Covering Radius of Matrix Codes Endowed with the Rank Metric

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Abstract

In this paper we study properties and invariants of matrix codes endowed with the rank metric, and relate them to the covering radius. We introduce new tools for the analysis of rank-metric codes, such as puncturing and shortening constructions. We give upper bounds on the covering radius of a code by applying different combinatorial methods. We apply the various bounds to the classes of maximal rank distance and quasi maximal rank distance codes.

Introduction

Rank-metric codes have featured prominently in the literature on algebraic codes in recent years and especially since their applications to error-correction in networks were understood. Such codes are subsets of the matrix ring $\mathbb{F}_q^{k\times m}$ endowed with the rank distance function, which measures the $\mathbb{F}_q$-rank of the difference of a pair of matrices. An analogue of the Singleton bound was given in [9]. If a code meets this bound it is referred to as a *maximum rank distance* (MRD) code. It is known that there exist codes meeting this bound for all values of q,k,m,d [9, 10, 19]. For this reason the *main coding problem* for rank metric codes, unlike the same problem for the Hamming metric, is closed: for any q,k,m,d the optimal size of a rank-metric code in $\mathbb{F}_q^{k\times m}$ of minimum rank distance d is known. There are very few classes of rank-metric codes known, due in part to the Delsarte-Gabidulin family and its generalizations [9, 10, 19], which are optimal and can be efficiently decoded [10, 14, 22].

The *covering radius* of a code is a fundamental parameter. It measures the maximum weight of any correctable error in the ambient space. It also characterizes the *maximality* property of a code, that is, whether or not the code is contained in another of the same minimum distance. The covering radius of a code measures the least integer r such that every element of the ambient space is within distance r of some codeword. This quantity is generally much harder to compute than the minimum distance of a code. There are numerous papers and books on this topic for classical codes with respect to the Hamming distance (see [1, 3, 4, 5, 13] and the references therein), but relatively little attention has been paid to it for rank-metric codes [11, 12].

In this paper we describe properties of rank-metric codes and relate these to the covering radius. We define new parameters and give tools for the analysis of such codes. In particular, we introduce new definitions for the puncturing and the shortening of a general rank-metric code. In many instances our tools are applied to establish new bounds on the rank-metric covering radius. Some of the derived bounds, such as the dual distance and external distance bounds, are analogues of known bounds for the Hamming distance. Others, such as the initial set bound, are unique to matrix codes. We apply our results to the classes of maximal rank distance and quasi maximal rank distance codes.

In Section 2 we consider the property of *maximality*. A code is maximal if it is not contained in another code of the same minimum distance. We introduce a new parameter, called the *maximality degree* of a code, and show that it is determined by minimum distance and covering radius of a code. These results are independent of the metric. In Section 3 we define shortened and punctured codes rank metric codes and describe their properties. We give a duality result relating a shortened and punctured code. In Section 4 we investigate translates of a code. We show that the weight enumerator of a coset of a linear code of rank weight is completely determined by the weights of first $n-d^{\perp}$ cosets, and establish this using Möbius inversion on the lattice of subspaces of $\mathbb{F}_q^k$. This is then applied to get the rank-metric analogue of the *dual distance bound*. We also give the rank-metric generalization of the *external distance bound*, which holds also for non-linear codes. In Section 5 we introduce the concept of the *initial set* of a matrix code and use this to derive a bound on the covering radius of a code. In Section 6 we apply previously derived bounds to maximum rank distance and quasi maximum rank distance codes.

1 Preliminaries

Throughout this paper, q is a fixed prime power, $\mathbb{F}_q$ is the finite field with q elements, and k,m are positive integers. We assume $k \leq m$ without loss of generality, and denote by $\mathbb{F}_q^{k\times m}$ the space of $k \times m$ matrices over $\mathbb{F}_q$. For any positive integer n we set $[n] := \{i \in \mathbb{N} : 1 \leq i \leq n\}$.

Definition 1. The **rank distance** between matrices $M,N \in \mathbb{F}_q^{k\times m}$ is $d(M,N) := \mathrm{rk}(M-N)$. A **rank-metric code** is a non-empty subset $\mathscr{C} \subseteq \mathbb{F}_q^{k\times m}$. When $|\mathscr{C}| \geq 2$, the **minimum rank distance** of $\mathscr{C}$ is the integer defined by $d(\mathscr{C}) := \min\{d(M,N) : M,N \in \mathscr{C},\ M \neq N\}$. The **weight** and **distance distribution** of a code $\mathscr{C} \subseteq \mathbb{F}_q^{k\times m}$ are the integer vectors $W(\mathscr{C}) = (W_i(\mathscr{C}) : 0 \leq i \leq k)$ and $B(\mathscr{C}) = (B_i(\mathscr{C}) : 0 \leq i \leq k)$, where, for all $i \in \{0,\ldots,k\}$,

$$W_i(\mathscr{C}) := |\{M \in \mathscr{C} : \mathrm{rk}(M) = i\}|, \qquad B_i(\mathscr{C}) := 1/|\mathscr{C}| \cdot |\{(M,N) \in \mathscr{C} \times \mathscr{C} : d(M,N) = i\}|.$$

It is easy to see that d defines a distance function on $\mathbb{F}_q^{k\times m}$.

Definition 2. A code $\mathscr{C} \subseteq \mathbb{F}_q^{k\times m}$ is **linear** if it is an $\mathbb{F}_q$-subspace of $\mathbb{F}_q^{k\times m}$. If this is the case, then the **dual code** of $\mathscr{C}$ is the linear code $\mathscr{C}^{\perp} := \{N \in \mathbb{F}_q^{k\times m} : \mathrm{Tr}(MN^t) = 0 \text{ for all } M \in \mathscr{C}\} \subseteq \mathbb{F}_q^{k\times m}$.

If $\mathscr{C} \subseteq \mathbb{F}_q^{k\times m}$ is a linear code then one can easily check that $d(\mathscr{C}) = \min\{\mathrm{rk}(M) : M \in \mathscr{C},\ M \neq 0\}$ and $W_i(\mathscr{C}) = B_i(\mathscr{C})$ for all $i \in \{0,\ldots,k\}$. Moreover, since the map $(M,N) \mapsto \mathrm{Tr}(MN^t)$ defines an inner product on the space $\mathbb{F}_q^{k\times m}$, we have $\dim(\mathscr{C}^{\perp}) = km - \dim(\mathscr{C})$ and $\mathscr{C}^{\perp\perp} = \mathscr{C}$.

Definition 3. The **covering radius** of a code $\mathscr{C} \subseteq \mathbb{F}_q^{k\times m}$ is the integer

$$\rho(\mathscr{C}) := \min\{i : \text{for all } X \in \mathbb{F}_q^{k\times m} \text{ there exists } M \in \mathscr{C} \text{ with } d(X,M) \leq i\}$$

In words, the covering radius of a code $\mathscr{C}$ is the maximum distance of $\mathscr{C}$ to any matrix in the ambient space, or the minimum value r such that the union of the spheres of radius r about each codeword cover the ambient space. The following result summarizes some simple properties of this invariant. These facts are known from studies of the Hamming distance covering radius and, being actually independent of the metric used, hold also in the rank metric case. For a comprehensive treatment of the covering problem for Hamming metric codes, see [4, 5].

Lemma 4. Let $\mathscr{C} \subseteq \mathbb{F}_q^{k\times m}$ be a code. The following hold.

1. $0 \leq \rho(\mathscr{C}) \leq k$. Moreover, $\rho(\mathscr{C}) = 0$ if and only if $\mathscr{C} = \mathbb{F}_q^{k\times m}$.
2. If $\mathscr{D} \subseteq \mathbb{F}_q^{k\times m}$ is a code with $\mathscr{C} \subseteq \mathscr{D}$, then $\rho(\mathscr{C}) \geq \rho(\mathscr{D})$.
3. If $\mathscr{D} \subseteq \mathbb{F}_q^{k\times m}$ is a code with $\mathscr{C} \subsetneq \mathscr{D}$, then $\rho(\mathscr{C}) \geq d(\mathscr{D})$.
4. $d(\mathscr{C}) - 1 < 2\rho(\mathscr{C})$, if $|\mathscr{C}| \geq 2$ and $\mathscr{C} \subsetneq \mathbb{F}_q^{k\times m}$.

Proof. To see that 3 holds, let $N \in \mathscr{D} \setminus \mathscr{C}$. By definition of covering radius, there exists a matrix $M \in \mathscr{C}$ with $d(M,N) \leq \rho(\mathscr{C})$. Thus $d(\mathscr{D}) \leq d(M,N) \leq \rho(\mathscr{C})$.

To see 4, observe that the packing radius $\lfloor (d(\mathscr{C})-1)/2 \rfloor$ of $\mathscr{C}$ cannot exceed the covering radius, and that equality occurs if and only if $\mathscr{C}$ is perfect, in which case we have $\lfloor (d(\mathscr{C})-1)/2 \rfloor = \rho(\mathscr{C})$. However there are no perfect codes for the rank metric [2]. $\square$

2 Maximality

In this short section we investigate some connections between the covering radius of a rank-metric code and the property of maximality. Recall that a code $\mathscr{C} \subseteq \mathbb{F}_q^{k \times m}$ is **maximal** if $|\mathscr{C}| = 1$ or $|\mathscr{C}| \geq 2$ and there is no code $\mathscr{D} \subseteq \mathbb{F}_q^{k \times m}$ with $\mathscr{D} \supsetneq \mathscr{C}$ and $d(\mathscr{D}) = d(\mathscr{C})$. In particular, $\mathbb{F}_q^{k \times m}$ is maximal.

Proposition 5 (see e.g. [4])**.** A code $\mathscr{C} \subseteq \mathbb{F}_q^{k \times m}$ with $|\mathscr{C}| \geq 2$ is maximal if and only if $\rho(\mathscr{C}) \leq d(\mathscr{C}) - 1$.

Proof. If $\mathscr{C}$ is not maximal, then there exists $\mathscr{C} \subsetneq \mathscr{D}$ with $d(\mathscr{D}) = d(\mathscr{C})$. Lemma 4 implies $\rho(\mathscr{C}) \geq d(\mathscr{C}) = d(\mathscr{D})$, i.e., $\rho(\mathscr{C}) > d(\mathscr{C}) - 1$. This shows $(\Leftarrow)$. Let us prove $(\Rightarrow)$. If $\mathscr{C} = \mathbb{F}_q^{k \times m}$ then the result is trivial. Therefore we assume $\mathscr{C} \subsetneq \mathbb{F}_q^{k \times m}$ and $\rho(\mathscr{C}) \geq d(\mathscr{C})$ by contradiction. By the definition of covering radius there exists $X \in \mathbb{F}_q^{k \times m} \setminus \mathscr{C}$ such that $d(M,X) \geq \rho(\mathscr{C})$ for all matrices $M \in \mathscr{C}$. Then the code $\mathscr{D} := \mathscr{C} \cup \{X\}$ strictly contains $\mathscr{C}$ and has $d(\mathscr{D}) = d(\mathscr{C})$. $\square$

We now propose a new natural parameter that measures the maximality of a code, and show how it relates to the covering radius.

Definition 6. The **maximality degree** of a code $\mathscr{C} \subseteq \mathbb{F}_q^{k \times m}$ with $|\mathscr{C}| \geq 2$ is the integer defined by

$$\mu(\mathscr{C}) := \begin{cases} \min\{d(\mathscr{C}) - d(\mathscr{D}) : \mathscr{D} \subseteq \mathbb{F}_q^{k \times m} \text{ is a code with } \mathscr{D} \supsetneq \mathscr{C}\} & \text{if } \mathscr{C} \subsetneq \mathbb{F}_q^{k \times m}, \\ 1 & \text{if } \mathscr{C} = \mathbb{F}_q^{k \times m}. \end{cases}$$

The maximality degree of a code $\mathscr{C} \subseteq \mathbb{F}_q^{k \times m}$ with $|\mathscr{C}| \geq 2$ satisfies $0 \leq \mu(\mathscr{C}) \leq d(\mathscr{C}) - 1$. Moreover, it is easy to see that $\mu(\mathscr{C}) > 0$ if and only if $\mathscr{C}$ is maximal. Notice that $\mu(\mathscr{C})$ can be interpreted as the minimum price (in terms of minimum distance) that one has to pay in order to enlarge $\mathscr{C}$ to a bigger code. We can derive a precise relation between the covering radius and the maximality degree of a code as follows.

Proposition 7. For any code $\mathscr{C} \subseteq \mathbb{F}_q^{k \times m}$ with $|\mathscr{C}| \geq 2$ we have $\mu(\mathscr{C}) = d(\mathscr{C}) - \min\{\rho(\mathscr{C}),\ d(\mathscr{C})\}$. In particular, if $\mathscr{C}$ is maximal then $\mu(\mathscr{C}) = d(\mathscr{C}) - \rho(\mathscr{C})$.

Proof. If $\mathscr{C}$ is not a maximal code, then by Proposition 5 we have $\mu(\mathscr{C}) = 0$ and $\rho(\mathscr{C}) \geq d(\mathscr{C})$. The result immediately follows.

Now assume that $\mathscr{C}$ is maximal. If $\mathscr{C} = \mathbb{F}_q^{k \times m}$ then the result is trivial. In the sequel we assume $\mathscr{C} \subsetneq \mathbb{F}_q^{k \times m}$. By Proposition 5 we have $\min\{\rho(\mathscr{C}),\ d(\mathscr{C})\} = \rho(\mathscr{C})$. We need to prove that

$$\mu(\mathscr{C}) = d(\mathscr{C}) - \rho(\mathscr{C}).$$

Take $X \in \mathbb{F}_q^{k \times m} \setminus \mathscr{C}$ with $\min\{d(X,M) : M \in \mathscr{C}\} = \rho(\mathscr{C})$. Define the code $\mathscr{D} := \mathscr{C} \cup \{X\} \supsetneq \mathscr{C}$. By definition of minimum distance we have $d(\mathscr{D}) = \min\{d(\mathscr{C}),\ \rho(\mathscr{C})\} = \rho(\mathscr{C})$, where the last equality again follows from Proposition 5. As a consequence, $\mu(\mathscr{C}) \leq d(\mathscr{C}) - d(\mathscr{D}) = d(\mathscr{C}) - \rho(\mathscr{C})$. Now assume by contradiction that $\mu(\mathscr{C}) < d(\mathscr{C}) - \rho(\mathscr{C})$. Let $\mathscr{D} \subseteq \mathbb{F}_q^{k \times m}$ be a code with $\mathscr{D} \supsetneq \mathscr{C}$ and $d(\mathscr{C}) - d(\mathscr{D}) = \mu(\mathscr{C})$. We have $d(\mathscr{C}) - d(\mathscr{D}) = \mu(\mathscr{C}) < d(\mathscr{C}) - \rho(\mathscr{C})$, and so $d(\mathscr{D}) > \rho(\mathscr{C})$. This contradicts Lemma 4. $\square$

3 Puncturing and shortening rank-metric codes

In this section we propose new definitions of puncturing and shortening of rank-metric codes, and show they relate to the minimum distance, the covering radius and the duality theory of codes endowed with the rank metric. Applications of our constructions will be discussed later.

Notation 8. Given a code $\mathscr{C} \subseteq \mathbb{F}_q^{k\times m}$ and an integer $1 \le u \le k-1$, we let

$$\mathscr{C}_u := \{M \in \mathscr{C} : M_{ij} = 0 \text{ whenever } i \le u\},$$

the set of matrices in $\mathscr{C}$ whose first u rows are zero. Moreover, if A is a $k \times k$ matrix over $\mathbb{F}_q$ we define the code $A\mathscr{C} := \{A \cdot M : M \in \mathscr{C}\} \subseteq \mathbb{F}_q^{k\times m}$. Finally, $\pi_u : \mathbb{F}_q^{k\times m} \to \mathbb{F}_q^{(k-u)\times m}$ denotes the projection on the last $k-u$ rows.

Notice that if $A \in \mathrm{GL}_k(\mathbb{F}_q)$ then the map $X \mapsto AX$ is a linear rank-metric isometry $\mathbb{F}_q^{k\times m} \to \mathbb{F}_q^{k\times m}$. In particular, if $\mathscr{C} \subseteq \mathbb{F}_q^{k\times m}$ is a code, then $A\mathscr{C}$ is a code with the same cardinality, minimum distance, covering radius and weight and distance distribution as $\mathscr{C}$.

Definition 9. Let $\mathscr{C} \subseteq \mathbb{F}_q^{k\times m}$ be a code, $A \in \mathrm{GL}_k(\mathbb{F}_q)$ an invertible matrix and $1 \le u \le k-1$ a positive integer. The **puncturing** of $\mathscr{C}$ with respect to A and u is the code

$$\Pi(\mathscr{C}, A, u) := \pi_u(A\mathscr{C}).$$

When $0 \in \mathscr{C}$, the **shortening** of $\mathscr{C}$ with respect to A and u is the code

$$\Sigma(\mathscr{C}, A, u) := \pi_u((A\mathscr{C})_u).$$

The shortening and puncturing of a code $\mathscr{C} \subseteq \mathbb{F}_q^{k\times m}$ are codes in the ambient space $\mathbb{F}_q^{(k-u)\times m}$. Notice moreover that linearity is preserved by puncturing and shortening.

It will be convenient for us to use the following notation in the sequel.

Notation 10. Given a code $\mathscr{C} \subseteq \mathbb{F}_q^{k\times m}$ and an $\mathbb{F}_q$-linear subspace $U \subseteq \mathbb{F}_q^k$, we denote by $\mathscr{C}(U)$ the set of matrices in $\mathscr{C}$ whose columnspace is contained in the space U.

Remark 11. It is easy to see that if $\mathscr{C}$ is linear, then $\mathscr{C}(U)$ is an $\mathbb{F}_q$-linear subspace of $\mathscr{C}$ for any U. Moreover, if $U \subseteq \mathbb{F}_q^k$ is a given subspace of dimension u, then $\mathscr{C}_{k-u} \cong (A\mathscr{C})(U)$ as $\mathbb{F}_q$-linear spaces, where $A \in \mathbb{F}_q^{k\times k}$ is any invertible matrix that maps $\langle e_{k-u+1}, \dots, e_k\rangle$ to U (here $\{e_1, \dots, e_k\}$ denotes the canonical basis of $\mathbb{F}_q^k$).

We now show an interesting relation between puncturing, shortening, and trace-duality.

Theorem 12 (duality of puncturing and shortening)**.** Let $\mathscr{C} \subseteq \mathbb{F}_q^{k\times m}$ be a linear code, $A \in \mathrm{GL}_k(\mathbb{F}_q)$ an invertible matrix and $1 \le u \le k-1$ an integer. Then

$$\Pi(\mathscr{C}, A, u)^\perp = \Sigma(\mathscr{C}^\perp, (A^t)^{-1}, u).$$

Proof. Let $M \in \Sigma(\mathscr{C}^\perp, (A^t)^{-1}, u) = \pi_u(((A^t)^{-1}\mathscr{C}^\perp)_u)$ and $N \in \Pi(\mathscr{C}, A, u) = \pi_u(A\mathscr{C})$. By definition, we can write $N = \pi_u(AN_1)$ with $N_1 \in \mathscr{C}$ and $M = \pi_u((A^t)^{-1}M_1)$ with $M_1 \in \mathscr{C}^\perp$ and $(A^t)^{-1}M_1 \in ((A^t)^{-1}\mathscr{C})_u$. Since the first u rows of $(A^t)^{-1}M_1$ are zero, by definition of trace we have

$$\mathrm{Tr}(\pi_u((A^t)^{-1}M_1)\pi_u(AN_1)^t) = \mathrm{Tr}((A^t)^{-1}M_1(AN_1)^t) = \mathrm{Tr}((A^t)^{-1}M_1N_1^tA^t) = \mathrm{Tr}(M_1N_1^t) = 0,$$

where the last equality follows from the fact that $M_1 \in \mathscr{C}^\perp$ and $N_1 \in \mathscr{C}$. This proves $(\supseteq)$. It suffices to show that the codes $\Pi(\mathscr{C}, A, u)^\perp$ and $\Sigma(\mathscr{C}^\perp, (A^t)^{-1}, u)$ have the same dimension over $\mathbb{F}_q$. Denote by $\{e_1, \dots, e_k\}$ the canonical basis of $\mathbb{F}_q^k$, and let $U := \langle e_1, \dots, e_u\rangle$. One has

$$\dim(\Pi(\mathscr{C}, A, u)^\perp) = m(k-u) - \dim(\Pi(\mathscr{C}, A, u)) = m(k-u) - (\dim(\mathscr{C}) - \dim((A\mathscr{C})(U))), \quad (1)$$

where the last equality follows from the $\mathbb{F}_q$-isomorphism $\Pi(\mathscr{C},A,u) \cong \mathscr{C}/(A\mathscr{C})(U)$. By [18, Lemma 28] we have

$$\dim((A\mathscr{C})(U))) = \dim(A\mathscr{C}) - m(k-u) + \dim((A\mathscr{C})^{\perp}(U^{\perp})). \tag{2}$$

Observe that $\dim(A\mathscr{C}) = \dim(\mathscr{C})$ and $(A\mathscr{C})^{\perp} = (A^t)^{-1}\mathscr{C}^{\perp}$. Moreover, since $U^{\perp} = \langle e_{u+1},\dots,e_k\rangle$, by definition of shortening we have $\pi_u(((A^t)^{-1}\mathscr{C}^{\perp})(U^{\perp})) = \Sigma(\mathscr{C}^{\perp},(A^t)^{-1},u)$. In particular, $\dim(\Sigma(\mathscr{C}^{\perp},(A^t)^{-1},u)) = \dim(((A^t)^{-1}\mathscr{C}^{\perp})(U^{\perp}))$. Thus Equation (2) can be written as

$$\dim((A\mathscr{C})(U))) = \dim(\mathscr{C}) - m(k-u) + \dim(\Sigma(\mathscr{C}^{\perp},(A^t)^{-1},u)). \tag{3}$$

Combining equations (1) and (3) we obtain

$$\dim(\Pi(\mathscr{C},A,u)^{\perp}) = \dim(\Sigma(\mathscr{C}^{\perp},(A^t)^{-1},u)).$$

This concludes the proof. $\square$

The following two propositions show how puncturing, shortening, cardinality, minimum distance and covering radius of rank-metric codes relate to each other.

Proposition 13. Let $\mathscr{C} \subseteq \mathbb{F}_q^{k\times m}$ be a code with $|\mathscr{C}| \geq 2$. Let $A \in \mathrm{GL}_k(\mathbb{F}_q)$ and $1 \leq u \leq k-1$.

1. $d(\Pi(\mathscr{C},A,u)) \geq d(\mathscr{C}) - 1$, if $|\Pi(\mathscr{C},A,u)| \geq 2$.
2. $d(\Sigma(\mathscr{C},A,u)) \geq d(\mathscr{C})$, if $0 \in \mathscr{C}$ and $|\Sigma(\mathscr{C},A,u)| \geq 2$.
3. Assume $u \leq d(\mathscr{C}) - 1$. Then $|\Pi(\mathscr{C},A,u)| = |\mathscr{C}|$. If $\mathscr{C}$ is linear, then $|\Sigma(\mathscr{C}^{\perp},A,u)| = q^{m(k-u)}/|\mathscr{C}|$.
4. Assume $u > d(\mathscr{C}) - 1$. Then $|\Pi(\mathscr{C},A,u)| \geq |\mathscr{C}|/q^{m(u-d(\mathscr{C})+1)}$. If $0 \in \mathscr{C}$, then $|\Sigma(\mathscr{C},A,k-u)| \leq q^{m(u-d(\mathscr{C})+1)}$.

Proof. Properties 1, 2 are simple and left to the reader. The first part of Property 3 follows from the definition of minimum distance, and the second part is a consequence of Theorem 12. Let us show Property 4. Write $u = d(\mathscr{C}) - 1 + v$ with $1 \leq v \leq k - d(\mathscr{C}) + 1$, and define the code $\mathscr{E} := \Pi(\mathscr{C},A,d(\mathscr{C})-1)$. By Property 3 we have $|\mathscr{C}| = |\Pi(\mathscr{C},A,d(\mathscr{C})-1| = |\mathscr{E}|$. It follows from the definitions that $\Pi(\mathscr{C},A,u) = \pi_v(\mathscr{E})$, where

$$\pi_v : \mathbb{F}_q^{(k-d(\mathscr{C})+1)\times m} \to \mathbb{F}_q^{(k-u)\times m}$$

denotes the projection on the last $k-u$ rows. For any $N \in \pi_v(\mathscr{E})$ let $[N] := \{M \in \mathscr{E} : \pi_v(M) = N\}$. Clearly, $[N] \cap [N'] = \emptyset$ whenever $N, N' \in \pi_v(\mathscr{E})$ and $N \neq N'$. Moreover, it is easy to see that $|[N]| \leq q^{mv}$ for all $N \in \pi_v(\mathscr{E})$. Therefore

$$|\mathscr{E}| \;=\; \left|\bigcup_{N\in\pi_v(\mathscr{E})} [N]\right| \;=\; \sum_{N\in\pi_v(\mathscr{E})} |[N]| \;\leq\; |\pi_v(\mathscr{E})|\cdot q^{mv},$$

and so $|\Pi(\mathscr{C},A,u)| = |\pi_v(\mathscr{E})| \geq |\mathscr{E}|/q^{mv}$. Let us prove the last part of Property 4. If $|\Sigma(\mathscr{C},A,k-u)| = 1$ then there is nothing to prove. Assume $|\Sigma(\mathscr{C},A,k-u)| \geq 2$. Then $\Sigma(\mathscr{C},A,k-u)$ has minimum distance at least $d(A\mathscr{C}) = d(\mathscr{C})$. Therefore by the Singleton-like bound [9] we have

$$|\Sigma(\mathscr{C},A,k-u)| \leq q^{m(u-d(\mathscr{C})+1)},$$

as claimed. $\square$

Proposition 14. Let $\mathscr{C} \subseteq \mathbb{F}_q^{k\times m}$ be a code. For all $A \in \mathrm{GL}_k(\mathbb{F}_q)$ and $1 \leq u \leq k-1$ we have

$$\rho(\mathscr{C}) \geq \rho(\Pi(\mathscr{C},A,u)) \geq \rho(\mathscr{C}) - u.$$

Proof. Let $\mathscr{D} := A\mathscr{C}$. Then $\Pi(\mathscr{C},A,u) = \pi_u(\mathscr{D})$. Let $X \in \mathbb{F}_q^{k\times m}$ be an arbitrary matrix. By definition of covering radius and punctured code there exists $M \in \mathscr{D}$ with $d(\pi_u(M),\pi_u(X)) \leq \rho(\pi_u(\mathscr{D}))$. Therefore $d(M,X) \leq d(\pi_u(M),\pi_u(X))+u \leq \rho(\pi_u(\mathscr{D}))+u$. Since X is arbitrary, this shows $\rho(\mathscr{D}) \leq \rho(\pi_u(\mathscr{D}))+u$, i.e., $\rho(\pi_u(\mathscr{D})) \geq \rho(\mathscr{D})-u = \rho(\mathscr{C})-u$.

Now let $X \in \mathbb{F}_q^{(k-u)\times m}$ be an arbitrary matrix. Complete X to a $k\times m$ matrix, say X', by adding u zero rows to the top. There exists $M \in \mathscr{D}$ with $d(X',M) \leq \rho(\mathscr{D})$. Thus

$$d(X,\pi_u(M)) = d(\pi_u(X'),\pi_u(M)) \leq d(X',M) \leq \rho(\mathscr{D}) = \rho(\mathscr{C}).$$

This shows $\rho(\pi_u(\mathscr{D})) \leq \rho(\mathscr{C})$, and concludes the proof. □

4 Translates of a rank-metric code

In this section we study the weight distribution of the translates of a code. As an application, we obtain two upper bound on the covering radius of a rank-metric code. Recall that the **translate** of a code $\mathscr{C} \subseteq \mathbb{F}_q^{k\times m}$ by a matrix $X \in \mathbb{F}_q^{k\times m}$ is the code

$$\mathscr{C}+X := \{M+X : M \in \mathscr{C}\} \subseteq \mathbb{F}_q^{k\times m}.$$

Clearly, full knowledge of the weight distribution of the translates of $\mathscr{C}$ tells us the covering radius, which is the maximum of the minimum weight of each translate of $\mathscr{C}$. Even partial information may yield a bound on the covering radius. More precisely, if $X \in \mathbb{F}_q^{k\times m}$ and $W_i(\mathscr{C}+X) \neq 0$, then $d(X,\mathscr{C}) := \min\{d(X,M) : M \in \mathscr{C}\} \leq i$. So if there exists r such that for each $X \in \mathbb{F}_q^{k\times m}$, $W_i(\mathscr{C}+X) \neq 0$ for some $i \leq r$ then, in particular, $\rho(\mathscr{C}) \leq r$. If such a value r can be determined, then we get an upper bound on the covering radius of $\mathscr{C}$.

The goal of this section is twofold. We first show that the weight distribution $W_0(\mathscr{C}+X),\dots,W_k(\mathscr{C}+X)$ of the translate $\mathscr{C}+X$ of a linear code $\mathscr{C} \subsetneq \mathbb{F}_q^{k\times m}$ is determined by the values of $W_0(\mathscr{C}+X),\dots,W_{k-d^\perp}(\mathscr{C}+X)$, where $d^\perp = d(\mathscr{C}^\perp)$. Moreover, we provide explicit formulas for $W_{k-d^\perp+1}(\mathscr{C}+X),\dots,W_k(\mathscr{C}+X)$ as linear functions of $W_0(\mathscr{C}+X),\dots,W_{k-d^\perp}(\mathscr{C}+X)$. As a simple application, we obtain an upper bound on the covering radius of a linear code in terms of the minimum distance of its dual code. Our proof uses combinatorial methods partly inspired by the theory of regular support functions on groups developed in [17].

In a second part, following work of Delsarte for the Hamming distance [7], we apply Fourier transform methods to obtain further results on the weight distributions of the translates of a (not necessarily linear) code $\mathscr{C} \subseteq \mathbb{F}_q^{k\times m}$. In particular, we obtain an upper bound for the covering radius of a general rank-metric code in terms of its external distance (defined below).

Throughout this section we follow Notation 10. We start with a preliminary lemma that describes some combinatorial properties of the translates of a linear code.

Lemma 15. Let $\mathscr{C} \subseteq \mathbb{F}_q^{k\times m}$ be a linear code, and let $U \subseteq \mathbb{F}_q^k$ be an $\mathbb{F}_q$-linear subspace of dimension u. Assume that $|\mathscr{C}(U)| = |\mathscr{C}|/q^{m(k-u)}$. Then for all matrices $X \in \mathbb{F}_q^{k\times m}$ we have

$$|(\mathscr{C}+X)(U)| = |\mathscr{C}|/q^{m(k-u)}.$$

Proof. Let $f : \mathbb{F}_q^k \to \mathbb{F}_q^k$ be a linear isomorphism such that $f(U) = V := \{(x_1,\dots,x_k) \in \mathbb{F}_q^k : x_i = 0 \text{ for all } i > u\}$. Let A be the matrix associated to f with respect to the canonical basis of $\mathbb{F}_q^k$. Define the linear code $\mathscr{D} := A\mathscr{C}$. The left-multiplication by A induces bijections

$$\mathscr{C}(U) \to \mathscr{D}(V), \qquad (\mathscr{C}+X)(U) \to (\mathscr{D}+AX)(V).$$

In particular, we have $|\mathscr{D}(V)| = |\mathscr{C}(U)|$, and it suffices to prove that

$$|(\mathscr{D}+AX)(V)| = |\mathscr{D}(V)|. \tag{4}$$

Let $\pi := \pi_u : \mathbb{F}_q^{k\times m} \to \mathbb{F}_q^{(k-u)\times m}$ denote the projection on the last $k-u$ rows. Throughout the proof we denote by π_1 and π_2 the restriction of π to $\mathscr{D}$ and to $\mathscr{D}+AX$, respectively. Clearly, π_1 is linear.

By definition of V we have $\ker(\pi_1) = \mathscr{D}(V)$. Therefore

$$|\pi_1(\mathscr{D}(V))| = |\mathscr{D}|/|\mathscr{D}(V)| = |\mathscr{C}|/|\mathscr{C}(U)| = q^{m(k-u)}.$$

In particular, π_1 is surjective. Again by definition of V, we have $(\mathscr{D}+AX)(V) = \pi_2^{-1}(0)$. Moreover, one can check that $|\pi_2^{-1}(0)| = |\pi_1^{-1}(-\pi(AX))|$. Thus

$$|(\mathscr{D}+AX)(V)| = |\pi_2^{-1}(0)| = |\pi_1^{-1}(-\pi(AX))|. \tag{5}$$

Since π_1 is surjective, there exists $N \in \mathscr{D}$ such that $\pi_1(N) = -\pi(AX)$. One can easily check that the map $\ker(\pi_1) \to \pi_1^{-1}(-\pi(AX))$ defined by $M \mapsto M+N$ is a bijection. Thus using Equation (5) and the fact that $\mathscr{D}(V) = \ker(\pi_1)$ we find

$$|\mathscr{D}(V)| = |\ker(\pi_1)| = |\pi_1^{-1}(-\pi(AX))| = |(\mathscr{D}+AX)(V)|.$$

This shows Equation (4), as desired. $\square$

A second preliminary result which will be needed later is the following.

Lemma 16. Let $\mathscr{C} \subsetneq \mathbb{F}_q^{k\times m}$ be a linear code. Then for all matrices $X \in \mathbb{F}_q^{k\times m}$ and for any subspace $U \subseteq \mathbb{F}_q$ with $u := \dim(U) \geq k - d(\mathscr{C}^\perp) + 1$ we have

$$(\mathscr{C}+X)(U) = |\mathscr{C}|/q^{m(k-u)}.$$

Proof. By Lemma 15 it suffices to prove the result for $X = 0$. By [18, Lemma 28], for any subspace $U \subseteq \mathbb{F}_q^k$ of dimension u we have

$$|\mathscr{C}(U)| = \frac{|\mathscr{C}|}{q^{m(k-u)}} |\mathscr{C}^\perp(U^\perp)|, \tag{6}$$

where $U^\perp$ denotes the orthogonal of U with respect to the standard inner product of $\mathbb{F}_q^k$. By definition of minimum distance we have $\mathscr{C}^\perp(U^\perp) = \{0\}$ for all $U \subseteq \mathbb{F}_q^k$ with $\dim(U^\perp) \leq d(\mathscr{C}^\perp) - 1$. Therefore the lemma immediately follows from Equation (6) and the fact that $\dim(U^\perp) = k - \dim(U)$. $\square$

We can now state our main result on the weight distribution of the translates of a linear rank-metric code.

Theorem 17. Let $\mathscr{C} \subsetneq \mathbb{F}_q^{k\times m}$ be a linear code, and let $X \in \mathbb{F}_q^{k\times m}$ be any matrix. Write $d^\perp := d(\mathscr{C}^\perp)$. Then for all $i \in \{k-d^\perp+1, ..., k\}$ we have

$$W_i(\mathscr{C}+X) = \sum_{u=0}^{k-d^\perp} (-1)^{i-u} q^{\binom{i-u}{2}} \begin{bmatrix} k-u \\ i-u \end{bmatrix}_q \sum_{j=0}^{u} W_j(\mathscr{C}+X) \begin{bmatrix} k-j \\ u-j \end{bmatrix}_q + \sum_{u=k-d^\perp+1}^{i} \begin{bmatrix} k \\ u \end{bmatrix}_q \frac{|\mathscr{C}|}{q^{m(k-u)}}.$$

In particular, the distance distribution of the translate $\mathscr{C}+X$ is completely determined by k, m, $|\mathscr{C}|$ and the weights $W_0(\mathscr{C}+X), ..., W_{k-d^\perp}(\mathscr{C}+X)$.

Proof. Recall from [20] that the set of subspaces of $\mathbb{F}_q^k$ is a graded lattice with respect to the partial order given by the inclusion. The rank function of this lattice is the dimension of vector spaces, and its Möbius function is given by

$$\mu(S,T) = (-1)^{t-s} q^{\binom{t-s}{2}}$$

for all subspaces $S \subseteq T \subseteq \mathbb{F}_q^k$ with $\dim(T) = t$ and $\dim(S) = s$. More details can be found on page 317 of [20]. Throughout the proof a sum over an empty set of indices is zero by definition. For any subspace $V \subseteq \mathbb{F}_q^k$ define

$$f(V) := |\{M \in \mathscr{C}+X : \text{columnspace}(M) = V\}| \quad \text{and} \quad g(V) := \sum_{U \subseteq V} f(V) = |(\mathscr{C}+X)(V)|.$$

By the Möbius inversion formula ([20], Proposition 3.7.1), for any subspace $V \subseteq \mathbb{F}_q^k$ we have

$$f(V) = \sum_{U \subseteq V} |(\mathscr{C}+X)(U)| \; \mu(U,V). \tag{7}$$

Fix any integer i with $k-d^\perp+1 \le i \le k$. By definition of weight distribution we have

$$W_i(\mathscr{C}+X) = \sum_{\substack{V \subseteq \mathbb{F}_q^k \\ \dim(V)=i}} f(V).$$

Therefore by Equation (7) the number $W_i(\mathscr{C}+X)$ can be expressed as

$$\begin{aligned}
W_i(\mathscr{C}+X) &= \sum_{\substack{V \subseteq \mathbb{F}_q^n \\ \dim(V)=i}} \sum_{U \subseteq V} |(\mathscr{C}+X)(U)| \; \mu(U,V) \\
&= \sum_{U \subseteq \mathbb{F}_q^k} \sum_{\substack{V \supseteq U \\ \dim(V)=i}} |(\mathscr{C}+X)(U)| \; \mu(U,V) \\
&= \sum_{U \subseteq \mathbb{F}_q^k} |(\mathscr{C}+X)(U)| \sum_{\substack{V \supseteq U \\ \dim(V)=i}} \mu(U,V) \\
&= \sum_{u=0}^{i} \sum_{\substack{U \subseteq \mathbb{F}_q^k \\ \dim(U)=u}} |(\mathscr{C}+X)(U)| \sum_{\substack{V \supseteq U \\ \dim(V)=i}} \mu(U,V) \\
&= \sum_{u=0}^{i} \sum_{\substack{U \subseteq \mathbb{F}_q^k \\ \dim(U)=u}} |(\mathscr{C}+X)(U)| \sum_{\substack{V \supseteq U \\ \dim(V)=i}} (-1)^{i-u} q^{\binom{i-u}{2}} \\
&= \sum_{u=0}^{i} (-1)^{i-u} q^{\binom{i-u}{2}} \begin{bmatrix} k-u \\ i-u \end{bmatrix}_q \sum_{\substack{U \subseteq \mathbb{F}_q^k \\ \dim(U)=u}} |(\mathscr{C}+X)(U)| \qquad (8)
\end{aligned}$$

We now re-write the quantity

$$\sum_{\substack{U \subseteq \mathbb{F}_q^k \\ \dim(U)=u}} |(\mathscr{C}+X)(U)|$$

in a more convenient form. By Lemma 16, for $u \ge k-d^\perp+1$ we have

$$\sum_{\substack{U \subseteq \mathbb{F}_q^k \\ \dim(U)=u}} |(\mathscr{C}+X)(U)| = \begin{bmatrix} k \\ u \end{bmatrix}_q |\mathscr{C}|/q^{m(k-u)}. \tag{9}$$

On the other hand, for $u \le k-d^\perp$ we have

$$\begin{aligned}
\sum_{\substack{U \subseteq \mathbb{F}_q^k \\ \dim(U)=u}} |(\mathscr{C}+X)(U)| &= |\{(U,M) : U \subseteq \mathbb{F}_q^k,\ \dim(U)=u,\ M \in \mathscr{C}+X,\ \text{columnspace}(M) \subseteq U\}| \\
&= \sum_{M \in \mathscr{C}+X} |\{U \subseteq \mathbb{F}_q^k : \dim(U)=u,\ U \supseteq \text{columnspace}(M)\}| \\
&= \sum_{j=0}^{u} \sum_{\substack{M \in \mathscr{C}+X \\ \text{rk}(M)=j}} |\{U \subseteq \mathbb{F}_q^k : \dim(U)=u,\ U \supseteq \text{columnspace}(M)\}|
\end{aligned}$$

$$= \sum_{j=0}^{u} W_j(\mathscr{C}+X) \begin{bmatrix} k-j \\ u-j \end{bmatrix}_q . \tag{10}$$

Combining equations (8), (9) and (10) one obtains the desired formula. $\square$

As a simple consequence of Theorem 17 we can obtain an upper bound on the covering radius of a linear code $\mathscr{C} \subsetneq \mathbb{F}_q^{k\times m}$ in terms of its dual distance, as we now show. Let $X \in \mathbb{F}_q^{k\times m} \notin \mathscr{C}$. Then $W_0(\mathscr{C}+X) = 0$. Theorem 17 with $i := k - d^{\perp} + 1$ gives

$$W_{k+d^{\perp}+1}(\mathscr{C}+X) = \sum_{u=1}^{k-d^{\perp}} (-1)^{i-u} q^{\binom{i-u}{2}} \begin{bmatrix} k-u \\ i-u \end{bmatrix}_q \sum_{j=1}^{u} W_j(\mathscr{C}+X) \begin{bmatrix} k-j \\ u-j \end{bmatrix}_q + \begin{bmatrix} k \\ k-d^{\perp}+1 \end{bmatrix}_q |\mathscr{C}|/q^{m(d^{\perp}-1)}.$$

In particular, $W_1(\mathscr{C}+X), \ldots, W_{k-d^{\perp}+1}(\mathscr{C}+X)$ cannot be all zero. This implies the following.

Corollary 18 (dual distance bound)**.** For any linear code $\mathscr{C} \subsetneq \mathbb{F}_q^{k\times m}$ we have $\rho(\mathscr{C}) \leq k - d(\mathscr{C}^{\perp}) + 1$.

We now relate the covering radius of a code with its external distance. In particular, we derive another upper bound on the covering radius of a linear code in terms of the rank distribution of the dual code. This is the rank distance analogue of Delsarte's external Hamming distance bound (c.f. [15, 7, 4]), and improves the dual distance bound of Corollary 18.

The approach uses q-Krawtchouk polynomials and Fourier transforms to obtain relations on the weight distribution of the translates of a code in $\mathbb{F}_q^{k\times m}$. The properties of q-Krawtchouk polynomials were described in [7, 8]. The Fourier transform arguments used are independent of the choice of metric used and so extend from the Hamming metric case. The principle novelty is the introduction of a q-annihilator polynomial, used in the proof of Lemma 23.

Throughout the reminder of this section $\mathscr{C} \subseteq \mathbb{F}_q^{k\times m}$ denotes a (possibly non-linear) code, and χ is a fixed non-trivial character of $(\mathbb{F}_q, +)$.

Definition 19. Let $Y \in \mathbb{F}_q^{k\times m}$. Define the **character map** on $(\mathbb{F}_q^{k\times m}, +)$ associated to Y by

$$\phi_Y : \mathbb{F}_q^{k\times m} \longrightarrow \mathbb{C}^{\times} : X \mapsto \chi(\mathrm{Tr}(YX^T)).$$

Clearly $\phi_X(Y) = \phi_Y(X)$ for all $X, Y \in \mathbb{F}_q^{k\times m}$. We denote by Φ the $km \times km$ symmetric matrix with values in $\mathbb{C}^{\times}$ defined as having entry $\phi_Y(X)$ in the column indexed by X and in the row indexed by Y. Define the $\mathbb{Q}$-module of length km: $\mathfrak{C} := \left\{ (\mathscr{A}_X : X \in \mathbb{F}_q^{k\times m}) : \mathscr{A}_X \in \mathbb{Q} \right\}$. For each Y, extend ϕ_Y to a character of $\mathfrak{C}$ as follows:

$$\phi_Y : \mathfrak{C} \longrightarrow \mathbb{C}^{\times} : \mathscr{A} = (\mathscr{A}_X : X \in \mathbb{F}_q^{k\times m}) \mapsto \sum_X \mathscr{A}_X \phi_Y(X).$$

Then $\Phi\mathscr{A} = (\phi_Y(\mathscr{A}) : Y \in \mathbb{F}_q^{k\times m}) \in \mathfrak{C}$. The rows of Φ are pairwise orthogonal, as can be seen from:

$$\sum_X \phi_Y(X)\phi_Z(X) = \sum_X \phi_X(Y)\phi_X(Z) = \sum_X \phi_X(Y-Z) = \sum_X \phi_{Y-Z}(X) = \begin{cases} q^{km} & \text{if } Y = Z, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $\Phi^2\mathscr{A} = \Phi^T\Phi\mathscr{A} = q^{km}\mathscr{A}$ and so $\mathscr{A}$ is determined completely by its transform

$$\mathscr{A}^* := \Phi\mathscr{A} = (\phi_Y(\mathscr{A}) : Y \in \mathbb{F}_q^{k\times m}).$$

Any subset $\mathscr{U} \subseteq \mathbb{F}_q^{k\times m}$ can be identified with the 0-1 vector $\overline{\mathscr{U}} = (\mathscr{U}_Z : Z \in \mathbb{F}_q^{k\times m}) \in \mathfrak{C}$, where

$$\mathscr{U}_Z = \begin{cases} 1 & \text{if } Z \in \mathscr{U}, \\ 0 & \text{otherwise.} \end{cases}$$

For any $X \in \mathbb{F}_q^{k\times m}$, the translate code $\mathscr{C}+X \subseteq \mathbb{F}_q^{k\times m}$ is then identified with $\overline{\mathscr{C}+X} = (\mathscr{C}_{Z-X} : Z \in \mathbb{F}_q^{k\times m})$. It is straightforward to show that $\phi_Y(\overline{\mathscr{C}+X}) = \phi_Y(\overline{\mathscr{C}})\phi_Y(X)$. This immediately yields the inversion formula

$$\mathscr{C}_X = \frac{1}{q^{km}}\sum_Y \phi_Y(\overline{\mathscr{C}+X}) = \frac{1}{q^{km}}\sum_Y \phi_Y(\overline{\mathscr{C}})\phi_Y(X).$$

For each $i \in [k]$ we let Ω^i be the set of matrices in $\mathbb{F}_q^{k\times m}$ of rank i.

Lemma 20 (see [9])**.** Let $Y \in \mathbb{F}_q^{k\times m}$. Then $\phi_Y(\overline{\Omega^i})$ depends only on the rank of Y. If Y has rank j, then this is given by

$$P_i(j) := \sum_{\ell=0}^{k} (-1)^{i-\ell} q^{\ell m+\binom{i-\ell}{2}} \begin{bmatrix} k-\ell \\ k-i \end{bmatrix}_q \begin{bmatrix} k-j \\ \ell \end{bmatrix}_q.$$

In terms of the transform of Ω^i this gives

$$\Phi\overline{\Omega^i} = (P_i(\mathrm{rk}(Y)) : Y \in \mathbb{F}_q^{k\times m}).$$

It is known [8, 9] that the $P_i(j)$ are orthogonal polynomials of degree i in the variable q^{-j}. Therefore, any rational polynomial γ of degree at most k in q^{-j} can be expressed as a $\mathbb{Q}$-linear combination of the q-Krawtchouck polynomials: $\gamma(x) = \sum_{j=0}^{k} \gamma_j P_j(x)$. Again, the orthogonality relations mean that the coefficients can be of γ can be retrieved as

$$\gamma_j = \frac{1}{q^{km}}\sum_{i=0}^{k} \gamma(i) P_i(j).$$

We let $P = (P_i(j))$ denote the $(k+1)\times(k+1)$ matrix with (j,i)-th component equal to $P_i(j)$. Then the **transform** of $B(\mathscr{C}) = (B_i(\mathscr{C}) : 0 \le i \le k)$ is defined as $B^*(\mathscr{C}) := |\mathscr{C}|^{-1}B(\mathscr{C})P$. The coefficents of $B^*(\mathscr{C})$ are non-negative [9, Theorem 3.2].

Let $\mathscr{D} := (D_Z : Z \in \mathbb{F}_q^{k\times m})$ where $D_Z = |\{(X,Y) : X,Y \in \mathscr{C}, X+Y = Z\}|$. It can be checked that

$$\phi_Y(\mathscr{D}) = \phi_Y(\mathscr{C})\phi_Y(\mathscr{C}) = \phi_Y(\mathscr{C})^2.$$

Then

$$\sum_{Y\in\Omega^i} \phi_Y(\mathscr{D}) = \sum_Z D_Z \sum_{Y\in\Omega^i} \phi_Y(Z) = \sum_Z D_Z \phi_Z(\Omega^i) = \sum_Z D_Z P_i(\mathrm{rk}(Z)) = |\mathscr{C}|\sum_{j=0}^{k} B_j(\mathscr{C})P_i(j) = |\mathscr{C}|(B(\mathscr{C})P)_i,$$

and in particular we have

$$|\mathscr{C}|B^*(\mathscr{C}) = (\sum_{Y\in\Omega^i} \phi_Y(\mathscr{D}) : 0 \le i \le k) = (\sum_{Y\in\Omega^i} \phi_Y(\mathscr{C})^2 : 0 \le i \le k).$$

Clearly $B_i^*(\mathscr{C}) = 0$ implies that $\phi_Y(\mathscr{C}) = 0$ for each $Y \in \Omega^i$.

Definition 21. The **external distance** of a code $\mathscr{C} \subseteq \mathbb{F}_q^{k\times m}$ is the integer

$$\sigma^*(\mathscr{C}) := |\{i \in [k] : B_i^*(\mathscr{C}) > 0\}|,$$

the number of non-zero coefficents of $B^*(\mathscr{C})$, excluding $B_0^*(\mathscr{C})$.

For ease of notation in the sequel we write $\sigma^* := \sigma^*(\mathscr{C})$. Let $0 < b_1 < ... < b_{\sigma^*} \le k$ denote the indices i of non-zero $B_i^*(\mathscr{C})$ for $i > 0$.

Definition 22. The **annihilator polynomial** of degree σ^* in the variable q^{-x} of $\mathscr{C}$ is

$$\alpha(x) := \frac{q^{mn}}{|\mathscr{C}|}\prod_{j=1}^{\sigma^*} \frac{1-q^{b_j-x}}{1-q^{b_j}} = \sum_{j=0}^{\sigma^*} \alpha_j P_j(x).$$

This is the q-analogue of the Hamming metric annihilator polynomial [15, pg. 168]. Notice that the b_j are the zeroes of α and $\alpha(0) = \frac{q^{mn}}{|\mathscr{C}|}$.

Lemma 23. Let $X \in \mathbb{F}_q^{k\times m}$ be an arbitrary matrix. Then

$$\sum_{j=1}^{\sigma^*} \alpha_j W_j(\mathscr{C}+X) = 1.$$

In particular, there exists some $j \in [\sigma^*]$ such that $W_j(\mathscr{C}+X) > 0$.

Proof. We must show that $\sum_{j=1}^{\sigma^*} \alpha_j (W_j(\mathscr{C}+X) : X \in \mathbb{F}_q^{k\times m}) = (1 : X \in \mathbb{F}_q^{k\times m})$. Since Φ is invertible, this holds if and only if for all Y,

$$\phi_Y\left(\sum_{j=1}^{\sigma^*} \alpha_j W_j(\mathscr{C}+X) : X \in \mathbb{F}_q^{k\times}\right) = \phi_Y(1 : X \in \mathbb{F}_q^{k\times m}) = \begin{cases} 0 & \text{if } Y \neq 0 \\ q^{km} & \text{if } Y = 0. \end{cases}$$

This was the approach taken, for example, in [15, Chapter 6, Lemma 18]. Let $Y \in \mathbb{F}_q^{k\times m}$. Then

$$\begin{aligned}
\phi_Y\left(\sum_{j=1}^{\sigma^*} \alpha_j W_j(\mathscr{C}+X) : X \in \mathbb{F}_q^{k\times m}\right) &= \sum_{j=1}^{\sigma^*} \alpha_j \sum_X W_j(\mathscr{C}+X)\phi_Y(X) \\
&= \sum_{j=1}^{\sigma^*} \alpha_j \sum_{X\in\Omega^j} \phi_Y(\overline{\mathscr{C}+X}) \\
&= \sum_{j=1}^{\sigma^*} \alpha_j \sum_{X\in\Omega^j} \phi_Y(\overline{\mathscr{C}})\phi_Y(X) \\
&= \sum_{j=1}^{\sigma^*} \alpha_j \phi_Y(\overline{\mathscr{C}}) \sum_{X\in\Omega^j} \phi_Y(X) \\
&= \sum_{j=1}^{\sigma^*} \alpha_j \phi_Y(\overline{\mathscr{C}}) \phi_Y(\overline{\Omega^j}) \\
&= \sum_{j=1}^{\sigma^*} \alpha_j \phi_Y(\overline{\mathscr{C}}) P_j(\mathrm{rk}(Y)) \\
&= \phi_Y(\overline{\mathscr{C}}) \sum_{j=1}^{\sigma^*} \alpha_j P_j(\mathrm{rk}(Y)) \\
&= \phi_Y(\overline{\mathscr{C}})\alpha(\ell),
\end{aligned}$$

where Y has rank ℓ.

Now $\alpha(0) = \frac{q^{mn}}{|\mathscr{C}|}$ and $\phi_0(\mathscr{C}) = |\mathscr{C}|$, so $\phi_0(\overline{\mathscr{C}})\alpha(0) = q^{km}$. Suppose that Y has rank $\ell > 0$. The roots of α are precisely those $j \geq 1$ such that $B_j^*(\mathscr{C})$ is non-zero. On the other hand, if $B_j^*(\mathscr{C}) = 0$ then $\phi_Y(\overline{\mathscr{C}}) = 0$. It follows that the product $\phi_Y(\overline{\mathscr{C}})\alpha(\ell) = 0$ and so

$$\Phi(\sum_{j=1}^{\sigma^*} \alpha_j W_j(\mathscr{C}+X) : X \in \mathbb{F}_q^{k\times m}) = \Phi(1 : X \in \mathbb{F}_q^{k\times m}),$$

as claimed. ☐

We can now upper-bound the covering radius of a general rank-metric code in terms of its external distance as follows.

Theorem 24 (external distance bound)**.** For any code $\mathscr{C} \subseteq \mathbb{F}_q^{m\times n}$ we have $\rho(C) \leq \sigma^*(\mathscr{C})$. Furthermore, if $\mathscr{C}$ is $\mathbb{F}_q$-linear then $\rho(\mathscr{C})$ is no greater than the number of non-zero weights of $\mathscr{C}^\perp$, excluding $W_0(\mathscr{C}^\perp)$.

Proof. The first part of the theorem is an immediate consequence of Lemma 23. The second part follows from the fact that $B_i^*(\mathscr{C}) = B_i(\mathscr{C}^\perp) = W_i(\mathscr{C}^\perp)$, provided that $\mathscr{C}$ is linear. This can be easily seen from the definition of $B^*(\mathscr{C})$ on page 10 and the MacWilliams identities for the rank metric [9]. $\square$

Example 25. Let $m = rs$ and let $\mathscr{C} = \{\sum_{i=0}^{r-1} f_i x^{q^{si}} : f_i \in \mathbb{F}_{q^m}\}$. Then $\mathscr{C}$ is the set of all $\mathbb{F}_{q^s}$-linear maps from $\mathbb{F}_{q^m}$ to itself. Therefore $\mathscr{C}$ has elements of $\mathbb{F}_{q^s}$-ranks $0,1,2,\ldots,r$. Let f have rank i over $\mathbb{F}_{q^s}$. Let $\mathrm{Im}\ f \subseteq \mathbb{F}_{q^m}$ have $\mathbb{F}_{q^s}$-basis $\{v_1,\ldots,v_i\}$ and let $\{u_1,\ldots,u_s\}$ be an $\mathbb{F}_q$-basis of $\mathbb{F}_{q^s}$. Then $\{u_i v_j : 1 \leq i \leq s,\ 1 \leq j \leq i\}$ is an $\mathbb{F}_q$-basis of $\mathrm{Im}\ f$ in $\mathbb{F}_{q^m}$, and so has dimension is. Then $\mathscr{C}$ has non-zero rank weights $\{s, 2s, \ldots, rs\}$ over $\mathbb{F}_q$, so that $\rho(\mathscr{C}^\perp) \leq r$.

5 Initial set bound

In this section we propose a definition of initial set of a linear rank-metric code inspired by [16]. Moreover we exploit the combinatorial structure of such set to derive an upper bound for the covering radius of the underlying code. Our technique relies on the specific "matrix structure" of rank-metric codes.

Notation 26. Given positive integers a,b and a set $S \subseteq [a]\times[b]$, we denote by $\mathbb{I}(S) \in \mathbb{F}_2^{a\times b}$ be the binary matrix defined by $\mathbb{I}(S)_{ij} := 1$ if $(i,j) \in S$, and $\mathbb{I}(S) := 0$ if $(i,j) \notin S$. Moreover, we denote by $\lambda(S)$ the minimum number of lines (rows or columns) required to cover all the ones in $\mathbb{I}(S)$.

The initial set of a linear code is defined as follows.

Definition 27. Let $\preceq$ denote the lexicographic order on $[k]\times[m]$. The **initial entry** of a non-zero matrix $M \in \mathbb{F}_q^{k\times m}$ is $\mathrm{in}(M) := \min_{\preceq}\{(i,j) : M_{ij} \neq 0\}$. The **initial set** of a non-zero linear code $\mathscr{C} \subseteq \mathbb{F}_q^{k\times m}$ is

$$\mathrm{in}(\mathscr{C}) := \{\mathrm{in}(M) : M \in \mathscr{C},\ M \neq 0\}.$$

We start with a preliminary lemma.

Lemma 28. Let $\mathscr{C} \subseteq \mathbb{F}_q^{k\times m}$ be a non-zero code. The following hold.

1. $\dim(\mathscr{C}) = |\mathrm{in}(\mathscr{C})|$,
2. $\mathrm{in}(\mathscr{C}) \subseteq [k - d(\mathscr{C}) + 1] \times [m]$.

Proof. Let $t := \dim(\mathscr{C})$, and let $\{M_1,\ldots,M_t\}$ be a basis of $\mathscr{C}$. Without loss of generality we may assume $(1,1) \preceq \mathrm{in}(M_1) \prec \cdots \prec \mathrm{in}(M_t)$. If $M \in \mathscr{C}\setminus\{0\}$, then there exist elements $a_1,\ldots,a_t \in \mathbb{F}_q$ such that $M = \sum_{i=1}^t a_i M_i$, hence $\mathrm{in}(M) \in \{\mathrm{in}(M_1),\ldots,\mathrm{in}(M_t)\}$. This shows $\mathrm{in}(\mathscr{C}) = \{\mathrm{in}(M_1),\ldots,\mathrm{in}(M_t)\}$. In particular, $|\mathrm{in}(\mathscr{C})| = t = \dim(\mathscr{C})$. Notice moreover that if $\mathrm{in}(M_t) \succ (k - d(\mathscr{C}) + 1, m)$, then clearly $\mathrm{rk}(M_t) \leq d(\mathscr{C}) - 1$, a contradiction. Therefore we have

$$(1,1) \preceq \mathrm{in}(M_1) \prec \cdots \prec \mathrm{in}(M_t) \preceq (k - d(\mathscr{C}) + 1, m).$$

This shows $\mathrm{in}(\mathscr{C}) \subseteq [k - d(\mathscr{C}) + 1] \times [m]$. $\square$

Remark 29. Let a,b be positive integers and let $S \subseteq [a]\times[b]$ be a set. Assume that $M \in \mathbb{F}_q^{a\times b}$ is a matrix with $M_{ij} = 0$ whenever $(i,j) \notin S$. Then $\mathrm{rk}(M) \leq \lambda(S)$. This can be proved by induction on $\lambda(S)$.

We can now state the main result of this section, which provides an upper bound on the covering radius of a linear rank-metric code $\mathscr{C}$ in terms of the combinatorial structure of its initial set.

Theorem 30 (initial set bound)**.** Let $\mathscr{C} \subseteq \mathbb{F}_q^{k\times m}$ be a non-zero linear code. We have $\rho(\mathscr{C}) \leq d(\mathscr{C}) - 1 + \lambda(S)$, where $S := [k - d(\mathscr{C}) + 1] \times [m] \setminus \mathrm{in}(\mathscr{C})$.

Proof. Let $X \in \mathbb{F}_q^{k\times m}$ be any matrix. It is easy to see that there exists a unique matrix $M \in \mathscr{C}$ such that $X_{ij} = M_{ij}$ for all $(i,j) \in \text{in}(\mathscr{C})$. Such matrix satisfies $(X-M)_{ij} = 0$ for all $(i,j) \in \text{in}(\mathscr{C})$. Let $\overline{X-M}$ be the matrix obtained from $X-M$ deleting the last $d(\mathscr{C})-1$ rows. We have

$$d(X,M) = \text{rk}(X-M) \le d(\mathscr{C}) - 1 + \text{rk}(\overline{X-M}) \le d(\mathscr{C}) - 1 + \lambda(S),$$

where S denotes the complement of $\text{in}(\mathscr{C})$ in $[k-d(\mathscr{C})+1]\times[m]$, and the last inequality follows from Remark 29. Since X is an arbitrary matrix, this shows $\rho(\mathscr{C}) \le d(\mathscr{C}) - 1 + \lambda(S)$. □

Remark 31. The initial set of a linear code $\mathscr{C} \subseteq \mathbb{F}_q^{k\times m}$ can be efficiently computed from any basis of $\mathscr{C}$ as follows. Denote by $w: \mathbb{F}_q^{k\times m} \to \mathbb{F}_q^{mk}$ the map that sends a matrix M to the mk-vector obtained concatenating the rows of M. Given a basis $\{M_1,\dots,M_t\}$ of $\mathscr{C}$, construct the vectors $v_1 := w(M_1),\dots,v_t := w(M_t)$. Perform Gaussian elimination on $\{v_1,\dots,v_t\}$ and obtain vectors $\overline{v}_1,\dots,\overline{v}_t$. Clearly, $\{w^{-1}(\overline{v}_1),\dots,w^{-1}(\overline{v}_t)\}$ is a basis of $\mathscr{C}$, and one can easily check that

$$\text{in}(\mathscr{C}) = \{\text{in}(w^{-1}(\overline{v}_1)),\dots,\text{in}(w^{-1}(\overline{v}_t))\}.$$

The following example shows that Theorem 30 gives in some cases a better bound than Corollary 24 for the covering radius of a linear code.

Example 32. Let $q=2$ and $k=m=3$. Denote by $\mathscr{C}$ the linear code generated over $\mathbb{F}_2$ by the four matrices

$$\begin{bmatrix}1&0&0\\0&0&1\\0&0&0\end{bmatrix}, \quad \begin{bmatrix}0&1&0\\0&0&0\\1&0&0\end{bmatrix}, \quad \begin{bmatrix}0&0&0\\1&0&0\\0&1&0\end{bmatrix}, \quad \begin{bmatrix}0&0&0\\0&1&1\\1&0&0\end{bmatrix}.$$

We have $d(\mathscr{C}) = 2$. Moreover, since

$$\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\1&0&0\\0&1&0\end{bmatrix}, \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix} \in \mathscr{C}^\perp,$$

we have $\sigma(\mathscr{C}^\perp) = 3$, and so Corollary 24 gives $\rho(\mathscr{C}) \le 3$. On the other hand, one can easily check that the initial set of $\mathscr{C}$ is $\text{in}(\mathscr{C}) = \{(1,1),(1,2),(2,1),(2,2)\}$. Thus following the notation of Theorem 30 we have $S = \{(1,3),(2,3)\}$ and $\lambda(S) = 1$. It follows $\rho(\mathscr{C}) \le d(\mathscr{C}) - 1 + \lambda(S) = 2$. Therefore Theorem 30 gives a better bound on $\rho(\mathscr{C})$ than Corollary 24. In fact, one can check that $\rho(\mathscr{C}) = 2$.

6 Covering radius of MRD and dually QMRD codes

It is well known [9] that if $\mathscr{C} \subseteq \mathbb{F}_q^{k\times m}$ is a code with $|\mathscr{C}| \ge 2$, then $\log_q|\mathscr{C}| \le m(k-d(\mathscr{C})+1)$. A code $\mathscr{C} \subseteq \mathbb{F}_q^{k\times m}$ is **MRD** if $|\mathscr{C}| = 1$ or $|\mathscr{C}| \ge 2$ and $\log_q|\mathscr{C}| = m(k-d(\mathscr{C})+1)$. MRD codes have the largest possible cardinality for their minimum distance. In particular, they are maximal. Therefore combining Proposition 5 and 7 we immediately obtain the following result.

Corollary 33. Let $\mathscr{C} \subseteq \mathbb{F}_q^{k\times m}$ be an MRD code with $|\mathscr{C}| \ge 2$. Then $\rho(\mathscr{C}) \le d(\mathscr{C}) - 1$. Moreover, equality holds if and only if the maximality degree of $\mathscr{C}$ is precisely 1.

The upper bound of Corollary 33 is not sharp in general, as we show in the following example. This proves in particular that not all MRD codes $\mathscr{C} \subsetneq \mathbb{F}_q^{k\times m}$ with $|\mathscr{C}| \ge 2$ can be nested into an MRD code $\mathscr{D} \supsetneq \mathscr{C}$ with $d(\mathscr{D}) = d(\mathscr{C}) - 1$.

Example 34. Take $q = 2$ and $k = m = 4$. Let $\mathscr{C}$ be the linear code generated over $\mathbb{F}_2$ by the following four matrices:

$$\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix}, \quad \begin{bmatrix}0&1&0&0\\0&0&1&1\\0&0&0&1\\1&1&0&0\end{bmatrix}, \quad \begin{bmatrix}0&0&1&0\\0&1&1&1\\1&0&1&0\\1&0&0&1\end{bmatrix}, \quad \begin{bmatrix}0&0&0&1\\1&1&1&0\\0&1&0&1\\0&1&1&1\end{bmatrix}.$$

We have $\dim(\mathscr{C}) = 4$ and $d(\mathscr{C}) = 4$. In particular, $\mathscr{C}$ is a linear MRD codes. On the other hand, one can check that $\rho(\mathscr{C}) = 2 \neq d(\mathscr{C}) - 1 = 3$, and that $\mu(\mathscr{C}) = 2$.

We conclude observing that combining properties 1, 2 and 4 of Proposition 13 one can easily obtain the following general result on the puncturing of an MRD code.

Corollary 35. Let $\mathscr{C} \subseteq \mathbb{F}_q^{k\times m}$ be an MRD code. Then for any $A \in \mathrm{GL}_k(\mathbb{F}_q)$ and for any $1 \leq u \leq k-1$ the punctured code $\Pi(\mathscr{C}, A, u)$ is MRD as well.

Dually QMRD codes were proposed in [6] as the best alternative to linear MRD codes for dimensions that are not multiples of m. A linear rank-metric code $\mathscr{C} \subseteq \mathbb{F}_q^{k\times m}$ is **dually QMRD** if $\dim(\mathscr{C}) \nmid m$ and the following two conditions hold:

$$d(\mathscr{C}) = k - \lceil \dim(\mathscr{C})/m \rceil + 1, \qquad d(\mathscr{C}^\perp) = k - \lceil \dim(\mathscr{C}^\perp)/m \rceil + 1.$$

Clearly, a code is dually QMRD if and only if its dual code is dually QMRD. The following proposition summarizes the most important properties of dually QMRD codes.

Lemma 36 (see Proposition 20 of [6])**.** Let $\mathscr{C} \subseteq \mathbb{F}_q^{k\times m}$ be a linear code. The following are equivalent.

1. $\mathscr{C}$ is dually QMRD,
2. $\mathscr{C}^\perp$ is dually QMRD,
3. $\dim(\mathscr{C}) \nmid m$ and $d(\mathscr{C}) + d(\mathscr{C}^\perp) = k + 1$.

Moreover, the weight distribution of a dually QMRD code $\mathscr{C}$ is determined by k, m and $\dim(\mathscr{C})$.

We now apply the external distance bound to derive an upper bound on the covering radius of dually QMRD codes. We start by computing the external distance, $\sigma^*(\mathscr{C})$, of a dually QMRD code $\mathscr{C}$ of given parameters. Since $\mathscr{C}$ is linear by definition, as in the proof of Corollary 24 we have $\sigma^*(\mathscr{C}) = |\{i \in [k] : W_i(\mathscr{C}^\perp) \neq 0\}|$. We will need the following preliminary lemma.

Lemma 37. Let $1 \leq t \leq km - 1$ be any integer. There exist linear codes $\mathscr{C} \subsetneq \mathscr{D} \subseteq \mathbb{F}_q^{k\times m}$ such that $\mathscr{C}$ is dually QMRD, $\mathscr{D}$ is MRD, $\dim(\mathscr{C}) = t$ and $d(\mathscr{C}) = d(\mathscr{D})$.

Proof. Let $\alpha := \lfloor t/m \rfloor$. It is well known (see e.g. the construction of [9, Section 6] or [19]) that there exist linear MRD codes $\mathscr{E} \subseteq \mathscr{D}$ with $\dim(\mathscr{E}) = m\alpha$ and $\dim(\mathscr{D}) = m(\alpha+1)$. Let $\mathscr{E} \subsetneq \mathscr{C} \subsetneq \mathscr{D}$ be a subspace with $\dim(\mathscr{C}) = t$. Since $\mathscr{E}$ is MRD, it is maximal. Therefore $d(\mathscr{C}) = d(\mathscr{D})$. Now consider the nested codes $\mathscr{D}^\perp \subsetneq \mathscr{C}^\perp \subsetneq \mathscr{E}^\perp$. Since $\mathscr{D}$ and $\mathscr{E}$ are MRD, their dual codes $\mathscr{D}^\perp$ and $\mathscr{E}^\perp$ are MRD as well (see [9, Theorem 5.5] or [18, Corollary 41] for a simpler proof). In particular, $\mathscr{D}^\perp$ is maximal, and so $d(\mathscr{C}^\perp) = d(\mathscr{E}^\perp)$. Since $\mathscr{D}$ and $\mathscr{E}^\perp$ are MRD, we have $d(\mathscr{D}) = k - (\alpha+1) + 1$ and $d(\mathscr{E}^\perp) = k - (k-\alpha) + 1$. Therefore

$$d(\mathscr{C}) + d(\mathscr{C}^\perp) = d(\mathscr{D}) + d(\mathscr{E}^\perp) = k - (\alpha+1) + 1 + k - (k-\alpha) + 1 = k+1,$$

and the result easily follows from Lemma 36. $\square$

We can now compute the external distance of a dually QMRD code.

Theorem 38. Let $\mathscr{C} \subseteq \mathbb{F}_q^{k\times m}$ be a dually QMRD code. Then $\sigma^*(\mathscr{C}) = d(\mathscr{C})$.

Proof. Since $\mathscr{C}$ is linear, as in the proof of Corollary 24 we have $\sigma^*(\mathscr{C}) = |\{i \in [k] : W_i(\mathscr{C}^\perp) > 0\}|$. By Lemma 37 there exist a dually QMRD code $\mathscr{C}_1$ and a linear MRD code $\mathscr{D}$ such that $\mathscr{C}_1 \subsetneq \mathscr{D}$, $\dim(\mathscr{C}) = \dim(\mathscr{C}_1)$ and $d(\mathscr{C}_1) = d(\mathscr{D})$. Since $\mathscr{C}$ and $\mathscr{C}_1$ have the same dimension and are both dually QMRD, by Lemma 36 the dual codes $\mathscr{C}^\perp$ and $\mathscr{C}_1^\perp$ have the same weight distribution. In particular, $\sigma^*(\mathscr{C}) = \sigma^*(\mathscr{C}_1)$. Therefore it suffices to prove the theorem for the code $\mathscr{C}_1$. By Lemma 36 we have $d(\mathscr{C}_1^\perp) = k+1-d(\mathscr{C}_1)$. This clearly implies

$$\sigma^*(\mathscr{C}_1) \leq k - (k+1-d(\mathscr{C}_1)) + 1 = d(\mathscr{C}_1). \tag{11}$$

On the other hand, by Corollary 24 we have $\sigma^*(\mathscr{C}_1) \geq \rho(\mathscr{C}_1)$, and by Lemma 4 we have $\rho(\mathscr{C}_1) \geq d(\mathscr{D})$. Therefore

$$\sigma^*(\mathscr{C}_1) \geq \rho(\mathscr{C}_1) \geq d(\mathscr{D}) = d(\mathscr{C}_1). \tag{12}$$

The theorem can now be easily obtained combining inequalities (11) and (12). □

Corollary 39. The covering radius of a dually QMRD code $\mathscr{C}$ satisfies $\rho(\mathscr{C}) \leq d(\mathscr{C})$. Moreover, equality holds if and only if $\mathscr{C}$ is not maximal.

Proof. Combine Corollary 24, Theorem 38, Proposition 7 and the fact that $\mathscr{C}$ is not maximal if and only if $\mu(\mathscr{C}) = 0$, by definition of maximality degree. □

The upper bound of Corollary 39 is not sharp in general, as we show in the following example. This proves in particular that there exist dually QMRD codes that are maximal. In particular, there exist dually QMRD codes that are not contained into an MRD code with the same minimum distance.

Example 40. Take $q = 2$ and $k = m = 4$. Let $\mathscr{C}$ be the linear code generated over $\mathbb{F}_2$ by the following three matrices:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

We have $\dim(\mathscr{C}) = 3$ and $d(\mathscr{C}) = 4$. Hence $\dim(\mathscr{C}^\perp) = 13$ and $d(\mathscr{C}^\perp) = 1$. Therefore $d(\mathscr{C}) + d(\mathscr{C}^\perp) = 5$, and $\mathscr{C}$ is dually QMRD by Lemma 36. One can check that $\rho(\mathscr{C}) = 3 \neq d(\mathscr{C}) = 4$, and that $\mu(\mathscr{C}) = 1$.

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