ON THE EXPONENTIAL STABILITY OF A STRATIFIED FLOW TO THE 2D IDEAL MHD EQUATIONS WITH DAMPING*

YI DU[†], WANG YANG[‡], AND YI ZHOU[§]

Abstract. We study the stability of a type of stratified flows of the two dimensional inviscid incompressible MHD equations with velocity damping. The exponential stability for the perturbation near certain stratified flow is investigated in a strip-type area $\mathbb{R} \times [0,1]$. Although the magnetic filed potential is governed by a transport equation, by using the algebraic structure of the incompressible condition, it turns out that the linearized MHD equations around the given stratified flow retain a non-local damping mechanism. After carefully analyzing the non-linear structure and introducing some suitable weighted energy norms, we get the exponential stability by combining the exponential decay in time in the lower order energy with that in the high order energy.

Key words. Exponential stability, Stratified flow, 2D ideal MHD equations with damping.

AMS subject classifications. 35Q35, 35L03.

1. Introduction. During the past decades, incompressible fluid equations have received much attention, because they are with mathematical challenge, and present many interesting phenomena, particularly the phenomena of the stability or instability related to the longtime behavior of solutions [8]. In recent years, researchers have discovered numerous new interesting phenomena, such as the stability for the planar Couette-flow in 2D Euler equations [4] and for the incompressible Navier-Stokes equations (n = 2, 3) with high Reynold's number [3]. To the half space case in \mathbb{R}^2 , with the assumption that the initial perturbation is small and periodic, the system is stable even if the basic flow is large [13, 25]. The stability for the compressible case with small Couette-type flow is discussed in [17]. Moreover, as to different Mach numbers, the stability or instability for a small steady Poiseuille-type flow in a layer area in \mathbb{R}^2 have been discussed in [18] and [19] respectively. For more relevant results on this topic, readers can see [9, 15, 16, 10, 11, 12] and the references therein.

Magneto-Hydrodynamics (MHD) equations describe the motion of an electrically conducting fluid in the presence of the magnetic field, underlying many physical phenomena such as the geomagnetic dynamo in geophysics and solar winds and solar flares in astrophysics ([7]). Mathematically, the MHD equations are extremely difficult to analyze due to the analogous nonlinear structure and the strong nonlinear coupling with the incompressible Navier-Stokes equations.

In this paper, we shall study the exponential stability for a type of stratified flow to the following 2D MHD equations with damping, namely,

(1.1)
$$\begin{cases} \partial_t U + (U \cdot \nabla)U + \kappa U + \nabla P = (B \cdot \nabla)B, & (x, y, t) \in \Omega \times \mathbb{R}^+, \\ \partial_t B + (U \cdot \nabla)B = (B \cdot \nabla)U, \\ \nabla \cdot U = \nabla \cdot B = 0, \end{cases}$$

where $U = (U_1, U_2)^T$, $B = (B_1, B_2)^T$ and the scalar P are the velocity, magnetic field

^{*}Submitted to the editors DATE.

[†]Department of Mathematical Sciences, Jinan University, Guangzhou 510632, China. (duyidy@gmail.com, duyidy@jnu.edu.cn).

[‡]Department of Mathematical Sciences, Jinan University, Guangzhou 510632, China (yang@stu2017.jnu.edu.cn).

[§]School of Mathematical Sciences, Fudan University, Shanghai 200433, China. Department of Mathematical Sciences, Jinan University, Guangzhou 510632, China. (yizhou@fudan.edu.cn) 1

and the pressure of the fluid respectively. κ is a positive constant. Here and hereafter, we denote $\nabla = (\partial_x, \partial_y)^T$ and the layer area

(1.2)
$$\Omega = \mathbb{R} \times [0, 1].$$

Noting the condition $\nabla \cdot B = 0$ and in 2D, we take

(1.3)
$$B = \nabla^{\perp} \Phi \triangleq (\partial_y \Phi, -\partial_x \Phi),$$

with Φ as a scalar function. The system (1.1) is equivalent to

(1.4)
$$\begin{cases} \partial_t \Phi + (U \cdot \nabla) \Phi = 0, \\ \partial_t U + (U \cdot \nabla) U + \kappa U + \nabla P = \nabla \cdot (\nabla^\perp \Phi \otimes \nabla^\perp \Phi), \quad (x, t) \in \Omega \times \mathbb{R}^+, \\ \nabla \cdot U = 0. \end{cases}$$

For completeness, we recall some related efforts on the global regularity problem for the 2D MHD system first. The dissipative and resistive MHD system has been well studied [26]. The dissipative and non-resistive MHD system has been first studied by Lin, Xu and Zhang [21], then the proof of [21] has been significantly simplified by Zhang [29] later. The inviscid and resistive case or mixed partial dissipation and partial magnetic diffusion 2D MHD system can be found in the series work of Cao and Wu, such as [5] and the references therein. Considering the ideal MHD system (i.e. inviscid and non-resistive), it will bring extra difficulty to bound the vortex stretching type terms or the nonlinear coupled terms even in 2D. Using the Elasser's variable $Z^{\pm} = U \pm B$, Bardos, Sulem and Sulem [2] prove the global existence of the classical solution when the initial data is close to the non-zero constant equilibrium state (\bar{B} , 0). Interested readers may see more results in [23, 27, 14, 24, 22, 28, 31, 30] et. al.

In this paper, we consider exponential stability for the MHD system (1.4) with the stratified flow of the form $\overline{U}(t, x, y) = (\Psi(t, y), 0)$ and the magnetic potential $\overline{\Phi}(t, x, y) = \overline{\Phi}(t, y) + C_0 x$. Here $(x, y) \in \Omega$ and C_0 is a non-zero constant. It turns out that functions $(\Psi(t, y), \overline{\Phi}(t, y))$ solve a 1D wave equation on the bounded interval $y \in [0, 1]$, which is similar to the case of visco-elasticity system given by Endo, Giga, Gotz and Liu [9]. We prove that if initially the equilibrium stratified flow is sufficiently smooth, then the pair of solutions $(\overline{U}, \overline{\Phi})$ is exponential stable as time tends to infinity.

Denote

(1.5)
$$u = U - \bar{U}, \quad \rho = \Phi - \bar{\Phi},$$

then the perturbed system (1.4) reads as

(1.6)
$$\partial_t \rho + u_2 \partial_y \bar{\Phi} + C_0 u_1 + (u \cdot \nabla) \rho + \Psi \partial_x \rho = 0,$$

and

(1.7)
$$\partial_t u + (u \cdot \nabla) u + (u_2 \partial_y \Psi, 0)^T + \kappa u + \Psi \partial_x u + \nabla P = (\nabla^\perp \rho \cdot \nabla) \nabla^\perp \rho - (\partial_x \rho \partial_y^2 \bar{\Phi}, 0)^T + (\partial_y \bar{\Phi} \partial_{xy} \rho, -\partial_y \bar{\Phi} \partial_x^2 \rho)^T - C_0 (\partial_y^2 \rho, -\partial_{xy} \rho)^T.$$

Here and hereafter, A^T means the transpose of the vector A.

We take the rigid boundary conditions for system (1.6)-(1.7) as

(1.8)
$$\begin{cases} \lim_{|x| \to +\infty} u(x, y, t) = 0, \quad u_2(x, y, t)|_{y=0,1} = 0, \\ (\partial_y u_1 - \partial_x u_2)(x, y, t)|_{y=0,1} = 0, \quad \partial_y \Psi(t, y)|_{y=0,1} = 0 \end{cases}$$

Now, our main theorem states as follows:

THEOREM 1.1. Let $\epsilon_0 > 0$ be a small constant and $k \ge 7$. For the system (1.5)-(1.8), taking the initial equilibrium $(\Psi, \partial_t \Psi, \bar{\Phi})|_{t=0} = (\Psi_0(y), \Psi_1(y), \bar{\Phi}_0(y)) \in H^{k+3}$, and the initial perturbation as $u_0 = (u_{10}, u_{20})$ and ρ_0 , if the following conditions hold:

(1.9)
$$\begin{cases} \partial_y^{2n-1}\rho_0(x,y)|_{y=0,1} = \partial_y^{2n-1}\Psi_0(x,y)|_{y=0,1} = \partial_y^{2n-1}\bar{\Phi}_0(x,y)|_{y=0,1} = 0, \\ \int_0^1 u_{10}(x,y)dy = \int_0^1 \rho_0(x,y)dy = 0, \end{cases}$$

and

(1.10)
$$\|u_0\|_{H^k(\Omega)} + \|\rho_0\|_{H^{k+1}(\Omega)} \lesssim \epsilon_0,$$

then the equilibrium flow $(\bar{U}, \bar{\Phi})$ of the system (1.6)-(1.8) is "exponentially stable" ¹ in the space of $H^k(\Omega)$.

One main difficulty in studying the system (1.6)-(1.8) is in the variable ρ , which is governed by a transport equation. It is a great challenging problem to establish the decay estimate to get our main result. Besides, due to the coupling terms there is a loss of derivative. Recently, in [6], authors study the stability property for the 2D incompressible Boussinesq equations with damping, where they introduce an interesting weighted estimate to handle the loss of derivative's problem. We can borrow the ideas from [6] to study our case.

Let us point out some technical difficulties in our proof. First, we consider a strongly nonlinear coupled system, which is much more complicated than the linear coupled system [6]. Besides, since we work in a sobolev space frame with high order derivatives, it needs a careful analysis of boundary conditions (see Section 2 for details). To get decay estimates for the perturbation of magnetic potential ρ , it needs a suitable background magnetic field, and this thought is inspired by the heuristic work of [2]. Actually, the linearized system turns out to be a 1D wave equation with damping on the bounded interval [0, 1], then we can get the exponential decay when we eliminate the zero mode of u_1 and ρ . Here we do not need the smallness for the equilibrium flow. More precisely, due to the damped wave equations on a 1D bounded interval, the solution possesses an exponential decay. With this decay rate, we can get our desired results without the smallness assumption of the background.

The strategy to prove our results is in the following two folds. First, the basic observation is the stratified flow actually plays a role of stability. Based on this fact, we exploit the structure of the system, and get good enough decay rates for both the perturbation and the basic flow. Second, we use the weighted energy estimates to overcome the loss of derivatives. In our proof, we use some ideas from [9, 27, 6].

This paper is organized as follows: in Section 2, we shall present some preliminaries and give the estimates for the basic flow. Moreover, the boundary conditions and our function space will also be presented. In Section 3, the energy estimates will be given. In Section 4, we shall give the decay estimates, and then the main theorem will be proved in Section 5. Throughout the paper, we sometimes use the notation $A \leq B$ as an equivalent notation to $A \leq CB$ with an uniform constant C.

2. Preliminaries. We begin with a well-known Sobolev inequality, see e.g. [1].

¹ Considering the abstract differential system $\dot{X} = g(X,t)$, there is a norm $\|\cdot\|_X$ and the solution X(t) is obtained. Meanwhile, if there exists a positive constant a, satisfying $\|X(t)\|_X \lesssim \|X(t_0)\|_X e^{-a(t-t_0)}$ for $t \ge t_0$, we call that the solution X(t) is exponentially stable in the sense of $\|\cdot\|_X$.

LEMMA 2.1. Let $\Omega_0 \subset \mathbb{R}^n$ and $s \in \mathbb{N}$, for $(f,g) \in H^s(\Omega_0)$ the following estimate holds:

(2.1)
$$\|fg\|_{H^s} \lesssim \|f\|_{H^s} \|g\|_{L^{\infty}} + \|g\|_{H^s} \|f\|_{L^{\infty}}.$$

2.1. Estimates for the basic flow. Since we take the basic flows as $\overline{U} = (\Psi(t,y),0)$ and $\overline{\Phi}(x,y,t) = \overline{\Phi}(t,y) + C_0 x$, then we have

(2.2)
$$\begin{cases} \partial_t \bar{U} + (\bar{U} \cdot \nabla) \bar{U} + \kappa \bar{U} + \nabla \bar{P} = \nabla \cdot (\nabla^\perp \bar{\bar{\Phi}} \otimes \nabla^\perp \bar{\bar{\Phi}}), \\ \partial_t \bar{\bar{\Phi}} + (\bar{U} \cdot \nabla) \bar{\bar{\Phi}} = 0. \end{cases}$$

Moreover, the system (2.2) can be simplified as

(2.3)
$$\begin{cases} \partial_t \Psi + \kappa \Psi + \partial_x \bar{P} = -C_0 \partial_y^2 \bar{\Phi}, & y \in [0, 1], \\ \partial_y \bar{P} = 0, \\ \partial_t \bar{\Phi} + C_0 \Psi = 0. \end{cases}$$

From $(2.3)_1$ and $(2.3)_2$, we conclude that $\partial_x \bar{P}$ only depends on the variable t. We write $\partial_x \bar{P} = h(t)$ as in [9] for a well picked function h(t), which does not play essential role in our proof. For simplicity, we take $\partial_x \bar{P} = h(t) = 0$. And then from (2.3), we have

(2.4)
$$\begin{cases} \partial_t^2 \bar{\Phi} + \kappa \partial_t \bar{\Phi} - C_0^2 \partial_y^2 \bar{\Phi} = 0, \\ \Psi = -\frac{1}{C_0} \partial_t \bar{\Phi}. \end{cases}$$

By using the boundary condition (1.8), we have $\partial_y \Psi(t, y)|_{y=0,1} = \partial_{ty} \Psi(t, y)|_{y=0,1} = 0$, which implies $\partial_y \overline{\Phi}(t, y)|_{y=0,1} = 0$. From (2.4), we have:

(2.5)
$$\partial_t \partial_y \Psi + k \partial_y \Psi = -C_0 \partial_y^3 \bar{\Phi},$$

then one has

(2.6)
$$\partial_y^3 \bar{\Phi}(t,y)|_{y=0,1} = 0.$$

Repeating this process, we get the following compatible conditions

(2.7)
$$\partial_y^{2n-1}\Psi(t,y)|_{y=0,1} = 0, \quad \partial_y^{2n-1}\bar{\Phi}(t,y)|_{y=0,1} = 0, \quad n \in \mathbb{N}.$$

PROPOSITION 2.2. For $2 \leq k \in \mathbb{N}$, and $(\bar{\Phi}(t,y), \Psi(t,y))$ is a pair of solutions to the system (2.4) and (2.7) with the initial data $(\bar{\Phi}_0(y), \Psi_0(y), \Psi_1(y)) \in H^k$, then there exists a uniform constant $\alpha > 0$, such that,

$$\|\bar{\Phi}(t,\cdot)\|_{H^{k}([0,1])} + \|\partial_{t}\bar{\Phi}(t,\cdot)\|_{H^{k-1}([0,1])} \lesssim (\|\bar{\Phi}_{0}\|_{H^{k}}^{2} + \|\Psi_{0}\|_{H^{k}}^{2} + \|\Psi_{1}\|_{H^{k-1}}^{2})e^{-\alpha t},$$

and

$$\|\Psi(t,\cdot)\|_{H^{k-1}([0,1])} + \|\partial_t \Psi(t,\cdot)\|_{H^{k-2}([0,1])} \lesssim (\|\bar{\Phi}_0\|_{H^k}^2 + \|\Psi_0\|_{H^k}^2 + \|\Psi_1\|_{H^{k-1}}^2)e^{-\alpha t}$$

Proof. By writing $\overline{\Phi}(t,y) = \overline{\Phi}_1(y)\overline{\Phi}_2(t)$, then from (2.4), we have

(2.8)
$$\begin{cases} \partial_y^2 \bar{\Phi}_1(y) = \frac{\lambda}{C_0^2} \bar{\Phi}_1(y), \\ \partial_t^2 \bar{\Phi}_2(t) + \kappa \partial_t \bar{\Phi}_2(t) = \lambda \bar{\Phi}_2(t), \\ \partial_y \bar{\Phi}(t,y)|_{y=0,1} = 0, \\ (\bar{\Phi}, \Psi)|_{t=0} = (\bar{\Phi}_0(y), \Psi_0(y)), \end{cases}$$

with λ being a constant.

Moreover, with boundary conditions (1.8) and (2.7), we have

(2.9)
$$\begin{cases} \bar{\Phi}_0 = \sum_{J=0}^{\infty} C_J \cos J \pi y, \\ \Psi_0 = \sum_{J=0}^{\infty} D_J \cos J \pi y, \end{cases} \quad y \in [0,1],$$

with

(2.10)
$$C_J = 2 \int_0^1 \bar{\Phi}_0 \cos(J\pi y) dy, \quad D_J = 2 \int_0^1 \Psi_0 \cos(J\pi y) dy.$$

For the case $\lambda \geq 0$, by noting the boundary condition (2.8)₃, it is obvious that $\bar{\Phi}_1(y) \equiv 0$, which concludes that $\bar{\Phi}(t,y) \equiv 0$.

When $\lambda < 0$, from $(2.8)_1$, by a standard procedure, we get

(2.11)
$$\bar{\Phi}_1(y) = C_1^1 \cos(\sqrt{\frac{-\lambda}{C_0^2}}y) + C_2^1 \sin(\sqrt{\frac{-\lambda}{C_0^2}}y).$$

Recalling the boundary condition $(2.8)_3$, we get $C_2^1 = 0$ and

(2.12)
$$\lambda = -(C_0 m \pi)^2, \quad \bar{\Phi}_1(y) = C_1^1 \cos(\sqrt{(m\pi)^2}y), \quad m \in \mathbb{Z}.$$

From $(2.8)_2$ and (2.11), we denote

$$\gamma_{1,2} = \frac{-\kappa \pm \sqrt{\kappa^2 - 4(C_0 m \pi)^2}}{2},$$

there are the following three different cases.

Case 1. $m^2 < \frac{\kappa^2}{4C_0^2\pi^2} = \mathbb{K}_0^2$. In this case, there are only finite terms of m. We get the solution of $(2.4)_1$ and (2.7) as

(2.13)
$$\bar{\Phi}^{(1)}(t,y) = \sum_{m < \mathbb{K}_0} (A_m e^{\gamma_1 t} + B_m e^{\gamma_2 t}) \cos m\pi y, \quad m \in \mathbb{N}.$$

Using the initial conditions (2.9) and $(2.4)_2$, then there holds

$$(2.14)A_m = \frac{2}{\sqrt{\kappa^2 - 4C_0^2 m^2 \pi^2}} \left(\gamma_2 \int_0^1 \bar{\Phi}_0(y) \cos m\pi y dy + C_0 \int_0^1 \bar{\Psi}_0(y) \cos m\pi y dy\right)$$
$$= \frac{1}{\sqrt{\kappa^2 - 4C_0^2 m^2 \pi^2}} \left(\gamma_2 C_m + C_0 D_m\right),$$

and

$$(2.15\mathcal{B}_m = \frac{-2}{\sqrt{\kappa^2 - 4C_0^2 m^2 \pi^2}} \left(\gamma_1 \int_0^1 \bar{\Phi}_0(y) \cos m\pi y dy + C_0 \int_0^1 \bar{\Psi}_0(y) \cos m\pi y dy\right) \\ = \frac{-1}{\sqrt{\kappa^2 - 4C_0^2 m^2 \pi^2}} \left(\gamma_1 C_m + C_0 D_m\right),$$

with C_m , D_m are given by (2.10) and $m \in \mathbb{N}$.

Therefore, by (2.9), (2.12), (2.14) and (2.15), we get

$$(2.16) \quad \bar{\Phi}^{(1)}(t,y) = \sum_{m < \mathbb{K}_0} \frac{1}{\sqrt{\kappa^2 - 4C_0^2 m^2 \pi^2}} (\gamma_2 C_m + C_0 D_m) (e^{2\gamma_1 t} + e^{2\gamma_2 t}) \\ + \frac{1}{\sqrt{\kappa^2 - 4C_0^2 m^2 \pi^2}} (C_0 D_m - \gamma_1 C_m) e^{\gamma_2 t} \cos m\pi y.$$
Thus taking $\alpha_0 = \min\{\frac{\kappa - \sqrt{\kappa^2 - 4(C_0 m\pi)^2}}{2}|_{m^2 < \frac{\kappa^2}{4C_0^2 \pi^2}}\}$, we have
$$(2.17) \qquad \|\partial_y^k \bar{\Phi}^{(1)}(t,y)\|_{L^2[0,1]}^2 \lesssim e^{-2\alpha_0 t} \sum_{m \le \mathbb{K}_0} (C_m^2 + C_0^2 D_m^2) m^{2k} \pi^{2k} \\ \lesssim e^{-2\alpha_0 t} (\|\bar{\Phi}_0\|_{H^k}^2 + C_0^2 \|\Psi_0\|_{H^k}^2).$$

Case 2. $m^2 = \frac{\kappa^2}{4C_0^2\pi^2} = \mathbb{K}_0^2$. In this case, $\gamma_1 = \gamma_2 = -\frac{\kappa}{2}$, we get the solution for $(2.4)_1$, (2.7) and (2.9) as

(2.18)
$$\bar{\Phi}^{(2)}(t,y) = (A^{(1)}_{\mathbb{K}_0} + B^{(1)}_{\mathbb{K}_0}t)e^{-\frac{\kappa}{2}t}\cos\mathbb{K}_0\pi y, \quad m \in \mathbb{N}.$$

Similarly to (2.14), we have

(2.19)
$$A_{\mathbb{K}_0}^{(1)} = C_{\mathbb{K}_0}, \quad B_{\mathbb{K}_0}^{(1)} = \frac{\kappa}{2} C_{\mathbb{K}_0} - C_0 D_{\mathbb{K}_0}.$$

From (2.18), (2.19) and (2.9), for this case we have

(2.20)
$$\|\partial_y^k \bar{\Phi}^{(2)}(t,y)\|_{L^2[0,1]}^2 \lesssim e^{-\frac{3}{4}\kappa t} (\|\bar{\Phi}_0\|_{H^k}^2 + C_0^2 \|\Psi_0\|_{H^k}^2).$$

Case 3. $m^2 > \frac{\kappa^2}{4C_0^2\pi^2} = \mathbb{K}_0^2$. In this case, we have

(2.21)
$$\bar{\Phi}^{(3)}(t,y) = \sum_{m>\mathbb{K}_0}^{\infty} \left(A_m^{(2)}\cos\sqrt{\frac{4(C_0m\pi)^2 - \kappa^2}{4}}t + B_m^{(2)}\sin\sqrt{\frac{4(C_0m\pi)^2 - \kappa^2}{4}}t\right)e^{-\frac{\kappa}{2}t}\cos m\pi y$$

Again, by using initial conditions (2.9) and (2.21), we get

$$(2.22) \ \bar{\Phi}^{(3)}(t,y) = e^{-\frac{\kappa}{2}t} \sum_{m>\mathbb{K}_0}^{\infty} \cos m\pi y \left(C_m \cos \sqrt{\frac{4(C_0 m\pi)^2 - \kappa^2}{4}} t + \frac{4}{\sqrt{4C_0^2 m^2 \pi^2 - \kappa^2}} \left(\frac{\kappa}{2} C_m - C_0 D_m \right) \sin \sqrt{\frac{4(C_0 m\pi)^2 - \kappa^2}{4}} t \right).$$

Furthermore, we have

$$(2.23) \quad \|\partial_y^k \bar{\Phi}^{(3)}\|_{L^2[0,1]}^2 \lesssim e^{-\kappa t} \sum_{m > \mathbb{K}_0}^\infty m^{2k} \pi^{2k} \left(C_m^2 \cos^2 \sqrt{\frac{4(C_0 m \pi)^2 - \kappa^2}{4}} t + (C_m^2 + C_0^2 D_m^2) \sin^2 \sqrt{\frac{4(C_0 m \pi)^2 - \kappa^2}{4}} t \right)$$
$$\lesssim e^{-\kappa t} (\|\bar{\Phi}_0\|_{H^k}^2 + C_0^2 \|\Psi_0\|_{H^k}^2).$$

In conclusion, taking $\alpha = \min\{\alpha_0, \frac{\kappa}{4}\}$, from (2.17), (2.20) and (2.23) we have

$$(2.24) \qquad \|\bar{\Phi}(t,\cdot)\|_{H^{k}([0,1])} \leq \sum_{i=1}^{3} \|\bar{\Phi}^{(i)}(t,\cdot)\|_{H^{k}([0,1])} \lesssim (\|\bar{\Phi}_{0}\|_{H^{k}}^{2} + C_{0}^{2}\|\Psi_{0}\|_{H^{k}}^{2})e^{-\alpha t},$$

which also implies that

(2.25)
$$\|\partial_t \Phi(t, \cdot)\|_{H^{k-1}([0,1])} \lesssim (\|\Psi_0\|_{H^k}^2 + \|\Psi_1\|_{H^k}^2) e^{-\alpha t}.$$

Similarly, by using $(2.4)_2$ and (2.8), we have

(2.26)
$$\|\Psi(t,\cdot)\|_{H^{k-1}([0,1])}$$
 and $\|\partial_t \Psi(t,\cdot)\|_{H^{k-2}([0,1])} \lesssim (\|\Psi_0\|_{H^k}^2 + \|\Psi_1\|_{H^k}^2)e^{-\alpha t}$.

2.2. Boundary conditions. Take ∂_y to equation (1.6) and recall $\partial_y u_1|_{y=0,1} = 0$, we have

(2.27)
$$(\partial_t \partial_y \rho + (u_1 + \Psi) \partial_x \partial_y \rho + \partial_y u_2 \partial_y \rho)|_{y=0,1} = 0.$$

Since we have the initial condition $\partial_y^{2n-1}\rho_0(x,y)|_{y=0,1}=0$, we obtain

(2.28)
$$\partial_y \rho(x,y)|_{y=0,1} = 0.$$

Writing $\omega = \nabla^{\perp} \cdot u$, then the boundary condition (1.8) implies $\omega|_{y=0,1} = 0$. Besides, from the incompressible constraint $\nabla \cdot u = 0$ we have

(2.29)
$$\partial_y^2 u_2|_{y=0,1} = 0.$$

From (1.7), we have

$$(2.30) \quad \partial_t \omega - \partial_t (\partial_y \Psi) + (u_1 + \Psi) \partial_x \omega + u_2 \partial_y (\omega - \partial_y \Psi) + \kappa (\omega - \partial_y \Psi) \\ = (\partial_y \rho + \partial_y \bar{\Phi}) (-\partial_x^3 \rho - \partial_x \partial_y^2 \rho) + (\partial_x \rho + C_0) (\partial_x^2 \partial_y \rho + \partial_y^3 \bar{\Phi} + \partial_y^3 \bar{\Phi}).$$

By using (1.6), (2.7) and (2.30), there holds

$$(2.31)\qquad\qquad\qquad\partial_u^3\rho|_{y=0,1}=0$$

Moreover, from (1.6) we also have

$$(2.32) \quad \partial_t \partial_y^3 \rho + \partial_y^3 u_2 \partial_y \bar{\Phi} + u_2 \partial_y^4 \bar{\Phi} + 3 \partial_y^2 u_2 \partial_y^2 \bar{\Phi} + 3 \partial_y u_2 \partial_y^3 \bar{\Phi} + C_0 \partial_y^3 u_1 + (\partial_y^3 u \cdot \nabla) \rho + (u \cdot \nabla) \partial_y^3 \rho + 3 (\partial_y^2 u \cdot \nabla) \partial_y \rho + 3 (\partial_y u \cdot \nabla) \partial_y^2 \rho + \partial_y^3 \Psi \partial_x \rho + \Psi \partial_x \partial_y^3 \rho + 3 \partial_y \Psi \partial_x \partial_y^2 \rho + 3 \partial_y^2 \Psi \partial_x y \rho = 0.$$

Combining (2.7), (2.30) with (2.28)-(2.32), we get

(2.33)
$$\partial_y^3 u_1|_{y=0,1} = 0.$$

Repeating the iterative process above, we get boundary compatible conditions as

(2.34)
$$\partial_y^{2n} u_2|_{y=0,1} = 0, \quad \partial_y^{2n-1} u_1|_{y=0,1} = 0, \quad \partial_y^{2n-1} \rho|_{y=0,1} = 0, \quad n \in \mathbb{N}^+.$$

2.3. Notations. In order to solve our problem in a certain Sobolev space, recalling the boundary conditions (2.34), we introduce following function spaces:

(2.35)
$$\begin{cases} X^k(\Omega) = \{ f \in H^k(\Omega) : \partial_y^{2n} f |_{\partial\Omega} = 0, n \in \mathbb{N}^+ \}, \\ Y^k(\Omega) = \{ f \in H^k(\Omega) : \partial_y^{2n-1} f |_{\partial\Omega} = 0, n \in \mathbb{N}^+ \}. \end{cases}$$

To our problem, we define the following functional space as:

(2.36)
$$\aleph^k(\Omega) := \{ v \in H^k(\Omega), v = (v_1, v_2) \in Y^k(\Omega) \times X^k(\Omega) \}$$

For convenience, in this paper, we use L^2 , \dot{H}^k and H^k to stand for $L^2(\Omega)$, $\dot{H}^k(\Omega)$ and $H^k(\Omega)$ respectively. We also use <,> as the scalar product in L^2 .

2.4. An orthonormal basis for $X^k(\Omega), Y^k(\Omega)$. Let us start by defining

(2.37)
$$a_q(y) := \begin{cases} \cos(qy\frac{\pi}{2}) & q \quad odd, \\ \sin(qy\frac{\pi}{2}) & q \quad even, \end{cases} \text{ with } y \in [0,1], \quad q \in \mathbb{N}^+ \cup \{0\},$$

and

(2.38)
$$b_q(y) := \begin{cases} \sin(qy\frac{\pi}{2}) & q \quad odd, \\ \cos(qy\frac{\pi}{2}) & q \quad even, \end{cases} \quad with \ y \in [0,1], \quad q \in \mathbb{N}^+ \cup \{0\},$$

where $(\{a_q\}_{q\in\mathbb{N}}, \{b_q\}_{q\in\mathbb{N}})$ is the orthonormal basis for $L^2([0,1])$.

Remark 2.3. Actually, the basis given above is chosen based on the boundary conditions and the Fourier series. More precisely, for $f \in L^2(\Omega)$, we have the $L^2(\Omega)$ -convergence series given by:

$$f(x,y) = \sum_{q \in \mathbb{N} \cup 0} \mathcal{F}_a[f](x,q)a_q(y) \quad where \quad \mathcal{F}_a[f](x,q) := \int_0^1 f(x,y)\overline{a_q(y)}dy$$

or

$$f(x,y) = \sum_{q \in \mathbb{N} \cup \{0\}} \mathcal{F}_b[f](x,q) b_q(y) \quad where \quad \mathcal{F}_b[f](x,q) := \int_0^1 f(x,y) \overline{b_q(y)} dy,$$

with $\overline{a_q(y)}$ and $\overline{b_q(y)}$ are the conjugate of $a_q(y)$ and $b_q(y)$ respectively.

3. The energy estimates. For simplicity, we denote

(3.1)
$$E_k^2(t) := \frac{1}{2} \{ \|u\|_{H^k}^2 + \|\partial_t u\|_{H^k}^2 + \|\rho\|_{H^{k+1}}^2 + \|\partial_t \rho\|_{H^{k+1}}^2 \},$$

and the following weighted energy, which plays a crucial role in our proof,

(3.2)
$$\Gamma_k^2(t) := \frac{1}{2} \{ \int_{\Omega} (1 + \partial_x \rho) |\nabla^k u|^2 + (1 + \partial_x \rho) |\nabla^k \nabla^\perp \rho|^2 dx dy \}.$$

3.1. Local well-posedness. To prove our main theorem, we first claim the local well-posedness for the system (1.6)-(1.8), which can be verified by a standard Galerkin procedure. Here, we omit the proof for simplicity, and give the results as follows:

PROPOSITION 3.1. For the system (1.6)-(1.8), with the initial data $(u_0, \rho_0) \in \aleph^k(\Omega) \times Y^{k+1}(\Omega)$ and $(\Psi_0, \bar{\Phi}_0) \in Y^{k+3}$, then there exists a positive constant T, such that the system (1.6)-(1.8) admits a unique pair of solutions (u, ρ) on [0, T]. Moreover, for $k \geq 3$ there holds

(3.3)
$$\sup_{0 \le t \le T} E_k^2(t) \le M E_k^2(0),$$

with M being a constant.

3.2. Energy Estimates.

PROPOSITION 3.2. (Energy Estimates) Under the condition of Proposition 3.1, we assume that the initial data $(u_0, \rho_0) \in \aleph^k(\Omega) \times Y^{k+1}(\Omega)$ is small enough, such that

(3.4)
$$\|\partial_x \rho(t,\cdot)\|_{H^2(\Omega)} < \frac{1}{4}, \quad t \in [0,T],$$

then we have the following energy estimates for the system (1.6)-(1.9):

(3.5)
$$\frac{d}{dt}E_k^2 \lesssim (E_k^2 + \Gamma_{k+1}^2)(E_4 + \|\bar{\Phi}\|_{H^{k+2}} + \|\Psi\|_{H^{k+2}} + \|\partial_t\Psi\|_{H^{k+1}}).$$

Proof. Through the standard energy estimates, noting $u = (u_1, u_2)^T$, we have

$$(3.6) \quad \frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 = \langle u, \partial_t u \rangle = -\kappa \|u\|_{L^2}^2 - \langle u_1, u_2 \partial_y \Psi + \partial_x \rho \partial_y^2 \bar{\Phi} \rangle + \langle u, (\nabla^\perp \rho \cdot \nabla) \nabla^\perp \rho + \partial_y \bar{\Phi} (\partial_{xy} \rho, -\partial_x^2 \rho)^T - C_0 (\partial_y^2 \rho, -\partial_{xy} \rho)^T \rangle .$$

Similarly, by using $\nabla \cdot u = 0$ and the boundary condition we have

$$(3.7)\frac{1}{2}\frac{d}{dt}\|u\|_{\dot{H}^{k}}^{2} = \langle \nabla^{k}u, \nabla^{k}\partial_{t}u \rangle$$

$$= \langle \nabla^{k}u, \nabla^{k}[-(u\cdot\nabla)u - \kappa u - \Psi\partial_{x}u] \rangle - \langle \nabla^{k}u_{1}, \nabla^{k}(u_{2}\partial_{y}\Psi + \partial_{x}\rho\partial_{y}^{2}\bar{\Phi}) \rangle$$

$$+ \langle \nabla^{k}u, \nabla^{k}[(\nabla^{\perp}\rho\cdot\nabla)\nabla^{\perp}\rho + \partial_{y}\bar{\Phi}(\partial_{xy}\rho, -\partial_{x}^{2}\rho)^{T} - C_{0}(\partial_{y}^{2}\rho, -\partial_{xy}\rho)^{T}] \rangle,$$

 $\quad \text{and} \quad$

$$(3.8) \quad \frac{1}{2} \frac{d}{dt} \|\partial_t u\|_{L^2}^2 = \langle \partial_t u, \partial_t [-(u \cdot \nabla)u - \kappa u - \Psi \partial_x u + (\nabla^\perp \rho \cdot \nabla)\nabla^\perp \rho] \rangle \\ + \partial_y \bar{\Phi} (\partial_{xy}\rho, -\partial_x^2 \rho)^T - C_0 (\partial_y^2 \rho, -\partial_{xy} \rho)^T] \rangle - \langle \partial_t u_1, \partial_t [u_2 \partial_y \Psi + \partial_x \rho \partial_y^2 \bar{\Phi}] \rangle,$$

as well as

$$(3.9)\frac{1}{2}\frac{d}{dt}\|\partial_t u\|^2_{\dot{H}^k} = \langle \nabla^k \partial_t u, \nabla^k \partial_t [-(u \cdot \nabla)u - \Psi \partial_x u - \kappa u \\ + (\nabla^\perp \rho \cdot \nabla)\nabla^\perp \rho + \partial_y \bar{\Phi} (\partial_{xy}\rho, -\partial_x^2 \rho)^T - C_0 (\partial_y^2 \rho, -\partial_{xy} \rho)^T] > \\ - \langle \partial_t \partial^k u_1, \partial_t \partial^k [u_2 \partial_y \Psi + \partial_x \rho \partial_y^2 \bar{\Phi}] > .$$

Applying ∇^{\perp} to (1.6) and then by a similar procedure, we have

(3.10)
$$\frac{1}{2} \frac{d}{dt} \|\nabla^{\perp}\rho\|_{\dot{H}^{k}}^{2} = \langle \nabla^{k}\nabla^{\perp}\rho, \nabla^{k}\partial_{t}\nabla^{\perp}\rho \rangle$$
$$= \langle \nabla^{k}\nabla^{\perp}\rho, \nabla^{k}\nabla^{\perp}[-u_{2}\partial_{y}\bar{\Phi} - C_{0}u_{1} - (u\cdot\nabla)\rho - \Psi\partial_{x}\rho] \rangle,$$

and

(3.11)
$$\frac{1}{2} \frac{d}{dt} \|\nabla^{\perp} \partial_t \rho\|_{\dot{H}^k}^2 = \langle \nabla^k \nabla^{\perp} \partial_t \rho, \nabla^k \partial_t^2 \nabla^{\perp} \rho \rangle$$
$$= \langle \nabla^k \nabla^{\perp} \partial_t \rho, \nabla^k \nabla^{\perp} \partial_t [-u_2 \partial_y \bar{\Phi} - C_0 u_1 - (u \cdot \nabla) \rho - \Psi \partial_x \rho] \rangle.$$

From (3.6)-(3.11), we get (3.12) $\frac{1}{2}\frac{d}{dt}(\|u\|_{\dot{H}^{k}}^{2}+\|\partial_{t}u\|_{\dot{H}^{k}}^{2}+\|\nabla^{\perp}\rho\|_{\dot{H}^{k}}^{2}+\|\nabla^{\perp}\partial_{t}\rho\|_{\dot{H}^{k}}^{2})+\kappa(\|u\|_{\dot{H}^{k}}+\|\partial_{t}u\|_{\dot{H}^{k}})\triangleq\sum_{i=1}^{3}G_{i},$

with

$$(3.13) \qquad G_{1} = -\langle u_{1}, u_{2}\partial_{y}\Psi + \partial_{x}\rho\partial_{y}^{2}\bar{\Phi} \rangle + \langle u, \partial_{y}\bar{\Phi}(\partial_{xy}\rho, -\partial_{x}^{2}\rho)^{T} \rangle \\ + \langle u, (\nabla^{\perp}\rho \cdot \nabla)\nabla^{\perp}\rho \rangle - \langle \partial_{t}u_{1}, \partial_{t}[u_{2}\partial_{y}\Psi + \partial_{x}\rho\partial_{y}^{2}\bar{\Phi}] \rangle \\ + \langle \partial_{t}u, \partial_{t}[\partial_{y}\bar{\Phi}(\partial_{xy}\rho, -\partial_{x}^{2}\rho)^{T}] \rangle - \langle \partial_{t}u, \partial_{t}(\Psi\partial_{x}u) \rangle \\ - \langle \partial_{t}u, \partial_{t}[(u \cdot \nabla)u] \rangle + \langle \partial_{t}u, \partial_{t}[(\nabla^{\perp}\rho \cdot \nabla)\nabla^{\perp}\rho] \rangle \\ - \langle \nabla^{\perp}\rho, \nabla^{\perp}(\Psi\partial_{x}\rho) \rangle - \langle \nabla^{\perp}\rho, \nabla^{\perp}[(u \cdot \nabla)\rho] \rangle \\ - \langle \nabla^{\perp}\rho, \nabla^{\perp}(u_{2}\partial_{y}\bar{\Phi}) \rangle + \langle \nabla^{\perp}\partial_{t}\rho, \nabla^{\perp}\partial_{t}[(u \cdot \nabla)\rho] \rangle \\ - \langle \nabla^{\perp}\partial_{t}\rho, \nabla^{\perp}\partial_{t}(u_{2}\partial_{y}\bar{\Phi}) \rangle - \langle \nabla^{\perp}\partial_{t}\rho, \nabla^{\perp}\partial_{t}(\Psi\partial_{x}\rho) \rangle,$$

and

$$(3.14) \quad G_{2} = -\langle \nabla^{k} u, \nabla^{k} [(u \cdot \nabla)u] \rangle - \langle \nabla^{k} u_{1}, \nabla^{k} [u_{2}\partial_{y}\Psi + \partial_{x}\rho\partial_{y}^{2}\bar{\Phi}] \rangle \\ - \langle \nabla^{k} u, \nabla^{k} [\Psi\partial_{x}u] \rangle - \langle \nabla^{k}\partial_{t} u_{1}, \nabla^{k}\partial_{t} [u_{2}\partial_{y}\Psi + \partial_{x}\rho\partial_{y}^{2}\bar{\Phi}] \rangle \\ - \langle \nabla^{k}\partial_{t} u, \nabla^{k}\partial_{t} (\Psi\partial_{x}u) \rangle - \langle \nabla^{k}\partial_{t} u, \nabla^{k}\partial_{t} [(u \cdot \nabla)u] \rangle \\ - \langle \nabla^{k}\nabla^{\perp}\rho, \nabla^{k}\nabla^{\perp} (\Psi\partial_{x}\rho) \rangle - \langle \nabla^{k}\nabla^{\perp}\partial_{t}\rho, \nabla^{k}\nabla^{\perp}\partial_{t} (\Psi\partial_{x}\rho) \rangle$$

as well as

$$\begin{aligned} (3.15) \\ G_3 &= < \nabla^k u, \nabla^k [(\nabla^\perp \rho \cdot \nabla) \nabla^\perp \rho] > - < \nabla^k \nabla^\perp \rho, \nabla^k \nabla^\perp [(u \cdot \nabla) \rho] > \\ &+ < \nabla^k \partial_t u, \nabla^k \partial_t [(\nabla^\perp \rho \cdot \nabla) \nabla^\perp \rho] > - < \nabla^k \nabla^\perp \partial_t \rho, \nabla^k \nabla^\perp \partial_t [(u \cdot \nabla) \rho] > \\ &+ < \nabla^k u, \nabla^k (\partial_y \bar{\Phi} \partial_{xy} \rho, -\partial_y \bar{\Phi} \partial_x^2 \rho)^T > - < \nabla^k \nabla^\perp \rho, \nabla^k \nabla^\perp (u_2 \partial_y \bar{\Phi}) > \\ &+ < \nabla^k \partial_t u, \nabla^k \partial_t (\partial_y \bar{\Phi} \partial_{xy} \rho, -\partial_y \bar{\Phi} \partial_x^2 \rho)^T > - < \nabla^k \nabla^\perp \partial_t \rho, \nabla^k \nabla^\perp \partial_t (u_2 \partial_y \bar{\Phi}) > \\ &- < \nabla^k u, C_0 \nabla^k (\partial_y^2 \rho, -\partial_{xy} \rho)^T > - < \nabla^k \nabla^\perp \rho, \nabla^k \nabla^\perp (C_0 u_1) > \\ &- < \nabla^k \partial_t u, C_0 \nabla^k \partial_t (\partial_y^2 \rho, -\partial_{xy} \rho)^T > - < \nabla^k \nabla^\perp \partial_t \rho, \nabla^k \nabla^\perp \partial_t (C_0 u_1) > \\ &= \sum_{i=1}^6 I_i. \end{aligned}$$

During the calculation given above, we used the incompressible condition and get

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the cancellation for some quadratic terms as

$$(3.16)$$

$$- \langle u_1, C_0 \partial_y^2 \rho \rangle + \langle u_2, C_0 \partial_{xy} \rho \rangle - \langle \nabla^\perp \rho, \nabla^\perp (C_0 u_1) \rangle$$

$$= - \langle u_1, C_0 \partial_y^2 \rho \rangle - \langle \partial_y u_2, C_0 \partial_x \rho \rangle - \langle \partial_y \rho, C_0 \partial_y u_1 \rangle - \langle \partial_x \rho, C_0 \partial_x u_1 \rangle$$

$$= - \langle u_1, C_0 \partial_y^2 \rho \rangle + \langle \partial_x u_1, C_0 \partial_x \rho \rangle + \langle u_1, C_0 \partial_y^2 \rho \rangle - \langle \partial_x \rho, C_0 \partial_x u_1 \rangle$$

$$= 0,$$

and

$$(3.17) \quad - \langle \partial_t u_1, C_0 \partial_y^2 \partial_t \rho \rangle - \langle \partial_t u_2, C_0 \partial_{xy} \partial_t \rho \rangle - \langle \nabla^\perp \partial_t \rho, \nabla^\perp \partial_t (C_0 u_1) \rangle = 0.$$

Noting that $k \geq 7$, therefore G_1 is a lower order term. It is easy to get that

(3.18)
$$G_1 \lesssim E_4^2(\|u\|_{H^3} + \|\nabla\rho\|_{H^3} + \|\Psi\|_{H^3} + \|\partial_t\Psi\|_{H^2} + \|\bar{\Phi}\|_{H^3}).$$

By using Lemma 2.1, for the term G_2 , we have

$$(3.19) \qquad G_2 \lesssim E_k^2 (E_4 + \|\bar{\Phi}\|_{H^3} + \|\Psi\|_{H^3} + \|\partial_t \Psi\|_{H^2}) \\ + (\|\bar{\Phi}\|_{H^{k+2}} + \|\Psi\|_{H^{k+1}} + \|\partial_t \Psi\|_{H^{k+1}}) E_k E_4 \\ + < \partial^k \partial_t u, \partial_t \Psi \partial^k \partial_x u > + < \partial^k \nabla^\perp \partial_t \rho, \partial_t \Psi \partial^k \nabla^\perp \partial_x \rho > \\ + < \partial^k \partial_t u, (\partial_t u \cdot \nabla) \partial^k u > .$$

Recalling (3.4), we have

$$\begin{aligned} (3.20) &|<\partial^{k}\partial_{t}u, \partial_{t}\Psi\partial^{k}\partial_{x}u>+<\partial^{k}\nabla^{\perp}\partial_{t}\rho, \partial_{t}\Psi\partial^{k}\nabla^{\perp}\partial_{x}\rho>\\ &+<\partial^{k}\partial_{t}u, (\partial_{t}u\cdot\nabla)\partial^{k}u>|\\ &\lesssim \frac{1}{(1-\|\partial_{x}\rho\|_{L^{\infty}})^{\frac{1}{2}}}\|\partial_{t}u\|_{H^{k}}\|\partial_{t}\Psi\|_{L^{\infty}}[\int_{\Omega}(1+\partial_{x}\rho)|\partial^{k+1}u|^{2})dxdy]^{\frac{1}{2}}\\ &+\frac{1}{(1-\|\partial_{x}\rho\|_{L^{\infty}})^{\frac{1}{2}}}\|\partial_{t}\rho\|_{H^{k+1}}\|\partial_{t}\Psi\|_{L^{\infty}}[\int_{\Omega}(1+\partial_{x}\rho)|\nabla^{k+1}\nabla^{\perp}\rho|^{2})dxdy]^{\frac{1}{2}}\\ &+\frac{1}{(1-\|\partial_{x}\rho\|_{L^{\infty}})^{\frac{1}{2}}}\|\partial_{t}u\|_{H^{k}}\|\partial_{t}u\|_{L^{\infty}}[\int_{\Omega}(1+\partial_{x}\rho)|\nabla^{k+1}u|^{2})dxdy]^{\frac{1}{2}}\\ &\lesssim \frac{\Gamma_{k+1}}{(1-\|\partial_{x}\rho\|_{L^{\infty}})^{\frac{1}{2}}}(\|\partial_{t}u\|_{H^{k}}+\|\partial_{t}\rho\|_{H^{k+1}})(\|\partial_{t}\Psi\|_{H^{1}}+\|\partial_{t}u\|_{H^{2}}).\end{aligned}$$

From (3.19) and (3.20), we have

(3.21)
$$G_2 \lesssim (\Gamma_{k+1}^2 + E_k^2)(E_4 + \|\bar{\Phi}\|_{H^{k+2}} + \|\Psi\|_{H^{k+1}} + \|\partial_t \Psi\|_{H^{k+1}}).$$

Next, for the term G_3 , we shall estimate I_i , $(1 \le i \le 6)$ term by term. For the term I_1 , by using Lemma 2.1 and the incompressible condition, we have:

$$\begin{aligned} (3.22) \\ I_1 &\lesssim (\|u\|_{H^k}^2 + \|\rho\|_{H^{k+1}}^2)(\|\nabla\rho\|_{H^3} + \|u\|_{H^3}) \\ &+ < \nabla^k u_1, \partial_y \rho \nabla^k \partial_{xy} \rho - \partial_x \rho \nabla^k \partial_y^2 \rho > - < \nabla^k u_2, \partial_y \rho \nabla^k \partial_x^2 \rho - \partial_x \rho \nabla^k \partial_{xy} \rho > \\ &- < \nabla^k \partial_y \rho, \nabla^k \partial_y u_1 \partial_x \rho + \nabla^k \partial_y u_2 \partial_y \rho > - < \nabla^k \partial_x \rho, \nabla^k \partial_x u_1 \partial_x \rho + \nabla^k \partial_x u_2 \partial_y \rho > . \end{aligned}$$

Due to $\nabla \cdot \boldsymbol{u} = \boldsymbol{0}$ and Lemma 2.1, we have

$$\begin{aligned} (3.23) < \nabla^{k} u_{1}, \partial_{y} \rho \nabla^{k} \partial_{xy} \rho - \partial_{x} \rho \partial^{k} \partial_{y}^{2} \rho > - < \nabla^{k} u_{2}, \partial_{y} \rho \nabla^{k} \partial_{x}^{2} \rho - \partial_{x} \rho \nabla^{k} \partial_{xy} \rho > \\ & - < \partial^{k} \nabla^{\perp} \rho, (\partial^{k} \nabla^{\perp} u \cdot \nabla) \rho > \\ & = - \int_{\Omega} \nabla^{k} \partial_{x} u_{1} \partial_{y} \rho \nabla^{k} \partial_{y} \rho \, dx dy - \int_{\Omega} \nabla^{k} \partial_{y} \rho \nabla^{k} \partial_{y} u_{2} \partial_{y} \rho \, dx dy \\ & + \int_{\Omega} \nabla^{k} \partial_{y} u_{1} \partial_{x} \rho \nabla^{k} \partial_{y} \rho \, dx dy - \int_{\Omega} \nabla^{k} \partial_{y} \rho \nabla^{k} \partial_{y} u_{1} \partial_{x} \rho \, dx dy \\ & + \int_{\Omega} \nabla^{k} \partial_{x} u_{2} \partial_{y} \rho \nabla^{k} \partial_{x} \rho \, dx dy - \int_{\Omega} \nabla^{k} \partial_{x} \rho \nabla^{k} \partial_{y} u_{2} \partial_{y} \rho \, dx dy \\ & - \int_{\Omega} \nabla^{k} \partial_{y} u_{2} \partial_{x} \rho \nabla^{k} \partial_{x} \rho \, dx dy - \int_{\Omega} \nabla^{k} \partial_{x} \rho \nabla^{k} \partial_{x} \rho \, dx dy \\ & - \int_{\Omega} \nabla^{k} u_{1} \partial_{xy} \rho \nabla^{k} \partial_{y} \rho \, dx dy + \int_{\Omega} \nabla^{k} u_{1} \partial_{xy} \rho \nabla^{k} \partial_{y} \rho \, dx dy \\ & + \int_{\Omega} \nabla^{k} u_{2} \partial_{xy} \rho \nabla^{k} \partial_{x} \rho \, dx dy - \int_{\Omega} \nabla^{k} u_{2} \partial_{xy} \rho \nabla^{k} \partial_{x} \rho \, dx dy = 0. \end{aligned}$$

Combining (3.22) with (3.23), we get

$$(3.24) I_1 \lesssim E_k^2 E_4.$$

As to the term I_2 , we have

$$(3.25) \quad I_{2} = \langle \nabla^{k} \partial_{t} u, \nabla^{k} [(\partial_{t} \nabla^{\perp} \rho \cdot \nabla) \nabla^{\perp} \rho + (\nabla^{\perp} \rho \cdot \nabla) \nabla^{\perp} \partial_{t} \rho] > \\ - \langle \nabla^{k} \nabla^{\perp} \partial_{t} \rho, \nabla^{k} [(\partial_{t} \nabla^{\perp} u \cdot \nabla) \rho + (\partial_{t} u \cdot \nabla) \nabla^{\perp} \rho \\ + (\nabla^{\perp} u \cdot \nabla) \partial_{t} \rho + (u \cdot \nabla) \partial_{t} \nabla^{\perp} \rho] > \\ \lesssim E_{k}^{2} E_{4} + \langle \nabla^{k} \partial_{t} u, (\partial_{t} \nabla^{\perp} \rho \cdot \nabla) \nabla^{k} \nabla^{\perp} \rho > \\ - \langle \nabla^{k} \nabla^{\perp} \partial_{t} \rho, (\partial_{t} u \cdot \nabla) \nabla^{k} \nabla^{\perp} \rho + \langle \nabla^{k} \partial_{t} u, (\nabla^{\perp} \rho \cdot \nabla) \nabla^{k} \nabla^{\perp} \partial_{t} \rho > \\ - \langle \nabla^{k} \nabla^{\perp} \partial_{t} \rho, (\nabla^{k} \partial_{t} \nabla^{\perp} u \cdot \nabla) \rho - \langle \nabla^{k} \nabla^{\perp} \partial_{t} \rho, (\nabla^{k} \nabla^{\perp} u \cdot \nabla) \partial_{t} \rho > . \end{cases}$$

Similar to (3.16) and (3.23), we have:

$$(3.26) < \nabla^{k}\partial_{t}u, (\nabla^{\perp}\rho\cdot\nabla)\nabla^{k}\nabla^{\perp}\partial_{t}\rho > - < \nabla^{k}\nabla^{\perp}\partial_{t}\rho, (\nabla^{k}\partial_{t}\nabla^{\perp}u\cdot\nabla)\rho > = -\int_{\Omega}\nabla^{k}\partial_{t}\partial_{x}u_{1}\partial_{y}\rho\nabla^{k}\partial_{t}y\rho\,dxdy - \int_{\Omega}\nabla^{k}\partial_{t}\partial_{y}\rho\nabla^{k}\partial_{t}\partial_{y}u_{2}\partial_{y}\rho\,dxdy + \int_{\Omega}\nabla^{k}\partial_{t}\partial_{y}u_{1}\partial_{x}\rho\nabla^{k}\partial_{t}y\rho\,dxdy - \int_{\Omega}\nabla^{k}\partial_{t}\partial_{y}\rho\nabla^{k}\partial_{t}\partial_{y}u_{1}\partial_{x}\rho\,dxdy + \int_{\Omega}\nabla^{k}\partial_{t}\partial_{x}u_{2}\partial_{y}\rho\nabla^{k}\partial_{t}x\rho\,dxdy - \int_{\Omega}\nabla^{k}\partial_{t}\partial_{x}\rho\nabla^{k}\partial_{t}\partial_{x}u_{2}\partial_{y}\rho\,dxdy - \int_{\Omega}\nabla^{k}\partial_{t}\partial_{y}u_{2}\partial_{x}\rho\nabla^{k}\partial_{t}x\rho\,dxdy - \int_{\Omega}\nabla^{k}\partial_{t}\partial_{x}\rho\nabla^{k}\partial_{t}\partial_{x}u_{1}\partial_{x}\rho\,dxdy - \int_{\Omega}\nabla^{k}\partial_{t}u_{1}\partial_{xy}\rho\nabla^{k}\partial_{t}y\rho\,dxdy + \int_{\Omega}\nabla^{k}\partial_{t}u_{1}\partial_{x}\rho\nabla^{k}\partial_{t}y\rho\,dxdy + \int_{\Omega}\nabla^{k}\partial_{t}u_{2}\partial_{x}\rho\nabla^{k}\partial_{t}x\rho\,dxdy - \int_{\Omega}\nabla^{k}\partial_{t}u_{2}\partial_{x}y\rho\nabla^{k}\partial_{t}x\rho\,dxdy = 0,$$

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and

$$(3.27) < \nabla^{k} \partial_{t} u, (\partial_{t} \nabla^{\perp} \rho \cdot \nabla) \nabla^{k} \nabla^{\perp} \rho > + < \nabla^{k} \nabla^{\perp} \partial_{t} \rho, (\partial_{t} u \cdot \nabla) \nabla^{k} \nabla^{\perp} \rho > + < \nabla^{k} \nabla^{\perp} \partial_{t} \rho, (\nabla^{k} \nabla^{\perp} u \cdot \nabla) \partial_{t} \rho > \lesssim \frac{\|\partial_{t} u\|_{H^{k}} \|\nabla \partial_{t} \rho\|_{L^{\infty}}}{(1 - \|\partial_{x} \rho\|_{L^{\infty}})^{\frac{1}{2}}} (\int_{\Omega} (1 + \partial_{x} \rho) |\nabla^{k+1} \nabla^{\perp} \rho|^{2} dx dy)^{\frac{1}{2}} + \frac{\|\partial_{t} \rho\|_{H^{k+1}} \|\partial_{t} u\|_{L^{\infty}}}{(1 - \|\partial_{x} \rho\|_{L^{\infty}})^{\frac{1}{2}}} (\int_{\Omega} (1 + \partial_{x} \rho) |\nabla^{k+1} \nabla^{\perp} \rho|^{2} dx dy)^{\frac{1}{2}} + \frac{\|\partial_{t} \rho\|_{H^{k+1}} \|\partial_{t} \nabla \rho\|_{L^{\infty}}}{(1 - \|\partial_{x} \rho\|_{L^{\infty}})^{\frac{1}{2}}} (\int_{\Omega} (1 + \partial_{x} \rho) |\partial^{k+1} u|^{2} dx dy)^{\frac{1}{2}}.$$

From (3.25)-(3.27), we have:

(3.28)
$$I_2 \lesssim E_k^2 E_4 + \frac{E_k E_4}{(1 - \|\partial_x \rho\|_{L^{\infty}})^{\frac{1}{2}}} \Gamma_{k+1} \lesssim (E_k^2 + \Gamma_{k+1}^2) E_4.$$

For I_3 , we have

$$(3.29) \quad I_3 \lesssim E_k^2 \|\bar{\Phi}\|_{H^3} + E_k \|\bar{\Phi}\|_{H^{k+2}} E_4 + \langle \nabla^k u, (\partial_y \Phi \nabla^k \partial_{xy} \rho, -\partial_y \Phi \nabla^k \partial_x^2 \rho) \rangle - \langle \nabla^k \nabla^\perp \rho, \nabla^k \nabla^\perp u_2 \partial_y \Phi \rangle.$$

Moreover, since $\overline{\Phi}$ is independent of x, we get

$$(3.30) < \nabla^{k} u_{,} (\partial_{y} \Phi \nabla^{k} \partial_{xy} \rho, -\partial_{y} \Phi \nabla^{k} \partial_{x}^{2} \rho)^{T} > - < \nabla^{k} \nabla^{\perp} \rho, \nabla^{k} \nabla^{\perp} u_{2} \partial_{y} \Phi >$$

$$= \int_{\Omega} \nabla^{k} u_{1} \partial_{y} \bar{\Phi} \nabla^{k} \partial_{xy} \rho \, dx dy - \int_{\Omega} \nabla^{k} u_{2} \partial_{y} \bar{\Phi} \nabla^{k} \partial_{x}^{2} \rho \, dx dy$$

$$- \int_{\Omega} \nabla^{k} \partial_{y} \rho \nabla^{k} \partial_{y} u_{2} \partial_{y} \bar{\Phi} dx dy - \int_{\Omega} \nabla^{k} \partial_{x} \rho \nabla^{k} \partial_{x} u_{2} \partial_{y} \bar{\Phi} dx dy$$

$$= - \int_{\Omega} \nabla^{k} \partial_{x} u_{1} \partial_{y} \bar{\Phi} \nabla^{k} \partial_{y} \rho \, dx dy + \int_{\Omega} \nabla^{k} \partial_{x} u_{2} \partial_{y} \bar{\Phi} \nabla^{k} \partial_{x} \rho \, dx dy$$

$$- \int_{\Omega} \nabla^{k} \partial_{y} \rho \nabla^{k} \partial_{y} u_{2} \partial_{y} \bar{\Phi} dx dy - \int_{\Omega} \nabla^{k} \partial_{x} \rho \nabla^{k} \partial_{x} u_{2} \partial_{y} \bar{\Phi} dx dy = 0,$$

and then

(3.31)
$$I_3 \lesssim E_k^2 \|\bar{\Phi}\|_{H^{k+2}}.$$

Similar to (3.26) and (3.30), we also have

$$(3.32) I_5 = I_6 = 0.$$

For the term I_4 , we have

$$(3.33) I_4 \lesssim E_k^2 (E_4 + \|\bar{\Phi}\|_{H^3} + \|\Psi\|_{H^3}) + E_k E_4 (\|\bar{\Phi}\|_{H^{k+2}} + \|\Psi\|_{H^{k+2}}) + < \nabla^k \partial_t u, \partial_y \Phi \nabla^k \partial_t (\partial_{xy}\rho, -\partial_x^2 \rho)^T > - < \nabla^k \nabla^\perp \partial_t \rho, \nabla^k \nabla^\perp \partial_t u_2 \partial_y \bar{\Phi} > - < \nabla^k \nabla^\perp \partial_t \rho, \nabla^k \nabla^\perp u_2 \partial_t \partial_y \Phi > + < \nabla^k \partial_t u, \partial_t \partial_y \Phi \nabla^k (\partial_{xy}\rho, -\partial_x^2 \rho)^T > .$$

Again, by the incompressible condition and integrating by parts, similar to (3.30), we have

$$(3.34) < \nabla^{k} \partial_{t} u, \partial_{y} \Phi \nabla^{k} \partial_{t} (\partial_{xy} \rho, -\partial_{x}^{2} \rho)^{T} > - < \nabla^{k} \nabla^{\perp} \partial_{t} \rho, \nabla^{k} \nabla^{\perp} \partial_{t} u_{2} \partial_{y} \bar{\Phi} > = \int_{\Omega} \partial_{y} \bar{\Phi} \nabla^{k} \partial_{t} \partial_{xy} \rho \nabla^{k} \partial_{t} u_{1} \, dx dy - \int_{\Omega} \partial_{y} \bar{\Phi} \nabla^{k} \partial_{t} \partial_{x}^{2} \rho \nabla^{k} \partial_{t} u_{2} \, dx dy - \int_{\Omega} \nabla^{k} \partial_{ty} u_{2} \partial_{y} \bar{\Phi} \nabla^{k} \partial_{yt} \rho \, dx dy - \int_{\Omega} \nabla^{k} \partial_{tx} u_{2} \partial_{y} \bar{\Phi} \nabla^{k} \partial_{xt} \rho \, dx dy = 0.$$

Next recalling (3.4), we have

$$(3.35) \qquad < \nabla^{k} \partial_{t} u_{1}, \partial_{t} \partial_{y} \Phi \nabla^{k} \partial_{xy} \rho > - < \nabla^{k} \nabla^{\perp} \partial_{t} \rho, \nabla^{k} \nabla^{\perp} u_{2} \partial_{t} \partial_{y} \Phi > - < \nabla^{k} \partial_{t} u_{2}, \partial_{t} \partial_{y} \Phi \nabla^{k} \partial_{x}^{2} \rho) > \lesssim \frac{\|\partial_{t} u\|_{H^{k}} \|\partial_{t} \Phi\|_{H^{2}}}{(1 - \|\partial_{x} \rho\|_{L^{\infty}})^{\frac{1}{2}}} (\int_{\Omega} (1 + \partial_{x} \rho) |\nabla^{k+1} \nabla^{\perp} \rho|^{2} dx dy)^{\frac{1}{2}} + \frac{\|\partial_{t} \rho\|_{H^{k+1}} \|\partial_{t} \Phi\|_{H^{2}}}{(1 - \|\partial_{x} \rho\|_{L^{\infty}})^{\frac{1}{2}}} (\int_{\Omega} (1 + \partial_{x} \rho) |\partial^{k+1} u|^{2} dx dy)^{\frac{1}{2}}.$$

Then from (3.33)-(3.35), we get

(3.36)
$$I_4 \lesssim (E_k^2 + \Gamma_{k+1}^2)(E_4 + \|\bar{\Phi}\|_{H^{k+2}} + \|\Psi\|_{H^{k+2}}).$$

Combing (3.24), (3.28), (3.31), (3.32) with (3.36), we get

(3.37)
$$G_3 \lesssim (E_k^2 + \Gamma_{k+1}^2)(E_4 + \|\bar{\Phi}\|_{H^{k+2}} + \|\Psi\|_{H^{k+2}}).$$

PROPOSITION 3.3. Under the assumptions of Proposition 3.2, we have the following weighted estimates:

(3.38)
$$\frac{d}{dt}\Gamma_{k+1}^{2} + \kappa \int_{\Omega} (1 + \partial_{x}\rho) |\partial^{k+1}u|^{2} dx dy$$
$$\lesssim (E_{k}^{2} + \Gamma_{k+1}^{2}) (E_{4} + E_{4}^{2} + \|\bar{\Phi}\|_{H^{k+3}} + \|\Psi\|_{H^{k+3}}).$$

Proof. Due to the equation (1.7), we have

$$\begin{aligned} &(3.39)\\ &\frac{1}{2}\frac{d}{dt}\int_{\Omega}|\partial^{k+1}u|^{2}(1+\partial_{x}\rho)dxdy\\ &=\int_{\Omega}(1+\partial_{x}\rho)\partial^{k+1}u\cdot\partial^{k+1}[-(u\cdot\nabla)u-(u_{2}\partial_{y}\Psi,0)^{T}-\kappa u\\ &-\Psi\partial_{x}u-\nabla P+(\nabla^{\perp}\rho\cdot\nabla)\nabla^{\perp}\rho-(\partial_{x}\rho\partial_{y}^{2}\bar{\Phi},0)^{T}\\ &+(\partial_{y}\bar{\Phi}\partial_{xy}\rho,-\partial_{y}\bar{\Phi}\partial_{x}^{2}\rho)^{T}-(C_{0}\partial_{y}^{2}\rho,-C_{0}\partial_{xy}\rho)^{T}]dxdy+\frac{1}{2}\int_{\Omega}|\partial^{k+1}u|^{2}\partial_{y}\partial_{t}\rho\,dxdy.\end{aligned}$$

Similarly, we also have:

$$(3.40)$$

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}|\partial^{k+1}\nabla^{\perp}\rho|^{2}(1+\partial_{x}\rho)dxdy$$

$$=\int_{\Omega}(1+\partial_{x}\rho)\partial^{k+1}\nabla^{\perp}\rho\cdot\partial^{k+1}\nabla^{\perp}(-u_{2}\partial_{y}\bar{\Phi}-C_{0}u_{1}-(u\cdot\nabla)\rho-\Psi\partial_{x}\rho)dxdy$$

$$+\frac{1}{2}\int_{\Omega}|\partial^{k+1}\nabla^{\perp}\rho|^{2}\partial_{xt}\rho\,dxdy.$$

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Then (3.39)-(3.40) imply that

$$(3.41) \qquad \frac{1}{2} \frac{d}{dt} \Gamma_{k+1}^{2} + \kappa \int_{\Omega} (1 + \partial_{x} \rho) |\partial^{k+1} u|^{2} dx dy$$

$$\lesssim \int_{\Omega} (1 + \partial_{x} \rho) \partial^{k+1} u \cdot \partial^{k+1} [-(u \cdot \nabla) u - (u_{2} \partial_{y} \Psi, 0)^{T} \\ -\Psi \partial_{x} u - (\partial_{x} \rho \partial_{y}^{2} \bar{\Phi}, 0)^{T}] dx dy$$

$$+ \int_{\Omega} (1 + \partial_{x} \rho) \partial^{k+1} \nabla^{\perp} \rho \cdot \partial^{k+1} \nabla^{\perp} (-\Psi \partial_{x} \rho) dx dy$$

$$- \int_{\Omega} (1 + \partial_{x} \rho) \partial^{k+1} u \cdot \nabla \partial^{k+1} P dx dy + \frac{\|\partial_{tx} \rho\|_{L^{\infty}}}{1 - \|\partial_{x} \rho\|_{L^{\infty}}} \Gamma_{k+1}^{2} + \sum_{i=7}^{9} I_{i},$$

with

$$(3.42) I_{7} = \int_{\Omega} (1 + \partial_{x}\rho)\partial^{k+1}u \cdot \partial^{k+1}(\partial_{y}\bar{\Phi}\partial_{xy}\rho, -\partial_{y}\bar{\Phi}\partial_{x}^{2}\rho)^{T}dxdy - \int_{\Omega} (1 + \partial_{x}\rho)\partial^{k+1}\nabla^{\perp}\rho \cdot \partial^{k+1}\nabla^{\perp}(u_{2}\partial_{y}\bar{\Phi})]dxdy, (3.43) I_{8} = \int_{\Omega} (1 + \partial_{x}\rho)\partial^{k+1}u \cdot \partial^{k+1}[(\nabla^{\perp}\rho \cdot \nabla)\nabla^{\perp}\rho]dxdy - \int_{\Omega} (1 + \partial_{x}\rho)\partial^{k+1}\nabla^{\perp}\rho \cdot \partial^{k+1}\nabla^{\perp}[(u \cdot \nabla)\rho]dxdy, (3.44) I_{9} = C_{0}\int_{\Omega} (1 + \partial_{x}\rho)\partial^{k+1}u \cdot \partial^{k+1}(-\partial_{y}^{2}\rho, \partial_{xy}\rho)dxdy - C_{0}\int_{\Omega} (1 + \partial_{x}\rho)\partial^{k+1}\nabla^{\perp}\rho \cdot \partial^{k+1}\nabla^{\perp}u_{1}dxdy.$$

Now we estimate the R.H.S. of (3.41) term by term.

$$(3.45) \qquad |\int_{\Omega} (1+\partial_x \rho) \partial^{k+1} u \cdot \partial^{k+1} [(u \cdot \nabla)u] dx dy| \\ \lesssim \Gamma_{k+1} E_k E_4 + |\int_{\Omega} (1+\partial_x \rho) \partial^{k+1} u \cdot (\partial^{k+1} u \cdot \nabla) u dx dy| \\ + |\int_{\Omega} (1+\partial_x \rho) \partial^{k+1} u \cdot (u \cdot \nabla) \partial^{k+1} u dx dy| \lesssim (\Gamma_{k+1}^2 + E_k^2) E_4.$$

Similarly, we have

$$(3.46) \qquad |\int_{\Omega} (1+\partial_{x}\rho)\partial^{k+1}u_{1} \cdot \partial^{k+1}(u_{2}\partial_{y}\Psi)dxdy| \\ \lesssim |\int_{\Omega} (1+\partial_{x}\rho)\partial^{k+1}u_{1} \cdot \partial^{k+1}u_{2}\partial_{y}\Psi dxdy| \\ + |\int_{\Omega} (1+\partial_{x}\rho)\partial^{k+1}u_{1} \cdot u_{2}\partial^{k+1}\partial_{y}\Psi dxdy| + (E_{k}^{2}+\Gamma_{k+1}^{2})\|\Psi\|_{H^{3}} \\ \lesssim (E_{k}^{2}+\Gamma_{k+1}^{2})(\|\Psi\|_{H^{k+2}}+E_{4})(1+\|\partial_{x}\rho\|_{H^{2}})^{\frac{1}{2}}.$$

Integrating by parts, we have:

(3.47)
$$|\int_{\Omega} (1+\partial_x \rho) \partial^{k+1} \nabla^{\perp} \rho \cdot \Psi \partial^{k+1} \nabla^{\perp} \partial_x \rho dx dy |$$
$$= |\int_{\Omega} \partial_x^2 \rho \Psi (\partial^{k+1} \nabla^{\perp} \rho)^2 dx dy| \lesssim \frac{\|\rho\|_{H^4} \|\Psi\|_{H^2}}{1-\|\partial_x \rho\|_{L^{\infty}}} \Gamma_{k+1}^2,$$

therefore we have

$$(3.48) \qquad |\int_{\Omega} (1+\partial_{x}\rho)\partial^{k+1}\nabla^{\perp}\rho \cdot \partial^{k+1}\nabla^{\perp}(-\Psi\partial_{x}\rho)dxdy| \\ \lesssim \Gamma_{k+1}(\|\Psi\|_{H^{k+2}}\|\rho\|_{H^{4}} + \|\Psi\|_{H^{3}}\|\rho\|_{H^{k+1}} + \Gamma_{k+1}\|\Psi\|_{H^{2}}) \\ + |\int_{\Omega} (1+\partial_{x}\rho)\partial^{k+1}\nabla^{\perp}\rho \cdot \Psi\partial^{k+1}\nabla^{\perp}\partial_{x}\rho dxdy|.$$

Similarly, we have

(3.49)

$$\int_{\Omega} (1 + \partial_x \rho) \partial^{k+1} \nabla^{\perp} \rho \cdot \partial^{k+1} \nabla^{\perp} (-\Psi \partial_x \rho) dx dy$$

$$\lesssim \Gamma_{k+1} [\|\Psi\|_{H^{k+2}} \|\nabla \rho\|_{H^3} + \|\Psi\|_{H^3} \|\rho\|_{H^{k+1}} + \Gamma_{k+1} (\|\Psi\|_{H^2} + \|\rho\|_{H^4} \|\Psi\|_{H^2})],$$

 $\quad \text{and} \quad$

$$(3.50) | \int_{\Omega} (1 + \partial_{x}\rho)\partial^{k+1}u \cdot \partial^{k+1}(\Psi\partial_{x}u) - (1 + \partial_{x}\rho)\partial^{k+1}u_{1}\partial^{k+1}(\partial_{x}\rho\partial_{y}^{2}\bar{\Phi})dxdy \lesssim (\|u\|_{H^{k}}\|\Psi\|_{H^{3}} + \|\rho\|_{H^{k+1}}\|\bar{\Phi}\|_{H^{4}} + \|\bar{\Phi}\|_{H^{k+3}}\|\rho\|_{H^{3}})\Gamma_{k+1} + \int_{\Omega} (1 + \partial_{x}\rho)\partial^{k+1}u \cdot (\partial^{k+1}\Psi\partial_{x}u + \Psi\partial^{k+1}\partial_{x}u)dxdy - \int_{\Omega} (1 + \partial_{x}\rho)\partial^{k+1}u_{1}\partial^{k+1}\partial_{x}\rho\partial_{y}^{2}\bar{\Phi}dxdy \lesssim \Gamma_{k+1}(\|\Psi\|_{H^{k+1}}\|u\|_{H^{3}} + \|u\|_{H^{k}}\|\Psi\|_{H^{3}} + \Gamma_{k+1}\|\Psi\|_{H^{2}}\|\rho\|_{H^{4}}) + \Gamma_{k+1}(\Gamma_{k+1}\|\Phi\|_{H^{3}} + \|\Phi\|_{H^{k+3}}\|\nabla\rho\|_{H^{2}} + \|\rho\|_{H^{k+1}}\|\bar{\Phi}\|_{H^{4}}) \lesssim (\Gamma_{k+1}^{2} + E_{k}^{2})(E_{4} + \|\Psi\|_{H^{k+1}} + \|\bar{\Phi}\|_{H^{k+3}}).$$

By using the equation (1.7) and the incompressible condition, we get

$$(3.51) P = -\Delta^{-1} \nabla \cdot [(u \cdot \nabla)u + (u_2 \partial_y \Psi, 0)^T + \Psi \partial_x u - (\nabla^{\perp} \rho \cdot) \nabla^{\perp} \rho].$$

Then by using Lemma 2.1, we have

$$(3.52) \qquad \|\nabla P\|_{\dot{H}^{k}} \leq \|(u \cdot \nabla)u + u_{2}\partial_{y}\Psi + \Psi\partial_{x}u - (\nabla^{\perp}\rho \cdot \nabla)\nabla^{\perp}\rho\|_{\dot{H}^{k}}$$
$$\lesssim (\|u\|_{H^{k}} + \|\rho\|_{H^{k+1}})E_{4} + \|\Psi\|_{H^{k+1}}E_{k}$$
$$+ \frac{\|\nabla\rho\|_{L^{\infty}}}{\sqrt{1 - \|\partial_{x}\rho\|_{L^{\infty}}}} \|\sqrt{1 + \partial_{x}\rho}\nabla^{k+1}\nabla^{\perp}\rho\|_{L^{2}}$$
$$\lesssim E_{4}E_{k} + E_{4}\Gamma_{k+1} + \|\Psi\|_{H^{k+1}}E_{k}.$$

Thus, we get

$$(3.53) \quad |\int_{\Omega} (1+\partial_x \rho) \partial^{k+1} u \cdot \nabla^{k+1} \nabla P dx dy| = |\int_{\Omega} \nabla \partial_x \rho \partial^{k+1} u \cdot \nabla^k \nabla P dx dy|$$
$$\lesssim \|\nabla \rho\|_{H^3} \frac{1}{\sqrt{1-\|\partial_x \rho\|_{L^{\infty}}}} \|\sqrt{1+\partial_x \rho} \nabla^{k+1} u\|_{L^2} \|\nabla P\|_{H^k}$$
$$\lesssim E_4 \Gamma_{k+1} (E_k + \Gamma_{k+1}) (E_4 + \|\Phi\|_{H^{k+1}} + \|\Psi\|_{H^{k+1}}).$$

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As to the term I_7 , there holds

$$(3.54) I_{7} \lesssim \Gamma_{k+1} (\|\Phi\|_{H^{k+2}} \|\rho\|_{H^{4}} + \|\Phi\|_{H^{k+3}} \|u\|_{H^{3}} + \|u\|_{H^{k}} \|\Phi\|_{H^{4}} + \|\Phi\|_{H^{4}} \|\rho\|_{H^{k+1}}) + \Gamma_{k+1}^{2} \|\Phi\|_{H^{3}} + |\int_{\Omega} (1 + \partial_{x}\rho)\partial^{k+1}u \cdot (\partial_{y}\bar{\Phi}\partial^{k+1}\partial_{xy}\rho, -\partial_{y}\bar{\Phi}\partial^{k+1}\partial_{x}^{2}\rho)^{T} dxdy - \int_{\Omega} (1 + \partial_{x}\rho)\partial^{k+1}\nabla^{\perp}\rho \cdot (\partial^{k+1}\nabla^{\perp}u_{2}\partial_{y}\bar{\Phi})dxdy|.$$

Integrating by parts, we have

$$\begin{aligned} (3.55) \\ &\int_{\Omega} (1+\partial_x \rho) \partial^{k+1} u \cdot (\partial_y \bar{\Phi} \partial^{k+1} \partial_{xy} \rho, -\partial_y \bar{\Phi} \partial^{k+1} \partial_x^2 \rho)^T dx dy \\ &- \int_{\Omega} (1+\partial_x \rho) \partial^{k+1} \nabla^\perp \rho \cdot (\partial^{k+1} \nabla^\perp u_2 \partial_y \bar{\Phi}) dx dy \\ &= -\int_{\Omega} (1+\partial_x \rho) \partial^{k+1} \partial_x u_1 \partial_y \bar{\Phi} \partial^{k+1} \partial_y \rho dx dy - \int_{\Omega} \partial_x^2 \rho \partial^{k+1} u_1 \partial_y \bar{\Phi} \partial^{k+1} \partial_y \rho dx dy \\ &+ \int_{\Omega} (1+\partial_x \rho) \partial^{k+1} \partial_x u_2 \partial_y \bar{\Phi} \partial^{k+1} \partial_x \rho dx dy + \int_{\Omega} \partial_x^2 \rho \partial^{k+1} u_2 \partial_y \bar{\Phi} \partial^{k+1} \partial_x \rho dx dy \\ &- \int_{\Omega} (1+\partial_x \rho) \partial^{k+1} \partial_y \rho \partial^{k+1} \partial_y u_2 \partial_y \bar{\Phi} dx dy - \int_{\Omega} (1+\partial_x \rho) \partial^{k+1} \partial_x \rho \partial^{k+1} \partial_x u_2 \partial_y \bar{\Phi} dx dy \\ &= \int_{\Omega} \partial_x^2 \rho \partial^{k+1} u_2 \partial^{k+1} \partial_x \rho dx dy - \int_{\Omega} \partial_x^2 \rho \partial^{k+1} u_1 \partial^{k+1} \partial_y \rho dx dy \lesssim \frac{\|\rho\|_{H^4}}{1-\|\partial_x \rho\|_{L^{\infty}}} \Gamma_{k+1}^2, \end{aligned}$$

which implies that

(3.56)

$$I_{7} \lesssim \Gamma_{k+1}(\|\Phi\|_{H^{k+2}}\|\rho\|_{H^{4}} + \|\Phi\|_{H^{k+3}}\|u\|_{H^{3}} + \|u\|_{H^{k}}\|\Phi\|_{H^{4}} + \|\Phi\|_{H^{4}}\|\rho\|_{H^{k+1}}) + \Gamma_{k+1}^{2}(\|\Phi\|_{H^{3}} + \frac{\|\rho\|_{H^{4}}}{1 - \|\partial_{x}\rho\|_{L^{\infty}}}).$$

Similar to (3.55), we get the bound for I_9

$$\begin{aligned} (3.57) \\ I_{9} &= -C_{0} \int_{\Omega} (1+\partial_{x}\rho)\partial^{k+1}u_{1}\partial^{k+1}\partial_{y}^{2}\rho dxdy + C_{0} \int_{\Omega} (1+\partial_{x}\rho)\partial^{k+1}u_{2}\partial^{k+1}\partial_{xy}\rho dxdy \\ &-C_{0} \int_{\Omega} (1+\partial_{x}\rho)\partial^{k+1}\nabla^{\perp}\rho \cdot \partial^{k+1}\nabla^{\perp}u_{1}dxdy \\ &= C_{0} \int_{\Omega} (1+\partial_{x}\rho)\partial^{k+1}\partial_{y}u_{1}\partial^{k+1}\partial_{y}\rho dxdy + C_{0} \int_{\Omega} \partial_{xy}\rho\partial^{k+1}u_{1}\partial^{k+1}\partial_{y}\rho dxdy \\ &-C_{0} \int_{\Omega} (1+\partial_{x}\rho)\partial^{k+1}\partial_{y}u_{2}\partial^{k+1}\partial_{x}\rho dxdy - C_{0} \int_{\Omega} \partial_{xy}\rho\partial^{k+1}u_{2}\partial^{k+1}\partial_{x}\rho dxdy \\ &-C_{0} \int_{\Omega} (1+\partial_{x}\rho)\partial^{k+1}\partial_{y}\rho\partial^{k+1}\partial_{y}u_{1}dxdy - C_{0} \int_{\Omega} (1+\partial_{x}\rho)\partial^{k+1}\partial_{x}\rho\partial^{k+1}\partial_{x}u_{1}dxdy \\ &= C_{0} \int_{\Omega} \partial_{xy}\rho\partial^{k+1}u_{1}\partial^{k+1}\partial_{y}\rho dxdy - C_{0} \int_{\Omega} \partial_{xy}\rho\partial^{k+1}u_{2}\partial^{k+1}\partial_{x}\rho dxdy \\ &\lesssim \frac{\|\rho\|_{H^{4}}}{1-\|\partial_{x}\rho\|_{L^{\infty}}}\Gamma_{k+1}^{2}. \end{aligned}$$

Now, let's turn to the term I_8 :

$$(3.58) I_8 \lesssim E_4 \Gamma_{k+1}^2 + \Gamma_{k+1}(\|\rho\|_{H^{k+1}} \|\nabla\rho\|_{H^4} + \|u\|_{H^k} \|\nabla\rho\|_{H^4} + \|\rho\|_{H^{k+1}} \|u\|_{H^4}) + |\int_{\Omega} (1 + \partial_x \rho) \partial^{k+1} u \cdot [(\nabla^{\perp} \rho \cdot \nabla) \partial^{k+1} \nabla^{\perp} \rho] dx dy - \int_{\Omega} (1 + \partial_x \rho) \partial^{k+1} \nabla^{\perp} \rho \cdot (\partial^{k+1} \nabla^{\perp} u \cdot \nabla) \rho dx dy|.$$

Integrating by part, we get

$$(3.59) \qquad \int_{\Omega} (1+\partial_{x}\rho)\partial^{k+1}u \cdot [(\nabla^{\perp}\rho\cdot\nabla)\partial^{k+1}\nabla^{\perp}\rho]dxdy -\int_{\Omega} (1+\partial_{x}\rho)\partial^{k+1}\nabla^{\perp}\rho \cdot (\partial^{k+1}\nabla^{\perp}u\cdot\nabla)\rho dxdy =\int_{\Omega} (1+\partial_{x}\rho)\partial^{k+1}u_{1}(\partial_{y}\rho\partial^{k+1}\partial_{xy}\rho - \partial_{x}\rho\partial^{k+1}\partial_{y}^{2}\rho)dxdy +\int_{\Omega} (1+\partial_{x}\rho)\partial^{k+1}u_{2}(\partial_{x}\rho\partial^{k+1}\partial_{xy}\rho - \partial_{y}\rho\partial^{k+1}\partial_{x}^{2}\rho) -\int_{\Omega} (1+\partial_{x}\rho)\partial^{k+1}\partial_{y}\rho(\partial^{k+1}\partial_{y}u_{1}\partial_{x}\rho + \partial^{k+1}\partial_{y}u_{2}\partial_{y}\rho) \lesssim E_{4}^{2}\Gamma_{k+1}^{2}$$

From (3.58)-(3.59) we get the bound of $I_8 \lesssim (E_4 + E_4^2)(E_k^2 + \Gamma_{k+1}^2)$. Taking (3.39)-(3.59) as a whole, we get

(3.60)
$$\frac{d}{dt}\Gamma_{k+1}^{2} + \kappa \int_{\Omega} (1 + \partial_{x}\rho) |\partial^{k+1}u|^{2} dx dy$$
$$\lesssim (E_{k}^{2} + \Gamma_{k+1}^{2})(E_{4} + E_{4}^{2} + \|\bar{\Phi}\|_{H^{k+3}} + \|\Psi\|_{H^{k+3}}).$$

4. Decay estimates. In this section, our goal is to get the decay rate in time of the perturbation (u, ρ) .

4.1. Linearized problem. At the beginning, our system (1.6)-(1.7) is rewritten as

(4.1)
$$\begin{cases} \partial_t u_1 + \kappa u_1 + C_0 \partial_y^2 \rho = F_1 - \partial_x P, \\ \partial_t u_2 + \kappa u_2 - C_0 \partial_{xy} \rho = F_2 - \partial_y P, \\ \partial_t \rho + C_0 u_1 = F_3. \end{cases}$$

Here, F_1 , F_2 , and F_3 are

$$(4.2) \quad \begin{cases} F_1 = (\nabla^{\perp} \rho \cdot \nabla) \partial_y \rho - \partial_x \rho \partial_y^2 \bar{\Phi} + \partial_y \bar{\Phi} \partial_{xy} \rho - \Psi \partial_x u_1 - u_2 \partial_y \Psi - (u \cdot \nabla) u_1, \\ F_2 = -(\nabla^{\perp} \rho \cdot \nabla) \partial_x \rho - \partial_y \bar{\Phi} \partial_x^2 \rho - \Psi \partial_x u_2 - (u \cdot \nabla) u_2, \\ F_3 = -u_2 \partial_y \bar{\Phi} - (u \cdot \nabla) \rho - \bar{\Phi} \partial_x \rho. \end{cases}$$

Recalling the incompressible condition, P is calculated as (3.51), the linearized equation of (4.1) is:

(4.3)
$$\begin{cases} \partial_t u_1 + \kappa u_1 = -C_0 \partial_y^2 \rho, \\ \partial_t u_2 + \kappa u_2 = C_0 \partial_{xy} \rho, \\ \partial_t \rho + C_0 u_1 = 0. \end{cases}$$

ON THE EXPONENTIAL STABILITY OF A STRATIFIED FLOW TO THE 2D IDEAL MHD EQUATIONS WITH DAMPING

Decoupling the system (4.3), we get

(4.4)
$$\begin{cases} \partial_t^2 u_1 + \kappa \partial_t u_1 - C_0^2 \partial_y^2 u_1 = 0, \\ \partial_t^2 u_2 + \kappa \partial_t u_2 - C_0^2 \partial_y^2 u_2 = 0, \\ \partial_t^2 \rho + \kappa \partial_t \rho - C_0^2 \partial_y^2 \rho = 0, \end{cases}$$

with the initial data $u|_{t=0} = (u_{10}, u_{20})$ and $\rho|_{t=0} = \rho_0$. Besides, boundary conditions are $u_2|_{y=0,1} = 0$ and $\partial_y u_1|_{y=0,1} = 0$.

Denoting \mathscr{F} as the Fourier transform, and noting boundary conditions, we write

 $(4.5) \quad (\mathcal{F}_{b_q}(u_{10}), \mathcal{F}_{a_q}(u_{20}), \mathcal{F}_{b_q}(\rho_0))(\xi, q) = \mathscr{F}(u_{10}, u_{20}, \rho_0), \quad (\xi, q) \in \mathbb{R} \times (\mathbb{N} \cup \{0\}).$

From the system (4.3), we have

(4.6)
$$\mathcal{F}_{b_q}(\partial_t u_1|_{t=0})(\xi,q) = -\kappa \mathcal{F}_{b_q}(u_{10})(\xi,q) + \frac{C_0 q^2 \pi^2}{4} \mathcal{F}_{b_q}(\rho_0)(\xi,q),$$

$$\mathcal{F}_{a_q}(\partial_t u_2|_{t=0})(\xi,q) = -\kappa \mathcal{F}_{a_q}(u_{20})(\xi,q) - C_0 \frac{q\xi\pi}{2} \mathcal{F}_{b_q}(\rho_0)(\xi,q),$$

$$\mathcal{F}_{b_q}(\partial_t \rho|_{t=0})(\xi,q) = -C_0 \mathcal{F}_{b_q}(u_{10})(\xi,q).$$

By applying the Fourier transform \mathscr{F} to (4.4) and noting initial conditions (4.5)-(4.6), it is sufficient to study the following ODE

(4.7)
$$\frac{d^2}{dt^2}\widehat{\Upsilon}(\xi,q,t) + \kappa \frac{d}{dt}\widehat{\Upsilon}(\xi,q,t) + \frac{C_0^2\pi^2}{4}q^2\widehat{\Upsilon}(\xi,q,t) = 0, \quad q \in \mathbb{N} \cup \{0\}$$

 $\widehat{\Upsilon}$ stands for the Fourier transform of Υ . Write

(4.8)
$$\delta(\kappa, C_0, q) = \kappa^2 - C_0^2 q^2 \pi^2,$$

the solution of (4.7) is given by

(4.9)
$$\widehat{\Upsilon}(\xi,q,t) = \begin{cases} \frac{\widehat{\Upsilon}'(\xi,q,0) - \phi_{-} \widehat{\Upsilon}(\xi,q,0)}{\sqrt{\delta}} e^{\phi_{+}t} + \frac{\phi_{+} \widehat{\Upsilon}(\xi,q,0) - \widehat{\Upsilon}'(\xi,q,t)}{\sqrt{\delta}} e^{\phi_{-}t}, & \delta \neq 0, \\ \widehat{\Upsilon}(\xi,q,0) e^{-\frac{\kappa}{2}t} + [\frac{\kappa}{2} \widehat{\Upsilon}(\xi,q,0) + \widehat{\Upsilon}'(\xi,q,0)] t e^{-\frac{\kappa}{2}t}, & \delta = 0, \end{cases}$$

where $\widehat{\Upsilon}'$ is $\frac{d}{dt}\widehat{\Upsilon}$, and $\phi_{\pm}(q) = \frac{-\kappa \pm \sqrt{\delta}}{2}$. Therefore, solutions to (4.3) with initial data (4.5)-(4.6), we have,

(4.10)
$$\mathcal{F}_{b_q}(u_1(t))(\xi,q) = \frac{C_0(q\frac{\pi}{2})^2 \mathcal{F}_{b_q}(\rho_0) + \phi_+ \mathcal{F}_{b_q}(u_{10})}{\sqrt{\delta}} e^{\phi_+ t} - \frac{C_0(q\frac{\pi}{2})^2 \mathcal{F}_{b_q}(\rho_0) + \phi_- \mathcal{F}_{b_q}(u_{10})}{\sqrt{\delta}} e^{\phi_- t},$$

(4.11)
$$\mathcal{F}_{b_q}(\rho(t))(\xi,q)) = \frac{-C_0 \mathcal{F}_{b_q}(u_{10}) - \phi_- \mathcal{F}_{b_q}(\rho_0)}{\sqrt{\delta}} e^{\phi_+ t} + \frac{\phi_+ \mathcal{F}_{b_q}(\rho_0) + C_0 \mathcal{F}_{b_q}(u_{10})}{\sqrt{\delta}} e^{\phi_- t},$$

(4.12)
$$\mathcal{F}_{a_q}(u_2(t))(\xi,q) = \frac{\phi_+ \mathcal{F}_{a_q}(u_{20}) - C_0 \xi q \frac{\pi}{2} \mathcal{F}_{b_q}(\rho_0)}{\sqrt{\delta}} e^{\phi_+ t} + \frac{C_0 \xi q \frac{\pi}{2} \mathcal{F}_{b_q}(\rho_0) - \phi_- \mathcal{F}_{a_q}(u_{20})}{\sqrt{\delta}} e^{\phi_- t},$$

where we use the fact $-\kappa - \phi_{-}(\xi, q) = \phi_{+}$.

4.2. Linear Decay. From (4.8) and (4.10)-(4.12), we know that the linearized solution of (4.3) does not decay when $\delta = \kappa^2$, which is equivalent to the case q = 0. Due to the initial conditions

(4.13)
$$\int_0^1 u_{10}(\xi, y) dy = \int_0^1 \rho_0(\xi, y) dy = 0,$$

from (4.10), we have

(4.14)
$$\mathcal{F}_{b_q}(u_1(t))(\xi,0) = \mathcal{F}_{b_q}(u_1(t))(\xi,0)e^{-\kappa t} = 0.$$

Similarly, from (4.11) we have

(4.15)
$$\mathcal{F}_{b_q}(\rho(t))(\xi,0) = \frac{C_0}{\kappa} \mathcal{F}_{b_q}(u_{10})e^{-\kappa t} + \mathcal{F}_{b_q}(\rho_0(\xi,0) - \frac{C_0}{\kappa} \mathcal{F}_{b_q}(u_{10})$$
$$= \frac{C_0}{\kappa} \mathcal{F}_{b_q}(u_{10})(\xi,0)(e^{-\kappa t} - 1) + \mathcal{F}_{b_q}(\rho_0(\xi,0) = 0,$$

as well as

(4.16)
$$\mathcal{F}_{a_q}(u_2(t))(\xi,0) = \mathcal{F}_{a_q}(u_{20})(\xi,0)e^{-\kappa t}.$$

From (4.14)-(4.16), we conclude the exponential decay when initial conditions (4.13) satisfies for q = 0.

When q > 0, there exists a uniform constant $c_{\kappa} > 0$

$$-\frac{\kappa}{2} < \sup_{q \in \mathbb{N}} Re\phi_+ = \sup_{q \in \mathbb{N}} Re\frac{-\kappa + \sqrt{\kappa^2 - C_0^2 q^2 \pi^2}}{2} = \frac{-\kappa + \sqrt{\kappa^2 - C_0^2 \pi^2}}{2} \triangleq -c_\kappa < 0.$$

Taking (4.13)-(4.17) as whole, we have the following decay estimates for the linearized system (4.3).

LEMMA 4.1. If (ρ, u) is the solution of (4.3) with the condition (1.7) and (4.13), then we have

(4.18)
$$\begin{cases} \|\rho\|_{H^{n}(\Omega)}(t) \leq e^{-c_{\kappa}t}[\|\rho_{0}\|_{H^{n}(\Omega)} + \|u_{10}\|_{H^{n}(\Omega)}], \\ \|u_{1}\|_{H^{n}(\Omega)}(t) \leq e^{-c_{\kappa}t}[\|\rho_{0}\|_{H^{n}(\Omega)} + \|u_{10}\|_{H^{n}(\Omega)}], \\ \|u_{2}\|_{H^{n}(\Omega)}(t) \leq e^{-c_{\kappa}t}[\|\rho_{0}\|_{H^{n}(\Omega)} + \|u_{20}\|_{H^{n}(\Omega)}]. \end{cases}$$

4.3. Non-Linear Decay.. Next, by using the Duhamel's principle, we denote

(4.19)
$$(w_1, w_2, w_3)(t, \xi, q) := (\mathcal{F}_{b_q} u_1, \mathcal{F}_{a_q} u_2, \mathcal{F}_{b_q} \rho),$$

and

(4.20)
$$(Q_1, Q_2, Q_3)(t, \xi, q) := (\mathcal{F}_{b_q} F_1, \mathcal{F}_{a_q} F_2, \mathcal{F}_{b_q} F_3),$$

with F_1, F_2, F_3 defined by (4.2).

Our system (4.1) can be rewritten as:

(4.21)
$$w'(t) = A \cdot w(t) + Q(t),$$

with

(4.22)
$$w(t) = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}, A = \begin{pmatrix} -\kappa & 0 & C_0 \frac{q^2 \pi^2}{4} \\ 0 & -\kappa & -C_0 \xi q \frac{\pi}{2} \\ -C_0 & 0 & 0 \end{pmatrix}, Q(t) = \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}.$$

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The solution of system (4.4) should be

(4.23)
$$w(t) = e^{At}w(0) + \int_0^t e^{A(t-s)}Q(s)ds.$$

Moreover, we get eigenvalues of the matrix A by letting

(4.24)
$$|\lambda I - A| = \begin{vmatrix} \lambda + k & 0 & -C_0 \frac{q^2 \pi^2}{4} \\ 0 & \lambda + k & C_0 \xi q \frac{\pi}{2} \\ C_0 & 0 & \lambda \end{vmatrix} = 0,$$

from which we get the eigenvalue of A as follows

(4.25)
$$\lambda_{\pm} = \frac{-\kappa \pm \sqrt{\kappa^2 - C_0^2 q^2 \pi^2}}{2}, \qquad \lambda = -\kappa.$$

And then we take the eigenvectors as

(4.26)
$$\begin{cases} v_{+} = \left(\frac{C_{0}(q\frac{\pi}{2})^{2}}{-\lambda_{-}}, \frac{C_{0}\xi q\frac{\pi}{2}}{\lambda_{-}}, 1\right)^{T}, \\ v_{-} = \left(\frac{C_{0}(q\frac{\pi}{2})^{2}}{-\lambda_{+}}, \frac{C_{0}\xi q\frac{\pi}{2}}{\lambda_{+}}, 1\right)^{T}, \\ v = (0, 1, 0)^{T}. \end{cases}$$

As proved in the linear system part, different results will appear when δ vanish or not. We shall discuss the decay rate in the following two cases:

Case 1($\delta \neq 0$): Denote

(4.27)
$$S = (v_+, v_-, v) = \begin{pmatrix} \frac{C_0(q\frac{\pi}{2})^2}{-\lambda_-} & \frac{C_0(q\frac{\pi}{2})^2}{-\lambda_+} & 0\\ \frac{C_0\xi q\frac{\pi}{2}}{\lambda_-} & \frac{C_0\xi q\frac{\pi}{2}}{\lambda_+} & 1\\ 1 & 1 & 0 \end{pmatrix},$$

and

(4.28)
$$D = \begin{pmatrix} \lambda_+(\xi, q) & 0 & 0\\ 0 & \lambda_-(\xi, q) & 0\\ 0 & 0 & \lambda \end{pmatrix},$$

there holds $A = SDS^{-1}$, with S^{-1} given by:

(4.29)
$$S^{-1} = \begin{pmatrix} \frac{\lambda - \lambda_+}{C_0(q\frac{\pi}{2})^2(-\sqrt{\delta})} & 0 & \frac{-\lambda_-}{\sqrt{\delta}} \\ \frac{\lambda - \lambda_+}{C_0(q\frac{\pi}{2})^2(\sqrt{\delta})} & 0 & \frac{\lambda_+}{\sqrt{\delta}} \\ \frac{\xi q\frac{\pi}{2}}{(q\frac{\pi}{2})^2} & 1 & 0 \end{pmatrix}.$$

Then we have

$$(4.30) \qquad e^{At} = S \begin{pmatrix} e^{\lambda+t} & 0 & 0\\ 0 & e^{\lambda-t} & 0\\ 0 & 0 & e^{\lambda t} \end{pmatrix} S^{-1} \\ = \begin{pmatrix} e^{\lambda+t} \frac{\lambda+}{\sqrt{\delta}} - e^{\lambda-t} \frac{\lambda-}{\sqrt{\delta}} & 0 & \frac{C_0(q\frac{\pi}{2})^2}{\sqrt{\delta}} (e^{\lambda+t} - e^{\lambda-t}) \\ e^{\lambda+t} \frac{-\xi\lambda+}{q\frac{\pi}{2}\sqrt{\delta}} + e^{\lambda-t} \frac{\xi\lambda-}{q\frac{\pi}{2}\sqrt{\delta}} + e^{\lambda t} \frac{\xi}{q\frac{\pi}{2}} & e^{\lambda t} & -\frac{C_0\xi q\frac{\pi}{2}}{\sqrt{\delta}} (e^{\lambda+t} - e^{\lambda-t}) \\ \frac{\lambda-\lambda+}{C_0(q\frac{\pi}{2})^2(-\sqrt{\delta})} (e^{\lambda+t} - e^{\lambda-t}) & 0 & \frac{-\lambda-}{\sqrt{\delta}} e^{\lambda+} + \frac{\lambda+}{\sqrt{\delta}} e^{\lambda-} \end{pmatrix},$$

which implies the following decay estimates (4.31)

$$\begin{split} \mathcal{F}_{b_{q}}(u_{1}(t)) &\approx e^{\lambda_{+}t} (\frac{\lambda_{+}}{\sqrt{\delta}} \mathcal{F}_{b_{q}}(u_{10}) + \frac{C_{0}(q\frac{\pi}{2})^{2}}{\sqrt{\delta}} \mathcal{F}_{b_{q}}(\rho_{0})) \\ &+ \frac{\lambda_{+}}{\sqrt{\delta}} \int_{0}^{t} e^{\lambda_{+}(t-s)} \mathcal{F}_{b_{q}}[F_{1}(s)] ds + \frac{C_{0}(q\frac{\pi}{2})^{2}}{\sqrt{\delta}} \int_{0}^{t} e^{\lambda_{+}(t-s)} \mathcal{F}_{a_{q}}[F_{3}(s)] ds, \\ \mathcal{F}_{a_{q}}(u_{2}(t)) &\approx e^{\lambda_{+}t} (\frac{-\xi}{q\frac{\pi}{2}} \frac{\lambda_{+}}{\sqrt{\delta}} \mathcal{F}_{b_{q}}(u_{10}) - \frac{C_{0}\xi q\frac{\pi}{2}}{\sqrt{\delta}} \mathcal{F}_{b_{q}}(\rho_{0})) \\ &+ \frac{-\xi}{q\frac{\pi}{2}} \frac{\lambda_{+}}{\sqrt{\delta}} \int_{0}^{t} e^{\lambda_{+}(t-s)} \mathcal{F}_{b_{q}}[F_{1}(s)] ds - \frac{C_{0}\xi q\frac{\pi}{2}}{\sqrt{\delta}} \int_{0}^{t} e^{\lambda_{+}(t-s)} \mathcal{F}_{b_{q}}[F_{3}(s)] ds, \\ \mathcal{F}_{a_{q}}(\rho(t)) &\approx e^{\lambda_{+}t} (\frac{\lambda_{-}\lambda_{+}}{C_{0}(q\frac{\pi}{2})^{2}(-\sqrt{\delta})} \mathcal{F}_{b_{q}}(u_{10}) - \frac{\lambda_{-}}{\sqrt{\delta}} \mathcal{F}_{a_{q}}(u_{20})) \\ &- \frac{\lambda_{-}\lambda_{+}}{C_{0}(q\frac{\pi}{2})^{2}(\sqrt{\delta})} \int_{0}^{t} e^{\lambda_{+}(t-s)} \mathcal{F}_{b_{q}}[F_{1}(s)] ds - \frac{\lambda_{-}}{\sqrt{\delta}} \int_{0}^{t} e^{\lambda_{+}(t-s)} \mathcal{F}_{a_{q}}[F_{3}(s)] ds. \end{split}$$

Case 2 ($\delta = 0$). In the case, $\lambda_{+} = \lambda_{-} = -\frac{\kappa}{2}$, taking

(4.32)
$$S_1 = \begin{pmatrix} \frac{2C_0(q\frac{\pi}{2})^2}{\kappa^2} & \frac{(2\kappa-4)C_0(q\frac{\pi}{2})^2}{\kappa^2} & 0\\ -\frac{2C_0^{\kappa}\xi q\frac{\pi}{2}}{\kappa} & \frac{(4-2\kappa)C_0\xi q\frac{\pi}{2}}{\kappa^2} & 1\\ 1 & 1 & 0 \end{pmatrix},$$

and

(4.33)
$$S_1^{-1} = \begin{pmatrix} \frac{\kappa^2}{4C_0(q\frac{\pi}{2})^2} & 0 & 1 - \frac{\kappa}{2} \\ -\frac{\kappa^2}{4C_0(q\frac{\pi}{2})^2} & 0 & \frac{\kappa}{2} \\ \frac{\xi q\frac{\pi}{2}}{(q\frac{\pi}{2})^2} & 1 & 0 \end{pmatrix},$$

then we get $A = S_1 J S_1^{-1}$ with the Jordan matrix J as

(4.34)
$$J = \begin{pmatrix} -\frac{\kappa}{2} & 1 & 0\\ 0 & -\frac{\kappa}{2} & 0\\ 0 & 0 & -\kappa \end{pmatrix}.$$

Now we write

(4.35)
$$J = \begin{pmatrix} -\frac{\kappa}{2} & 0 & 0\\ 0 & -\frac{\kappa}{2} & 0\\ 0 & 0 & -\kappa \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{pmatrix} = D + N,$$

hence, we get

$$(4.36) e^{At} = e^{-\frac{\kappa}{2}t} \begin{pmatrix} 1 - \frac{\kappa}{2}t & 0 & C_0(q\frac{\pi}{2})^2t \\ \frac{-\kappa\xi}{2q\frac{\pi}{2}}t - \frac{\xi}{q\frac{\pi}{2}}(1 - e^{-\frac{\kappa}{2}t}) & e^{-\frac{\kappa}{2}t} & -C_0\xi q\frac{\pi}{2}t \\ -\frac{\kappa^2}{4C_0(q\frac{\pi}{2})^2}t & 0 & \frac{\kappa}{2}t + 1 \end{pmatrix}.$$

From (4.31) and (4.36), we have:

$$\mathcal{F}_{b_{q}}(u_{1}(t)) \lesssim e^{-c_{k}t} \left[\frac{\lambda_{+}}{\sqrt{\delta}} \mathcal{F}_{b_{q}}(u_{10}) + \frac{C_{0}(q\frac{\pi}{2})^{2}}{\sqrt{\delta}} \mathcal{F}_{b_{q}}(\rho_{0})\right] \\ + \int_{0}^{t} e^{-c_{k}(t-s)} \mathcal{F}_{b_{q}}[F_{1}(s)] ds + \int_{0}^{t} e^{-c_{k}(t-s)} \mathcal{F}_{a_{q}}[F_{3}(s)] ds, \\ \mathcal{F}_{a_{q}}(u_{2}(t)) \lesssim e^{-c_{k}t} \left[\frac{-\xi}{q\frac{\pi}{2}} \frac{\lambda_{+}}{\sqrt{\delta}} \mathcal{F}_{b_{q}}(u_{10}) - \frac{C_{0}\xi q\frac{\pi}{2}}{\sqrt{\delta}} \mathcal{F}_{b_{q}}(\rho_{0})\right] \\ + \int_{0}^{t} e^{-c_{k}(t-s)} \mathcal{F}_{b_{q}}[F_{1}(s)] ds - \int_{0}^{t} e^{-c_{k}(t-s)} \mathcal{F}_{b_{q}}[F_{3}(s)] ds, \\ \mathcal{F}_{a_{q}}(\rho(t)) \lesssim e^{-c_{k}t} \left[\frac{\lambda_{-}\lambda_{+}}{C_{0}(q\frac{\pi}{2})^{2}(-\sqrt{\delta})} \mathcal{F}_{b_{q}}(u_{10}) - \frac{\lambda_{-}}{\sqrt{\delta}} \mathcal{F}_{a_{q}}(u_{20})\right] \\ + \int_{0}^{t} e^{-c_{k}(t-s)} \mathcal{F}_{b_{q}}[F_{1}(s)] ds - \int_{0}^{t} e^{-c_{k}(t-s)} \mathcal{F}_{a_{q}}[F_{3}(s)] ds.$$

Furthermore, similar to the process of (4.31)-(4.36), we also have (4.38)

$$\begin{split} \mathcal{F}_{b_{q}}(\partial_{t}u_{1}(t)) &\lesssim e^{-c_{k}t} [\frac{\lambda_{+}}{\sqrt{\delta}} \mathcal{F}_{b_{q}}(u_{10}) + \frac{C_{0}(q\frac{\pi}{2})^{2}}{\sqrt{\delta}} \mathcal{F}_{b_{q}}(\rho_{0})] + \mathcal{F}_{b_{q}}[F_{1}(t)] \\ &+ \int_{0}^{t} e^{-c_{k}(t-s)} \mathcal{F}_{b_{q}}[F_{1}(s)] ds + \int_{0}^{t} e^{-c_{k}(t-s)} \mathcal{F}_{a_{q}}[F_{3}(s)] ds + \mathcal{F}_{a_{q}}[F_{3}(t)], \\ \mathcal{F}_{a_{q}}(\partial_{t}u_{2}(t)) &\lesssim e^{-c_{k}t} [\frac{-\xi}{(q\frac{\pi}{2})} \frac{\lambda_{+}}{\sqrt{\delta}} \mathcal{F}_{b_{q}}(u_{10}) - \frac{C_{0}\xi q\frac{\pi}{2}}{\sqrt{\delta}} \mathcal{F}_{b_{q}}(\rho_{0})] + \mathcal{F}_{b_{q}}[F_{1}(t)] \\ &+ \int_{0}^{t} e^{-c_{k}(t-s)} \mathcal{F}_{b_{q}}[F_{1}(s)] ds - \int_{0}^{t} e^{-c_{k}(t-s)} \mathcal{F}_{b_{q}}[F_{3}(s)] ds - \mathcal{F}_{a_{q}}[F_{3}(t)], \\ \mathcal{F}_{a_{q}}(\partial_{t}\rho(t)) &\lesssim e^{-c_{k}t} [\frac{\lambda_{-}\lambda_{+}}{C_{0}(q\frac{\pi}{2})^{2}(-\sqrt{\delta})} \mathcal{F}_{b_{q}}(u_{10}) - \frac{\lambda_{-}}{\sqrt{\delta}} \mathcal{F}_{a_{q}}(u_{20})] + \mathcal{F}_{b_{q}}[F_{1}(t)] \\ &+ \int_{0}^{t} e^{-c_{k}(t-s)} \mathcal{F}_{b_{q}}[F_{1}(s)] ds - \int_{0}^{t} e^{-c_{k}(t-s)} \mathcal{F}_{a_{q}}[F_{3}(s)] ds - \mathcal{F}_{a_{q}}[F_{3}(t)]. \end{split}$$

5. The proof of the main results. Recalling the Proposition 3.2 and Proposition 3.3, it is sufficient to get the bound of $\int_0^T E_4 dt$. We shall prove this bound by a bootstrap procedure. For simplicity, when $k \ge 7$, we denote

(5.1)
$$\|\bar{\Phi}_0\|_{H^{k+3}} + \|\Psi_0\|_{H^{k+3}} \triangleq M_0,$$

we have the following decay estimates:

LEMMA 5.1. Assuming that $E_k^2 + \Gamma_{k+1}^2 \lesssim \varepsilon_0^2$, for $t \in [0,T]$ and $k \ge 7$, then we have

$$(5.2) E_4 \le 4M_0 e^{-\beta t},$$

with $\beta = \min\{\alpha, \frac{c_{\kappa}}{2}\}.$

Proof. From (4.21), (4.37) and (4.38), by using the Plancherel's theorem, we have

(5.3)
$$E_4(t) \lesssim e^{-c_k t} E_4(0) + \int_0^t e^{-c_k (t-s)} \|F(\cdot,s)\|_{H^5} ds + \|F(\cdot,t)\|_{H^5}.$$

Noting that $k \geq 7$ and $E_k^2 + \Gamma_{k+1}^2 \lesssim \varepsilon_0^2$, we have

(5.4)
$$\|F(\cdot,t)\|_{H^5} \lesssim E_6 E_4 + E_6 (\|\bar{\Phi}\|_{H^7} + \|\Psi\|_{H^6}).$$

By using Proposition 2.2, there exists a uniform constant C_1 , such that

(5.5)
$$E_{4}(t) \leq C_{1}e^{-c_{k}t}\epsilon_{0}^{2} + C_{1}\left(E_{6}E_{4} + E_{6}(\|\bar{\Phi}\|_{H^{7}} + \|\Psi\|_{H^{6}})\right) + C_{1}\int_{0}^{t}e^{-c_{k}(t-s)}(E_{4}E_{k} + E_{k}(\|\bar{\Phi}\|_{H^{7}} + \|\Psi\|_{H^{6}}))ds \leq C_{1}\left(e^{-c_{k}t}\epsilon_{0}^{2} + \varepsilon_{0}\left(E_{4} + M_{0}e^{-\alpha t}\right) + \int_{0}^{t}\varepsilon_{0}e^{-c_{k}(t-s)}(E_{4} + M_{0}e^{-\alpha s})ds\right).$$

Noting $\beta = \min\{\alpha, \frac{c_{\kappa}}{2}\}$, by using the continuity procedure, we now claim that

(5.6)
$$E_4(t) \le 4M_0 e^{-\beta t}.$$

Actually, from (5.5), we have

(5.7)
$$\int_0^t \varepsilon_0 e^{-c_\kappa (t-s)} (E_4 + M_0 e^{-\alpha s}) ds \leq M_0 \varepsilon_0 e^{-c_\kappa t} \int_0^t e^{c_\kappa s} (4e^{-\beta s} + e^{-\alpha s}) ds$$
$$\lesssim M_0 \varepsilon_0 e^{-c_\kappa t} (e^{(c_\kappa - \beta)t} + e^{(c_\kappa - \alpha)t}) + M_0 \varepsilon_0 e^{-c_\kappa t} \leq C_2 M_0 \varepsilon_0 e^{-\beta t},$$

with C_2 be a uniform constant.

Then from (5.5) and (5.7), we get

(5.8)
$$E_4(t) \le e^{-\beta t} \varepsilon_0 \left(C_1 \varepsilon_0 + 5C_1 M_0 + C_1 C_2 M_0 \right)$$

Noting that ε_0 is small enough, by taking $\varepsilon_0 (C_1 \varepsilon_0 + 5C_1 M_0 + C_1 C_2 M_0) < 2M_0$, which proves our claim (5.6).

Proof of our main theorem.

From the Proposition 2.2, Proposition 3.2 and Proposition 3.3, by using the Gronwall's inequality and Lemma 5.1, we have

(5.9)
$$E_k^2(t) + \Gamma_{k+1}^2(t)$$

 $\lesssim (E_k(0)^2 + \Gamma_{k+1}^2(0)) e^{\int_0^t E_4 + \|\bar{\Phi}\|_{H^{k+3}} + \|\Psi\|_{H^{k+2}} + \|\partial_t \Psi\|_{H^{k+2}} ds} \lesssim \epsilon_0^2 e^{(\frac{4}{\beta} + \frac{1}{\alpha})M_0}$

Thus by (5.9), we have

(5.10)
$$\|u\|_{H^k}^2 + \|\partial_t u\|_{H^k}^2 + \kappa \int_0^T \|u\|_{H^k}^2 + \|\partial_t u\|_{H^k}^2 ds \lesssim \varepsilon_0^2$$

from which we complete our proofs.

Acknowledgement. Yi Du and Wang Yang were supported by NSFC (grant No. 11471126; 11971199). Yi Zhou was supported by Key Laboratory of Mathematics for Nonlinear Sciences (Fudan University), Ministry of Education of China, P.R.China. Shanghai Key Laboratory for Contemporary Applied Mathematics, School of Mathematical Sciences, Fudan University, P.R. China, NSFC (grants No. 11421061), 973 program (grant No. 2013CB834100) and 111 project.

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