions $f : \mathbb{R}^n \mapsto \mathbb{R}$ with continuous partial derivatives up to the kth order while $C_c^k(\mathbb{R}^n)$ denotes the space of C^k functions with compact support. If B is a set, we use B^o and I_B to denote the interior and indicator function of B, respectively. Throughout the paper, we adopt the conventions that $\sup \emptyset = -\infty$ and $\inf \emptyset = +\infty$.

2 Formulation

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual condition on which is defined an n-dimensional standard $\mathcal{F}_t$-adapted Brownian motion $W(\cdot)$. Let $\{\psi(\cdot)\}$ be an $\mathcal{F}_t$-adapted Lévy process with Lévy measure $\nu(\cdot)$. Denote by $N(\cdot,\cdot)$ the corresponding $\mathcal{F}_t$-adapted Poisson random measure defined on $\mathbb{R}_+ \times \mathbb{R}^n_0$:

$$N(t, U) := \sum_{0<s\leq t} I_U(\Delta\psi_s) = \sum_{0<s\leq t} I_U(\psi(s) - \psi(s-)),$$

where $t \geq 0$ and U is a Borel subset of $\mathbb{R}^n_0 = \mathbb{R}^n - \{0\}$. The compensator $\tilde{N}$ of N is given by

$$\tilde{N}(\mathrm{d}t, \mathrm{d}z) := N(\mathrm{d}t, \mathrm{d}z) - \nu(\mathrm{d}z)\mathrm{d}t. \tag{2.1}$$

Assume that $W(\cdot)$ and $N(\cdot,\cdot)$ are independent and that $\nu(\cdot)$ is a Lévy measure satisfying

$$\int_{\mathbb{R}^n_0} (1 \wedge |z|^2)\nu(\mathrm{d}z) < \infty, \tag{2.2}$$

where $a_1 \wedge a_2 = \min\{a_1, a_2\}$ for $a_1, a_2 \in \mathbb{R}$.

We consider a stochastic differential equation with regime-switching together with Lévy-type jumps of the form

$$\begin{aligned} \mathrm{d}X(t) = & b(X(t), \alpha(t))\mathrm{d}t + \sigma(X(t), \alpha(t))\mathrm{d}W(t) \\ & + \int_{\mathbb{R}^n_0} \gamma(X(t-), \alpha(t-), z)\tilde{N}(\mathrm{d}t, \mathrm{d}z), \quad t \geq 0, \end{aligned} \tag{2.3}$$

with initial conditions

$$X(0) = x_0 \in \mathbb{R}^n, \quad \alpha(0) = \alpha_0 \in \mathcal{M}, \tag{2.4}$$

where $b(\cdot,\cdot) : \mathbb{R}^n \times \mathcal{M} \mapsto \mathbb{R}^n$, $\sigma(\cdot,\cdot) : \mathbb{R}^n \times \mathcal{M} \mapsto \mathbb{R}^{n\times n}$, and $\gamma(\cdot,\cdot,\cdot) : \mathbb{R}^n \times \mathcal{M} \times \mathbb{R}^n_0 \mapsto \mathbb{R}^n$ are measurable functions, and $\alpha(\cdot)$ is a switching component with a finite state space $\mathcal{M} := \{1, \ldots, m\}$ and infinitesimal generator $Q = (q_{ij}(x)) \in \mathbb{R}^{m\times m}$. That is, $\alpha(\cdot)$ satisfies

$$\mathbb{P}\left\{\alpha(t+\delta) = j | X(t) = x, \alpha(t) = i, \alpha(s), s \leq t\right\} = \begin{cases} q_{ij}(x)\delta + o(\delta), & \text{if } j \neq i, \\ 1 + q_{ii}(x)\delta + o(\delta), & \text{if } j = i, \end{cases} \tag{2.5}$$

as $\delta \downarrow 0$, where $q_{ij}(x) \geq 0$ for $i, j \in \mathcal{M}$ with $j \neq i$ and $q_{ii}(x) = -\sum_{j\neq i} q_{ij}(x) < 0$ for each $i \in \mathcal{M}$.

The evolution of the discrete component $\alpha(\cdot)$ in (2.3) can be represented by a stochastic integral with respect to a Poisson random measure; see, for example, Skorokhod (1989). In fact, for $x \in \mathbb{R}^n$ and $i, j \in \mathcal{M}$ with $j \neq i$, let $\Delta_{ij}(x)$ be the consecutive left-closed, right-open intervals of the half real line $\mathbb{R}_+ := [0, \infty)$, each having length $q_{ij}(x)$. In case $q_{ij}(x) = 0$, we set $\Delta_{ij}(x) = \emptyset$. Define a function $h : \mathbb{R}^n \times \mathcal{M} \times \mathbb{R} \mapsto \mathbb{R}$ by

$$h(x, i, z) = \sum_{j=1}^{m} (j - i) I_{\{z\in\Delta_{ij}(x)\}}. \tag{2.6}$$

Then the evolution of the switching process (2.5) can be represented by the stochastic differential equation

$$\mathrm{d}\alpha(t) = \int_{\mathbb{R}_+} h(X(t-), \alpha(t-), z) N_1(\mathrm{d}t, \mathrm{d}z), \tag{2.7}$$

where $N_1(\mathrm{d}t, \mathrm{d}z)$ is a Poisson random measure (corresponding to a random point process $\mathfrak{p}(\cdot)$) with intensity $\mathrm{d}t \times \lambda(\mathrm{d}z)$, and $\lambda(\cdot)$ is the Lebesgue measure on $\mathbb{R}$. Denote the compensated Poisson random measure of $N_1(\cdot)$ by $\tilde{N}_1(\mathrm{d}t, \mathrm{d}z) := N_1(\mathrm{d}t, \mathrm{d}z) - \mathrm{d}t \times \lambda(\mathrm{d}z)$. Throughout this paper, we assume that the Lévy process $\psi(\cdot)$, the random point process $\mathfrak{p}(\cdot)$, and the Brownian motion $W(\cdot)$ are independent.

We make the following assumptions throughout the paper:

Assumption 2.1. Assume

$$b(0, i) = \sigma(0, i) = \int_{\mathbb{R}_0^n} \gamma(0, i, z)\nu(\mathrm{d}z) = 0 \ \text{ for all } i \in \mathcal{M}. \tag{2.8}$$

Assumption 2.2. For some positive constant κ, we have

$$\begin{aligned} &|b(x, i) - b(y, i)|^2 + |\sigma(x, i) - \sigma(y, i)|^2 \\ &\qquad + \int_{\mathbb{R}_0^n} |\gamma(x, i, z) - \gamma(y, i, z)|^2 \nu(\mathrm{d}z) \le \kappa |x - y|^2, \end{aligned} \tag{2.9}$$

$$\int_{\mathbb{R}_0^n} \left[|\gamma(x, i, z)|^2 + |x \cdot \gamma(x, i, z)| \right] \nu(\mathrm{d}z) \le \kappa |x|^2 \tag{2.10}$$

for all $x, y \in \mathbb{R}^n$ and $i \in \mathcal{M} = \{1, \dots, m\}$, and that

$$\sup \{q_{ij}(x) : x \in \mathbb{R}^n, i \neq j \in \mathcal{M}\} \le \kappa < \infty. \tag{2.11}$$

Under Assumptions 2.1 and 2.2, $X(t) \equiv 0$ is an equilibrium point of (2.3). Moreover, in view of Zhu et al. (2015), for each initial condition $(x_0, \alpha_0) \in \mathbb{R}^n \times \mathcal{M}$, the system represented by (2.3) and (2.5) (or equivalently, (2.3) and (2.7)) has a unique strong solution $(X(\cdot), \alpha(\cdot)) = (X^{x_0,\alpha_0}(\cdot), \alpha^{x_0,\alpha_0}(\cdot))$; the solution does not explode in finite time with probability one. In addition, the generalized Itô lemma reads

$$f(X(t), \alpha(t)) - f(x_0, \alpha_0) = \int_0^t \mathcal{L} f(X(s-), \alpha(s-))\mathrm{d}s + M_1^f(t) + M_2^f(t) + M_3^f(t),$$

for $f \in C_c^2(\mathbb{R}^n \times \mathcal{M})$, where $\mathcal{L}$ is the operator associated with the process (X, α) defined by:

$$\begin{aligned} \mathcal{L} f(x, i) &= Df(x, i) \cdot b(x, i) + \frac{1}{2}\mathrm{tr}((\sigma\sigma')(x, i) D^2 f(x, i)) + \sum_{j \in \mathcal{M}} q_{ij}(x)[f(x, j) - f(x, i)] \\ &\quad + \int_{\mathbb{R}_0^n} [f(x + \gamma(x, i, z), i) - f(x, i) - Df(x, i) \cdot \gamma(x, i, z)]\nu(\mathrm{d}z), \ (x, i) \in \mathbb{R}^d \times \mathcal{M}, \end{aligned}$$

and

$$M_1^f(t) = \int_0^t Df(X(s-), \alpha(s-)) \cdot \sigma(X(s-), \alpha(s-))\mathrm{d}W(s),$$

$$M_2^f(t) = \int_0^t \int_{\mathbb{R}_+} \big[f(X(s-), \alpha(s-) + h(X(s-), \alpha(s-), z)) - f(X(s-), \alpha(s-))\big] \tilde{N}_1(\mathrm{d}s, \mathrm{d}z),$$
$$M_3^f(t) = \int_0^t \int_{\mathbb{R}_0^n} [f(X(s-) + \gamma(X(s-), \alpha(s-), z), \alpha(s-)) - f(X(s-), \alpha(s-))] \tilde{N}(\mathrm{d}s, \mathrm{d}z).$$

Similar to the terminologies in Khasminskii (2012), we have

Definition 2.3. The equilibrium point of (2.3) is said to be

(i) *almost surely exponentially stable* if there exists a $\delta > 0$ independent of $(x_0, \alpha_0) \in \mathbb{R}_0^n \times \mathcal{M}$ such that
$$\limsup_{t\to\infty} \frac{1}{t} \log |X^{x_0,\alpha_0}(t)| \le -\delta \text{ a.s.}$$

(ii) *exponentially stable in the pth moment* if there exists a $\delta > 0$ independent of $(x_0, \alpha_0) \in \mathbb{R}_0^n \times \mathcal{M}$ such that
$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}[|X^{x_0,\alpha_0}(t)|^p] \le -\delta.$$

To study stability of the equilibrium point of (2.3), we first present the following "nonzero" property, which asserts that almost all sample paths of all solutions to (2.3) starting from a nonzero initial condition will never reach the origin. This phenomenon was first established for diffusion processes in Khasminskii (2012) and later extended to regime-switching diffusions in Mao and Yuan (2006), Yin and Zhu (2010) under fairly general conditions. For processes with Lévy type jumps, additional assumptions are needed to handle the jumps.

Assumption 2.4. Assume there exists a constant $\varrho > 0$ such that
$$|x + \gamma(x, i, z)| \ge \varrho |x|, \text{ for all } (x, i) \in \mathbb{R}_0^n \times \mathcal{M} \text{ and } \nu\text{-almost all } z \in \mathbb{R}_0^n. \tag{2.12}$$

Remark 2.5. From Mao and Yuan (2006), Yin and Zhu (2010), we know that under Assumptions 2.1 and 2.2, a regimes-switching diffusion without jumps cannot "diffuse" from a nonzero state to zero a.s. Assumption 2.4 further prevents the process X of (2.3) jumps from a nonzero state to zero.

Also a sufficient condition for (2.12) is
$$2x \cdot \gamma(x, i, z) + |\gamma(x, i, z)|^2 \ge 0,$$
for ν-almost all $z \in \mathbb{R}_0^n$ and all $(x, i) \in \mathbb{R}^n \times \mathcal{M}$. Indeed, under such a condition, we have $|x + \gamma(x, i, z)|^2 = |x|^2 + 2x \cdot \gamma(x, i, z) + |\gamma(x, i, z)|^2 \ge |x|^2$ for ν-almost all $z \in \mathbb{R}_0^n$. This, of course, implies (2.12).

Lemma 2.6. *Suppose Assumptions 2.1, 2.2, and 2.4 hold. Then for any $(x, i) \in \mathbb{R}_0^n \times \mathcal{M}$, we have*
$$\mathbb{P}_{x,i}\{X(t) \ne 0 \text{ for all } t \ge 0\} = 1. \tag{2.13}$$

Proof. Consider the function $V(x,i) := |x|^{-2}$ for $x \neq 0$ and $i \in \mathcal{M}$. Direct calculations reveal that $DV(x,i) = -2|x|^{-4}x$, and $D^2V(x,i) = -2|x|^{-4}I + 8|x|^{-6}xx'$. Next we prove that for all $x, y \in \mathbb{R}^n$ with $x \neq 0$ and $|x+y| \geq \varrho|x|$, we have

$$V(x+y,i) - V(x,i) - DV(x,i)\cdot y = \frac{1}{|x+y|^2} - \frac{1}{|x|^2} + \frac{2x'y}{|x|^4} \leq K\frac{|y|^2 + |x'y|}{|x|^4}, \tag{2.14}$$

where K is some positive constant. Let us prove (2.14) in several cases:

Case 1: $x'y \geq 0$. In this case, it is easy to verify that for any $\theta \in [0,1]$, we have $|x+\theta y|^2 = |x|^2 + 2\theta x'y + \theta^2|y|^2 \geq |x|^2$. Therefore we can use the Taylor expansion with integral reminder to compute

$$\begin{aligned}|x+y|^{-2} - |x|^{-2} + 2|x|^{-4}x'y &= \int_0^1 \frac{1}{2} y\cdot D^2V(x+\theta y)y\,\mathrm{d}\theta \\ &= \int_0^1 \left[-\frac{|y|^2}{|x+\theta y|^4} + 4\frac{y'(x+\theta y)(x+\theta y)'y}{|x+\theta y|^6}\right]\mathrm{d}\theta \\ &\leq 4\int_0^1 \frac{|y|^2}{|x+\theta y|^4}\mathrm{d}\theta \leq 4\int_0^1 \frac{|y|^2}{|x|^4}\mathrm{d}\theta = \frac{4|y|^2}{|x|^4}.\end{aligned}$$

Case 2: $x'y < 0$ **and** $2x'y + |y|^2 \geq 0$. In this case, we have $|x+y|^2 = |x|^2 + 2x'y + |y|^2 \geq |x|^2$ and hence $|x+y|^{-2} - |x|^{-2} \leq 0$; which together with $x'y \leq 0$ implies that

$$|x+y|^{-2} - |x|^{-2} + 2|x|^{-4}x'y \leq 0.$$

Case 3: $x'y < 0$ **and** $2x'y + |y|^2 < 0$. In this case, we use the bound $|x+y| \geq \varrho|x|$ to compute

$$\begin{aligned}|x+y|^{-2} - |x|^{-2} + 2|x|^{-4}x'y &= \frac{1}{|x+y|^2} - \frac{1}{|x|^2} - \frac{|y|^2}{|x|^4} + \frac{2x'y + |y|^2}{|x|^4} \\ &= \frac{-2x'y}{|x|^2|x+y|^2} - \frac{|y|^2}{|x|^2|x+y|^2} - \frac{|y|^2}{|x|^4} + \frac{2x'y+|y|^2}{|x|^4} \\ &\leq \frac{2|x'y|}{\varrho^2|x|^4}.\end{aligned}$$

Combining the three cases gives (2.14).

Observe that (2.12) of Assumption 2.4 implies that if $x \neq 0$, then $x + c(x,i,z) \neq 0$ for ν-almost all $z \in \mathbb{R}_0^n$. Therefore we use Assumptions 2.1 and 2.2 and (2.14) to compute

$$\begin{aligned}\mathcal{L}V(x,i) &= -2\,|x|^{-4}\,x\cdot b(x,i) + \frac{1}{2}\mathrm{tr}\left[\sigma\sigma'(x,i)|x|^{-6}\left(-2\,|x|^2\,I + 8xx'\right)\right] \\ &\quad + \int_{\mathbb{R}_0^n}\left[|x+\gamma(x,i,z)|^{-2} - |x|^{-2} + 2|x|^{-4}x\cdot\gamma(x,i,z)\right]\nu(\mathrm{d}z) \\ &\leq 2\kappa|x|^{-2} + 4|\sigma(x,i)|^2|x|^{-4} + K|x|^{-4}\int_{\mathbb{R}_0^n}\left[|\gamma(x,i,z)|^2 + |x\cdot\gamma(x,i,z)|\right]\nu(\mathrm{d}z) \\ &\leq K|x|^{-2} = KV(x,i),\end{aligned} \tag{2.15}$$

where K is a positive constant.

Now consider the process (X, α) with initial condition $(X(0), \alpha(0)) = (x, i) \in \mathbb{R}^n_0 \times \mathcal{M}$. Define for $0 < \varepsilon < |x| < R$, $\tau_\varepsilon := \inf\{t \geq 0 : |X(t)| \leq \varepsilon\}$ and $\tau_R := \inf\{t \geq 0 : |X(t)| \geq R\}$. Then (2.15) allows us to derive

$$\begin{aligned}
&\mathbb{E}_{x,i}[e^{-K(t\wedge\tau_\varepsilon\wedge\tau_R)} V(X(t \wedge \tau_\varepsilon \wedge \tau_R), \alpha(t \wedge \tau_\varepsilon \wedge \tau_R))] \\
&\quad = V(x, i) + \mathbb{E}_{x,i}\left[\int_0^{t\wedge\tau_\varepsilon\wedge\tau_R} e^{-Ks}(-K + \mathcal{L})V(X(s), \alpha(s))\mathrm{d}s\right] \\
&\quad \leq V(x, i) = |x|^{-2}, \quad \text{for all } t \geq 0.
\end{aligned}$$

Note that on the set $\{\tau_\varepsilon < t\wedge\tau_R\}$, we have $V(X(t\wedge\tau_\varepsilon\wedge\tau_R), \alpha(t\wedge\tau_\varepsilon\wedge\tau_R)) = |X(t\wedge\tau_\varepsilon\wedge\tau_R)|^{-2} \geq \varepsilon^{-2}$. Thus it follows that

$$e^{-Kt}\varepsilon^{-2}\mathbb{P}_{x,i}\{\tau_\varepsilon < t \wedge \tau_R\} \leq \mathbb{E}_{x,i}[e^{-K(t\wedge\tau_\varepsilon\wedge\tau_R)} V(X(t \wedge \tau_\varepsilon \wedge \tau_R), \alpha(t \wedge \tau_\varepsilon \wedge \tau_R))] \leq |x|^{-2}.$$

It is well known that under Assumptions 2.1 and 2.2, the process X has no finite explosion time and hence $\tau_R \to \infty$ a.s. as $R \to \infty$. Therefore for any $t > 0$, we have $\mathbb{P}_{x,i}\{\tau_\varepsilon < t\} \leq e^{Kt}\varepsilon^2|x|^{-2}$. Passing to the limit as $\varepsilon \downarrow 0$, we obtain $\mathbb{P}_{x,i}\{\tau_0 < t\} = 0$ for any $t > 0$, where $\tau_0 := \inf\{t \geq 0 : X(t) = 0\}$. This gives (2.13) and hence completes the proof. □

3 Stability of Nonlinear Systems: General Results

This section is devoted to establishing sufficient conditions in terms of the existence of appropriate Lyapunov functions for stability of the equilibrium point of the system (2.3). Section 3.1 considers almost surely exponential stability while Section 3.2 studies pth moment exponential stability and demonstrates that pth moment exponential stability implies almost surely exponential stability under certain conditions. Finally we present a sufficient condition for stability using M-matrices in Section 3.3.

3.1 Almost Sure Exponential Stability

Theorem 3.1. *Suppose Assumptions 2.1, 2.2, and 2.4 hold. Let $V : \mathbb{R}^n \times \mathcal{M} \mapsto \mathbb{R}^+$ be such that $V(\cdot, i) \in C^2(\mathbb{R}^n)$ for each $i \in \mathcal{M}$. Suppose there exist $p > 0$, $c_1(i) > 0$, $c_2(i) \in \mathbb{R}$, and nonnegative constants $c_3(i)$, $c_4(i)$ and $c_5(i)$ such that for all $x \neq 0$ and $i \in \mathcal{M}$,*

(i) $c_1(i)|x|^p \leq V(x, i)$,

(ii) $\mathcal{L}V(x, i) \leq c_2(i)V(x, i)$,

(iii) $|DV(x, i) \cdot \sigma(x, i)|^2 \geq c_3(i)V(x, i)^2$,

(iv) $\displaystyle\int_{\mathbb{R}^n_0}\left[\log\left(\frac{V(x + \gamma(x, i, z), i)}{V(x, i)}\right) - \frac{V(x + \gamma(x, i, z), i)}{V(x, i)} + 1\right]\nu(\mathrm{d}z) \leq -c_4(i)$,

(v) $\displaystyle\sum_{j\in\mathcal{M}} q_{ij}(x)\left(\log V(x, j) - \frac{V(x, j)}{V(x, i)}\right) \leq -c_5(i)$.

Then

$$\limsup_{t\to\infty} \frac{1}{t} \log |X(t)| \le \frac{1}{p} \max_{i\in\mathcal{M}} \big\{ c_2(i) - 0.5c_3(i) - c_4(i) - c_5(i) \big\} =: \delta \quad a.s. \tag{3.1}$$

In particular, if $\delta < 0$ then the trivial solution of (2.3) is a.s. exponentially stable.

Remark 3.2. Conditions (iv) and (v) of Theorem 3.1 are natural because of the following observations. At one hand, using the elementary inequality $\log y \le y - 1$ for $y > 0$ we derive

$$\int_{\mathbb{R}_0^n} \left[\log\left(\frac{V(x+\gamma(x,i,z),i)}{V(x,i)} \right) - \frac{V(x+\gamma(x,i,z),i)}{V(x,i)} + 1 \right] \nu(\mathrm{d}z) \le 0;$$

this leads us to assume that the left-hand side of the above equation is bounded above by a nonpositive constant $-c_4(i)$ in condition (iv); however, the constant may depend on $i \in \mathcal{M}$. On the other hand, the inequality $\log y \le y - 1$ for $y > 0$ also leads to

$$\log V(x,j) - \frac{V(x,j)}{V(x,i)} \le \log V(x,i) - 1.$$

Then it follows that for every $i \in \mathcal{M}$,

$$\begin{aligned} &\sum_{j\in\mathcal{M}} q_{ij}(x) \left(\log V(x,j) - \frac{V(x,j)}{V(x,i)} \right) \\ &\quad = \sum_{j\neq i} q_{ij}(x) \left(\log V(x,j) - \frac{V(x,j)}{V(x,i)} \right) + q_{ii}(x)(\log V(x,i) - 1) \\ &\quad \le \sum_{j\neq i} q_{ij}(x) \left(\log V(x,i) - 1 \right) + q_{ii}(x)(\log V(x,i) - 1) = 0. \end{aligned}$$

In view of this observation, a nonpositive constant $-c_5(i)$ in condition (v) is therefore reasonable; again, this constant may depend on $i \in \mathcal{M}$. In fact, the constants $c_1(i), \dots, c_5(i)$ in Conditions (i)–(iv) may all depend on $i \in \mathcal{M}$; this allows for some extra flexibility for the selection of the Lyapunov function V and more importantly, the sufficient condition for a.s. exponential stability in Theorem 3.1 and Corollary 3.3.

It is also worth poiting out that conditions (i)–(iv) are similar to those in Theorem 3.1 of Applebaum and Siakalli (2009). Condition (v) is needed so that we can control the fluctuations of $\frac{1}{t}\log|X(t)|$ due to the presence of regime switching.

The proof of Theorem 3.1 is a straightforward extension of that of Theorem 3.1 of Applebaum and Siakalli (2009); some additional care are needed due to the presence of regime-switching. For completeness and also to preserve the flow of presentation, we relegate the proof to the Appendix A.

Corollary 3.3. *In addition to the conditions of Theorem 3.1, suppose also that the discrete component α in (2.3) and (2.7) is an irreducible continuous-time Markov chain with an invariant distribution $\pi = (\pi_i, i \in \mathcal{M})$, then (3.1) can be strengthened to*

$$\limsup_{t\to\infty} \frac{1}{t} \log |X(t)| \le \frac{1}{p} \sum_{i\in\mathcal{M}} \pi_i [c_2(i) - 0.5\, c_3(i) - c_4(i) - c_5(i)] \quad a.s. \tag{3.2}$$

Proof. This follows from applying the ergodic theorem of continuous-time Markov chain (see, for example, (Norris, 1998, Theorem 3.8.1)) to the right-hand side of (A.6):

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \big[c_2(\alpha(s)) - 0.5c_3(\alpha(s)) - c_4(\alpha(s)) - c_5(\alpha(s))\big]\mathrm{d}s$$
$$= \sum_{i=1}^m \pi_i[c_2(i) - 0.5\, c_3(i) - c_4(i) - c_5(i)] \ \text{ a.s.}$$

Then (3.2) follows directly. □

3.2 Exponential pth-Moment Stability

Theorem 3.4. *Suppose Assumptions 2.1 and 2.2. Let p, c_1, c_2, c_3 be positive constants. Assume that there exists a function $V : \mathbb{R}^n \times \mathcal{M} \mapsto \mathbb{R}^+$ such that $V(\cdot, i) \in C^2(\mathbb{R}^n)$ for each $i \in \mathcal{M}$ satisfying*

(i) $c_1|x|^p \le V(x,i) \le c_2|x|^p$,

(ii) $\mathcal{L}V(x,i) \le -c_3V(x,i)$,

for all $(x,i) \in \mathbb{R}^n \times \mathcal{M}$. Let $(X(0), \alpha(0)) = (x,i) \in \mathbb{R}^n \times \mathcal{M}$. Then we have

(a) $\mathbb{E}[|X(t)|^p] \le \frac{c_2}{c_1}|x|^p e^{-c_3t}$. *In particular, the equilibrium point of (2.3) is exponentially stable in the pth moment with Lyapunov exponent less than or equal to $-c_3$.*

(b) Assume in addition that $p \in (0,2]$. Then there exists an almost surely finite and positive random variable Ξ such that

$$|X(t)|^p \le \frac{\Xi}{c_1}e^{-c_3t} \text{ for all } t \ge 0 \text{ a.s.} \tag{3.3}$$

In particular, the equilibrium point of (2.3) is almost sure exponentially stable with Lyapunov exponent less than or equal to $-\frac{c_3}{p}$.

Proof. The proof of part (a) is very similar to that of Theorem 3.1 in Mao (1999); see also Theorem 4.1 in Applebaum and Siakalli (2009). For brevity, we shall omit the details here.

Part (b) is motivated by (Khasminskii, 2012, Theorem 5.15). For any $t \ge 0$ and $(x,i) \in \mathbb{R}^n \times \mathcal{M}$, we consider the function $f(t,x,i) := e^{c_3t}V(x,i)$. Condition (i) and Lemma 3.1 of Zhu et al. (2015) imply that

$$\mathbb{E}[f(t, X(t), \alpha(t))] = \mathbb{E}[e^{c_3t}V(X(t), \alpha(t))] \le c_2e^{c_3t}\mathbb{E}[|X(t)|^p] < \infty.$$

On the other hand, thanks to Itô's formula, we have for all $0 \le s < t$

$$f(t, X(t), \alpha(t))$$
$$= f(s, X(s), \alpha(s)) + \int_s^t e^{c_3r}(c_3 + \mathcal{L})V(X(r), \alpha(r))\mathrm{d}r$$

$$
\begin{aligned}
&+\int_s^t e^{c_3 r} DV(X(r),\alpha(r))\cdot\sigma(X(r),\alpha(r))\mathrm{d}W(r)\\
&+\int_s^t\int_{\mathbb{R}_+} e^{c_3 r}\big[V(X(r-),\alpha(r-)+h(X(r-),\alpha(r-),z))-V(X(r-),\alpha(r-))\big]\tilde{N}_1(\mathrm{d}r,\mathrm{d}z)\\
&+\int_s^t\int_{\mathbb{R}_0^n} e^{c_3 r}[V(X(r-)+\gamma(X(r-),\alpha(r-),z),\alpha(r-))-V(X(r-),\alpha(r-))]\widetilde{N}(\mathrm{d}r,\mathrm{d}z)\\
\le\ & f(s,X(s),\alpha(s))+\int_s^t e^{c_3 r} DV(X(r),\alpha(r))\cdot\sigma(X(r),\alpha(r))\mathrm{d}W(r)\\
&+\int_s^t\int_{\mathbb{R}_+} e^{c_3 r}\big[V(X(r-),\alpha(r-)+h(X(r-),\alpha(r-),z))-V(X(r-),\alpha(r-))\big]\tilde{N}_1(\mathrm{d}r,\mathrm{d}z)\\
&+\int_s^t\int_{\mathbb{R}_0^n} e^{c_3 r}\big[V(X(r-)+\gamma(X(r-),\alpha(r-),z),\alpha(r-))-V(X(r-),\alpha(r-))\big]\widetilde{N}(\mathrm{d}r,\mathrm{d}z),
\end{aligned}
$$

where we used condition (ii) to obtain the inequality. Let $\tau_n := \inf\{t\ge 0 : |X(t)|\ge n\}$. Then we have $\lim_{n\to\infty}\tau_n=\infty$ a.s. and $\mathbb{E}[f(t\wedge\tau_n, X(t\wedge\tau_n),\alpha(t\wedge\tau_n))|\mathcal{F}_s]\le f(s\wedge\tau_n, X(s\wedge\tau_n),\alpha(s\wedge\tau_n))$ a.s. Passing to the limit as $n\to\infty$, and noting that f is positive, we obtain from Fatou's lemma that $\mathbb{E}[f(t,X(t),\alpha(t))|\mathcal{F}_s]\le f(s,X(s),\alpha(s))$ a.s. Therefore it follows that the process $\{f(t,X(t),\alpha(t)), t\ge 0\}$ is a positive supermartingale. The martingale convergence theorem (see, for example, Theorem 3.15 and Problem 3.16 in Karatzas and Shreve (1991)) then implies that $f(t,X(t),\alpha(t))$ converges a.s. to a finite limit as $t\to\infty$. Consequently there exists an a.s. finite and positive random variable Ξ such that

$$
\sup_{t\ge 0}\{e^{c_3 t}V(X(t),\alpha(t))\}=\sup_{t\ge 0}\{f(t,X(t),\alpha(t))\}\le\Xi<\infty,\ \text{ a.s.}
$$

Furthermore, it follows from condition (i) that $|X(t)|^p\le\frac{1}{c_1}V(X(t),\alpha(t))$. Putting this observation into the above displayed equation yields (3.3). □

Theorem 3.5. *Let Assumptions 2.1 and 2.2 hold. Suppose the equilibrium point of* (2.3) *is pth moment exponentially stable for some* $p\ge 2$ *and that for some positive constant* $\hat{\kappa}$*, we have*

$$
\int_{\mathbb{R}_0^n}|\gamma(x,i,z)|^p\,\nu(dz)\le\hat{\kappa}|x|^p,\quad (x,i)\in\mathbb{R}^n\times\mathcal{M}. \tag{3.4}
$$

Then the equilibrium point of (2.3) *is almost surely exponentially stable.*

The proof of Theorem 3.5 is very similar to the proofs of Theorem 4.2 of Applebaum and Siakalli (2009) and Theorem 5.9 of Mao and Yuan (2006) and is deferred to Appendix A. Note that Theorem 4.4 in Applebaum and Siakalli (2009) requires a condition (Assumption 4.1) similar to (3.4) to hold for all $q\in[2,p]$. Here we observe that it is enough to have (3.4) for a single p, as long as $p\ge 2$. Also notice that in the special case when $p=2$, then (3.4) is already contained in Assumption 2.2.

3.3 Criteria for Stability Using M-Matrices

In this subsection, we assume that in (2.5), $Q(x)=Q\in\mathbb{R}^{m\times m}$, a constant matrix. Consequently, the switching component $\alpha(\cdot)$ in (2.3) is a continuous-time Markov chain. Let us also assume

Assumption 3.6. There exist a positive number $p > 0$ and a positive definite matrix $G \in \mathcal{S}^{n\times n}$ such that for all $x \neq 0$ and $i \in \mathcal{M}$, we have

$$\langle Gx, b(x,i)\rangle + \frac{1}{2}\langle \sigma(x,i), G\sigma(x,i)\rangle \le \varrho_i \langle x, Gx\rangle, \tag{3.5}$$

$$(\langle x, G\sigma(x,i)\rangle)^2 \begin{cases} \le \delta_i(\langle x, Gx\rangle)^2, & \text{if } p \ge 2, \\ \ge \delta_i(\langle x, Gx\rangle)^2, & \text{if } 0 < p < 2, \end{cases} \tag{3.6}$$

and

$$\int_{\mathbb{R}_0^n}\left[\left(\frac{\langle x+\gamma(x,i,z), G(x+\gamma(x,i,z))\rangle}{\langle x, Gx\rangle}\right)^{\frac{p}{2}} - 1 - \frac{p\langle \gamma(x,i,z), Gx\rangle}{\langle x, Gx\rangle}\right]\nu(\mathrm{d}z) \le \lambda_i, \tag{3.7}$$

where ϱ_i, δ_i, and λ_i, $i \in \mathcal{M}$ are constants.

Corresponding to the infinitesimal generator Q of (2.5) and $p > 0$, $G \in \mathcal{S}^{n\times n}$ in Assumption 3.6, we define an $m \times m$ matrix

$$\mathcal{A} := \mathcal{A}(p, G) = \mathrm{diag}(\theta_1, \dots, \theta_m) - Q, \tag{3.8}$$

where $\theta_i := -[p\varrho_i + p(p-2)\delta_i + \lambda_i]$, $i = 1, \dots, m$.

Theorem 3.7. *Suppose Assumptions 2.1, 2.2, 2.4, and 3.6 hold and that the matrix $\mathcal{A}$ defined in (3.8) is a nonsingular M-matrix, then the equilibrium point of (2.3) is exponentially stable in the pth moment. In addition, if either $p \in (0, 2]$ or $p > 2$ with (3.4) valid, then then the equilibrium point of (2.3) is a.s. exponentially stable.*

Recall from Mao and Yuan (2006) that a square matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is a nonsingular M-matrix if A can be expressed in the form $A = sI - G$ with some $G \ge 0$ and $s > \rho(G)$, where I is the identity matrix and $\rho(G)$ denotes the spectral radius of G.

Proof. Since $\mathcal{A}$ of (3.8) is a nonsingular M-matrix, by Theorem 2.10 of Mao and Yuan (2006), there exists a vector $(\beta_1, \dots, \beta_m)' \gg 0$ such that $(\bar\beta_1, \dots, \bar\beta_m)' := \mathcal{A}(\beta_1, \dots, \beta_m)' \gg 0$. Componentwise, we can write

$$\bar\beta_i := \theta_i\beta_i - \sum_{j\in\mathcal{M}} q_{ij}\beta_j > 0, \quad i \in \mathcal{M}.$$

Define $V(x,i) := \beta_i(x'Gx)^{\frac{p}{2}}$ for $(x,i) \in \mathbb{R}^n \times \mathcal{M}$. Then condition (i) of Theorem 3.4 is trivially satisfied. Moreover, we can use Assumption 3.6 to compute

$$\begin{aligned}\mathcal{L}V(x,i) &= \beta_i p(x'Gx)^{\frac{p}{2}-1}\langle b(x,i), Gx\rangle + \sum_{j=1}^m q_{ij}\beta_j(x'Gx)^{\frac{p}{2}} \\ &\quad + \frac{1}{2}\beta_i p(x'Gx)^{\frac{p}{2}-2}\mathrm{tr}\big(\sigma(x,i)\sigma'(x,i)[(p-2)Gxx'G + x'GxG]\big)\end{aligned}$$

$$
\begin{aligned}
&+\beta_i\int_{\mathbb{R}^n_0}\Big[\big(\langle x+\gamma(x,i,z),G(x+\gamma(x,i,z))\rangle\big)^{\frac{p}{2}}-(x'Gx)^{\frac{p}{2}}\\
&\qquad\qquad -p(x'Gx)^{\frac{p}{2}-1}\langle\gamma(x,i,z),Gx\rangle\Big]\nu(\mathrm{d}z)\\
&\le\sum_{j=1}^m q_{ij}\beta_j(x'Gx)^{\frac{p}{2}}+\beta_i\big[p\varrho_i+p(p-2)\delta_i+\lambda_i\big](x'Gx)^{\frac{p}{2}}\\
&=\Bigg[\sum_{j=1}^m q_{ij}\beta_j-\theta_i\beta_i\Bigg](x'Gx)^{\frac{p}{2}}=-\bar\beta_i(x'Gx)^{\frac{p}{2}}\le-\varsigma V(x,i),
\end{aligned}
$$

where $\varsigma=\min_{1\le i\le m}\frac{\bar\beta_i}{\beta_i}>0$. This verifies condition (ii) of Theorem 3.4. Therefore by Theorem 3.4, part (a), we conclude that the equilibrium point of (2.3) is exponentially stable in the pth moment.

The assertion on a.s. exponential stability follows from Theorem 3.4 part (b) for the case $p\in(0,2]$ and Theorem 3.5 for the case $p>2$. $\square$

4 Stability of Linear Markovian Regime-Switching Jump Diffusion Systems

In this section, we consider a linear regime-switching jump diffusion

$$
\mathrm{d}X(t)=A(\alpha(t))X(t)\mathrm{d}t+B(\alpha(t))X(t)\mathrm{d}W(t)+\int_{\mathbb{R}^n_0}C(\alpha(t-),z)X(t-)\tilde N(\mathrm{d}t,\mathrm{d}z), \tag{4.1}
$$

where $\alpha(\cdot)$ is an irreducible continuous-time Markov chain taking values in $\mathcal{M}=\{1,\dots,m\}$. Consequently we assume that $q_{ij}(\cdot)$ in (2.5) are constants for all $i,j\in\mathcal{M}$. In addition, unless otherwise mentioned, we assume that $\alpha(\cdot)$ has an invariant distribution $\pi=(\pi_i,i\in\mathcal{M})$ throughout the section. In (4.1), for each $i\in\mathcal{M}$ and $z\in\mathbb{R}^n_0$, $A_i=A(i),B_i=B(i)$ and $C_i(z)=C(i,z)$ are $n\times n$ matrices satisfying the following condition

$$
\begin{aligned}
&\max_{i\in\mathcal{M}}\int_{\mathbb{R}^n_0}\big[|C_i(z)|^2+|C_i(z)|\big]\nu(\mathrm{d}z)<\infty,\ \text{and}\\
&\langle\xi,(I+C_i(z)')(I+C_i(z))\xi\rangle\ge\varrho^2|\xi|^2,\ \text{for all }\xi\in\mathbb{R}^n\text{ and }\nu\text{-almost all }z\in\mathbb{R}^n_0,
\end{aligned}\tag{4.2}
$$

where ϱ is a positive constant. Apparently (4.1) satisfies Assumption 2.1. In addition, the first equation of (4.2) guarantees that Assumption 2.2 is satisfied as well. Finally, since $|x+C_i(z)x|=|(I+C_i(z))x|=|\langle x,(I+C_i(z)')(I+C_i(z))x\rangle|^{\frac12}$, the uniform ellipticity condition on the matrix $(I+C_i(z)')(I+C_i(z))$ in the second equation of (4.2) implies that Assumption 2.4 holds.

We will deal with almost sure exponential stability in Section 4.1 and moment exponential stability in Section 4.2. In both sections, we will treat one-dimensional and multidimensional systems separately. Finally Section 4.3 presents several examples.

4.1 Almost Sure Exponential Stability

4.1.1 One Dimensional System

Let us first consider the one-dimensional regime-switching jump diffusion

$$\mathrm{d}x(t) = a(\alpha(t))x(t)\mathrm{d}t + b(\alpha(t))x(t)\mathrm{d}W(t) + \int_{\mathbb{R}_0} c(\alpha(t-), z)x(t-)\tilde{N}(\mathrm{d}t, \mathrm{d}z). \tag{4.3}$$

Suppose for each $i \in \mathcal{M}$, $a_i = a(i)$, $b_i = b(i)$ are real numbers, and $c_i(\cdot) = c(i, \cdot)$ is a measurable function from $\mathbb{R}_0$ to $\mathbb{R} - \{-1\}$ satisfying (4.2). Notice that (4.3) can be regarded as an extended jump type Black-Scholes model with regime switching; this is motivated by the jump diffusion models in Cont and Tankov (2004), Cont and Voltchkova (2005) as well as the regime-switching models as in Barone-Adesi and Whaley (1987), Zhang (2001).

Proposition 4.1. *Suppose*

$$\max_{i\in\mathcal{M}}\left\{\int_{\mathbb{R}_0} \big(\log|1 + c_i(z)|\big)^2\nu(\mathrm{d}z) + \int_{\mathbb{R}_0} \big|\log|1 + c_i(z)| - c_i(z)\big|\nu(\mathrm{d}z)\right\} < \infty, \tag{4.4}$$

then the solution to (4.3) satisfies the following property:

$$\lim_{t\to+\infty} \frac{1}{t}\log|x(t)| = \delta := \sum_{i=1}^{m} \pi_i\left[a_i - \frac{1}{2}b_i^2 + \int_{\mathbb{R}_0} \big(\log|1 + c_i(z)| - c_i(z)\big)\nu(\mathrm{d}z)\right] \quad a.s.$$

In particular, the equilibrium point of (4.3) is almost surely exponentially stable if and only if $\delta < 0$.

Proof. As in the proof of Theorem 3.1, we need only to consider the case when $x(t) \neq 0$ for all $t \geq 0$ with probability 1. Let $x(0) = x \neq 0$ and $\alpha(0) = i \in \mathcal{M}$. Then by Itô's formula we have

$$\begin{aligned}\log|x(t)| = \log|x| + \int_0^t \Big[a(\alpha(s)) - \frac{1}{2}b^2(\alpha(s))\\ + \int_{\mathbb{R}_0} [\log|1 + c(\alpha(s-), z)| - c(\alpha(s-), z)]\nu(\mathrm{d}z)\Big]\mathrm{d}s + M_1(t) + M_2(t),\end{aligned}$$

where

$$M_1(t) = \int_0^t b(\alpha(s))\mathrm{d}W(s), \text{ and } M_2(t) = \int_0^t\int_{\mathbb{R}_0} \log|1 + c(\alpha(s-), z)|\widetilde{N}(\mathrm{d}s, \mathrm{d}z).$$

Obviously M_1 is a martingale vanishing at 0 with quadratic variation $\langle M_1, M_1\rangle_t = \int_0^t b^2(\alpha(s))\mathrm{d}s \leq t\max_{i\in\mathcal{M}} b_i^2$. On the other hand, (4.4) implies that M_2 is a martingale vanishing at 0. In addition, the quadratic variation of M_2 is given by

$$\langle M_2, M_2\rangle_t = \int_0^t\int_{\mathbb{R}_0} (\log|1 + c(\alpha(s-), z)|)^2\nu(\mathrm{d}z)\mathrm{d}s \leq Kt,$$

where $K = \max_{i\in\mathcal{M}} \int_{\mathbb{R}_0} (\log|1+c_i(z)|)^2 \nu(\mathrm{d}z) < \infty$. Therefore we can apply the strong law of large numbers for martingales (see, for example, (Mao and Yuan, 2006, Theorem 1.6)) to conclude

$$\lim_{t\to+\infty} \frac{M_1(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to+\infty} \frac{M_2(t)}{t} = 0 \quad \text{a.s.}$$

Then the ergodic theorem for continuous-time Markov chain leads to the desired assertion. □

4.1.2 Multidimensional Systems

Let us now focus on the multidimensional system (4.1). As before, we suppose that the discrete component α in (4.1) and (2.7) is an irreducible continuous-time Markov chain with an invariant distribution $\pi = (\pi_i, i \in \mathcal{M})$. For notational simplicity, define the column vector $\mu = (\mu_1, \mu_2, \ldots, \mu_m)' \in \mathbb{R}^m$ with

$$\mu_i := \mu_i(G) = \frac{1}{2}\lambda_{\max}(GA_iG^{-1} + G^{-1}A_i'G + G^{-1}B_i'G^2B_iG^{-1}), \tag{4.5}$$

where $G \in \mathcal{S}^{n\times n}$ is a positive definite matrix. Also let

$$\beta := -\pi\mu = -\sum_{i=1}^{m} \pi_i\mu_i. \tag{4.6}$$

Then it follows from Lemma A.12 of Yin and Zhu (2010) that the equation

$$Q\zeta = \mu + \beta\mathbb{1} \tag{4.7}$$

has a solution $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_m)' \in \mathbb{R}^m$, where $\mathbb{1} := (1, 1, \ldots, 1)' \in \mathbb{R}^m$. Thus we have from (4.7) that

$$\mu_i - \sum_{j=1}^{m} q_{ij}\zeta_j = -\beta, \quad i \in \mathcal{M}. \tag{4.8}$$

Before we state the main result of this section, let us introduce some more notation. If A is a square matrix, then $\rho(A)$ denotes the spectral radius of A. Furthermore, if A is symmetric, we denote

$$\widehat{\lambda}(A) := \begin{cases} \lambda_{\max}(A), & \text{if } \lambda_{\max}(A) < 0, \\ 0 \vee \lambda_{\min}(A), & \text{otherwise.} \end{cases} \tag{4.9}$$

Proposition 4.2. *The trivial solution of (4.1) is a.s. exponentially stable if there exist a positive definite matrix $G \in \mathcal{S}^{n\times n}$, positive numbers h_i and p such that $h_i - p\zeta_i > 0$ for each $i \in \mathcal{M}$ and that*

$$\sum_{i\in\mathcal{M}} \pi_i[c_2(i) - 0.5c_3(i) - c_4(i) - c_5(i)] < 0, \tag{4.10}$$

where

$$c_2(i) := p\mu_i + \frac{p-2}{8}\Lambda^2(GB_iG^{-1} + G^{-1}B_i'G) - \frac{p}{h_i - p\zeta_i}(\mu_i + \beta) + \frac{1}{h_i - p\zeta_i}\sum_{j\in\mathcal{M}} q_{ij}h_j + \eta_i,$$

$$
\begin{aligned}
c_3(i) &:= \frac{p^2}{4}\widehat{\lambda}^2(GB_iG^{-1}+G^{-1}B_i'G),\\
c_4(i) &:= -\bigg\{0\wedge\int_{\mathbb{R}_0^n}\bigg[\frac{p}{2}\log\lambda_{\max}(G^{-1}(I+C_i(z))'G^2(I+C_i(z))G^{-1}) \qquad (4.11)\\
&\qquad\qquad - \big(\lambda_{\min}(G^{-1}(I+C_i(z))'G^2(I+C_i(z))G^{-1})\big)^{\frac{p}{2}}+1\bigg]\nu(\mathrm{d}z)\bigg\},\\
c_5(i) &:= -\sum_{j\in\mathcal{M}} q_{ij}\left(\log(h_j-p\zeta_j)-\frac{h_j-p\zeta_j}{h_i-p\zeta_i}\right),
\end{aligned}
$$

in which μ_i, β, and ζ_i are defined in (4.5), (4.6), and (4.7), respectively, and

$$
\begin{aligned}
\eta_i &:= \int_{\mathbb{R}_0^n}\Big[\big(\lambda_{\max}(G^{-1}(I+C_i(z))'G^2(I+C_i(z))G^{-1}\big)^{\frac{p}{2}}-1 \qquad (4.12)\\
&\qquad\qquad -\frac{p}{2}\lambda_{\min}(GC_i(z)G^{-1}+G^{-1}C_i'(z)G)\Big]\nu(\mathrm{d}z),
\end{aligned}
$$

$$
\Lambda(GB_iG^{-1}+G^{-1}B_i'G) := \begin{cases}\widehat{\lambda}(GB_iG^{-1}+G^{-1}B_i'G), & \text{if } 0<p\le 2,\\ \rho(GB_iG^{-1}+G^{-1}B_i'G), & \text{if } p>2.\end{cases} \qquad (4.13)
$$

In the above, we require that the integrals with respect to ν in (4.11) and (4.12) are well-defined.

Remark 4.3. Note that the constants $c_2(i)$ and $c_5(i)$ in the statement of Proposition 4.2 actually depend on the choice of the solution ζ to equation (4.7). Nevertheless, for notational simplicity, we write $c_2(i), c_5(i)$ instead of $c_2(i;\zeta), c_5(i;\zeta)$. Since Q is a singular matrix, and $\pi(\mu+\beta\mathbb{1})=0$, in view of Lemma A.12 of Yin and Zhu (2010), (4.7) has infinitely many solutions and any two solutions ζ^1,ζ^2 of (4.7) satisfy $\zeta^1-\zeta^2=\varrho\mathbb{1}$ for some $\varrho\in\mathbb{R}$. Hence Proposition 4.2 and in particular (4.10) can be strengthened as: If

$$
\min\left\{\sum_{i\in\mathcal{M}}\pi_i\big[c_2(i)-0.5\,c_3(i)-c_4(i)-c_5(i)\big]\,\bigg|\,\zeta\in\mathbb{R}^m, Q\zeta=\mu+\beta\mathbb{1}\right\}<0, \qquad (4.14)
$$

then the trivial solution of (4.1) is a.s. exponentially stable.

The proof of Proposition 4.2 follows from a direct application of Theorem 3.1 and Corollary 3.3. The idea is to construct an appropriate Lyapunov function V that satisfies conditions (i)–(v) of Theorem 3.1. To preserve the flow of presentation, we arrange the proof to the Appendix A.

Next we present a sufficient condition for a.s. exponential stability for the equilibrium point of a linear stochastic differential equation *without switching*.

Corollary 4.4. *Let $i\in\mathcal{M}$. Suppose there exist a positive definite matrix $G_i\in\mathcal{S}^{n\times n}$ and a positive number $p\in(0,2]$ such that*

$$
\tilde{c}_2(i)-0.5c_3(i)-c_4(i)<0, \qquad (4.15)
$$

where $c_3(i), c_4(i)$ are defined in (4.11), and

$$\tilde{c}_2(i) := p\mu_i + \frac{p-2}{8}\Lambda^2(G_iB_iG_i^{-1} + G_i^{-1}B_i'G_i) + \eta_i,$$

where μ_i, η_i, and $\Lambda(G_iB_iG_i^{-1} + G_i^{-1}B_i'G_i)$ are similarly defined in (4.5), (4.12) and (4.13) respectively, then the equilibrium point of the stochastic differential equation

$$\mathrm{d}X^{(i)}(t) = A(i)X^{(i)}(t)\mathrm{d}t + B(i)X^{(i)}(t)\mathrm{d}W(t) + \int_{\mathbb{R}_0^n} C(i,z)X^{(i)}(t-)\tilde{N}(\mathrm{d}t,\mathrm{d}z), \tag{4.16}$$

is a.s. exponentially stable.

In addition, if $G_i = G$ and (4.15) holds for every $i \in \mathcal{M}$, then the equilibrium point of (4.1) is a.s. exponentially stable.

Proof. This follows from Proposition 4.2 directly. □

4.2 pth Moment Exponential Stability

4.2.1 One-Dimensional System

As in Section 4.1, let us first derive a necessary and sufficient condition for the pth moment exponential stability for the one-dimensional linear system (4.3). To this end, we need to introduce some notations. Let $\mathcal{P}$ be the set of probability measures on the state space $\mathcal{M}$; then under the irreducibility and ergodicity assumptions, the empirical measure of the continuous-time Markov chain $\alpha(\cdot)$ satisfies the large deviation principle with the rate function

$$\mathbf{I}(\mu) := -\inf_{u_1,\dots,u_m>0} \sum_{i,j\in\mathcal{M}} \frac{\mu_i q_{ij} u_j}{u_i}, \tag{4.17}$$

where $\mu = (\mu_1, \dots, \mu_m) \in \mathcal{P}$; we refer to Donsker and Varadhan (1975) for details. It is known that $\mathbf{I}(\mu)$ is lower semicontinuous and $\mathbf{I}(\mu) = 0$ if and only if $\mu = \pi$. In addition, by virtue of Zong et al. (2014), if $a = (a_1, \dots, a_m)' \in \mathbb{R}^m$, then we have

$$\Upsilon(a) := \lim_{t\to\infty} \frac{1}{t} \log\left(\mathbb{E}\left[\exp\left\{\int_0^t a(\alpha(s))\mathrm{d}s\right\}\right]\right) = \sup_{\mu\in\mathcal{P}} \left\{\sum_{i\in\mathcal{M}} a_i\mu_i - \mathbf{I}(\mu)\right\}. \tag{4.18}$$

Note that $\sum_{i\in\mathcal{M}} a_i\pi_i \le \Upsilon(a) \le \max_{i\in\mathcal{M}} a_i$.

Proposition 4.5. *Assume the conditions of Proposition 4.1. In addition, assume that there exists some $p > 0$ such that for each $i \in \mathcal{M}$, $\int_{\mathbb{R}_0} ||1 + c(i,z)|^p - pc(i,z) - 1|\,\nu(\mathrm{d}z) < \infty$. Denote $f = (f(1), \dots, f(m))$ with*

$$f(i) = f_p(i) = pa(i) + \frac{1}{2}p(p-1)b^2(i) + \int_{\mathbb{R}_0} [|1 + c(i,z)|^p - pc(i,z) - 1]\nu(\mathrm{d}z).$$

Then we have

$$\lim_{t\to\infty} \frac{1}{t} \log(\mathbb{E}[|x(t)|^p]) = \Upsilon(f), \tag{4.19}$$

where $\Upsilon(f)$ is similarly defined as in (4.18). Therefore, the trivial solution of (4.3) is pth moment exponentially stable if and only if $\Upsilon(f) < 0$.

Proof. See Appendix A. □

4.2.2 Multidimensional System

Now let's focus on establishing a sufficient condition for the p-th moment exponential stability of the trivial solution of the multidimensional system (4.1). In view of Theorem 3.4 and the calculations in Proposition 4.2, we have the following proposition:

Proposition 4.6. *If there exist a positive definite matrix $G \in \mathcal{S}^{n\times n}$, positive numbers p and $h_i, i \in \mathcal{M}$ such that*

$$\delta := \min\left\{ \max_{i\in\mathcal{M}} c(i;h,\zeta) \middle| \zeta \in \mathbb{R}^m, Q\zeta = \mu + \beta \mathbb{1}, h_i - p\zeta_i > 0 \text{ for each } i \in \mathcal{M} \right\} < 0, \quad (4.20)$$

then the equilibrium point of (4.1) is exponentially stable in the pth moment with Lyapunov exponent less than or equal to δ, where $\mu \in \mathbb{R}^m$ and $\beta \in \mathbb{R}$ are defined in (4.5) and (4.6), respectively, and

$$c(i;h,\zeta) := p\mu_i + \frac{p-2}{8}\Lambda^2(GB_iG^{-1} + G^{-1}B_i'G) - \frac{p}{h_i - p\zeta_i}(\mu_i + \beta) + \frac{1}{h_i - p\zeta_i}\sum_{j\in\mathcal{M}} q_{ij}h_j + \eta_i,$$

in which η_i and $\Lambda(GB_iG^{-1} + G^{-1}B_i'G)$ are defined in (4.12) and (4.13), respectively.

Proof. Let $p, h = (h_1, \dots, h_m)'$ and G be as in the statement of the corollary and consider the function $V(x,i) = (h_i - p\zeta_i)(x'G^2x)^{p/2}$, $(x,i) \in \mathbb{R}^n \times \mathcal{M}$. Then we have

$$0 < \min_{i\in\mathcal{M}}(h_i - p\zeta_i)(\lambda_{\min}(G^2))^{p/2}|x|^p \le V(x,i) \le \max_{i\in\mathcal{M}}(h_i - p\zeta_i)(\lambda_{\max}(G^2))^{p/2}|x|^p.$$

Moreover, the detailed calculations in the proof of Proposition 4.2 reveal that

$$\mathcal{L}V(x,i) \le c(i;h,\zeta)V(x,i) \le \max_{i\in\mathcal{M}} c(i;h,\zeta)V(x,i).$$

Then condition (4.20) and Theorem 3.4 lead to the conclusion. □

Finally we apply Theorem 3.7 to derive a sufficient condition for a.s. and moment exponential stability for the equilibrium point of (4.1). Note that in Proposition 4.7 below, the infinitesimal generator $Q = (q_{ij})$ of the continuous-time Markov chain α need not to be irreducible and ergodic.

Proposition 4.7. *Suppose that there exist a positive constant p and a positive definite matrix $G \in \mathcal{S}^{n\times n}$ such that the $m \times m$ matrix $\mathcal{A} := \operatorname{diag}(\theta_1, \dots \theta_m) - Q$ is a nonsingular M-matrix, then the equilibrium point of (4.1) is pth moment exponentially stable, where for each $i \in \mathcal{M}$,*

$$\theta_i := -\left[\frac{p}{2}\lambda_{\max}(GA_iG^{-1} + G^{-1}A_iG + G^{-1}B_i'G^2B_iG^{-1}) \right. \\ \left. + \frac{p(p-2)}{4}\rho^2(GB_iG^{-1} + G^{-1}B_iG) + \eta_i\right],$$

and η_i is defined in (4.12). If in addition that either $p \in (0,2]$ or else $p > 2$ with

$$\max_{i\in\mathcal{M}} \int_{\mathbb{R}_0^n} |C_i(z)|^p \nu(\mathrm{d}z) < \infty,$$

then the equilibrium point of (4.1) is also a.s. exponentially stable.

The proof of Proposition 4.7 consists of straightforward verifications of (3.5)–(3.7) of Assumption 3.6. Theorem 3.7 then leads to the assertions on almost sure and moment stability. Again we shall arrange the proof to the appendix A.

4.3 Examples

Example 4.8. In this example, we consider the one-dimension linear system given in (4.3), in which $\alpha \in \mathcal{M} = \{1,2,3\}$ is a continuous-time Markov chain with generator $Q = \begin{bmatrix} -3 & 1 & 2 \\ 2 & -2 & 0 \\ 4 & 0 & -4 \end{bmatrix}$, $a_1 = 4, a_2 = 2, a_3 = 3,\ b_1 = 1, b_2 = 3, b_3 = 1$, and $c_i(z) = 1 \wedge z^2$ for $i = 1,2,3$. In addition, suppose that the characteristic measure of the Poisson random measure N is given by the Lévy measure $\nu(\mathrm{d}z) = \frac{\mathrm{d}z}{z^{4/3}}$, $z \in \mathbb{R}_0$. Note that ν is an infinite Lévy measure, i.e., $\nu(\mathbb{R}_0) = \infty$.

By direct computations, we get

$$\delta = \sum_{i=1}^{3} \pi_i \left[a_i - \frac{1}{2}b_i^2 + \int_{\mathbb{R}_0} \big(\log|1 + c_i(z)| - c_i(z)\big)\nu(\mathrm{d}z)\right] = -0.2867.$$

Then Proposition 4.1 implies that the trivial solution of (4.3) is almost surely exponentially stable.

However, if the jumps are excluded from the system (4.3), that is, if $c_i(z) = 0$ for $i = 1,2,3$, then

$$\sum_{i=1}^{3} \pi_i \left(a_i - \frac{1}{2}b_i^2\right) = 1.75,$$

which implies that the trivial solution of (4.3) is almost surely exponentially unstable. This example indicates that the jumps can contribute to the stability of the equilibrium point.

Example 4.9. Consider the linear system

$$\mathrm{d}X(t) = A(\alpha(t))X(t)\mathrm{d}t + B(\alpha(t))X(t)\mathrm{d}W(t) + \int_{\mathbb{R}_0} C(\alpha(t-), z)X(t-)\tilde{N}(\mathrm{d}t, \mathrm{d}z), \quad (4.21)$$

in which $\alpha \in \mathcal{M} = \{1,2,3\}$ is a continuous-time Markov chain with generator $Q = \begin{bmatrix} -3 & 1 & 2 \\ 2 & -2 & 0 \\ 4 & 0 & -4 \end{bmatrix}$, N is a Poisson random measure on $[0,\infty) \times \mathbb{R}_0$ whose corresponding Lévy measure is given by $\nu(\mathrm{d}z) = \frac{1}{2}(e^{-z} \wedge e^{z})\mathrm{d}z$, $z \in \mathbb{R}$, and

$$A_1 = \begin{bmatrix} 10 & 1 & 8 \\ -3 & 10 & 2 \\ -1 & -8 & 12 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 17 & 5 & 8 \\ -1 & 11 & -3 \\ 4 & -5 & 13 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 10 & -4 & 12 \\ 8 & 10 & -8 \\ 3 & -9 & 11 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 4 \\ -1 & -2 & 1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} -1 & 2 & 1 \\ -3 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 & 2 & 2 \\ -1 & 1 & 4 \\ -3 & -2 & 1 \end{bmatrix},$$
$$C_i(z) = 0 \in \mathbb{R}^{3\times 3}.$$

Here ν belongs to the class of double exponential distributions; we refer to Kou (2002) for applications of such distributions in math finance.

Let us take

$$G = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad h = [20, 20, 20], \quad p = 0.1.$$

Then by direct calculation, we get

$$\pi = (0.5, 0.25, 0.25), \quad \mu = (23.7194, 34.0899, 28.3542)', \qquad \beta = -27.4707, \quad c_3 = c_4 = \mathbf{0},$$

and

$$\min_{\zeta \in D} \sum_{i \in \mathcal{M}} \pi_i [c_2(i) - c_5(i)] = 2.7422 > 0, \tag{4.22}$$

where $D := \{\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{R}^3 |\ Q\zeta = \mu + \beta \mathbb{1}\}$, and the minimizer in (4.22) is given by $\zeta = (131.65, 112, 142.5264)'$, and

$$c_2(1) - c_5(1) = 6.5169, \quad c_2(2) - c_5(2) = 9.8252, \quad c_2(3) - c_5(3) = 2.7433.$$

Thus we cannot apply Proposition 4.2 to determine the almost surely exponential stability of the trivial solution of (4.21).

Next we observe that

$$\sum_{i \in \mathcal{M}} \pi_i \left[\lambda_{\min}(B_i B_i') + \frac{1}{2} \lambda_{\min}(A_i + A_i') - \max\left\{ \lambda^2_{\min}(B_i B_i'), \lambda^2_{\max}(B_i B_i') \right\} \right] = 0.3541.$$

Therefore by virtue of Theorem 4.3 in Khasminskii et al. (2007), the trivial solution of (4.21) is unstable in probability, which, in turn, implies that the trivial solution cannot be almost surely exponential stable.

Example 4.10. Again consider the linear system (4.21) with the same Q, A_i, B_i, G, h, and p as given in Example 4.9, but with

$$C_1 = \begin{bmatrix} 15 & 1 & 2 \\ -1 & 9 & 1 \\ 7 & -1 & 10 \end{bmatrix}, \ C_2 = \begin{bmatrix} 20 & 6 & -3 \\ 1 & 14 & 2 \\ 3 & 2 & 8 \end{bmatrix}, \ C_3 = \begin{bmatrix} 7 & 1 & 4 \\ 2 & 10 & 1 \\ -1 & 5 & 11 \end{bmatrix}.$$

Then we have

$$\pi = (0.5, 0.25, 0.25), \quad \mu = (23.7194, 34.0899, 28.3542)', \quad \beta = -27.4707, \quad c_3 = c_4 = \mathbf{0},$$

and

$$\min_{\zeta \in D} \sum_{i \in \mathcal{M}} \pi_i [c_2(i) - c_5(i)] = -0.0939, \tag{4.23}$$

where the minimizer of (4.23) is $\zeta = (116.2209, 112.9113, 116)'$, and

$$c_2(1) - c_5(1) = -0.3687, \quad c_2(2) - c_5(2) = -0.1647, \quad c_2(3) - c_5(3) = 0.3463.$$

Therefore thanks to Proposition 4.2, the trivial solution of (4.21) is almost surely exponentially stable. A comparison between Examples 4.9 and 4.10 shows that in some cases, the jumps can suppress the growth of the solution. In addition, we notice that the switching mechanism also contributes to the almost surely exponential stability.

5 Conclusions and Further Remarks

Motivated by the emerging applications of complex stochastic systems in areas such as finance and energy spot price modeling, this paper is devoted to almost sure and pth moment exponential stabilities of regime-switching jump diffusions. The main results include sufficient conditions for almost sure and pth moment exponential stabilities of the equilibrium point of nonlinear and linear regime-switching jump diffusions. For general nonlinear systems, the sufficient conditions for stability are expressed in terms of the existence of appropriate Lyapunov functions; from which we also derive a condition using M-matrices. In addition, we show that pth moment stability implies almost sure exponential stability. For one-dimensional linear regime-switching jump diffusions, we obtain necessary and sufficient conditions for almost sure and pth moment exponential stabilities. For the multidimensional system, we present verifiable sufficient conditions in terms of the eigenvalues of certain matrices for stability. Several examples are provided to illustrate the results.

In this work, the switching component α has a finite state space. A relevant question is: Can we allow α to have an infinite countable space? In addition, the jump part is driven by a Poisson random measure associated with a Lévy process. A worthwhile future effort is to treat systems in which the random driving force is an alpha-stable process that has finite pth moment with $p < 2$. This requires more work and careful consideration.

Acknowledgements

We would like to thanks the anonymous reviewers for their useful comments and suggestions.

The research of Zhen Chao was supported in part by the NSFC (No. 11471122), the NSF of Zhejiang Province (No. LY15A010016) and ECNU reward for Excellent Doctoral Students in Academics (No. xrzz2014020). The research of Kai Wang and Yanling Zhu was supported in part by the NSF of Anhui Province (No. 1708085MA17 and No. 1508085QA13), the Key

NSF of Education Bureau of Anhui Province (No. KJ2013A003) and the Support Plan of Excellent Youth Talents in Colleges and Universities in Anhui Province (2014). The research of Chao Zhu was supported in part by the NSFC (No. 11671034) and the Simons Foundation (award number 523736).

A Several Technical Proofs

Proof of Theorem 3.1. Recall that thanks to Lemma 2.6, for every $(x_0, \alpha_0) \in \mathbb{R}_0^n \times \mathcal{M}$, $X(t) := X^{x_0,\alpha_0}(t) \neq 0$ for all $t \geq 0$ a.s. Let $U(x,i) = \log V(x,i)$ for $(x,i) \in \mathbb{R}_0^n \times \mathcal{M}$. Since

$$DU(x,i) = \frac{DV(x,i)}{V(x,i)} \text{ and } D^2U(x,i) = \frac{D^2V(x,i)}{V(x,i)} - \frac{DV(x,i)DV(x,i)'}{V^2(x,i)},$$

we have

$$\begin{aligned}
&\mathcal{L}U(x,i) \\
&= \frac{\langle DV(x,i), b(x,i)\rangle}{V(x,i)} + \frac{1}{2V(x,i)}\mathrm{tr}\big(\sigma\sigma'(x,i)D^2V(x,i)\big) - \frac{|\langle DV(x,i), \sigma(x,i)\rangle|^2}{2V^2(x,i)} \\
&+ \sum_{j\in\mathcal{M}} q_{ij}(x)\log V(x,j) + \int_{\mathbb{R}_0^n}\left[\log\frac{V(x+\gamma(x,i,z),i)}{V(x,i)} - \frac{DV(x,i)\cdot\gamma(x,i,z)}{V(x,i)}\right]\nu(\mathrm{d}z)\mathrm{d}s \\
&= \frac{\mathcal{L}V(x,i)}{V(x,i)} - \frac{|\langle DV(x,i), \sigma(x,i)\rangle|^2}{2V^2(x,i)} + \sum_{j\in\mathcal{M}} q_{ij}(x)\left(\log V(x,j) - \frac{V(x,j)}{V(x,i)}\right) \\
&+ \int_{\mathbb{R}_0^n}\left[\log\frac{V(x+\gamma(x,i,z),i)}{V(x,i)} - \frac{V(x+\gamma(x,i,z),i)}{V(x,i)} + 1\right]\nu(\mathrm{d}z). \qquad \text{(A.1)}
\end{aligned}$$

Now we apply Itô's formula to the process $U(X(t),\alpha(t))$:

$$U(X(t),\alpha(t)) = U(x_0,\alpha_0) + \int_0^t \mathcal{L}U(X(s),\alpha(s))\mathrm{d}s + M(t), \qquad \text{(A.2)}$$

where $M(t) = M_1(t) + M_2(t) + M_3(t)$, and

$$\begin{aligned}
M_1(t) &= \int_0^t \frac{\langle DV(X(s),\alpha(s)), \sigma(X(s),\alpha(s))\rangle}{V(X(s),\alpha(s))}\mathrm{d}W(s), \\
M_2(t) &= \int_0^t\int_{\mathbb{R}_0^n}\log\frac{V(X(s-)+\gamma(X(s-),\alpha(s-),z),\alpha(s-))}{V(X(s-),\alpha(s-))}\widetilde{N}(\mathrm{d}s,\mathrm{d}z), \\
M_3(t) &= \int_0^t\int_{\mathbb{R}}\log\frac{V(X(s-),\alpha(s-)+h(X(s-),\alpha(s-),y))}{V(X(s-),\alpha(s-))}\widetilde{N}_1(\mathrm{d}s,\mathrm{d}y).
\end{aligned}$$

By the exponential martingale inequality (Applebaum, 2009, Theorem 5.2.9), for any $k \in \mathbb{N}$ and $\theta \in (0,1)$, we have

$$\mathbb{P}\bigg\{\sup_{0\leq t\leq k}\bigg[M(t) - \frac{\theta}{2}\int_0^t \frac{|\langle DV(X(s),\alpha(s)), \sigma(X(s),\alpha(s))\rangle|^2}{|V(X(s),\alpha(s))|^2}\mathrm{d}s$$

$$- f_{1,\theta}(t) - f_{2,\theta}(t)\Big] > \theta\sqrt{k}\Big\} \leq e^{-\theta^2\sqrt{k}},$$

where

$$\begin{aligned} f_{1,\theta}(t) = \frac{1}{\theta}\int_0^t \int_{\mathbb{R}_0^n} \Bigg[\bigg(\frac{V(X(s-)+\gamma(X(s-),\alpha(s-),z),\alpha(s-))}{V(X(s-),\alpha(s-))}\bigg)^\theta - 1 \\ - \theta \log \frac{V(X(s-)+\gamma(X(s-),\alpha(s-),z),\alpha(s-))}{V(X(s-),\alpha(s-))}\Bigg]\nu(\mathrm{d}z)\mathrm{d}s, \\ f_{2,\theta}(t) = \frac{1}{\theta}\int_0^t \int_{\mathbb{R}} \Bigg[\bigg(\frac{V(X(s-),\alpha(s-)+h(X(s-),\alpha(s-),y))}{V(X(s-),\alpha(s-))}\bigg)^\theta - 1 \\ - \theta \log \frac{V(X(s-),\alpha(s-)+h(X(s-),\alpha(s-),y))}{V(X(s-),\alpha(s-))}\Bigg]\lambda(\mathrm{d}y)\mathrm{d}s. \end{aligned}$$

We can verify that $\sum_k e^{-\theta^2\sqrt{k}} < \infty$. Therefore the Borel-Cantelli lemma implies that there exists an $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that for every $\omega \in \Omega_0$, there exists an integer $k_0 = k_0(\omega)$ so that for all $k \geq k_0$ and $0 \leq t \leq k$, we have

$$M(t) \leq \frac{\theta}{2}\int_0^t \frac{|\langle DV(X(s),\alpha(s)),\sigma(X(s),\alpha(s))\rangle|^2}{|V(X(s),\alpha(s))|^2}\mathrm{d}s + \theta\sqrt{k} + f_{1,\theta}(t) + f_{2,\theta}(t). \tag{A.3}$$

Now putting (A.3) and (A.1) into (A.2), it follows that for all $\omega \in \Omega_0$ and $0 \leq t \leq k$, we have

$$\begin{aligned} &U(X(t),\alpha(t)) - U(x_0,\alpha_0) \\ &\leq \int_0^t \frac{\mathcal{L}V(X(s),\alpha(s))}{V(X(s),\alpha(s))}\mathrm{d}s - \frac{1-\theta}{2}\int_0^t \frac{|\langle DV(X(s),\alpha(s)),\sigma(X(s),\alpha(s))\rangle|^2}{|V(X(s),\alpha(s))|^2}\mathrm{d}s \\ &\quad + \theta\sqrt{k} + f_{1,\theta}(t) + f_{2,\theta}(t) + \int_0^t \sum_{j\in\mathcal{M}} q_{\alpha(s),j}(X(s))\bigg(\log V(X(s),j) - \frac{V(X(s),j)}{V(X(s),\alpha(s))}\bigg)\mathrm{d}s \\ &\quad + \int_0^t \int_{\mathbb{R}_0^n} \Bigg[\log \frac{V(X(s-)+\gamma(X(s-),\alpha(s-),z),\alpha(s-))}{V(X(s-),\alpha(s-))} + 1 \\ &\qquad - \frac{V(X(s-)+\gamma(X(s-),\alpha(s-),z),\alpha(s-))}{V(X(s-),\alpha(s-))}\Bigg]\nu(\mathrm{d}z)\mathrm{d}s \\ &\leq \int_0^t \Big[c_2(\alpha(s)) - \frac{1-\theta}{2}c_3(\alpha(s)) - c_4(\alpha(s)) - c_5(\alpha(s))\Big]\mathrm{d}s + \theta\sqrt{k} + f_{1,\theta}(t) + f_{2,\theta}(t). \end{aligned} \tag{A.4}$$

Next we argue that for any $t \geq 0$, $f_{1,\theta}(t) + f_{2,\theta}(t) \to 0$ as $\theta \downarrow 0$. To this end, we first use the elementary inequality $e^a \geq a + 1$ for $a \in \mathbb{R}$ to obtain

$$\frac{1}{\theta}\Bigg[\bigg(\frac{V(x+\gamma(x,i,z),i)}{V(x,i)}\bigg)^\theta - 1 - \theta\log\frac{V(x+\gamma(x,i,z),i)}{V(x,i)}\Bigg] \geq 0 \text{ for any } (x,i) \in \mathbb{R}^n \times \mathcal{M}.$$

Next the inequality $x^r \le 1 + r(x-1)$ for $0 \le r \le 1$ and $x > 0$ leads to

$$\begin{aligned}
&\frac{1}{\theta}\bigg[\bigg(\frac{V(x+\gamma(x,i,z),i)}{V(x,i)}\bigg)^{\theta} - 1 - \theta \log \frac{V(x+\gamma(x,i,z),i)}{V(x,i)}\bigg] \\
&\quad \le \frac{1}{\theta}\bigg[1 + \theta\bigg(\frac{V(x+\gamma(x,i,z),i)}{V(x,i)} - 1\bigg) - 1 - \theta \log \frac{V(x+\gamma(x,i,z),i)}{V(x,i)}\bigg] \\
&\quad = \frac{V(x+\gamma(x,i,z),i)}{V(x,i)} - 1 - \log \frac{V(x+\gamma(x,i,z),i)}{V(x,i)};
\end{aligned}$$

notice that the last expression in the above equation is nonnegative thanks to the inequality $a - 1 - \log a \ge 0$ for $a > 0$. Next by virtue of (2.10), we can slightly modify the proof of Lemma 3.3 in Applebaum and Siakalli (2009) to obtain

$$\begin{aligned}
\int_0^t \int_{\mathbb{R}_0^n} \bigg[&\frac{V(X(s-)+\gamma(X(s-),\alpha(s-),z),\alpha(s-))}{V(X(s-),\alpha(s-))} - 1 \\
&- \log \frac{V(X(s-)+\gamma(X(s-),\alpha(s-),z),\alpha(s-))}{V(X(s-),\alpha(s-))}\bigg]\nu(\mathrm{d}z)\mathrm{d}s < \infty \quad \text{a.s.}
\end{aligned}$$

In addition, we can verify that $\lim_{\theta\downarrow 0}[\frac{1}{\theta}(a^{\theta}-1) - \log a] = 0$ for $a > 0$. Then the dominated convergence theorem leads to

$$\begin{aligned}
\lim_{\theta\downarrow 0} f_{1,\theta}(t) = \int_0^t \int_{\mathbb{R}_0^n} \lim_{\theta\downarrow 0} \frac{1}{\theta}\bigg[&\bigg(\frac{V(X(s-)+\gamma(X(s-),\alpha(s-),z),\alpha(s-))}{V(X(s-),\alpha(s-))}\bigg)^{\theta} - 1 \\
&- \theta \log \frac{V(X(s-)+\gamma(X(s-),\alpha(s-),z),\alpha(s-))}{V(X(s-),\alpha(s-))}\bigg]\nu(\mathrm{d}z)\mathrm{d}s = 0.
\end{aligned}$$

On the other hand, using (2.11), we can readily verify that

$$\begin{aligned}
\int_0^t \int_{\mathbb{R}} \bigg[&\frac{V(X(s-),\alpha(s-)+h(X(s-),\alpha(s-),y))}{V(X(s-),\alpha(s-))} - 1 \\
&- \log \frac{V(X(s-),\alpha(s-)+h(X(s-),\alpha(s-),y))}{V(X(s-),\alpha(s-))}\bigg]\lambda(\mathrm{d}y)\mathrm{d}s < \infty \quad \text{a.s.}
\end{aligned}$$

Therefore using exactly the same argument as above, we derive $\lim_{\theta\downarrow 0} f_{2,\theta}(t) = 0$.

Now passing to the limit as $\theta \downarrow 0$ in (A.4) leads to

$$U(X(t),\alpha(t)) - U(x_0,\alpha_0) \le \int_0^t \big[c_2(\alpha(s)) - 0.5c_3(\alpha(s)) - c_4(\alpha(s)) - c_5(\alpha(s))\big]\mathrm{d}s \tag{A.5}$$

for all $\omega \in \Omega_0$, $k \ge k_0 = k_0(\omega)$ and $0 \le t \le k$. Recall that $U(x,i) = \log V(x,i)$. Then inserting condition (i) into (A.5) yields that for almost all $\omega \in \Omega$, $k \ge k_0$, and $k-1 \le t \le k$, we have

$$\frac{1}{t}[p \log |X(t)| + \log c_1(\alpha(t))] \le \frac{1}{t} \log V(X(t),\alpha(t))$$

$$
\begin{aligned}
&\leq \frac{1}{t}\int_0^t \big[c_2(\alpha(s)) - 0.5c_3(\alpha(s)) - c_4(\alpha(s)) - c_5(\alpha(s))\big]\mathrm{d}s + \frac{\log V(x_0,\alpha_0)}{t}\\
&\leq \frac{1}{t}\int_0^t \big[c_2(\alpha(s)) - 0.5c_3(\alpha(s)) - c_4(\alpha(s)) - c_5(\alpha(s))\big]\mathrm{d}s + \frac{\log V(x_0,\alpha_0)}{k-1}\\
&\leq \max_{i\in\mathcal{M}}\big\{c_2(i) - 0.5c_3(i) - c_4(i) - c_5(i)\big\} + \frac{\log V(x_0,\alpha_0)}{k-1};
\end{aligned}
\tag{A.6}
$$

the last inequality yields (3.1) by letting $t \to \infty$. □

Proof of Theorem 3.5. Fix some $(x_0,\alpha_0) \in \mathbb{R}^n_0 \times \mathcal{M}$ and denote by $(X(t),\alpha(t))$ the unique solution to (2.3)–(2.7) with initial condition $(X(0),\alpha(0)) = (x_0,\alpha_0)$. Suppose that for some $\varrho > 0$, we have

$$
\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}[|X(t)|^p] \leq -\varrho < 0.
$$

Then for any $\varrho > \varepsilon > 0$, there exists a positive constant T such that $\mathbb{E}[|X(t)|^p] \leq e^{-(\varrho-\varepsilon)t}$ for all $t \geq T$. This, together with Lemma 3.1 of Zhu et al. (2015), implies that there exists some positive number M so that

$$
\mathbb{E}[|X(t)|^p] \leq Me^{-(\varrho-\varepsilon)t} \text{ for all } t \geq 0. \tag{A.7}
$$

Let $\delta > 0$. Then we have for any $k \in \mathbb{N}$,

$$
\begin{aligned}
\mathbb{E}\bigg[\sup_{(k-1)\delta\leq t\leq k\delta}|X(t)|^p\bigg] \leq 4^p\mathbb{E}\bigg[&|X((k-1)\delta)|^p + \sup_{(k-1)\delta\leq t\leq k\delta}\bigg|\int_{(k-1)\delta}^t b(X(s),\alpha(s))\mathrm{d}s\bigg|^p\\
&+ \sup_{(k-1)\delta\leq t\leq k\delta}\bigg|\int_{(k-1)\delta}^t \sigma(X(s),\alpha(s))\mathrm{d}W(s)\bigg|^p\\
&+ \sup_{(k-1)\delta\leq t\leq k\delta}\bigg|\int_{(k-1)\delta}^t\int_{\mathbb{R}^n_0}\gamma(X(s-),\alpha(s-),z)\tilde{N}(\mathrm{d}s,\mathrm{d}z)\bigg|^p\bigg].
\end{aligned}
\tag{A.8}
$$

Using (2.8) and (2.9), we have

$$
\begin{aligned}
\mathbb{E}\bigg[\sup_{(k-1)\delta\leq t\leq k\delta}\bigg|\int_{(k-1)\delta}^t b(X(s),\alpha(s))\mathrm{d}s\bigg|^p\bigg] &\leq \mathbb{E}\bigg[\bigg|\int_{(k-1)\delta}^{k\delta}|b(X(s),\alpha(s))|\mathrm{d}s\bigg|^p\bigg]\\
&\leq (\sqrt{\kappa}\delta)^p\mathbb{E}\bigg[\sup_{(k-1)\delta\leq t\leq k\delta}|X(t)|^p\bigg],
\end{aligned}
\tag{A.9}
$$

where $\kappa > 0$ is the constant appearing in (2.9). On the other hand, by the Burkhoder-Davis-Gundy inequality (see, e.g., Theorem 2.13 on p. 70 of Mao and Yuan (2006)) and (2.8), (2.9), we have

$$
\begin{aligned}
\mathbb{E}\bigg[\sup_{(k-1)\delta\leq t\leq k\delta}\bigg|\int_{(k-1)\delta}^t \sigma(X(s),\alpha(s))\mathrm{d}W(s)\bigg|^p\bigg] &\leq C_p\mathbb{E}\bigg[\bigg(\int_{(k-1)\delta}^{k\delta}|\sigma(X(s),\alpha(s))|^2\mathrm{d}s\bigg)^{p/2}\bigg]\\
&\leq C_p(\kappa\delta)^{p/2}\mathbb{E}\bigg[\sup_{(k-1)\delta\leq t\leq k\delta}|X(t)|^p\bigg],
\end{aligned}
\tag{A.10}
$$

where C_p is a positive constant depending only on p. Next we use Kunita's first inequality (see, e.g., Theorem 4.4.23 on p. 265 of Applebaum (2009)), (2.9), (2.10) and (3.4) to estimate

$$\begin{aligned}
&\mathbb{E}\bigg[\sup_{(k-1)\delta\le t\le k\delta}\bigg|\int_{(k-1)\delta}^{t}\int_{\mathbb{R}_0^n}\gamma(X(s-),\alpha(s-),z)\tilde{N}(\mathrm{d}s,\mathrm{d}z)\bigg|^p\bigg]\\
&\le D_p\mathbb{E}\bigg[\int_{(k-1)\delta}^{k\delta}\bigg(\int_{\mathbb{R}_0^n}|\gamma(X(s-),\alpha(s-),z)|^2\nu(\mathrm{d}z)\bigg)^{p/2}\mathrm{d}s\bigg]\\
&\quad + D_p\mathbb{E}\bigg[\int_{(k-1)\delta}^{k\delta}\int_{\mathbb{R}_0^n}|\gamma(X(s-),\alpha(s-),z)|^p\nu(\mathrm{d}z)\mathrm{d}s\bigg]\\
&\le D_p[\kappa^{p/2}+\hat{\kappa}]\delta\mathbb{E}\bigg[\sup_{(k-1)\delta\le t\le k\delta}|X(t)|^p\bigg],
\end{aligned}\tag{A.11}$$

where $\hat{\kappa}>0$ is the constant appearing in (3.4) and D_p is a positive constant depending only on p.

Now we plug (A.9), (A.10), and (A.11) into (A.8) to derive

$$\begin{aligned}
&\mathbb{E}\bigg[\sup_{(k-1)\delta\le t\le k\delta}|X(t)|^p\bigg]\\
&\le 4^p\mathbb{E}[|X((k-1)\delta)|^p]+4^p\big((\sqrt{\kappa}\delta)^p+C_p(\kappa\delta)^{p/2}+D_p[\kappa^{p/2}+\hat{\kappa}]\delta\big)\mathbb{E}\bigg[\sup_{(k-1)\delta\le t\le k\delta}|X(t)|^p\bigg].
\end{aligned}$$

Now we choose a $\delta>0$ sufficiently small so that

$$4^p\big((\sqrt{\kappa}\delta)^p+C_p(\kappa\delta)^{p/2}+D_p[\kappa^{p/2}+\hat{\kappa}]\delta\big)<\frac{1}{2}.$$

Then it follows from (A.7) that

$$\mathbb{E}\bigg[\sup_{(k-1)\delta\le t\le k\delta}|X(t)|^p\bigg]\le 2M4^pe^{-(\varrho-\varepsilon)(k-1)\delta}.\tag{A.12}$$

The rest of the proof uses the same arguments as those in the proof of Theorem 5.9 of Mao and Yuan (2006). For completeness, we include the details here. Thanks to (A.12), we have from the Chebyshev inequality that

$$\mathbb{P}\bigg\{\omega\in\Omega:\sup_{(k-1)\delta\le t\le k\delta}|X(t)|>e^{-(\varrho-2\varepsilon)(k-1)\delta/p}\bigg\}\le 2M4^pe^{-\varepsilon(k-1)\delta}.$$

Then by the Borel-Cantelli lemma, there exists an $\Omega_0\subset\Omega$ with $\mathbb{P}(\Omega_0)=1$ such that for all $\omega\in\Omega_0$, there exists a $k_0=k_0(\omega)\in\mathbb{N}$ such that

$$\sup_{(k-1)\delta\le t\le k\delta}|X(t,\omega)|\le e^{-(\varrho-2\varepsilon)(k-1)\delta/p},\qquad \text{for all } k\ge k_0=k_0(\omega).$$

Consequently for all $\omega\in\Omega_0$, if $(k-1)\delta\le t\le k\delta$ and $k\ge k_0(\omega)$, we have

$$\frac{1}{t}\log(|X(t,\omega)|)\le-\frac{(\varrho-2\varepsilon)(k-1)\delta}{pt}\le-\frac{(\varrho-2\varepsilon)(k-1)}{pk}.$$

This implies that

$$\limsup_{t\to\infty}\frac{1}{t}\log(|X(t,\omega)|)\le -\frac{\varrho-2\varepsilon}{p},\qquad \text{for all } \omega\in\Omega_0.$$

Now letting $\varepsilon\downarrow 0$, we obtain that $\limsup_{t\to\infty}\frac{1}{t}\log(|X(t)|)\le -\frac{\varrho}{p}$ a.s. This completes the proof. □

Proof of Proposition 4.2. We need to find an appropriate Lyapunov function V so that all conditions of Theorem 3.1 are satisfied. In addition, since α is an irreducible continuous-time Markov chain with a stationary distribution $\pi=(\pi_i,i\in\mathcal{M})$, the assertion on a.s. exponential stability under (4.10) follows from Corollary 3.3. To this end, let $G\in\mathcal{S}^{n\times n}$ and $p>0$ be as in the statement of the theorem. We consider the Lyapunov function

$$V(x,i)=(h_i-p\zeta_i)(x'G^2x)^{p/2},\quad (x,i)\in\mathbb{R}^n\times\mathcal{M},$$

where $h_i>0$ such that $h_i-p\xi_i>0$. Let us now verify that V satisfies conditions (i)–(v) of Theorem 3.1.

It is readily seen that for each $i\in\mathcal{M}$, $V(\cdot,i)$ is continuous, nonnegative, and vanishes only at $x=0$. Also observe that condition (i) of Theorem 3.1 is satisfied with $c_1(i):=(h_i-p\zeta_i)(\lambda_{\min}(G^2))^{\frac{p}{2}}$. We can verify for $x\neq 0$ that

$$\begin{aligned}DV(x,i)&=(h_i-p\zeta_i)p(x'G^2x)^{p/2-1}G^2x,\\ D^2V(x,i)&=(h_i-p\zeta_i)p(x'G^2x)^{p/2-2}[(p-2)G^2xx'G^2+x'G^2xG^2].\end{aligned}$$

Then we compute

$$\begin{aligned}&\frac{1}{h_i-p\zeta_i}\mathcal{L}V(x,i) \qquad\qquad (A.13)\\ &=p(x'G^2x)^{\frac{p}{2}-1}x'G_i^2A_ix+\frac{1}{2}\mathrm{tr}\big(p(x'G^2x)^{\frac{p}{2}-2}[(p-2)G^2xx'G^2+x'G^2xG^2]B_ixx'B_i'\big)\\ &\quad+\int_{\mathbb{R}_0^n}\Big[(x'(I+C_i(z))'G^2(I+C_i(z))x)^{\frac{p}{2}}-(x'G^2x)^{\frac{p}{2}}-p(x'G^2x)^{\frac{p}{2}-1}x'G^2C_i(z)x\Big]\nu(\mathrm{d}z)\\ &\quad+\sum_{j\in\mathcal{M}}q_{ij}\frac{h_j-h_i-p\zeta_j+p\zeta_i}{h_i-p\zeta_i}(x'G^2x)^{\frac{p}{2}}\\ &=p(x'G^2x)^{p/2}\Bigg[\frac{x'(G^2A_i+A_i'G^2+B_i'G^2B_i)x}{2x'G^2x}+(p-2)\frac{(x'B_i'G^2x)^2}{2(x'G^2x)^2}+\frac{1}{p(h_i-p\zeta_i)}\sum_{j\in\mathcal{M}}q_{ij}h_j\\ &\quad-\frac{1}{h_i-p\zeta_i}\sum_{j\in\mathcal{M}}q_{ij}\zeta_j+\int_{\mathbb{R}_0^n}\Big[\frac{(x'(I+C_i(z))'G^2(I+C_i(z))x)^{\frac{p}{2}}}{p(x'G^2x)^{\frac{p}{2}}}-\frac{1}{p}-\frac{x'G^2C_i(z)x}{x'G^2x}\Big]\nu(\mathrm{d}z)\Bigg].\end{aligned}$$

Note that

$$\frac{x'(G^2A_i+A_i'G^2+B_i'G^2B_i)x}{2x'G^2x}=\frac{x'G(GA_iG^{-1}+G^{-1}A_i'G+G^{-1}B_i'G^2B_iG^{-1})Gx}{2|Gx|^2}$$

$$\leq \frac{1}{2}\lambda_{\max}(GA_iG^{-1} + G^{-1}A_i'G + G^{-1}B_i'G^2B_iG^{-1}) = \mu_i. \tag{A.14}$$

In addition, we have

$$\begin{aligned}
\frac{p-2}{2}\left(\frac{x'B_i'G^2x}{x'G^2x}\right)^2 &= \frac{p-2}{8}\left(\frac{x'G(G^{-1}B_i'G + GB_iG^{-1})Gx}{x'G^2x}\right)^2 \\
&\leq \begin{cases} \frac{p-2}{8}\widehat{\lambda}^2(G^{-1}B_i'G + GB_iG^{-1}), & \text{if } 0 < p \leq 2, \\ \frac{p-2}{8}\rho^2(G^{-1}B_i'G + GB_iG^{-1}), & \text{if } p > 2. \end{cases} \\
&= \frac{p-2}{8}\Lambda^2(G^{-1}B_i'G + GB_iG^{-1}).
\end{aligned} \tag{A.15}$$

On the other hand, since

$$\begin{aligned}
\frac{(x'(I + C_i(z))'G^2(I + C_i(z))x)^{\frac{p}{2}}}{(x'G^2x)^{\frac{p}{2}}} &= \left(\frac{x'(I + C_i(z))'G^2(I + C_i(z))x}{x'G^2x}\right)^{\frac{p}{2}} \\
&\leq \big(\lambda_{\max}(G^{-1}(I + C_i(z))'G^2(I + C_i(z))G^{-1})\big)^{\frac{p}{2}},
\end{aligned}$$

and

$$\frac{x'G^2C_i(z)x}{x'G^2x} = \frac{x'G(GC_i(z)G^{-1} + G^{-1}C_i'(z)G)Gx}{2|Gx|^2} \geq \frac{1}{2}\lambda_{\min}(GC_i(z)G^{-1} + G^{-1}C_i'(z)G),$$

it follows that

$$\begin{aligned}
&\int_{\mathbb{R}_0^n}\left[\frac{(x'(I + C_i(z))'G^2(I + C_i(z))x)^{\frac{p}{2}}}{p(x'G^2x)^{\frac{p}{2}}} - \frac{1}{p} - \frac{x'G^2C_i(z)x}{x'G^2x}\right]\nu(\mathrm{d}z) \\
&\leq \frac{1}{p}\int_{\mathbb{R}_0^n}\bigg[\big(\lambda_{\max}(G^{-1}(I + C_i(z))'G^2(I + C_i(z))G^{-1})\big)^{\frac{p}{2}} \\
&\qquad\qquad - 1 - \frac{p}{2}\lambda_{\min}(GC_i(z)G^{-1} + G^{-1}C_i'(z)G)\bigg]\nu(\mathrm{d}z) \\
&= \frac{\rho_i}{p}.
\end{aligned} \tag{A.16}$$

Then upon putting the estimates (A.14)–(A.16) into (A.13), we have

$$\begin{aligned}
\mathcal{L}V(x,i) &\leq (h_i - p\zeta_i)(x'G^2x)^{\frac{p}{2}} \cdot \bigg[p\mu_i + \frac{p-2}{8}\Lambda^2(GB_iG^{-1} + G^{-1}B_i'G) \\
&\qquad + \frac{1}{h_i - p\zeta_i}\sum_{j\in\mathcal{M}} q_{ij}h_j - \frac{p}{h_i - p\zeta_i}\sum_{j\in\mathcal{M}} q_{ij}\zeta_j + \eta_i\bigg] \\
&= V(x,i)\bigg[p\mu_i + \frac{p-2}{8}\Lambda^2(GB_iG^{-1} + G^{-1}B_i'G) \\
&\qquad + \frac{1}{(h_i - p\zeta_i)}\sum_{j\in\mathcal{M}} q_{ij}h_j - \frac{p}{h_i - p\zeta_i}(\mu_i + \beta) + \eta_i\bigg],
\end{aligned}$$

where we used (4.8) to derive the last step. Thus condition (ii) of Theorem 3.1 is satisfied with $c_2(i) = p\mu_i + \frac{p-2}{8}\Lambda^2(GB_iG^{-1} + G^{-1}B_i'G) - \frac{p}{h_i - p\zeta_i}(\mu_i + \beta) + \frac{1}{(h_i - p\zeta_i)}\sum_{j\in\mathcal{M}} q_{ij}h_j + \eta_i$. In view of

$$|\langle DV(x,i), B_ix\rangle|^2 = p^2V^2(x,i)\left(\frac{x'G^2B_ix}{x'G^2x}\right)^2 = \frac{p^2}{4}V^2(x,i)\left(\frac{x'G(GB_iG^{-1} + G^{-1}B_i'G)Gx}{x'G^2x}\right)^2$$
$$\geq \frac{p^2}{4}\widehat{\lambda}^2(GB_iG^{-1} + G^{-1}B_i'G)V^2(x,i),$$

Note that

$$\begin{aligned}
&\int_{\mathbb{R}_0^n}\left[\log\left(\frac{V(x + C_i(z)x, i)}{V(x,i)}\right) - \frac{V(x + C_i(z)x, i)}{V(x,i)} + 1\right]\nu(\mathrm{d}z)\\
&= \int_{\mathbb{R}_0^n}\left[\frac{p}{2}\log\frac{x'(I + C_i(z))'G^2(I + C_i(z))x}{x'G^2x} - \frac{(x'(I + C_i(z))'G^2(I + C_i(z))x)^{\frac{p}{2}}}{(x'G^2x)^{\frac{p}{2}}} + 1\right]\nu(\mathrm{d}z)\\
&\leq 0 \wedge \int_{\mathbb{R}_0^n}\left[\frac{p}{2}\log\lambda_{\max}(G^{-1}(I + C_i(z))'G^2(I + C_i(z))G^{-1})\right.\\
&\qquad\left. - \left(\lambda_{\min}(G^{-1}(I + C_i(z))'G^2(I + C_i(z))G^{-1})\right)^{\frac{p}{2}} + 1\right]\nu(\mathrm{d}z)\\
&= -c_4(i).
\end{aligned}$$

This establishes Condition (iv). Likewise, we can verify condition (v) as follows.

$$\begin{aligned}
&\sum_{j\in\mathcal{M}} q_{ij}\left(\log V(x,j) - \frac{V(x,j)}{V(x,i)}\right)\\
&= \sum_{j\in\mathcal{M}} q_{ij}\left(\log(h_j - p\zeta_j) + \frac{p}{2}\log(x'G^2x) - \frac{h_j - p\zeta_j}{h_i - p\zeta_i}\right)\\
&= \sum_{j\in\mathcal{M}} q_{ij}\left(\log(h_j - p\zeta_j) - \frac{h_j - p\zeta_j}{h_i - p\zeta_i}\right) = -c_5(i).
\end{aligned}$$

Thus we have verified all conditions of Theorem 3.1 and hence in view of Corollary 3.3, (4.10) implies the desired conclusion. □

Proof of Proposition 4.5. This proof is motivated by the proofs of Theorem 5.24 in Mao and Yuan (2006) and Theorem 3.3 in Zong et al. (2014). As in the proof of Proposition 4.1, let us assume $x(0) = x \neq 0$ and $\alpha(0) = i \in \mathcal{M}$. Then by the Itô formula, we have

$$\begin{aligned}
&|x(t)|^p\\
&= |x|^p\exp\left\{\int_0^t\left[pa(\alpha(s)) - \frac{p}{2}b^2(\alpha(s)) + p\int_{\mathbb{R}_0}[\log|1 + c(\alpha(s-),z)| - c(\alpha(s-),z)]\nu(\mathrm{d}z)\right]\mathrm{d}s\right.\\
&\qquad\left. + \int_0^t pb(\alpha(s))\mathrm{d}W(s) + \int_0^t\int_{\mathbb{R}_0} p\log|1 + c(\alpha(s-),z)|\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\right\}
\end{aligned}$$

$$= |x|^p \exp\bigg\{\int_0^t f(\alpha(s))\mathrm{d}s\bigg\}\mathcal{E}(t),$$

where $\mathcal{E}(t) := \exp\{g(t)\}$ with

$$g(t) = \int_0^t pb(\alpha(s))\mathrm{d}W(s) - \frac{1}{2}\int_0^t p^2b^2(\alpha(s))\mathrm{d}s + \int_0^t\int_{\mathbb{R}_0} p\log|1+c(\alpha(s-),z)|\widetilde{N}(\mathrm{d}s,\mathrm{d}z)$$
$$- \int_0^t\int_{\mathbb{R}_0} [|1+c(\alpha(s-),z)|^p - p\log|1+c(\alpha(s-),z)| - 1]\nu(\mathrm{d}z)\mathrm{d}s.$$

For each $t \geq 0$, let $\mathcal{G}_t := \sigma(\alpha(s) : 0 \leq s \leq t)$, $\mathcal{G} := \bigvee_{t\geq 0}\mathcal{G}_t$, and $\mathcal{H}_t := \sigma(W(s), N([0,s)\times A), 0 \leq s \leq t, A \in \mathcal{B}(\mathbb{R}_0))$. Denote $\mathcal{D}_t := \mathcal{G}\bigvee\mathcal{H}_t$. Let $\{\tau_k, k = 1,2,\dots\}$ denote the sequence of switching times for the continuous-time Markov chain $\alpha(\cdot)$; that is, we define $\tau_1 := \inf\{t \geq 0 : \alpha(t) \neq \alpha(0)\}$ and $\tau_{k+1} := \inf\{t \geq \tau_k : \alpha(t) \neq \alpha(\tau_k)\}$ for $k = 1,2,\dots$ It is well-known that $\tau_k \to \infty$ a.s. as $k \to \infty$. Write $\tau_0 := 0$. Then we can compute

$$\begin{aligned}\mathbb{E}[|x(t)|^p] &= \mathbb{E}\Bigg[|x|^p\sum_{k=0}^\infty I_{\{\tau_k\leq t<\tau_{k+1}\}}\exp\bigg\{\int_0^t f(\alpha(s))\mathrm{d}s\bigg\}\mathcal{E}(t)\Bigg]\\ &= \sum_{k=0}^\infty \mathbb{E}\Bigg[|x|^p\mathbb{E}\bigg[I_{\{\tau_k\leq t<\tau_{k+1}\}}\exp\bigg\{\int_0^t f(\alpha(s))\mathrm{d}s\bigg\}\mathcal{E}(t)\Big|\mathcal{D}_{\tau_k}\bigg]\Bigg]\\ &= \sum_{k=0}^\infty \mathbb{E}\Bigg[|x|^p I_{\{\tau_k\leq t<\tau_{k+1}\}}\exp\bigg\{\int_0^t f(\alpha(s))\mathrm{d}s\bigg\}\mathcal{E}(\tau_k)\mathbb{E}\big[\exp\{g(t)-g(\tau_k)\}\big|\mathcal{D}_{\tau_k}\big]\Bigg].\end{aligned}$$

Note that on the event $\{\tau_k \leq t < \tau_{k+1}\}$, we have

$$\begin{aligned}g(t) - g(\tau_k) &= \int_{\tau_k}^t pb(\alpha(\tau_k))\mathrm{d}W(s) - \frac{1}{2}\int_{\tau_k}^t p^2b^2(\alpha(\tau_k))\mathrm{d}s\\ &\quad + \int_{\tau_k}^t\int_{\mathbb{R}_0} p\log|1+c(\alpha(\tau_k-),z)|\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\\ &\quad - \int_{\tau_k}^t\int_{\mathbb{R}_0}[|1+c(\alpha(\tau_k-),z)|^p - p\log|1+c(\alpha(\tau_k-),z)| - 1]\nu(\mathrm{d}z)\mathrm{d}s.\end{aligned}$$

Then it follows from the definition of the σ-algebra $\mathcal{D}_{\tau_k}$ and Corollary 5.2.2 of Applebaum (2009) that $\mathbb{E}\big[\exp\{g(t)-g(\tau_k)\}|\mathcal{D}_{\tau_k}\big] = 1$. Consequently, we have

$$\begin{aligned}&\mathbb{E}[|x(t)|^p]\\ &= \sum_{k=0}^\infty \mathbb{E}\Bigg[|x|^p I_{\{\tau_k\leq t<\tau_{k+1}\}}\exp\bigg\{\int_0^t f(\alpha(s))\mathrm{d}s\bigg\}\mathcal{E}(\tau_k)\Bigg]\\ &= \sum_{k=0}^\infty \mathbb{E}\Bigg[\mathbb{E}\bigg[|x|^p I_{\{\tau_k\leq t<\tau_{k+1}\}}\exp\bigg\{\int_0^t f(\alpha(s))\mathrm{d}s\bigg\}\mathcal{E}(\tau_k)\Big|\mathcal{D}_{\tau_{k-1}}\bigg]\Bigg]\\ &= \sum_{k=0}^\infty \mathbb{E}\Bigg[|x|^p I_{\{\tau_k\leq t<\tau_{k+1}\}}\exp\bigg\{\int_0^t f(\alpha(s))\mathrm{d}s\bigg\}\mathcal{E}(\tau_{k-1})\mathbb{E}[\exp\{g(\tau_k)-g(\tau_{k-1})|\mathcal{D}_{\tau_{k-1}}\}]\Bigg].\end{aligned}$$

As argued before, we have $\mathbb{E}[\exp\{g(\tau_k) - g(\tau_{k-1})|\mathcal{D}_{\tau_{k-1}}\}] = 1$ and hence

$$\mathbb{E}[|x(t)|^p] = \sum_{k=0}^{\infty} \mathbb{E}\bigg[|x|^p I_{\{\tau_k \le t < \tau_{k+1}\}} \exp\bigg\{\int_0^t f(\alpha(s))\mathrm{d}s\bigg\}\mathcal{E}(\tau_{k-1})\bigg].$$

Continue in this fashion and we derive that

$$\begin{aligned}\mathbb{E}[|x(t)|^p] &= \sum_{k=0}^{\infty} \mathbb{E}\bigg[|x|^p I_{\{\tau_k \le t < \tau_{k+1}\}} \exp\bigg\{\int_0^t f(\alpha(s))\mathrm{d}s\bigg\}\mathcal{E}(\tau_0)\bigg] \\ &= \sum_{k=0}^{\infty} \mathbb{E}\bigg[|x|^p I_{\{\tau_k \le t < \tau_{k+1}\}} \exp\bigg\{\int_0^t f(\alpha(s))\mathrm{d}s\bigg\}\bigg] \\ &= \mathbb{E}\bigg[|x|^p \exp\bigg\{\int_0^t f(\alpha(s))\mathrm{d}s\bigg\}\bigg].\end{aligned}$$

Then it follows (4.18) that

$$\lim_{t\to\infty} \frac{1}{t}\log(\mathbb{E}[|x(t)|^p]) = \lim_{t\to\infty} \frac{1}{t}\log(|x|^p) + \lim_{t\to\infty} \frac{1}{t}\log\bigg(\mathbb{E}\bigg[\exp\bigg\{\int_0^t f(\alpha(s))\mathrm{d}s\bigg\}\bigg]\bigg) = \Upsilon(f).$$

This completes the proof. □

Proof of Proposition 4.7. In view of Theorem 3.7, we only need to verify Assumption 3.6 for the positive definite matrix G^2. But as observed in the proof of Proposition 4.2, we have

$$\langle G^2 x, A_i x\rangle + \frac{1}{2}\langle B_i x, G^2 B_i x\rangle \le \frac{1}{2}\lambda_{\max}(GA_iG^{-1} + G^{-1}A_iG + G^{-1}B_i'G^2B_iG^{-1})\langle x, G^2 x\rangle,$$

and (A.16) shows that

$$\int_{\mathbb{R}_0^n}\bigg[\frac{(x'(I + C_i(z))'G^2(I + C_i(z))x)^{\frac{p}{2}}}{(x'G^2x)^{\frac{p}{2}}} - 1 - p\frac{x'G^2C_i(z)x}{x'G^2x}\bigg]\nu(\mathrm{d}z) \le \eta_i.$$

Finally we observe that

$$\frac{(\langle x, G^2 B_i x\rangle)^2}{(\langle x, G^2 x\rangle)^2} = \frac{1}{4}\bigg(\frac{x'G(G^{-1}B_i'G + GB_iG^{-1})Gx}{x'G^2x}\bigg)^2 \le \frac{1}{4}\rho^2(G^{-1}B_i'G + GB_iG^{-1}).$$

Therefore we have verified (3.5)–(3.7) of Assumption 3.6. Then the assertions of the proposition follows from Theorem 3.7 directly. □

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