Pattern formation for a local/nonlocal interaction functional arising in colloidal systems

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Abstract

In this paper we study pattern formation for a physical local/nonlocal interaction functional where the local attractive term is given by the 1-perimeter and the nonlocal repulsive term is the Yukawa (or screened Coulomb) potential. This model is physically interesting as it is the Γ-limit of a double Yukawa model used to explain and simulate pattern formation in colloidal systems [\[2,](#page-30-0) [9,](#page-30-1) [23,](#page-31-0) [19\]](#page-31-1). Following a strategy introduced in [\[11\]](#page-30-2) we prove that in a suitable regime minimizers are periodic stripes, in any space dimension.

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1 Introduction

The ability of matter to arrange itself in periodic structures is often referred to as spontaneous pattern formation. This phenomenon is of fundamental importance in Science, Technology, Engineering and Mathematics and it is often caused by the interaction between local attractive and nonlocal repulsive forces.

Instances of spontaneous pattern formation at a mesoscopic scale are that showed by certain suspensions of charged colloids, polymers and also by protein solutions, when the attractive and repulsive forces compete at some strength ratio. The phenomenon is since at least two decades object of experimental and computational investigation (see among the various papers on the subject [\[30,](#page-32-0) [33,](#page-32-1) [2,](#page-30-0) [3,](#page-30-3) [9,](#page-30-1) [19,](#page-31-1) [23\]](#page-31-0)). In particular, one can observe gathering of the particles in lamellas (stripes) or bubbles (clusters) according to the different regimes between the two mutual interactions. These self-assembly processes play a crucial role in applications such as the production of photonic crystals, the possibility to control the formation of clusters in various diseases, nanolithography or gelation processes.

For colloidal systems, the long-range repulsive forces have been shown on theoretical grounds to be represented by a Yukawa (or screened Coulomb) potential [\[12,](#page-30-4) [34\]](#page-32-2) (the so-called DLVO Theory).

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The kernel of the Yukawa potential is the following

$$
K_{\mu}(\zeta) := \frac{e^{-\mu|\zeta|}}{|\zeta|^{d-2}}, \quad \mu > 0
$$
\n(1.1)

if $d \geq 3$ and $K_{\mu}(\zeta) := -e^{-\mu|\zeta|} \ln(|\zeta|)$ if $d = 2$.

The Yukawa potential was introduced in the 30s by Yukawa in particle physics [\[35\]](#page-32-3). Other than for electrolytes and colloids, it is used also in plasma physics, where it represents the potential of a charged particle in a weakly nonideal plasma, in solid state physics, where it describes the effects of a charged particle in a sea of conduction electrons, and in quantum mechanics.

Pattern formation in models for colloid particles involving the Yukawa potential as repulsive term has been numerically studied in several papers (see e.g. [\[2,](#page-30-0) [3,](#page-30-3) [9,](#page-30-1) [19,](#page-31-1) [23\]](#page-31-0)) and lamellar (striped) phases has been observed in suitable regimes.

In particular, some of these models (see e.g. [\[2,](#page-30-0) [9,](#page-30-1) [19,](#page-31-1) [23\]](#page-31-0)) use as short range attractive term the Yukawa potential with opposite sign and parameter μ much larger than the one appearing in the repulsive Yukawa. A model of this kind is the following: for $d \geq 1$, $\beta > 1$, $L > 0$, $E \subset \mathbb{R}^d$ $[0, L)^d$ -periodic and $J > 0$, consider

$$
\tilde{\mathcal{E}}_{\beta,J,L}(E) := \frac{1}{L^d} \Big(J C_{\beta,L} \int_{[0,L)^d} \int_{\mathbb{R}^d} |\chi_E(x+\zeta) - \chi_E(x)| K_{\beta}(\zeta) d\zeta dx \n- \int_{[0,L)^d} \int_{\mathbb{R}^d} |\chi_E(x+\zeta) - \chi_E(x)| K_1(\zeta) d\zeta dx \Big),
$$
\n(1.2)

where $C_{\beta,L}$ is a positive normalization constant depending on β and L.

In Section [6](#page-24-0) we show that, for a natural choice of the constant $C_{\beta,L}$ (see [\(6.2\)](#page-24-1)) and substituting the Euclidean norm $|\cdot|$ with the 1-norm $|z|_1 = \sum_{i=1}^d |z_i|$ in [\(1.1\)](#page-1-0), the functionals $\tilde{\mathcal{E}}_{\beta,J,L}$ Γ-converge as $\beta \to +\infty$ to the following functional

$$
\tilde{\mathcal{F}}_{J,L}(E) := \frac{1}{L^d} \Big(J \text{Per}_1(E, [0, L)^d) - \int_{[0, L)^d} \int_{\mathbb{R}^d} |\chi_E(x + \zeta) - \chi_E(x)| K_1(\zeta) d\zeta dx \Big), \tag{1.3}
$$

where

$$
\operatorname{Per}_1(E, [0, L)^d) := \int_{\partial E \cap [0, L)^d} |\nu^E(x)|_1 d\mathcal{H}^{d-1}(x), \quad |z|_1 = \sum_{i=1}^d |z_i|,
$$

with $\nu^{E}(x)$ exterior normal to E in x, is the 1-perimeter of E, and K_1 is from now on the Yukawatype potential

$$
K_1(\zeta) := \frac{e^{-|\zeta|_1}}{|\zeta|_1^{d-2}},\tag{1.4}
$$

namely the potential obtained substituting the Euclidean norm in [\(1.1\)](#page-1-0) with the 1-norm. Notice that, since we assume periodicity of the sets w.r.t. $[0,L)^d$, the choice of the norm does not reduce the underlying symmetries of the problem, namely those w.r.t. permutation of coordinates.

A functional of this kind (namely with the perimeter as attractive term and the Yukawa as repulsive term), but in a different regime, appears also in a series of papers [\[27,](#page-31-2) [20,](#page-31-3) [21\]](#page-31-4) in connection with the sharp interface limit of the Ohta-Kawasaki model for small volume fractions.

The potential [\(1.4\)](#page-1-1) behaves, for short range interactions, like the Coulomb potential and, for long range interactions, like a strong decaying potential, analogously to the generalized antiferromagnetic potentials considered in [\[11,](#page-30-2) [22,](#page-31-5) [18\]](#page-31-6).

Our aim in this paper is to characterize minimizers of (1.3) for a suitable range of J .

Evolutionary problems for attractive-repulsive models have been studied by various authors (see e.g. [\[6,](#page-30-5) [7,](#page-30-6) [8,](#page-30-7) [5\]](#page-30-8)), and in [\[8,](#page-30-7) [5\]](#page-30-8) exponentially decaying kernels have been considered.

The main difficulty in showing pattern formation lies in the fact that the functional exhibits a larger group of symmetries than the expected minimizers.

For one-dimensional models, where the symmetry breaking does not occur, pattern formation has been proved among others in [\[28,](#page-31-7) [16\]](#page-31-8), using either convexity methods or reflection positivity techniques.

In more space dimensions, in the discrete setting, pattern formation was shown in [\[18\]](#page-31-6) for the kernel $\tilde{K}(\zeta) = \frac{1}{(|\zeta|_1 + 1)^p}$ and $p > 2d$.

In the continuous setting, pattern formation was shown for the first time for a functional which is invariant under permutation of coordinates in [\[11\]](#page-30-2). There the kernel \tilde{K} can have any exponent $p \geq d+2$.

Both in the discrete and continuous case, one can show that there is a critical constant J_c such that for every $J > J_c$ the minimizers are trivial, namely the minimizers are either the empty set or the whole \mathbb{R}^d . In both [\[18,](#page-31-6) [11\]](#page-30-2), the authors show that for J close enough to J_c the minimizers are a union of periodic stripes. For a comparision of the two papers see [\[11\]](#page-30-2). From now on we will concentrate with the continous case.

In order to be more precise, let us recall that in the continuous setting a union of stripes is a $[0, L)^d$ periodic set which is, up to Lebesgue null sets, of the form $V_i^{\perp} + \hat{E}e_i$ for some $i \in \{1, \ldots, d\}$, where V_i^{\perp} is the $(d-1)$ -dimensional subspace orthogonal to e_i and $\hat{E} \subset \mathbb{R}$ with $\hat{E} \cap [0,\hat{L}) = \cup_{k=1}^N (s_i,t_i)$. A union of stripes is periodic if $\exists h > 0$, $\nu \in \mathbb{R}$ s.t. $\hat{E} \cap [0, L) = \cup_{k=0}^{N} (2kh + \nu, (2k + 1)h + \nu)$. In this paper we prove an analogous result for the functional [\(1.3\)](#page-1-2).

While for the power-like potential K the physical exponents $p = d + 1$ (thin magnetic films), $p = d$ (3D-micromagnetics) and $p = d - 2$ (Coulomb potential) remain excluded by the results in [\[18,](#page-31-6) [11\]](#page-30-2), here we are able to prove pattern formation for a physical model.

Let us now state our results precisely. First of all, there exists also in this case a critical constant \tilde{J}_{∞} such that if $J > \tilde{J}_{\infty}$ then the only minimizers are \mathbb{R}^d and the empty set. Such a constant is given by

$$
\tilde{J}_{\infty} := \int_{\mathbb{R}^d} |\zeta_1| K_1(\zeta) \, \mathrm{d}\zeta.
$$

What one expects is that for values of J strictly below \tilde{J}_{∞} minimizers are periodic unions of stripes of optimal period.

Therefore one sets

$$
\tilde{J}_M = \int_{\mathbb{R}^{d-1}} \int_{-M}^M |\zeta_1| K_1(\zeta) d\zeta
$$

and considers the functional

$$
\tilde{\mathcal{F}}_{\tilde{J}_M,L}(E) = \frac{1}{L^d} \Big(\tilde{J}_M \text{Per}_1(E, [0, L)^d) - \int_{[0, L)^d} \int_{\mathbb{R}^d} |\chi_E(x + \zeta) - \chi_E(x)| K_1(\zeta) d\zeta dx \Big). \tag{1.5}
$$

One can see that minimizing [\(1.5\)](#page-2-0) in the class of periodic unions of stripes, for $1 \ll 2M \leq L$, those with optimal energy have width and distance of order $h_M \leq M$, $\frac{h_M}{M} \to 1$ as $M \to +\infty$ and energy of order $e_M^* \ge -e^{-\alpha_M M}$, with $\alpha_M \le 1$, $\alpha_M \to 1$ as $M \to +\infty$.

Therefore it is natural to rescale the spacial variables and the functional so that the optimal width and distance for unions of stripes is $O(1)$ and the energy is $O(1)$.

Setting $M\zeta' = \zeta$, $Mx' = x$, and $\tilde{\mathcal{F}}_{\tilde{J}_M,L}(E) = -e_M^* \mathcal{F}_{M,L/M}(E/M)$ one ends up considering the rescaled functional

$$
\mathcal{F}_{M,L}(E) = \frac{M^2}{L^d} \Big(J_M \text{Per}_1(E, [0, L)^d) - \int_{[0, L)^d} \int_{\mathbb{R}^d} |\chi_E(x + \zeta) - \chi_E(x)| \bar{K}_M(\zeta) \, dx \, d\zeta \Big) \tag{1.6}
$$

where

$$
\bar{K}_M(\zeta) = \frac{-1}{e_M^*|\zeta|^{d-2}} e^{-M|\zeta|}.
$$

and

$$
J_M = \int_{\mathbb{R}^{d-1}} \int_{-1}^1 \bar{K}_M(\zeta) |\zeta_1| \,\mathrm{d}\zeta.
$$

For fixed $M > 0$, consider first for all $L > 0$ the minimal value obtained by $\mathcal{F}_{M,L}$ on $[0, L)^d$ -periodic unions of stripes and then the minimal among these values as L varies in $(0, +\infty)$. We will denote this value by e_M^* .

By the reflection positivity technique (see Section [3\)](#page-12-0), this value is attained on periodic stripes of width and distance $h_M^* > 0$, which is unique provided M is large enough (see Theorem [3.1\)](#page-12-1). Our main theorem is the following:

Theorem 1.1. There exists a constant M_0 such that for every $M > M_0$ and $L = 2kh_M^*$ for some $k \in \mathbb{N}$, then the minimizer of $\mathcal{F}_{M,L}$ are optimal stripes of width and distance h_M^* .

In Theorem [1.1](#page-3-0) notice that M_0 is independent of L.

Notice that the $[0, L)^d$ -periodic boundary conditions were imposed in order to give sense to the functional which is otherwise not well-defined. If one is interested to show that optimal periodic stripes of width and distance h_M^* are "optimal" if one varies also the periodicity, then it is not difficult to see that Theorem 1.3 is sufficient. This process is similar to the "thermodynamic limit" which is of particular relevance in physics.

By adapting some of the arguments in [\[11\]](#page-30-2), one can provide a characterization of minimizers of $\mathcal{F}_{M,L}$ also for arbitrary L, but this time with M larger than a constant depending on L. Namely, one has the following

Theorem 1.2. Let $L > 0$. Then there exists $\overline{M} > 0$ such that $\forall M \geq \overline{M}$ there exists $h_{M,L}$ such that the minimizers of $\mathcal{F}_{M,L}$ are periodic stripes of width and distance $h_{M,L}$.

According to the next theorem, when L is large then $h_{M,L}$ is close to h_M^* .

Theorem 1.3. There exists $C > 0$ and $\hat{M} > 0$ such that for every $M > \hat{M}$ the width and distance $h_{M,L}$ of minimizers of $\mathcal{F}_{M,L}$ satisfies

$$
\left| h_{M,L} - h_M^* \right| \leq \frac{C}{L}.
$$

As discussed in [\[2,](#page-30-0) [9,](#page-30-1) [23,](#page-31-0) [19\]](#page-31-1), one of the possible models used to show gelification in charged colloids and pattern formation is to consider both as attractive and as repulsive term the Yukawa potential, with different signs and appropriate rescaling (see (1.2)).

Therefore, the following Γ-convergence result connects our analytical results with the ones obtained in the above cited experiments and simulations for the functional [\(1.2\)](#page-1-3).

Theorem 1.4. The functionals $\tilde{\mathcal{E}}_{\beta,J,L}$ defined in [\(1.2\)](#page-1-3) Γ -converge in the L^1 topology as $\beta \to +\infty$ and up to subsequences to the functional $\tilde{F}_{J,L}$ defined in [\(1.3\)](#page-1-2).

The main theorems in this paper are Theorem [1.1](#page-3-0) and Theorem [1.4.](#page-3-1) The proofs of Theorem [1.2](#page-3-2) and Theorem [1.3](#page-3-3) can be obtained by using the new estimates present in this paper and the strategy contained in [\[22,](#page-31-5) [11\]](#page-30-2). For this reason in this paper we will focus on the proof of Theorem [1.1](#page-3-0) and Theorem [1.4.](#page-3-1)

The general strategy of the proof is similar to the one introduced in [\[11\]](#page-30-2) and consists in the following steps: decomposition of the functional in terms which penalize deviations from being a stripe([\[22,](#page-31-5) [11\]](#page-30-2)); decomposition of \mathbb{R}^d in different regions according to how much in a region the set E"resembles" a stripe ([\[18,](#page-31-6) [11\]](#page-30-2)); rigidity estimate to prove that in the limit $M \to +\infty$ minimizers approacha striped structure $(22, 11)$; stability estimates to prove that once close to a stripe the mostconvenient thing is to be flat $([11])$ $([11])$ $([11])$; use of the reflection positivity technique.

However, the rigidity estimate (see Lemma [4.1\)](#page-14-0) and the stability lemma (see Lemma [5.2\)](#page-21-0) base on the specific properties of the Yukawa kernel and are therefore different. Finally, in Section [6](#page-24-0) we prove the Theorem [1.4.](#page-3-1)

1.1 Structure of the paper

This paper is organized as follows: in Section [2,](#page-4-0) after setting the notation, we estimate the energy and width of optimal stripes and we rescale the functional accordingly. Then, we identify suitable quantities that penalize deviations from being a union of stripes. In Section [3](#page-12-0) we show that in the interesting regime the width of minimizers is uniquely determined and minimizers are periodic. Section [4](#page-14-1) contains the main rigidity estimate and Section [5](#page-21-1) the proof of Theorem [1.1.](#page-3-0) In Section [6](#page-24-0) we prove the Γ-convergence of the double Yukawa functional [\(1.2\)](#page-1-3) to [\(1.3\)](#page-1-2) as $\beta \to +\infty$.

2 Setting and preliminary results

In this section, we set the notation and we introduce some preliminary results in the spirit of those given in [\[22\]](#page-31-5) and [\[11\]](#page-30-2), which will be necessary to carry on our analysis.

2.1 Notation and preliminary definitions

In the following, we let $\mathbb{N} = \{1, 2, \dots\}$, $d \geq 1$. On \mathbb{R}^d , we let $\langle \cdot, \cdot \rangle$ be the Euclidean scalar product and $|\cdot|$ be the Euclidean norm. We let (e_1, \ldots, e_d) be the canonical basis in \mathbb{R}^d and for $y \in \mathbb{R}^d$ we let $y_i = \langle y, e_i \rangle e_i$ and $y_i^{\perp} := y - y_i$. For $y \in \mathbb{R}^d$, let $|y|_1 = \sum_{i=1}^d |y_i|$ be its 1-norm and $|y|_{\infty} = \max_i |y_i|$ its ∞-norm.

Given a measurable set $A \subset \mathbb{R}^d$, let us denote by $\mathcal{H}^{d-1}(A)$ its $(d-1)$ -dimensional Hausdorff measure and |A| its Lebesgue measure. Moreover, let $\chi_A : \mathbb{R}^d \to \mathbb{R}$ the function defined by

$$
\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in \mathbb{R}^d \setminus A \end{cases}
$$
 (2.1)

and by $\mathbb{1}_{A}^{\infty} : \mathbb{R}^{d} \to \mathbb{R} \cup \{+\infty\}$ the function

$$
\mathbb{1}_A^{\infty}(x) = \begin{cases} +\infty & \text{if } x \in A \\ 0 & \text{if } x \in \mathbb{R}^d \setminus A \end{cases}
$$
 (2.2)

A set $E \subset \mathbb{R}^d$ is of (locally) finite perimeter if the distributional derivative of χ_E is a (locally) finite measure. We let ∂E be the reduced boundary of E. We call ν^E the exterior normal to E. The first term of our functional is, up to a constant, the 1-perimeter of a set relative to $[0, L)^d$, namely

$$
\operatorname{Per}_1(E, [0, L)^d) := \int_{\partial E \cap [0, L)^d} |\nu^E(x)|_1 d\mathcal{H}^{d-1}(x)
$$

and, for $i \in \{1, \ldots, d\}$

$$
\text{Per}_{1i}(E, [0, L)^d) = \int_{\partial E \cap [0, L)^d} |\nu_i^E(x)| \, d\mathcal{H}^{d-1}(x),\tag{2.3}
$$

thus $\text{Per}_1(E, [0, L)^d) = \sum_{i=1}^d \text{Per}_{1i}(E, [0, L)^d)$. Notice that in the definition of Per_1 the norm applied to the exterior normal ν_E is the 1-norm, instead of the Euclidean norm used to define the standard perimeter.

Because of periodicity, w.l.o.g. we always assume that $|D\chi_E|(\partial [0,L)^d) = 0$, being χ_E the characteristic function of E and $D\chi_E$ its distributional derivative.

When $d = 1$ one can define

$$
Per_1(E, [0, L)) = Per(E, [0, L)) = \#(\partial E \cap [0, L)),
$$

where ∂E is the reduced boundary of E.

While writing slicing formulas, with a slight abuse of notation we will sometimes identify $x_i \in [0, L)^d$ with its coordinate in \mathbb{R} w.r.t. e_i and $\{x_i^{\perp} : x \in [0, L)^d\}$ with $[0, L)^{d-1} \subset \mathbb{R}^{d-1}$.

In Section [4](#page-14-1) we will have to apply slicing on small cubes around a point. Therefore we need to introduce the following notation. For $z \in [0, L)^d$ and $r > 0$, we define $Q_r(z) = \{x \in \mathbb{R}^d : |x - z|_{\infty} \leq$ r}. For $r > 0$ and x_i^{\perp} we let $Q_r^{\perp}(x_i^{\perp}) = \{z_i^{\perp} : |x_i^{\perp} - z_i^{\perp}|_{\infty} \leq r\}$ or we think of $x_i^{\perp} \in [0, L)^{d-1}$ and $Q_r^{\perp}(x_i^{\perp})$ as a subset of \mathbb{R}^{d-1} . Since the subscript i will be always present in the centre (namely x_i^{\perp}) of such $(d-1)$ -dimensional cube, the implicit dependence on i of $Q_r^{\perp}(x_i^{\perp})$ should be clear. We denote also by $Q_r^i(t_i) \subset \mathbb{R}$ the interval of length r centred in t_i .

Now, let us turn to the elements defining the nonlocal term in [\(1.3\)](#page-1-2). The kernel is the so called Yukawa kernel

$$
K_1(\zeta) = \frac{e^{-|\zeta|_1}}{|\zeta|_1^{d-2}}, \quad \zeta \in \mathbb{R}^d,
$$
\n(2.4)

up to considering the 1-norm in the exponential instead of the Euclidean norm.

As commented in the Introduction, such a kernel is both physical and reflection positive, namely it satisfies the following property (see property (2.6) in [\[11\]](#page-30-2)): the function

$$
\widehat{K}_1(t) := \int_{\mathbb{R}^{d-1}} K_1(t, \zeta_2, \dots, \zeta_d) \, d\zeta_2 \cdots d\zeta_d.
$$

is the Laplace transform of a nonnegative function (see Lemma [3.2\)](#page-13-0). Notice that

$$
\widehat{K}_1(t) = e^{-|t|} C(t),
$$
\n(2.5)

where $0 < C(t) \leq \overline{C}$ and $C(t) \to 0$ as $|t| \to +\infty$.

As in [\[18,](#page-31-6) [22,](#page-31-5) [11\]](#page-30-2), there exists a critical constant \tilde{J}_{∞} such that if $J > \tilde{J}_{\infty}$, the functional [\(1.3\)](#page-1-2) is nonnegative and therefore has trivial minimizers. Such a constant is given by

$$
\tilde{J}_{\infty} := \int_{\mathbb{R}^d} |\zeta_1| K_1(\zeta) \, \mathrm{d}\zeta.
$$

A proof of this fact is analogous to [\[22\]](#page-31-5)[Proposition 3.5], so we omit it. Letting, for $M \geq 0$,

$$
\tilde{J}_M := \int_{\mathbb{R}^{d-1}} \int_{-M}^M |\zeta_1| K_1(\zeta) \, \mathrm{d}\zeta = \int_{-M}^M |\zeta_1| \hat{K}_1(\zeta_1) \, \mathrm{d}\zeta_1
$$

and using the fact that, for all $J < \tilde{J}_{\infty}$, $J = \tilde{J}_M$ for some $M \geq 0$, we come to the functional

$$
\tilde{\mathcal{F}}_{\tilde{J}_M,L}(E) = \frac{1}{L^d} \Big[\text{Per}_1(E, [0, L)^d) \int_{\mathbb{R}^{d-1}} \int_{-M}^M |\zeta_1| K_1(\zeta) d\zeta - \int_{[0, L)^d} \int_{\mathbb{R}^d} |\chi_E(x + \zeta) - \chi_E(x)| K_1(\zeta) d\zeta dx \Big].
$$
\n(2.6)

2.2 Energy and width of optimal stripes

We are interested in showing pattern formation for $J = \tilde{J}_M$, with M large but finite. Since we expect the width of the stripes to become larger and larger and the value of the functional to approach 0 as M tends to $+\infty$, it is convenient to find the optimal energy and width among all the $[0, L)^d$ -periodic stripes so as to find suitable rescaling parameters. Let

$$
E_h = \bigcup_{j=1}^{N} ((2j - 1)h, 2jh) \times \mathbb{R}^{d-1}
$$

be a periodic union of stripes of width and distance h (in particular, $L = Nh$) and assume that

$$
1 \ll 2M \leq L,
$$

The energy of E_h , that we denote by $e_M(h)$, is given by the following formula:

$$
e_M(h) = \tilde{\mathcal{F}}_{\tilde{J}_M, L}(E_h) = 2 \left[\frac{1}{h} \int_0^M \zeta_1 \hat{K}_1(\zeta_1) \, d\zeta_1 - \frac{1}{h} \int_0^{+\infty} \min\{h, \zeta_1\} \hat{K}(\zeta_1) \, d\zeta_1 - \frac{1}{h} \sum_{k \in \mathbb{N}} \int_0^h \int_{2kh}^{(2k+1)h} \hat{K}_1(u-v) \, dv \, du \right].
$$
 (2.7)

In particular, if $h < M$ then

$$
e_M(h) = \frac{1}{h} \int_h^M \zeta_1 \widehat{K}_1(\zeta_1) \, d\zeta_1 - \int_h^{+\infty} \widehat{K}_1(\zeta_1) \, d\zeta_1 - \frac{1}{h} \sum_{k \in \mathbb{N}} \int_0^h \int_{2kh}^{(2k+1)h} \widehat{K}_1(u-v) \, dv \, du,\tag{2.8}
$$

while if $h > M$

$$
e_M(h) = -\frac{1}{h} \int_M^{+\infty} \min\{h, \zeta_1\} \widehat{K}_1(\zeta_1) d\zeta_1 - \frac{1}{h} \sum_{k \in \mathbb{N}} \int_0^h \int_{2kh}^{(2k+1)h} \widehat{K}_1(u-v) dv du. \tag{2.9}
$$

Let us check that

$$
e_M(h) > e_M(M), \quad \forall \, h > M.
$$

First of all notice that, since

$$
\int_M^h \frac{\rho}{h} \widehat{K}_1(\rho) d\rho < \int_M^h \widehat{K}_1(\rho) d\rho \quad \forall \, h > M,
$$

the first term in [\(2.9\)](#page-6-0) is smaller if $h = M$. Then we show that, setting

$$
g(h) := \frac{1}{h} \sum_{k \in \mathbb{N}} \int_0^h \int_{2kh}^{(2k+1)h} \widehat{K}_1(u-v) \, \mathrm{d}u \, \mathrm{d}v,
$$

 $g'(h) < 0$ if $h > M$. From these two facts one deduces immediately that M is minimal among all $h \geq M$. In the estimates below, one can assume w.l.o.g. that

$$
\widehat{K}_1(z) \sim e^{-|z|}.\tag{2.10}
$$

One has that

$$
g'(h) = -\frac{1}{h^2} \sum_{k \in \mathbb{N}} \int_0^h \int_{2kh}^{(2k+1)h} \widehat{K}_1(u-v) du dv + \frac{1}{h} \sum_{k \in \mathbb{N}} \int_{2kh}^{(2k+1)h} \widehat{K}_1(h-v) dv + \frac{1}{h} \sum_{k \in \mathbb{N}} (2k+1) \int_0^h \widehat{K}_1(u-(2k+1)h) du - \frac{1}{h} \sum_{k \in \mathbb{N}} (2k) \int_0^h \widehat{K}_1(u-(2k)h) du \sim -\frac{1}{h^2} \sum_{k \in \mathbb{N}} \int_0^h \int_{2kh}^{(2k+1)h} \widehat{K}_1(u-v) du dv + \sum_{k \in \mathbb{N}} \frac{-(2k-1)e^{-(2k-1)h} + 2(2k)e^{-2kh} - (2k+1)e^{-(2k+1)h}}{h} < 0.
$$
\n(2.11)

Let us deal now with the case $h < M$. As for the first two terms in [\(2.8\)](#page-6-1), making the same approximation as in [\(2.10\)](#page-7-0), one obtains

$$
\frac{1}{h} \int_{h}^{M} \rho \widehat{K}_{1}(\rho) d\rho - \int_{h}^{+\infty} \widehat{K}_{1}(\rho) d\rho \sim \frac{e^{-h}(h+1) - e^{-M}(M+1)}{h} - e^{-h} \sim \frac{e^{-h} - e^{-M}(M+1)}{h}.
$$
\n(2.12)

The last term of [\(2.8\)](#page-6-1) can be estimated as follows:

$$
-\frac{1}{h} \sum_{k \in \mathbb{N}} \int_0^h \int_{2kh}^{(2k+1)h} \hat{K}_1(u-v) dv du \sim -\sum_{k \in \mathbb{N}} \frac{(e^{-2kh} - e^{-(2k+1)h})(e^h - 1)}{h} \sim -\frac{e^{-h}(1 - e^{-h})}{(1 + e^{-h})h}.
$$
\n(2.13)

Therefore, for $h < M$

$$
e_M(h) \sim \frac{2e^{-2h} - e^{-M}(M+1) - e^{-h}e^{-M}(M+1)}{h(1 + e^{-h})}.
$$
\n(2.14)

Hence we have the following: for any $C < 1/2$, if $h \leq CM$ and M is sufficiently large depending on C, then $e_M(h) > 0$, thus h cannot be optimal. Finally, one has that

$$
e'_{M}(h) \sim \frac{-4e^{-2h}h(1+e^{-h}) - 2e^{-2h} + e^{-M}(M+1)[-(1+e^{-h})^2 + 3h(1+e^{-h})e^{-h}]}{h^2(1+e^{-h})^2},
$$
 (2.15)

which is negative if $h \in [M/2, M)$ and M is large enough. Therefore, if h_M is the optimal width, then

$$
h_M \leq M
$$
 and $h_M/M \to 1$ as $M \to +\infty$.

As a consequence, the following holds.

$$
e_M(h) \sim -e^{-\alpha_M M} \tag{2.16}
$$

.

for some $\alpha_M \leq 1$, $\alpha_M \to 1$ as $M \to +\infty$.

2.3 Rescaling

Let us denote by $h_{M,L}$ an optimal width and distance and by e_M^* the minimal energy for the functional $\mathcal{F}_{\tilde{J}_M,L}$. As we already saw

$$
M/2 \le h_{M,L} \le M \quad \text{and} \quad e_M^* \ge -e^{-\alpha_M M}
$$

where $\alpha_M \to 1$ and $h_{M,L}/M \to 1$ as $M \to +\infty$ for $1 << M << L$. In this section we will rescale the spacial variables and the functional so that the optimal width and distance for unions of stripes is $O(1)$ and the energy is $O(1)$. By the change of variables $M\zeta' = \zeta$, $Mx' = x$ we have that

$$
\begin{aligned} \text{Per}_1(E/M,[0,L/M)^d) &= M^{1-d} \text{Per}_1(E,[0,L)^d), \\ J_M &:= -\int_{\mathbb{R}^{d-1}} \int_{-1}^1 \frac{|\zeta_1'|}{e_M^* |\zeta'|^{d-2}} \exp(-M|\zeta'|) \, \mathrm{d}\zeta' = -e_M^{*-1} M^{-3} \int_{\mathbb{R}^{d-1}} \int_{-M}^M |\zeta_1| K_1(\zeta) \, \mathrm{d}\zeta = -e_M^{*-1} M^{-3} \tilde{J}_M \end{aligned}
$$

and

$$
\int_{[0,L)^d} \int_{\mathbb{R}^d} |\chi_E(x+\zeta) - \chi_E(x)| K_1(\zeta) \, dx \, d\zeta = M^{d+2} \int_{[0,L/M)^d} \int_{\mathbb{R}^d} \frac{|\chi_{E/M}(x'+\zeta') - \chi_{E/M}(x')|}{|\zeta'|^{d-2}} e^{-M|\zeta'|} \, dx' \, d\zeta'
$$

Thus we have that

$$
\tilde{J}_M \text{Per}_1(E, [0, L)^d) = -M^{d+2} J_M e_M^* \text{Per}_1(E/M, [0, L/M)^d).
$$

Finally defining

$$
\bar{K}_M(\zeta) = \frac{-1}{e_M^* |\zeta|_1^{d-2}} e^{-M|\zeta|_1}
$$

and putting everything together we have that

$$
\tilde{\mathcal{F}}_{\tilde{J}_M,L}(E) = \frac{-M^{d+2}e_M^*}{L^d} \Big(J_M \text{Per}_1(E/M, [0, L/M)^d) - \int_{[0, L/M)^d} \int_{\mathbb{R}^d} |\chi_{E/M}(x' + \zeta') - \chi_{E/M}(x')| \bar{K}_M(\zeta') \, dx' \, d\zeta' \Big).
$$

Then let us set

$$
\tilde{\mathcal{F}}_{\tilde{J}_M,L}(E) = -e_M^* \mathcal{F}_{M,\tilde{L}}(\tilde{E})
$$
\n(2.17)

where $\tilde{L}=L/M$ and $\tilde{E}=E/M,$ and let us drop the tildes in the r.h.s.. Hence

$$
\mathcal{F}_{M,L}(E) = \frac{M^2}{L^d} \Big(J_M \text{Per}_1(E, [0, L)^d) - \int_{[0, L)^d} \int_{\mathbb{R}^d} |\chi_E(x + \zeta) - \chi_E(x)| \bar{K}_M(\zeta) \, dx \, d\zeta \Big). \tag{2.18}
$$

Notice also that, due to [\(2.16\)](#page-8-0), there exists a constant $1 > \gamma_M > 0$ such that

$$
\bar{K}_M(\zeta) \ge \frac{1}{|\zeta|_1^{d-2}} e^{-M(|\zeta|_1 - \gamma_M)}\tag{2.19}
$$

with $\gamma_M \to 1$ as $M \to +\infty$. For simplicity of notation we define

 $\widehat{K}_{M}(t) := \int_{\mathbb{R}^{d-1}} \bar{K}_{M}(t, \zeta_{2}, \ldots, \zeta_{d}) \, d\zeta_{2} \cdots d\zeta_{d}.$ (2.20)

2.4 Splitting and lower bounds

Using the equality

$$
|\chi_E(x) - \chi_E(x + \zeta)| = |\chi_E(x) - \chi_E(x + \zeta_i)| + |\chi_E(x + \zeta) - \chi_E(x + \zeta_i)|
$$

- 2 $|\chi_E(x) - \chi_E(x + \zeta_i)||\chi_E(x + \zeta) - \chi_E(x + \zeta_i)|$

one splits the nonlocal term getting the following lower bound:

$$
\mathcal{F}_{M,L}(E) \geq \frac{M^2}{L^d} \sum_{i=1}^d \Big[\int_{[0,L)^d \cap \partial E} \int_{\mathbb{R}^{d-1}} \int_{-1}^1 |\nu_i^E(x)| |\zeta_i| \bar{K}_M(\zeta) d\zeta d\mathcal{H}^{d-1}(x) \n- \int_{[0,L)^d} \int_{\mathbb{R}^d} |\chi_E(x+\zeta_i) - \chi_E(x)| \bar{K}_M(\zeta) d\zeta dx \Big] \n+ \frac{2}{d} \frac{M^2}{L^d} \sum_{i=1}^d \int_{[0,L)^d} \int_{\mathbb{R}^d} |\chi_E(x+\zeta_i) - \chi_E(x)| |\chi_E(x+\zeta_i^{\perp}) - \chi_E(x)| \bar{K}_M(\zeta) d\zeta dx \n= \frac{M^2}{L^d} \Big(\sum_{i=1}^d \mathcal{G}_{M,L}^i(E) + \sum_{i=1}^d I_{M,L}^i(E) \Big), \tag{2.21}
$$

where

$$
\mathcal{G}_{M,L}^{i}(E) := \int_{[0,L)^d \cap \partial E} \int_{\mathbb{R}^{d-1}} \int_{-1}^1 |\nu_i^E(x)| |\zeta_i| \bar{K}_M(\zeta) d\zeta d\mathcal{H}^{d-1}(x) - \int_{[0,L)^d} \int_{\mathbb{R}^d} |\chi_E(x+\zeta_i) - \chi_E(x)| \bar{K}_M(\zeta) d\zeta dx
$$

and

$$
I_{M,L}^i(E) := \frac{2}{d} \int_{[0,L)^d} \int_{\mathbb{R}^d} |\chi_E(x + \zeta_i) - \chi_E(x)| |\chi_E(x + \zeta_i^{\perp}) - \chi_E(x)| \bar{K}_M(\zeta) d\zeta dx.
$$

Moreover, let

$$
I_{M,L}(E) := \sum_{i=1}^{d} I_{M,L}^{i}(E).
$$

One can further express $\mathcal{G}_{M,L}^{i}(E)$ as a sum of contributions obtained by first slicing and then considering interactions with neighbouring points on the slice lying on ∂E , namely

$$
\mathcal{G}_{M,L}^{i}(E) = \int_{[0,L)^{d-1}} \sum_{s \in \partial E_{t_i^{\perp}} \cap [0,L)} r_{i,M}(E, t_i^{\perp}, s) dt_i^{\perp}
$$
\n(2.22)

where for $s \in \partial E_{t_i^{\perp}}$

$$
r_{i,M}(E, t_i^{\perp}, s) := \int_{-1}^1 |\zeta_i| \widehat{K}_M(\zeta_i) d\zeta_i - \int_{s^-}^s \int_0^{+\infty} |\chi_{E_{t_i^{\perp}}} (u + \rho) - \chi_{E_{t_i^{\perp}}} (u)| \widehat{K}_M(\rho) d\rho du - \int_s^{s^+} \int_{-\infty}^0 |\chi_{E_{t_i^{\perp}}} (u + \rho) - \chi_{E_{t_i^{\perp}}} (u)| \widehat{K}_M(\rho) d\rho du
$$
(2.23)

and

$$
s^{+} = \inf\{t' \in \partial E_{t_i^{\perp}}, \text{with } t' > s\}
$$

$$
s^{-} = \sup\{t' \in \partial E_{t_i^{\perp}}, \text{with } t' < s\}. \tag{2.24}
$$

Setting

$$
f_E(t_i^{\perp}, t_i, t_i^{\prime \perp}, t_i^{\prime}) := |\chi_E(t_i^{\perp} + t_i + t_i^{\prime}) - \chi_E(t_i + t_i^{\perp})| |\chi_E(t_i^{\perp} + t_i + t_i^{\prime \perp}) - \chi_E(t_i + t_i^{\perp})|,
$$
 (2.25)

we can rewrite the last term in the r.h.s. of [\(2.21\)](#page-9-0) as

$$
I_{M,L}^{i}(E) = \frac{2}{d} \int_{[0,L)^{d}} \int_{\mathbb{R}^{d}} f_{E}(t_{i}^{\perp}, t_{i}, \zeta_{i}^{\perp}, \zeta_{i}) \bar{K}_{M}(\zeta) d\zeta dt = \int_{[0,L)^{d-1}} \sum_{s \in \partial E_{t_{i}^{\perp}} \cap [0,L)} v_{i,M}(E, t_{i}^{\perp}, s) dt_{i}^{\perp} + \int_{[0,L)^{d}} w_{i,M}(E, t_{i}^{\perp}, t_{i}) dt
$$
\n(2.26)

where

$$
w_{i,M}(E, t_i^{\perp}, t_i) = \frac{1}{d} \int_{\mathbb{R}^d} f_E(t_i^{\perp}, t_i, \zeta_i^{\perp}, \zeta_i) \bar{K}_M(\zeta) \, d\zeta.
$$
 (2.27)

and

$$
v_{i,M}(E, t_i^{\perp}, s) = \frac{1}{2d} \int_{s^-}^{s^+} \int_{\mathbb{R}^d} f_E(t_i^{\perp}, u, \zeta_i^{\perp}, \zeta_i) \bar{K}_M(\zeta) d\zeta du \qquad (2.28)
$$

and s^+, s^- as in (2.24) .

Notice that the term $r_{i,M}$ is penalizing sets whose slices in direction i have boundary points at distance smaller than some given constant (see Lemma [4.2\)](#page-14-2).

The term $v_{i,M}$ penalizes oscillations in direction e_i whenever the neighbourhood of the point $(t_i^{\perp} +$ se_i) is close in L^1 to a stripe oriented along e_j . This statement is made precise in Lemma [5.2.](#page-21-0)

2.5 Averaging

We now define the "local contribution" to the energy in a cube $Q_l(z)$, where $z \in [0, L)^d$ and $0 < l < L$ is a length which in the Section [4](#page-14-1) will be fixed independently of L (l will depend only on the dimension).

We let

$$
\bar{F}_{i,M}(E, Q_l(z)) := \frac{1}{l^d} \Big[\int_{Q_l^{\perp}(z_i^{\perp})} \sum_{s \in \partial E_{t_i^{\perp}} \atop t_i^{\perp} + se_i \in Q_l(z)} (v_{i,M}(E, t_i^{\perp}, s) + r_{i,M}(E, t_i^{\perp}, s)) dt_i^{\perp} + \int_{Q_l(z)} w_{i,M}(E, t_i^{\perp}, t_i) dt \Big]
$$
\n
$$
\bar{F}_M(E, Q_l(z)) := \sum_{i=1}^d \bar{F}_{i,M}(E, Q_l(z)). \tag{2.29}
$$

Thanks to Lemma 7.2 in [\[11\]](#page-30-2), one has that the r.h.s. of [\(2.21\)](#page-9-0) is equal to

$$
\frac{M^2}{L^d} \int_{[0,L)^d} \bar{F}_M(E, Q_l(z)) \, \mathrm{d}z. \tag{2.30}
$$

,

This implies that

$$
\mathcal{F}_{M,L}(E) \ge \frac{M^2}{L^d} \int_{[0,L)^d} \bar{F}_M(E, Q_l(z)) \, \mathrm{d}z. \tag{2.31}
$$

Given that, in the above inequality, equality holds for stripes, if we show that the minimizers of [\(2.30\)](#page-11-0) are periodic optimal stripes, then the same claim holds for $\mathcal{F}_{M,L}$.

2.6 A distance from unions of stripes

The purpose of the next definition is to introduce a quantity which measures the distance of a given set E from being a union of stripes.

Definition 2.1. For every $\eta > 0$ we denote by \mathcal{A}^i_{η} the family of all sets F such that

- (i) they are union of stripes oriented along the direction e_i
- (ii) their connected components of the boundary are distant at least η .

We denote by

$$
D_{\eta}^{i}(E,Q) := \inf \left\{ \frac{1}{\text{vol}(Q)} \int_{Q} |\chi_{E} - \chi_{F}| : F \in \mathcal{A}_{\eta}^{i} \right\} \quad and \quad D_{\eta}(E,Q) = \inf_{i} D_{\eta}^{i}(E,Q). \tag{2.32}
$$

Finally, we let $A_{\eta} := \cup_i A_{\eta}^i$.

We recall now some properties of the distance (2.32) (see Remark 7.4 in [\[11\]](#page-30-2)).

Lemma 2.2.

- (i) Let $E, F \subset \mathbb{R}^d$. Then, the map $z \mapsto D_\eta(E, Q_l(z))$ is Lipschitz. The Lipschitz constant can be shown to be C_d/l , where C_d is a constant depending only on the dimension d.
- (ii) For every $\varepsilon > 0$, there exists $\delta_0 = \delta_0(\varepsilon)$, such that for every $\delta \leq \delta_0$ and whenever $D^j_\eta(E, Q_l(z)) \leq$ δ and $D^i_\eta(E, Q_l(z)) \leq \delta$ for $i \neq j$ and $\eta > 0$, the following hold

$$
\min\left(|Q_l(z)\setminus E|, |E\cap Q_l(z)|\right) \le \varepsilon. \tag{2.33}
$$

3 The one-dimensional problem

We consider the following one-dimensional functional: on an L-periodic set $E \subset \mathbb{R}$ of locally finite perimeter

$$
\mathcal{F}_{M,L}^1(E) = \frac{M^2}{L} \Big(\int_{-1}^1 \widehat{K}_M(\rho) \Big[\text{Per}(E,[0,L)) |\rho| - \int_0^L |\chi_E(s) - \chi_E(s+\rho)| \, ds \Big] \, d\rho \Big),
$$

where \widehat{K}_M has been defined in [\(2.20\)](#page-9-1).

The functional $\mathcal{F}_{M,L}^1$ corresponds to $\mathcal{F}_{M,L}(E)$ when the set E is a union of stripes. Namely, given $E \subset \mathbb{R}^d$ and such that $E = \hat{E} \times \mathbb{R}^{d-1}$ where E is L-periodic, then

$$
\mathcal{F}_{M,L}(E) = \mathcal{F}_{M,L}^1(\hat{E}).
$$

The purpose of this section is to show that the periodic sets are minimizers among the sets composed of stripes, whenever M is large enough. For the above one-dimensional problem there are some standard techniques available in the literature. In particular, our proof will rely on the reflection positivity technique, introduced in the context of quantum field theory by Osterwalder and Schrader and then applied for the first time in statistical mechanics by Fröhlich, Simon and Spencer. For works where the reflection positivity is used in models with competing interactions, see e.g. [\[13\]](#page-31-9), [\[14,](#page-31-10) [16,](#page-31-8) [17,](#page-31-11) [15\]](#page-31-12), [\[22,](#page-31-5) [11\]](#page-30-2). As the technique is nowadays standard, we will only outline briefly the steps and show some of the differences with respect to the literature.

Before showing optimality of periodic stripes, we show that there exists a unique optimal period h_M^* , provided M is large enough. For $h > 0$, let $E_h = \bigcup_{j \in \mathbb{Z}} [(2j)h, (2j + 1)h]$ and define the energy

$$
e_M(h) = \mathcal{F}^1_{M,2h}(E_h) = \frac{M^2}{h} \Big[\int_h^1 (\rho - h) \hat{K}_M(\rho) d\rho - \int_1^{+\infty} h \hat{K}_M(\rho) d\rho - \sum_{k \in \mathbb{N}} \int_0^h \int_{2kh}^{(2k+1)h} \hat{K}_M(u-v) dv du \Big].
$$

We prove the following

Theorem 3.1. There exists $\tilde{M} > 0$ such that $\forall M > \tilde{M}$, there exists a unique minimizer of $e_M(\cdot)$, h_M^* .

Proof. In Section [2.2](#page-6-2) we showed that, as $M \to +\infty$, minimizers to e_M tend to 1. So, in order to prove the theorem it is sufficient to show that there exist $\delta > 0$, M such that $e''_M(1) \ge \delta$ for all

 $M > \tilde{M}$. Computing $e''_M(1)$, one finds

$$
\frac{e''_M(1)}{M^2} = \hat{K}_M(1)
$$
\n
$$
-2\sum_{k\in\mathbb{N}} \int_0^1 \int_{2k}^{2k+1} \hat{K}_M(u-v) dv du + 2\sum_{k\in\mathbb{N}} \int_{2k}^{2k+1} \hat{K}_M(h-v) dv
$$
\n
$$
+2\sum_{k\in\mathbb{N}} \int_0^1 [(2k+1)\hat{K}_M(u-(2k+1)) - 2k\hat{K}_M(u-2k)] du
$$
\n
$$
-2\sum_{k\in\mathbb{N}} [(2k+1)\hat{K}_M(2k) - 2k\hat{K}_M(2k-1)]
$$
\n
$$
-\sum_{k\in\mathbb{N}} \int_{2k}^{2k+1} \hat{K}'_M(h-v) dv
$$
\n
$$
+ \sum_{k\in\mathbb{N}} \int_0^1 [(2k+1)^2 \hat{K}'_M(u-(2k+1)) - (2k)^2 \hat{K}'_M(u-2k)] du.
$$
\n(3.1)

One has the following lower estimate:

$$
\frac{e''_M(1)}{M^2} \ge 2 \sum_{k \in \mathbb{N}} [(2k+1)\hat{K}_M(2k+1) - (2k)\hat{K}_M(2k) - (2k+1)\hat{K}_M(2k) - (2k)\hat{K}_M(2k-1) + M \sum_{k \in \mathbb{N}} \hat{K}_M(2k-1) - M \sum_{k \in \mathbb{N}} [(2k+1)^2 \hat{K}_M(2k+1) - (2k)^2 \hat{K}_M(2k)].
$$
\n(3.2)

To conclude, observe now that the lower bound can be rewritten as

$$
\sum_{k \in 2\mathbb{N}} e^{-Mk} \{ M[-(k+1)^2 e^{-M} + k^2 + e^M] + 2[(k+1)e^{-M} - (2k+1) + ke^M] \},
$$
 (3.3)

which is positive for M sufficiently large.

Let us now return to the issue of showing that optimal periodic stripes are optimal among all stripes. This fact follows from the reflection positivity technique and is well-known in the literature. We refer the reader to the references given at the beginning of this section. Only for notational reasons we also refer to [\[11,](#page-30-2) [22\]](#page-31-5). The only part needed is the reflection positivity of the Yukawa kernel.

Lemma 3.2. The Yukawa kernel is reflection positive, namely there exists a positive Borel measure μ such that

$$
\widehat{K}_1(t) = \int_0^{+\infty} e^{-\alpha t} \, \mathrm{d}\mu(\alpha).
$$

Proof. Due to the Hausdorff-Bernstein-Widder theorem, one has that a function φ is reflection positive if and only if it is completely monotone, namely $(-1)^n \frac{d^n}{dt^n} \varphi \ge 0$.

 \Box

By using the complete monotone property one has that the set of functions which are reflection positive is an algebra. Indeed, given two functions φ , ψ which are completely monotone by the Leibniz rule one has that

$$
(-1)^n \frac{d^n}{dt^n} (\varphi \psi) = \sum_{k=0}^n (-1)^k \frac{d^k \varphi}{dt^k} \cdot (-1)^{n-k} \frac{d^{n-k} \psi}{dt^{n-k}} \ge 0
$$

In order to conclude the proof, we need to show that the map

$$
t \mapsto e^{-t} \int_{\mathbb{R}^{d-1}} \frac{1}{(t + |\zeta_2| + \dots + |\zeta_d|)^{d-2}} e^{-(|\zeta_2| + \dots + |\zeta_d|)}
$$

is completely monotone. Given the complete monotone functions are an algebra, we need to check that the single terms in the product above are completely monotone. This can be done easily with explicit calculations.

Once reflection positivity of the kernel is shown the proof follows by standard means in the literature. We refer the reader to [\[22,](#page-31-5) [11\]](#page-30-2) where further details are given and a similar notation is used.

4 A local rigidity estimate

The core of the proof of Theorem [1.1](#page-3-0) is the following proposition (see alse [\[11\]](#page-30-2) for a similar proposition in a different setting):

Proposition 4.1 (Local Rigidity). For every $N > 1, l, \delta > 0$, there exist $\overline{M}, \overline{\eta} > 0$ such that whenever $M > \overline{M}$ and $\overline{F}_M(E, Q_l(z)) < N$ for some $z \in [0, L)^d$ and $E \subset \mathbb{R}^d$ $[0, L)^d$ -periodic, with $L > l$, then it holds $D_n(E, Q_l(z)) \leq \delta$ for every $\eta < \bar{\eta}$. Moreover $\bar{\eta}$ can be chosen independent on δ. Notice that \overline{M} and $\overline{\eta}$ are independent of L.

In particular, such a rigidity estimate tells us that on small cubes minimizers of the functional are, for M large enough, close to stripes of a given minimal width.

In order to prove Proposition [4.1,](#page-14-0) we will need to analize the behaviour of F_M for large M. First of all, we start with the following lemma, about the term $r_{i,M}$. In particular this tells us that given a sequence of sets ${E_M}_{M>0} \subset \mathbb{R}^d$ of bounded local energy $\bar{F}_M(E, Q_l(z))$, if M is large enough their boundary points on the slices have distance at least η_0 where η_0 is fixed arbitrarily close to 1 and then they converge to a set of locally finite perimeter E_0 .

Lemma 4.2. There exists a function $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that, for all $E \subset \mathbb{R}^d$ of locally finite perimeter, $t_i^{\perp} \in [0, L)^{d-1}$, $s \in \partial E_{t_i^{\perp}}$

$$
r_{i,M}(E, t_i^{\perp}, s) \ge g\big((\gamma_M - \min(|s - s^-|, |s - s^+|), M\big) \tag{4.1}
$$

with γ_M defined in [\(2.19\)](#page-9-2). The function g satisfies the following: $g(v, M) \ge g(v', M)$ whenever $v > v'$, $g(v, M) \ge -e^{-cM}$ for some $c > 0$ and $g(v, M) \to +\infty$ as $M \to +\infty$ provided $v > 0$.

In particular, for every $0 < \eta_0 < 1$ there exists M_0 such that, for all $M > M_0$ if $\min\{|s - s^-|, |s - s^-|, s - t\}$ s^+ |} $< \eta_0$ then $r_{i,M}(E, t_i^{\perp}, s) > 0$.

 \Box

Proof. The proof of this lemma uses the following inequality: for every $E \subset \mathbb{R}^d$ of locally finite perimeter, $\forall t_i^{\perp} \subset [0, L)^{d-1}$,

$$
\forall \rho > 0, \quad \int_{s^{-}}^{s} |\chi_{E_{t_{i}^{\perp}}}(u) - \chi_{E_{t_{i}^{\perp}}}(u+\rho)| du \le \min(\rho, s-s^{-})
$$

$$
\forall \rho < 0, \quad \int_{s}^{s^{+}} |\chi_{E_{t_{i}^{\perp}}}(u) - \chi_{E_{t_{i}^{\perp}}}(u+\rho)| du \le \min(-\rho, s^{+}-s).
$$
\n(4.2)

Indeed,

$$
\int_{0}^{1} \zeta_{i} \widehat{K}_{M}(\zeta_{i}) d\zeta_{i} - \int_{s^{-}}^{s} \int_{0}^{+\infty} |\chi_{E_{t_{i}^{\perp}}} (u + \rho) - \chi_{E_{t_{i}^{\perp}}} (u) | \widehat{K}_{M}(\rho) d\rho du
$$
\n
$$
\stackrel{(4.2)}{\geq} \int_{0}^{1} \zeta_{i} \widehat{K}_{M}(\zeta_{i}) d\zeta_{i} - \int_{0}^{+\infty} \min(|s - s^{-}|, \zeta_{i}) \widehat{K}_{M}(\zeta_{i}) d\zeta_{i}
$$
\n
$$
\geq \int_{|s - s^{-}|}^{1} (\zeta_{i} - |s - s^{-}|) \widehat{K}_{M}(\zeta_{i}) d\zeta_{i} - \int_{1}^{+\infty} |s - s^{-}| \widehat{K}_{M}(\zeta_{i}) d\zeta_{i} \qquad (4.3)
$$

and analogously

$$
\int_{-1}^{0} |\zeta_{i}| \hat{\overline{K}}_{M}(\zeta_{i}) d\zeta_{i} - \int_{s}^{s^{+}} \int_{-\infty}^{0} |\chi_{E_{t_{i}^{\perp}}}(u+\rho) - \chi_{E_{t_{i}^{\perp}}}(u)| \hat{\overline{K}}_{M}(\rho) d\rho du
$$
\n
$$
\geq \int_{-1}^{-|s-s^{+}|} (|\zeta_{i}| - |s-s^{+}|) \hat{\overline{K}}_{M}(\zeta_{i}) d\zeta_{i} - \int_{-\infty}^{-1} |s-s^{+}| \hat{\overline{K}}_{M}(\zeta_{i}) d\zeta_{i}. \tag{4.4}
$$

Then, by [\(2.23\)](#page-10-1) and using [\(2.19\)](#page-9-2), since $\gamma_M < 1$ one gets [\(4.1\)](#page-14-3) and the statement of the lemma. \Box **Remark 4.3.** From Lemma [4.2,](#page-14-2) since $\gamma_M \to 1$ as $M \to +\infty$, it follows as well that the function

$$
r_{i,\infty}(E,t_i^{\perp},s) := \liminf_{M \to +\infty} r_{i,M}(E,t_i^{\perp},s)
$$

satisfies

$$
r_{i,\infty}(E, t_i^{\perp}, s) = +\infty \quad whenever \quad \min(|s - s^-|, |s - s^+|) < 1. \tag{4.5}
$$

 In particular, if $\{E_M\}_{M>0} \subset \mathbb{R}^d$ is a family of sets of locally finite perimeter with $\sup_M \bar{F}_M(E_M, Q_l(z)) \leq$ N, then for a.e. $t_i^{\perp} \in Q_l^{\perp}(z_i^{\perp})$ and for every $I \subset \mathbb{R}$ open interval,

$$
\liminf_{M \to +\infty} \min \{|s_i^M - s_{i+1}^M| : \partial E_{M, t_i^\perp} \cap I = \{s_i^M\}_{i=1}^{m(M)}\} \ge 1
$$
\n(4.6)

In particular, E_M converges in L^1_{loc} to a set E_{∞} of locally finite perimeter such that

$$
\min\{|s_i^{\infty} - s_j^{\infty}| : \partial E_{\infty, t_i^{\perp}} \cap I = \{s_k^{\infty}\}_{k=1}^{m(\infty)}\} \ge 1.
$$
\n(4.7)

For details of how to deduce (4.7) form (4.6) see the proof of Lemma 7.5 in [\[11\]](#page-30-2).

Remark 4.4. Let us notice the following: the family of kernels K_M is monotone increasing in M as $M \to +\infty$. Let K_{∞} be defined by

$$
\bar{K}_{\infty}(\zeta) := \liminf_{M \to +\infty} \bar{K}_M(\zeta). \tag{4.8}
$$

From [\(2.19\)](#page-9-2) we get that

$$
\bar{K}_{\infty} \ge 0 \quad and \quad \bar{K}_{\infty}(\zeta) = +\infty \text{ whenever } |\zeta| < 1,
$$
\n(4.9)

where the last statement comes from the fact that $\gamma_M \to 1$ as $M \to +\infty$.

Let us now proceed to the proof of Proposition [4.1.](#page-14-0) The main steps can be summarized as follows. Given a sequence of sets $E_M \subset \mathbb{R}^d$ of bounded local energy, by Remark [4.3](#page-15-3) their boundary points on the slices are not too close (they have mutual distance at least 1) and then they converge to a set of locally finite perimeter E_{∞} . Then, using the monotonicity in M of the kernel one gets as $M \to +\infty$ a bound on the liminf of the cross interaction terms $I_{M,L}$ on E_{∞} (see [\(4.15\)](#page-18-0)). Thanks to the fact that boundary points on the slices of E_{∞} have mutual distance at least η_0 with η_0 close to 1 and that [\(4.9\)](#page-16-0) holds, one gets that boundary points in $\partial E_{\infty,t_i^{\perp}}$ are a constant function of t_i^{\perp} . Therefore the only shapes admissible for E_{∞} are checkerboards or stripes, and finally by an analogous energetic argument we rule out checkerboards.

As a consequence, for M sufficiently large but depending only on l, the sets E_M will be close to E_{∞} in the sense of Definition [2.1](#page-11-2) and therefore to stripes of a minimal given width.

Proof. The proof will follow by contradiction. Indeed, assume that the claim is false. This implies that there exists $N > 1, l, \delta > 0$ and sequences $\{M_k\}, \{\eta_k\}, \{L_k\}, \{z_k\}, \{E_{M_k}\}\$ such that:

- (i) one has that $M_k \to +\infty$, $L_k > l$, $\eta_k \downarrow 0$, $z_k \in [0, L_k)^d$;
- (ii) the family of sets E_{M_k} is $[0, L_k)^d$ -periodic
- (iii) one has that $D_{\eta_k}(E_{M_k}, Q_l(z_k)) > \delta$ and $\bar{F}_{M_k}(E_{M_k}, Q_l(z_k)) < N$.

Given that $\eta \mapsto D_{\eta}(E, Q_l(z))$ is monotone increasing, we can fix $\bar{\eta}$ sufficiently small instead of η_k with $D_{\bar{\eta}}(E_{M_k}, Q_l(z_k)) > \delta$. In particular, $\bar{\eta}$ will be chosen at the end of the proof depending only on N, l .

W.l.o.g. (taking e.g. $E_{M_k} - z_k$ instead of E_{M_k}) we can assume there exists $z \in \mathbb{R}^d$ such that $z_k = z$ for all $k \in \mathbb{N}$.

Because of Remark [4.3,](#page-15-3) one has that $\sup_k \text{Per}_1(E_{M_k}, Q_l(z)) < +\infty$. Thus up to subsequences there exists E_{∞} such that $E_{M_k} \to E_{\infty}$ in $L^1(Q_l(z))$ with

$$
D_{\bar{\eta}}(E_{\infty}, Q_l(z)) > \delta \tag{4.10}
$$

In order to keep the notation simpler, we will write $M \to +\infty$ instead of $M_k \to +\infty$ and $E_M \to E_\infty$ instead of $E_{M_k} \to E_{\infty}$.

Define $J_i := (z_i - l/2, z_i + l/2).$

By Lebesgue's theorem, there exists a subsequence of M such that for almost every $t_i^{\perp} \in Q_l^{\perp}(z_i^{\perp})$ one has that $E_{M,t_i^{\perp}} \cap J_i$ converges to $E_{\infty,t_i^{\perp}} \cap J_i$ in $L^1(Q_l(z))$.

By using [\(2.29\)](#page-11-3) and the fact that $v_{i,M} \ge 0$, we have that

$$
N \ge \bar{F}_M(E_M, Q_l(z)) \ge \frac{1}{l^d} \sum_{i=1}^d \int_{Q_l^{\perp}(z_i^{\perp})} \sum_{\substack{s \in \partial E_{M, t_i^{\perp}} \\ s \in J_i}} r_{i,M}(E_M, t_i^{\perp}, s) dt_i^{\perp} + \int_{Q_l(z)} w_{i,M}(E_M, t_i^{\perp}, t_i) dt_i^{\perp} dt_i.
$$
\n(4.11)

By the Fatou lemma, we have that

$$
l^d M \ge \liminf_{M \to +\infty} \sum_{i=1}^d \int_{Q_l^{\perp}(z_i^{\perp})} \sum_{\substack{s \in \partial E_{M, t_i^{\perp}} \\ s \in J_i}} r_{i,M}(E_M, t_i^{\perp}, s) dt_i^{\perp} \ge \sum_{i=1}^d \int_{Q_l^{\perp}(z_i^{\perp})} \liminf_{M \to +\infty} \sum_{\substack{s \in \partial E_{M, t_i^{\perp}} \\ s \in J_i}} r_{i,M}(E_M, t_i^{\perp}, s) dt_i^{\perp}
$$

$$
\ge \sum_{i=1}^d \int_{Q_l^{\perp}(z_i^{\perp})} \mathbb{1}_{B_i}^{\infty}(t_i^{\perp}) dt_i^{\perp},
$$

where

$$
B_i = \Big\{ t_i^{\perp} \in Q_l^{\perp}(z_i^{\perp}) : \min\{|s_i^{\infty} - s_j^{\infty}| : \partial E_{\infty, t_i^{\perp}} = \{s_k^{\infty}\}_{k=1}^{m(\infty, t_i^{\perp})}\} < 1\Big\},
$$

and in the last inequality we have used Remark [4.3.](#page-15-3) For the last term in [\(4.11\)](#page-17-0), namely

$$
\liminf_{M \to +\infty} \int_{Q_l(z)} w_{i,M}(E_M, t_i^{\perp}, t_i) dt_i^{\perp} dt_i
$$
\n
$$
\geq \liminf_{M \to +\infty} \frac{1}{d} \int_{Q_l(z)} \int_{Q_l(z)} f_{E_M}(t_i^{\perp}, t_i, t_i'^{\perp}, t_i') \bar{K}_M(t) dt dt'
$$
\n
$$
= \liminf_{M \to +\infty} \frac{1}{d} \int_{Q_l(z)} \int_{Q_l(z)} f_{E_M}(t_i^{\perp}, t_i, t_i'^{\perp} - t_i^{\perp}, t_i' - t_i) \bar{K}_M(t - t') dt dt'
$$
\n
$$
\geq \frac{1}{d} \int_{Q_l(z)} \int_{Q_l(z)} f_{E_\infty}(t_i^{\perp}, t_i, t_i'^{\perp} - t_i^{\perp}, t_i' - t_i) \bar{K}_\infty(t - t') dt dt',
$$
\n
$$
(4.12)
$$

where in the third line we have used a change of variables.

In order to prove [\(4.12\)](#page-17-1) we fix $M' > 0$ and by using initially $E_M \to E_\infty$ in $L^1(Q_l(z))$ and afterwards the monotonicity of $M \mapsto \bar{K}_M(\zeta)$ we have that

$$
\liminf_{M \to +\infty} \int_{Q_l(z)} w_{i,M}(E_M, t_i^{\perp}, t_i) dt_i^{\perp} dt_i \ge \sup_{M'} \liminf_{M \to +\infty} \int_{Q_l(z)} w_{i,M'}(E_M, t_i^{\perp}, t_i) dt_i^{\perp} dt_i
$$
\n
$$
\ge \sup_{M'} \frac{1}{d} \int_{Q_l(z)} \int_{Q_l(z)} f_{E_\infty}(t_i^{\perp}, t_i, t_i'^{\perp} - t_i'^{\perp}, t_i - t_i') \bar{K}_{M'}(t - t') dt dt'
$$
\n
$$
\ge \frac{1}{d} \int_{Q_l(z)} \int_{Q_l(z)} f_{E_\infty}(t_i^{\perp}, t_i, t_i'^{\perp} - t_i'^{\perp}, t_i - t_i') \bar{K}_{\infty}(t - t') dt dt'.
$$

Thus, we have shown that

$$
\sum_{i=1}^{d} \frac{1}{d} \int_{Q_l(z)} \int_{Q_l(z)} f_{E_{\infty}}(t_i^{\perp}, t_i, t_i'^{\perp} - t_i'^{\perp}, t_i - t_i') \bar{K}_{\infty}(t - t') dt dt' + \sum_{i=1}^{d} \int_{Q_l^{\perp}(z_i^{\perp})} \mathbb{1}_{B_i}^{\infty}(t_i^{\perp}) dt_i^{\perp} \lesssim l^d N.
$$
\n(4.13)

Defining

$$
\text{Int}(t_i^{\perp}, t_i'^{\perp}) := \int_{Q_l^i(z_i)} \int_{Q_l^i(z_i)} f_{E_{\infty}}(t_i^{\perp}, t_i, t_i'^{\perp} - t_i^{\perp}, t_i' - t_i) \bar{K}_{\infty}(t - t') dt_i dt_i', \tag{4.14}
$$

one has

$$
\int_{Q_l^\perp(z_i^\perp)} \int_{Q_l^\perp(z_i^\perp)} \operatorname{Int}(t_i^\perp, t_i'^\perp) \, \mathrm{d} t_i^\perp \, \mathrm{d} t_i'^\perp \lesssim l^d N < +\infty \tag{4.15}
$$

Given $\lambda \in (0, \frac{l}{2})$ $\frac{l}{2}$, $u \in (z_i - l + \lambda, z_i + l - \lambda)$ and $t_i^{\perp} \in Q_l^{\perp}(z_i^{\perp})$, we denote by

$$
r_{\lambda}^{i}(u, t_{i}^{\perp}) := \min \left\{ \inf \{ |u - s| : s \in \partial E_{\infty, t_{i}^{\perp}} \text{ and } s \in (z_{i} - l + \lambda, z_{i} + l - \lambda) \}, |u - z_{i} + l - \lambda|, |z_{i} + l - \lambda - u| \right\}
$$

$$
r_{o}^{i}(t_{i}^{\perp}) := \inf_{s \in \partial E_{\infty, t_{i}^{\perp}} \cap Q_{i}^{i}(z_{i})} \min(s^{+} - s, s - s^{-}),
$$
(4.16)

where s^+, s^- are defined in [\(2.24\)](#page-10-0). Notice that, since $\int_{Q_l^{\perp}(z_i^{\perp})} \mathbb{1}_{B_i}^{\infty}(t_i^{\perp}) dt_i^{\perp} < +\infty$, for a.e. t_i^{\perp} $r_o^i(t_i^{\perp}) \geq 1.$

Notice that the map $r^i_\lambda(\cdot,t_i^{\perp})$ is well-defined for almost every t_i^{\perp} and measurable. The role of $\lambda > 0$ is to deal with the boundary, since E_{∞} is not $[0, l)^d$ -periodic.

Suppose that, for every u, one has that $r_{\lambda}^{i}(u, \cdot)$ is constant almost everywhere: if this holds for every i, then it is not difficult to see that E_{∞} is (up to null sets) either a union of stripes or a checkerboards, where by checkerboards we mean any set whose boundary is the union of affine subspace orthogonal to coordinate axes, and there are at least two of these directions.

The checkerboards can be ruled out via an energetic argument (see the comment at the end of this section).

In order to obtain that $r^i_\lambda(u, \cdot)$ is constant almost everywhere we proceed in the following way. In the next lemma we give a lower bound for the interaction term.

Lemma 4.5. Let $\lambda \in (0, L/2)$ and let $t_i^{\perp}, t_i^{\perp} \in Q_l^{\perp}(z_i^{\perp}), t_i^{\perp} \neq t_i^{\prime \perp}$ be such that $\min(r_o^i(t_i^{\perp}), r_o^i(t_i^{\perp})) >$ $|t_i^{\perp} - t_i^{\perp}|$ and $|t_i^{\perp} - t_i^{\perp}| \le \min\{\lambda, 1/2\}$. Then for every $u \in (z_i - l + \lambda, z_i + l - \lambda)$ it holds

$$
Int(t_i^{\perp}, t_i^{\perp}) \ge \mathbb{1}_{\{(t_i^{\perp}, t_i^{\perp}) : r_{\lambda}^i(u, t_i^{\perp}) \neq r_{\lambda}^i(u, t_i^{\perp})\}}^{*}(t_i^{\perp}, t_i^{\perp}). \tag{4.17}
$$

Proof. In this lemma we use a slicing argument similar to Lemma 3.5 in [\[11\]](#page-30-2). However the presence of a different kernel gives a different quantitative estimate.

W.l.o.g. let us assume that $r_\lambda^i(u, t_i^{\perp}) < r_\lambda^i(u, t_i'^{\perp})$. In particular this implies that $r_\lambda^i(u, t_i^{\perp}) <$ $\min(|u-z_i+l-\lambda|, |z_i+l-\lambda-u|)$, and hence there exists a point $s_o \in (z_i - l + \lambda, z_i + l - \lambda)$ such that

$$
|u - s_o| = \inf\{|u - s| : s \in \partial E_{\infty, t_i^{\perp}}, s \in (z_i - l + \lambda, z_i + l - \lambda)\}.
$$

Let us denote by $\delta = |t_i^{\perp} - t_i^{\perp}|$ and by $r = |r_\lambda^i(u, t_i^{\perp}) - r_\lambda^i(u, t_i^{\perp})|$. Given that $r_o(t_i^{\perp}) > \delta$, the following holds

$$
(s_o - \delta, s_o + \delta) \cap E_{\infty, t_i^{\perp}} = (s_o, s_o + \delta) \qquad \text{or} \qquad (s_o - \delta, s_o + \delta) \cap E_{\infty, t_i^{\perp}} = (s_o - \delta, s_o).
$$

Notice that since $\lambda \geq \delta$, we have that $(s_o - \delta, s_o + \delta) \subset Q_l^i(z_i)$. In the following, we will assume that

$$
(s_o - \delta, s_o + \delta) \cap E_{\infty, t_i^{\perp}} = (s_o, s_o + \delta)
$$
\n
$$
(4.18)
$$

The other case is analogous.

We will distinguish two subcases:

(i) Suppose $r > \delta/2$. From the definition of δ and r , for every slice in t_i^{\perp} it holds

$$
(s_o - \delta/2, s_o + \delta/2) \cap E_{\infty, t_i^{\perp}} = (s_o - \delta/2, s_o + \delta/2) \qquad \text{or} \qquad (s_o - \delta/2, s_o + \delta/2) \cap E_{\infty, t_i^{\perp}} = \emptyset.
$$

Indeed on the slice $E_{\infty,t_i^{\prime\perp}}$, the closest jump point to s_o is at least r distant and $r > \delta/2$. We will assume the first of the alternatives above. The other case is analogous.

For every $a \in (s_o - \delta/2, s_o)$ and $a' \in (s_o, s_o + \delta/2)$, one has that

$$
f_{E_{\infty}}(t_i^{\perp}, a, t_i'^{\perp} - t_i^{\perp}, a' - a) = 1.
$$

Given that $r_{\lambda}^{i}(u, t_{i}^{\perp}) \leq L$, one hs that

$$
\begin{split} \text{Int}(t_i^{\perp}, t_i'^{\perp}) &= \int_{Q_l^i(z_i)} \int_{Q_l^i(z_i)} f_{E_{\infty}}(t_i^{\perp}, t_i, t_i'^{\perp} - t_i^{\perp}, t_i' - t_i) \bar{K}_{\infty}(t' - t) \, \text{d}t_i \, \text{d}t_i' \\ &\geq \int_{s_o - \delta/2}^{s_o} \int_{s_o}^{s_o + \delta/2} f_{E_{\infty}}(t_i^{\perp}, t_i, t_i'^{\perp} - t_i^{\perp}, t_i' - t_i) \bar{K}_{\infty}(t' - t) \, \text{d}t_i' \, \text{d}t_i \\ &\geq \int_{s_o - \delta/2}^{s_o} \int_{s_o}^{s_o + \delta/2} \bar{K}_{\infty}(t' - t) \, \text{d}t_i' \, \text{d}t_i = +\infty, \end{split}
$$

since $|t-t'| < 1$ if $\max(|t_i^{\perp} - t_i'^{\perp}|, |t_i - t_i'|) \leq 1/2$ and therefore $\bar{K}_{\infty}(t-t') = +\infty$.

(ii) Let us assume now that $r \leq \delta/2$. Given that $r_o(t_i^{\perp}), r_o(t_i^{\perp}) > \delta$, one has that either

$$
(s_o - r, s_o + \delta/2) \cap E_{\infty, t_i'} = (s_o - r, s_o + \delta/2) \quad \text{or} \quad (s_o - r, s_o + \delta/2) \cap E_{\infty, t_i'} = \emptyset
$$

or
$$
(s_o - \delta/2, s_o + r) \cap E_{\infty, t_i'} = (s_o - \delta/2, s_o + r) \quad \text{or} \quad (s_o - \delta/2, s_o + r) \cap E_{\infty, t_i'} = \emptyset.
$$

Indeed if none of the above were true we would have that $\#(\partial E_{\infty,t_i'}\cap (s_o-\delta/2,s_o+\delta/2))\geq 2$, which contradicts $r_o(t_i^{\perp}) > \delta$. W.l.o.g. we will assume

$$
(s_o - r, s_o + \delta/2) \cap E_{\infty, t_i'} = (s_o - r, s_o + \delta/2).
$$

The other cases are similar.

Then for every $a \in (s_o-r, s_o)$ and $a' \in (s_o, s_o+\delta/2)$, one has that $f_{E_{\infty}}(t_i^{\perp}, a, t_i'^{\perp} - t_i^{\perp}, a' - a) = 1$. Thus

$$
\begin{split} \mathrm{Int}(t_i^{\perp},t_i'^{\perp}) &= \int_{Q_l^i(z_i)} \int_{Q_l^i(z_i)} f_{E_{\infty}}(t_i^{\perp},t_i,t_i'^{\perp}-t_i^{\perp},t_i'-t_i) \bar{K}_{\infty}(t'-t) \, \mathrm{d}t_i \, \mathrm{d}t_i' \\ &\geq \int_{s_o-r}^{s_o} \int_{s_o}^{s_o+\delta/2} f_{E_{\infty}}(t_i^{\perp},t_i,t_i'^{\perp}-t_i^{\perp},t_i'-t_i) \bar{K}_{\infty}(t'-t) \, \mathrm{d}t_i' \, \mathrm{d}t_i \\ &\geq \int_{s_o-r}^{s_o} \int_{s_o}^{s_o+\delta/2} \bar{K}_{\infty}(t'-t) \, \mathrm{d}t_i' \, \mathrm{d}t_i = \mathbb{1}_{\{(t_i'^{\perp},t_i^{\perp}): \, r_\lambda^i(u,t_i'^{\perp}) \neq r_\lambda^i(u,t_i^{\perp})\}}(t_i'^{\perp},t_i^{\perp}), \end{split}
$$

following at the end the same argument as in case (i).

 \Box

 \Box

Lemma 4.6. Assume that $E_{\infty} \subset \mathbb{R}^d$ is a set of locally finite perimeter such that [\(4.13\)](#page-18-1) holds. Let $\lambda \in (0, l/2)$ and let $r_{\lambda}^{i}(u, \cdot)$ be as defined in [\(4.16\)](#page-18-2). Then, we have that $r_{\lambda}^{i}(u, \cdot)$ is constant almost everywhere.

Proof. First of all, since $\int_{Q_l^{\perp}(z_i^{\perp})} \mathbb{1}_{B_i}^{\infty}(t_i^{\perp}) dt_i^{\perp} < +\infty$, $r_o(t_i^{\perp}) \ge 1$ for a.e. $t_i^{\perp} \in Q_l^{\perp}(z_i^{\perp})$. Let B be the set defined by

$$
B:=\left\{(t_i'^\perp,t_i^\perp)\in [0,L)^{d-1}\times [0,L)^{d-1}: r^i_\lambda(u,t_i^\perp)\neq r^i_\lambda(u,t_i'^\perp),\, |t_i'^\perp-t_i^\perp|\leq \min(\lambda,1/2)\right\}.
$$

Then, by [\(4.15\)](#page-18-0) and Lemma [4.5,](#page-18-3)

$$
\int_{Q_l^{\perp}(z_i^{\perp})} \int_{Q_l^{\perp}(z_i^{\perp})} \mathbb{1}_B^{\infty}(t_i^{\perp}, t_i'^{\perp}) \, \mathrm{d}t_i \perp \, \mathrm{d}t_i'^{\perp} \le \int_{Q_l^{\perp}(z_i^{\perp})} \int_{Q_l^{\perp}(z_i^{\perp})} \mathrm{Int}(t_i^{\perp}, t_i'^{\perp}) \, \mathrm{d}t_i^{\perp} \, \mathrm{d}t_i'^{\perp} \lesssim l^d N < +\infty. \tag{4.19}
$$

Hence, $r_{\lambda}^{i}(u, t_{i}^{\perp}) = r_{\lambda}^{i}(u, t_{i}^{\prime \perp})$ whenever $|t_{i}^{\perp} - t_{i}^{\prime \perp}| \leq \min(\lambda, 1/2)$ and therefore the statement of the lemma follows.

From the fact that $r^i_\lambda(u, \cdot)$ is constant almost everywhere for every u and for every i, one can deduce that E_{∞} must be a checkerboard or a union of stripes. We recall that by a checkerboard we mean any set whose boundary is the union of affine hyperplanes orthogonal to coordinate axes, and there are at least two of these directions.

However, the checkerboard can be ruled out immediately. To see this we consider the contribution to the energy given in a neighbourhood of an edge. W.l.o.g. we may assume that around this edge the set E_{∞} is of the following form $-\varepsilon \leq x_1 \leq 0$ and $-\varepsilon \leq x_2 \leq 0$ and $x_i \in (-\varepsilon, \varepsilon)$ for $i \neq 1, 2$. Notice that for every $|\zeta| < 1$ such that $\zeta_1 + x_1 > 0$, $\zeta_2 + x_2 > 0$ and $\zeta_i \in (-\varepsilon, \varepsilon)$ for $i \neq 1, 2$, the

integrand in $Int(x_1^{\perp}, (x_1 + \zeta_1)^{\perp})$ is equal to $+\infty$. Therefore also the first term in [\(4.13\)](#page-18-1) must be $+\infty$, which contradicts our assumptions.

Moreover, since the second term in the l.h.s. of [\(4.13\)](#page-18-1) explodes for stripes with minimal width tending to zero, one has that there exists $\bar{\eta} = \bar{\eta}(N, l) \ge 1$ such that $D_{\bar{\eta}}(E_{\infty}, Q_l(z)) = 0$. This contradicts that $D_{\bar{\eta}}(E_{\infty}, Q_l(z)) > \delta$, which was assumed at the beginning of the proof. contradicts that $D_{\bar{\eta}}(E_{\infty}, Q_l(z)) > \delta$, which was assumed at the beginning of the proof.

5 Proof of Theorem [1.1](#page-3-0)

The main purpose of this section is to prove Theorem [1.1.](#page-3-0)

The general strategy of the proof is similar to the one used to prove Theorem 1.4 in [\[11\]](#page-30-2). We refer to the outline of the proof of Theorem 1.4 in [\[11\]](#page-30-2) for a detailed overview of the ideas of the proof. Whenever a needed result is already present in [\[11\]](#page-30-2), we will refer to the appropriate lemma or proposition in [\[11\]](#page-30-2).

In order to simplify notation, we will use $A \lesssim B$, whenever there exists a constant \bar{C}_d depending only on the dimension d such that $A \leq \overline{C}_d B$.

For notational reasons it is convenient to introduce the one-dimensional analogue of [\(2.23\)](#page-10-1). Namely, let $E \subset \mathbb{R}$ be a set of locally finite perimeter and let $s^-, s, s^+ \in \partial E$. We define

$$
r_M(E,s) := -1 + \int_{\mathbb{R}} |\rho| \widehat{K}_M(\rho) d\rho - \int_{s^-}^s \int_0^{+\infty} |\chi_E(\rho + u) - \chi_E(u)| \widehat{K}_M(\rho) d\rho du - \int_s^{s^+} \int_{-\infty}^0 |\chi_E(\rho + u) - \chi_E(u)| \widehat{K}_M(\rho) d\rho du.
$$
 (5.1)

The quantities defined in [\(2.23\)](#page-10-1) and [\(5.1\)](#page-21-2) are related via $r_{i,M}(E, t_i^{\perp}, s) = r_M(E_{t_i^{\perp}}, s)$.

The following is a technical lemma needed in the proof of Lemma [5.3,](#page-23-0) analogous to Lemma 7.7 in [\[11\]](#page-30-2). It says that given a set ^E [⊂] **^R**, and ^I [⊂] **^R** an interval, then the one-dimensional contribution to the energy, namely $\sum_{s \in \partial E \cap I} r_M(E, s)$, is comparable to the periodic case up to a constant C_0 depending only on the dimension.

Lemma 5.1. There exists $C_0 > 0$ such that the following holds. Let $E \subset \mathbb{R}$ be a set of locally finite perimeter and $I \subset \mathbb{R}$ be an open interval. Let s^{-} , s and s^{+} be three consecutive points on the boundary of E and $r_M(E, s)$ defined as in [\(5.1\)](#page-21-2). Then there exists $M_0 > 0$ such that for all $M > M_0$ it holds

$$
\sum_{\substack{s \in \partial E \\ s \in I}} r_M(E, s) \ge e_M^* |I| - C_0. \tag{5.2}
$$

The proof is analogous to that of Lemma 7.7 in [\[11\]](#page-30-2) and therefore we omit it.

The next lemma is the so called local stability Lemma. Informally, it shows that if we are in a cube where the set $E \subset \mathbb{R}^d$ is close to a set E' which is a union of stripes in direction e_i (according to Definition [2.1\)](#page-11-2), then it is not convenient to oscillate in direction e_i with $j \neq i$ (namely, on the slices in direction e_i to have points in $\partial E_{t_i^{\perp}}$).

Lemma 5.2 (Local Stability). Let $(t_i^{\perp} + s e_i) \in (\partial E) \cap [0, l)^d$ and $0 < \eta_0 < 1$ and M_0 as Lemma [4.2.](#page-14-2) Then, for every $\varepsilon < \eta_0$ there exists $\tilde{M} = \tilde{M}(\tilde{\varepsilon}) > M_0$ such that for every $M > \tilde{M}$ the following holds: assume that

(a) min(|s – l|, |s|) > η_0

(b) $D_{\eta}^{j}(E, [0, l)^{d}) \leq \frac{\varepsilon^{d}}{16l}$ $\frac{\varepsilon^a}{16l^d}$ for some $\eta > 0$ and with $j \neq i$ (this condition expresses that $E \cap [0, l)^d$ is close to stripes oriented along a direction orthogonal to e_i)

Then
$$
r_{i,M}(E, t_i^{\perp}, s) + v_{i,M}(E, t_i^{\perp}, s) \geq 0.
$$

In the proof one uses a lower bound on the term $v_{i,M}$ which, basing on the Yukawa kernel \bar{K}_M , is different from the one obtained in [\[11\]](#page-30-2) and therefore we report it.

Proof. Let s^-, s, s^+ be three consecutive points for $\partial E_{t_i^{\perp}}$. By Lemma [4.2,](#page-14-2) for all $0 < \eta_0 < 1$, there exists $M_0 > 0$ such that if $M > M_0$

$$
\min(|s - s^{-}|, |s^{+} - s|) < \eta_0
$$
 then $r_{i,M}(E, t_i^{\perp}, s) > 0$.

Thus without loss of generality we may assume that $\min(|s - s^-|, |s^+ - s|) \ge \eta_0$. Thus, given that, for every $s, r_{i,M}(E, t_i^{\perp}, s) > -e^{-cM}$ for some $c > 0$ (see Lemma [4.2\)](#page-14-2), one has that

$$
r_{i,M}(E, t_i^{\perp}, s) + v_{i,M}(E, t_i^{\perp}, s) \ge -e^{-cM} + \frac{1}{2d} \int_{s^-}^{s^+} \int_{\mathbb{R}^d} f_E(t_i^{\perp}, u, \zeta_i^{\perp}, \zeta_i) K_M(\zeta) d\zeta du \tag{5.3}
$$

Let now $0 < \varepsilon < \eta_0$. By assumption, for some $t_i \in \partial E_{t_i^{\perp}}$ one of the following holds:

- (i) $(t_i \varepsilon, t_i) \subset E_{t_i^{\perp}}$ and $(t_i, t_i + \varepsilon) \subset E_{t_i^{\perp}}^c$
- (ii) $(t_i \varepsilon, t_i) \subset E_{t_i^{\perp}}^c$ and $(t_i, t_i + \varepsilon) \subset E_{t_i^{\perp}}$.

W.l.o.g., we may assume that (i) above holds and that $i = d$. As shown in [\[11,](#page-30-2) Lemma 6.1, Lemma 7.8], hypothesis (b) implies that

$$
\max\left(\frac{|Q_{\varepsilon}^{\perp}(t_d^{\perp}) \times (t_d - \varepsilon, t_d) \cap E^c|}{|Q_{\varepsilon}^{\perp}(t_d^{\perp}) \times (t_d - \varepsilon, t_d)|}, \frac{|Q_{\varepsilon}^{\perp}(t_d^{\perp}) \times (t_d, t_d + \varepsilon) \cap E|}{|Q_{\varepsilon}^{\perp}(t_d^{\perp}) \times (t_d - \varepsilon, t_d)|}\right) \ge \frac{7}{16}.
$$
\n(5.4)

Thus, we can further assume that

$$
(t_d - \varepsilon, t_d) \subset E_{t_d^{\perp}} \quad \text{and} \quad \frac{|Q_{\varepsilon}^{\perp}(t_d^{\perp}) \times (t_d - \varepsilon, t_d) \cap E^c|}{|Q_{\varepsilon}^{\perp}(t_d^{\perp}) \times (t_d - \varepsilon, t_d)|} \ge \frac{7}{16}.
$$

For every $s \in (t_d - \varepsilon, t_d)$, $(\zeta_d^{\perp}, s) \notin E$ and $\zeta_d + s \in (t_d, t_d + \varepsilon)$ we have that $f_E(t_d^{\perp}, s, \zeta_d^{\perp}, \zeta_d) = 1$. Thus by integrating initially in ζ_d and using [\(2.19\)](#page-9-2), we have that

$$
\int_{t_d-\varepsilon}^{t_d+\varepsilon} \int_{t_d-s}^{t_d+\varepsilon-s} \int_{Q_{\varepsilon}^{\perp}(t_d^{\perp})} f_E(t_d^{\perp}, s, \zeta_d^{\perp}, \zeta_d) K_M(\zeta) d\zeta_d^{\perp} d\zeta_d ds \ge
$$
\n
$$
\geq \frac{e^{M(\gamma_M-\varepsilon)}}{\varepsilon^{d-2}} \varepsilon \int_{Q_{\varepsilon}^{\perp}(t_d^{\perp})} \int_{t_d-\varepsilon}^{t_d} |\chi_{E_{t_d^{\perp}}} (s) - \chi_{E_{t_d^{\perp}+\zeta_d^{\perp}}} (s)| ds d\zeta_d^{\perp}
$$
\n
$$
\geq \frac{e^{M(\gamma_M-\varepsilon)}}{\varepsilon^{d-2}} \varepsilon \int_{Q_{\varepsilon}^{\perp}(t_d^{\perp})} \int_{t_d-\varepsilon}^{t_d} |1 - \chi_{E_{t_d^{\perp}+\zeta_d^{\perp}}} (s)| ds d\zeta_d^{\perp}
$$
\n
$$
\geq \frac{e^{M(\gamma_M-\varepsilon)}}{\varepsilon^{d-2}} \varepsilon |Q_{\varepsilon}^{\perp}(t_d^{\perp}) \times (t_d - \varepsilon, t_d) \cap E^c| \geq \frac{7e^{M(\gamma_M-\varepsilon)}\varepsilon^{d+1}}{16\varepsilon^{d-2}},
$$

which tends to $+\infty$ as $M \to +\infty$.

Therefore, for \tilde{M} sufficiently large depending on ε the r.h.s. of [\(5.3\)](#page-22-0) is positive. Up to a permutation of coordinates, this naturally holds also for $i = 2, \ldots, d - 1$. Therefore the lemma is proved.

The following Lemma, analogue of Lemma 7.9 in [\[11\]](#page-30-2), gives an estimate from below to the contribution of the energy on a segment of a slice in direction e_i .

Lemma 5.3. Let $0 < \eta_0 < 1$, \tilde{M} as in Lemma [5.2.](#page-21-0) Let $\delta = \varepsilon^d/(16l^d)$ with $0 < \varepsilon \leq \eta_0$, $M > \tilde{M}$ and $l > C_0/(-e_M^*)$, where C_0 is the constant appearing in Lemma [5.1.](#page-21-3) Let $t_i^{\perp} \in [0, L)^{d-1}$ and $\eta > 0$. The following statements hold: there exists C_1 constant independent of l (but depending on the dimension) such that

(i) Given $J \subset \mathbb{R}$ such that for every $s \in J$ it holds $D_{\eta}^{j}(E, Q_{l}(t_{i}^{\perp} + s e_{i})) \leq \delta$ with $j \neq i$, then

$$
\int_{J} \bar{F}_{i,M}(E, Q_l(t_i^{\perp} + s e_i)) \, \mathrm{d}s \ge -\frac{C_1}{l}.\tag{5.6}
$$

Moreover, if $J = [0, L)$, then

$$
\int_{J} \bar{F}_{i,M}(E, Q_l(t_i^{\perp} + s e_i)) \, \mathrm{d}s \ge 0. \tag{5.7}
$$

(ii) Given $J = (a, b) \subset \mathbb{R}$. If for $s = a$ and $s = b$ it holds $D_{\eta}^{j}(E, Q_{l}(t_{i}^{\perp} + s e_{i})) \leq \delta$ with $j \neq i$, then

$$
\int_{J} \bar{F}_{i,M}(E, Q_l(t_i^{\perp} + s e_i)) \, \mathrm{d}s \ge |J| e_M^* - \frac{C_1}{l},\tag{5.8}
$$

otherwise

$$
\int_{J} \bar{F}_{i,M}(E, Q_l(t_i^{\perp} + s e_i)) \, \mathrm{d}s \ge |J| e_M^* - C_1 l. \tag{5.9}
$$

Moreover, if $J = [0, L)$, then

$$
\int_{J} \bar{F}_{i,M}(E, Q_l(t_i^{\perp} + s e_i)) \, \mathrm{d}s \ge |J| e_M^*.
$$
\n(5.10)

Proof. For the proof we refer to Lemma 7.9 in [\[11\]](#page-30-2). C_1 correspond to M_0 there, M to τ , η_0 to $\tilde{\varepsilon}$ and e_M^* to C^*_τ . In the proof one uses Lemma [4.2](#page-14-2) and Lemma [5.2.](#page-21-0) Here, the estimate $r_{i,\tau}(E, t_i^{\perp}, s) \geq -1$ is replaced by $r_{i,M}(E,t_i^{\perp},s) \geq -e^{-cM}$ (see Lemma [4.2\)](#page-14-2).

The purpose of the next lemma is to give a lower bound on the energy in the case that almost all the volume of $Q_l(z)$ is filled by E or E^c .

Lemma 5.4. Let E be a set of locally finite perimeter such that $\min(|Q_l(z) \setminus E|, |E \cap Q_l(z)|) \leq \nu l^d$, for some $\nu > 0$. Then

$$
\bar{F}_M(E, Q_l(z)) \geq -\frac{e^{-cM}\nu d}{\eta_0},
$$

where $\eta_0 < 1$, provided $M \geq M_0(\eta_0)$ as in Lemma [4.2.](#page-14-2)

 \Box

For the proof we refer to Lemma 7.11 in [\[11\]](#page-30-2), substituting the lower bound $r_{i,\tau}(E, t_i^{\perp}, s) \ge -1$ with $r_{i,M}(E,t_i^{\perp},s) \geq -e^{-cM}$ and δ with ν .

Given the preliminary lemmas, the proof of Theorem [1.1](#page-3-0) is analogous to that of [\[11\]](#page-30-2)[Theorem 1.4]. The sets defined in the proof and the main estimates depend on a set of parameters l, δ, ρ, N, η and M. The validity of the theorem relies as in [\[11\]](#page-30-2) on a suitable choice of such parameters. Since at this point the proof does not present novelties w.r.t. [\[11\]](#page-30-2), we omit it referring to [\[11\]](#page-30-2)[Section 7.3] with the following substitutions: τ is replaced by M and whenever τ in [\[11\]](#page-30-2) has to be chosen smaller than some quantity, M has to be larger than some quantity; C^* is replaced by e^* and C^*_{τ} by e_M^* . The other parameters remain the same.

6 A nonlocal to local Γ-limit

As discussed in [\[2,](#page-30-0) [9,](#page-30-1) [23,](#page-31-0) [19\]](#page-31-1), one of the possible models used to show gelification in charged colloids and pattern formation is to consider both as attractive and as repulsive term the Yukawa potential, with different signs and appropriate rescaling.

We therefore consider the following functional: for $E \subset \mathbb{R}^d$, $d \geq 3$, $L > 0$, $J > 0$ and $\beta > 1$, let

$$
\tilde{\mathcal{E}}_{\beta,J,L}(E) := \frac{1}{L^d} \Big(J C_{\beta,L} \int_{[0,L)^d} \int_{\mathbb{R}^d} |\chi_E(x+\zeta) - \chi_E(x)| K_{\beta}(\zeta) d\zeta dx \n- \int_{[0,L)^d} \int_{\mathbb{R}^d} |\chi_E(x+\zeta) - \chi_E(x)| K_1(\zeta) d\zeta dx \Big),
$$
\n(6.1)

where $C_{\beta,L}$ is a positive normalization constant defined in [\(6.2\)](#page-24-1) depending on β and L and K_{β}, K_1 are the Yukawa kernels with parameters β and 1 and with the 1-norm. The aim of this section is to prove Theorem [1.4.](#page-3-1)

6.1 The normalization constant

We compute here the normalization constant $C_{\beta,L}$ which allows the first term of [\(6.1\)](#page-24-2) to Γ-converge to the 1-perimeter as $\beta \to +\infty$. Let $\overline{E} \subset \mathbb{R}^d \ [0, L)^d$ -periodic be such that $\overline{E} \cap [0, L)^d = [L/2, L) \times [0, L)^{d-1}$. Let $H^- := [0, L/2) \times$ $[0, L)^{d-1}$ and $H^+ := [L/2, L) \times [-L/2, 3/2L)^{d-1}.$ We define $C_{\beta,L}$ as

$$
C_{\beta,L} := L^{d-1} \Big(\int_{H^-} \int_{H^+} |\chi_E(x) - \chi_E(y)| K_{\beta}(x - y) \, dx \, dy \Big)^{-1} \tag{6.2}
$$

We give now bounds from above and from below for $C_{\beta,L}$ which are independent of L. By definition,

$$
\int_{H^{-}}\int_{H^{+}}|\chi_{E}(x)-\chi_{E}(y)|K_{\beta}(x-y)\,dx\,dy =
$$
\n
$$
=\int_{0}^{L/2}\int_{L/2}^{L}e^{-\beta|x_{1}-y_{1}|}\int_{[0,L)^{d-1}}\int_{[-L/2,3/2L)^{d-1}}\frac{e^{-\beta|x_{1}^{\perp}-y_{1}^{\perp}|_{1}}}{(|x_{1}-y_{1}|+|x_{1}^{\perp}-y_{1}^{\perp}|_{1})^{d-2}}\,dy_{1}^{\perp}dx_{1}^{\perp}dy_{1}dx_{1}.
$$

Therefore,

$$
\int_{H^{-}} \int_{H^{+}} |\chi_{E}(x) - \chi_{E}(y)| K_{\beta}(x - y) dx dy \le
$$
\n
$$
\leq L^{d-1} \int_{0}^{L/2} \int_{L/2}^{L} e^{-\beta(y_{1} - x_{1})} \int_{[0,2L)^{d-1}} \frac{e^{-\beta|\zeta_{1}^{+}|_{1}}}{((y_{1} - x_{1}) + |\zeta_{1}^{+}|_{1})^{d-2}} d\zeta_{1}^{+} dy_{1} dx_{1}
$$
\n
$$
\leq L^{d-1} \int_{0}^{L/2} \int_{L/2}^{L} e^{-\beta(y_{1} - x_{1})} \int_{0}^{2L/(y_{1} - x_{1})} \left(\frac{1}{1+t}\right)^{d-2} e^{-\beta t(y_{1} - x_{1})} (y_{1} - x_{1}) dt dy_{1} dx_{1}
$$
\n
$$
\leq L^{d-1} \int_{0}^{L/2} \int_{L/2}^{L} e^{-\beta(y_{1} - x_{1})} \int_{0}^{2L/(y_{1} - x_{1})} e^{-\beta t(y_{1} - x_{1})} (y_{1} - x_{1}) dt dy_{1} dx_{1}
$$
\n
$$
\leq \frac{L^{d-1}}{\beta^{3}} (1 - e^{-2L\beta}) (1 - e^{-\beta L/2})^{2} \tag{6.3}
$$

On the other hand,

$$
\int_{H^{-}} \int_{H^{+}} |\chi_{E}(x) - \chi_{E}(y)| K_{\beta}(x - y) dx dy \ge
$$
\n
$$
\geq L^{d-1} \int_{0}^{L/2} \int_{L/2}^{L} e^{-\beta(y_{1} - x_{1})} \int_{[0, L/2)^{d-1}} \frac{e^{-\beta|\zeta_{1}^{+}|_{1}}}{((y_{1} - x_{1}) + |\zeta_{1}^{+}|_{1})^{d-2}} d\zeta_{1}^{+} dy_{1} dx_{1}
$$
\n
$$
\geq L^{d-1} \int_{0}^{L/2} \int_{L/2}^{L} e^{-\beta(y_{1} - x_{1})} \int_{1}^{(L)/(2(y_{1} - x_{1}))} \left(\frac{1}{1+t}\right)^{d-2} e^{-\beta t(y_{1} - x_{1})} (y_{1} - x_{1}) dt dy_{1} dx_{1}
$$
\n
$$
\geq \frac{L^{d-1}}{\beta^{3}} \alpha(\beta, L) \tag{6.4}
$$

where $1 \ge \alpha(\beta, L) \ge \bar{\alpha} > 0$ for all $\beta \ge 1, L \ge \bar{L} > 0$. Therefore, $C_{\beta,L}$ satisfies

$$
0 < \bar{c}\beta^3 \le C_{\beta,L} \le \beta^3 \bar{C} < +\infty \tag{6.5}
$$

with \bar{c}, \bar{C} independent of L, β provided $\beta \geq 1, L \geq \bar{L} > 0$.

6.2 Γ-convergence

The main result of this section is the following

Theorem 6.1. The functionals $\tilde{\mathcal{E}}_{\beta,J,L}$ defined in [\(6.1\)](#page-24-2) Γ -convergence in the L^1 topology as $\beta \to +\infty$ and up to subsequences to the functional $\tilde{F}_{J,L}$ defined in [\(1.3\)](#page-1-2).

Since the second term in (6.1) is continuous w.r.t. L^1 -convergence, in order to prove the theorem it is sufficient to show that

$$
\mathcal{P}_{\beta}(\cdot) = C_{\beta,L} \int_{[0,L)^d} \int_{\mathbb{R}^d} |\chi_{(\cdot)}(x+\zeta) - \chi_{(\cdot)}(x)| K_{\beta}(\zeta) d\zeta dx \quad \xrightarrow[\beta \to +\infty]{\Gamma} \text{Per}_1(\cdot,[0,L)^d). \tag{6.6}
$$

W.l.o.g. we consider $d = 2$.

Let $\{E_{\beta}\}_\beta$ be a sequence of $[0, L)^2$ -periodic sets with $\sup_{\beta} \mathcal{P}_{\beta}(E_{\beta}) < +\infty$.

Let then $\alpha \in (0, 1/2)$ and for all $\beta > 1$ define

$$
A_{\alpha,\beta}:=\Big\{t\in[0,L)^2:\,\frac{|E_\beta\cap Q_{1/\beta}(t)|}{|Q_{1/\beta}(t)|}\in(\alpha,1-\alpha)\Big\}.
$$

For every $t \in A_{\alpha,\beta}$

$$
\mathcal{P}_{\beta}(E_{\beta}) \ge C_{\beta,L} \int_{Q_{1/\beta}(t)} \int_{Q_{1/\beta}(t)} |\chi_{E_{\beta}}(x) - \chi_{E_{\beta}}(y)| K_{\beta}(x-y) dx dy \ge c \frac{\alpha(1-\alpha)}{\beta}, \qquad (6.7)
$$

since $C_{\beta,L}$ goes like β^3 (see [\(6.5\)](#page-25-0)) and since when $|x-y| \leq 1/\beta$ the kernel K_β is bounded from below by a constant.

Let now $N(\beta,\alpha) \in \mathbb{N}$ be the maximal number of disjoint cubes $Q_{1/\beta}(t_i)$ centred in $t_i \in A_{\alpha,\beta}$ of side length $1/\beta$.

One has that

$$
\mathcal{P}_{\beta}(E_{\beta}) \ge cN(\alpha, \beta)\frac{\alpha(1-\alpha)}{\beta}
$$

and from the uniform upper bound on $\mathcal{P}_{\beta}(E_{\beta})$

$$
N(\alpha, \beta) \le c(\alpha)\beta.
$$

For sure $A_{\alpha,\beta} \subset \bigcup_{i=1}^{N(\alpha,\beta)} Q_{4/\beta}(t_i)$, from which it follows that

$$
|A_{\alpha,\beta}| \le c(\alpha)\beta \frac{16}{\beta^2} \quad \longrightarrow \quad 0 \quad \text{ as } \beta \to +\infty. \tag{6.8}
$$

Let g be any weak^{*}-L[∞] limit of subsequences of χ_{E_β} , as $\beta \to +\infty$. Then for any $t \in \mathbb{R}$ one has that $g \in [0,1]$. Moreover, from the above reasoning, for every $\alpha \in (0,1/2)$ there exists a null set X_{α} such that, for all $t \in [0,L)^2 \setminus X_{\alpha}$ either $g(t) \geq 1 - \alpha$ or $g(t) \leq \alpha$. From this it follows that $g = \chi_E$ for some $[0, L)^2$ -periodic set E.

In particular, the weak^{*}- L^{∞} convergence of E_{β} to E can be upgraded to strong L^{1} convergence. We claim that E is of finite perimeter in $[0, L)^2$. Indeed, consider the set

$$
E^{1/2} := \left\{ t \in \mathbb{R}^2 : \exists \lim_{r \to 0} \int_{B_r(t)} \chi_E(u) \, \mathrm{d}u = 1/2 \right\}.
$$

By Federer's characterization of the sets of finite perimeter, one has that $\mathcal{H}^1(E^{1/2} \cap [0, L)^2) < +\infty$ if and only if the set E is of finite perimeter.

Let us consider a fine covering of $E^{1/2} \cap [0, L)^2$ with cubes $\{Q_{r(t)}(t)\}_{t \in \mathcal{T}}$ such that

$$
\frac{|E \cap Q_{r(t)}(t)|}{|Q_{r(t)}(t)|} \in \left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right). \tag{6.9}
$$

Thanks to the covering Theorem of Besicovitch, there exist $N = N(d)$ collections of disjoint cubes $\{Q_{r(t_i^j)}(t_i^j)$ $\{G_i\}\}_{i\in\mathcal{T}_j} \subset \{Q_{r(t)}(t)\}_{t\in\mathcal{T}}, \ j=1,\ldots,N$ such that

$$
E^{1/2} \subset \bigcup_{j=1,\ldots,N} \bigcup_{i \in \mathcal{T}_j} Q_{r(t_i^j)}(t_i^j).
$$

As a consequence,

$$
\mathcal{H}^{1}(E^{1/2}) \leq \sum_{j=1}^{N} \sum_{i \in \mathcal{T}_{j}} \sqrt{2} r(t_{i}^{j})
$$
\n(6.10)

In order to prove that the r.h.s. of (6.10) is bounded, we claim the following: for β large enough, the number T_i^j ⁱ of disjoint cubes of side length $1/\beta$, $\{Q_{1/\beta}(t_m)\}_{m=1}^{T_i^j}$ contained in $Q_{r(t_i^j)}(t_i^j)$ i, j such that $\frac{|E \cap Q_{1/\beta}(t_m)|}{|Q_{1/\beta}(t_m)|} \in \left[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right]$ is bigger or equal than $cr(t_i^j)\beta$ for some con i) for fixed $\left[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right]$ is bigger or equal than $cr(t_i^j)$ i^j) β for some constant $c > 0$. Before proving the claim, let us see how this gives an upper bound for [\(6.10\)](#page-27-0). Since the sets E_β converge in L^1 to E, then for β sufficiently large and independent of i, j

$$
\frac{|E_{\beta} \cap Q_{1/\beta}(t_m)|}{|Q_{1/\beta}(t_m)|} \in \left(\frac{1}{2} - 2\varepsilon, \frac{1}{2} + 2\varepsilon\right).
$$

Therefore,

$$
\mathcal{P}_{\beta}(E_{\beta}) \geq \sum_{i \in \mathcal{T}_j} \frac{\bar{c}}{4\beta^4} C_{\beta,L} T_i^j \geq \sum_{i \in \mathcal{T}_j} \frac{\bar{c}}{4\beta^4} C_{\beta,L} c r(t_i^j) \beta \geq \sum_{i \in \mathcal{T}_j} \tilde{c} r(t_i^j),
$$

from which by the upper bound on $\mathcal{P}_{\beta}(E_{\beta})$ the finiteness of [\(6.10\)](#page-27-0) follows. We now prove the lower bound on T_i^j i contained in the claim. Define the following sets:

$$
A_{-} := \left\{ x \in Q_{r(t_i^j)}(t_i^j) : \frac{|E \cap Q_{1/\beta}(x)|}{|Q_{1/\beta}(x)|} < \frac{1}{2} - \varepsilon \right\},\newline A_{+} := \left\{ x \in Q_{r(t_i^j)}(t_i^j) : \frac{|E \cap Q_{1/\beta}(x)|}{|Q_{1/\beta}(x)|} > \frac{1}{2} + \varepsilon \right\},\newline A := \left\{ x \in Q_{r(t_i^j)}(t_i^j) : \frac{|E \cap Q_{1/\beta}(x)|}{|Q_{1/\beta}(x)|} \in \left[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right] \right\}.
$$

The set A separates A₋ and A₊, meaning that for every segment [x, y] connecting $x \in A$ - with $y \in A_+$ there exists $z \in [x, y]$ with $z \in A$. Therefore if we show that

$$
\operatorname{Per}(\partial A_{\pm}) \ge cr(t_i^j),\tag{6.11}
$$

the claim is proved. The lower bound [\(6.11\)](#page-27-1) is a consequence of the isoperimetric inequality applied to A_{-} or A_{+} . Indeed, the measure of each of A_{\pm} is bigger or equal than $r(t_i^j)$ (i) ²/4 for β sufficiently large depending only on $E \cap [0, L)^2$, being almost all points in the sets $E \cap Q_{r(t_i^j)}$ and $E^c \cap Q_{r(t_i^j)}$ respectively of density 1 and 0 and of total measure bigger or equal than $(1/2 - \varepsilon)r(t_i^j)$ $i^j)^2.$ Let us call $\mathcal P$ a Γ-limit, up to subsequences, of the l.h.s. of [\(6.6\)](#page-25-1). The next step to prove Theorem [6.1](#page-25-2) is the following

Lemma 6.2.

$$
\mathcal{P}(E) = \int_{\partial E \cap [0,L)^d} \phi(\nu_E(x)) \, d\mathcal{H}^{d-1}(x),\tag{6.12}
$$

where ∂E is the reduced boundary of E, $\phi(\nu) = \lim_{\varepsilon \to 0}$ $\mathcal{P}(E_{\nu}\cap [0,\varepsilon)^d)$ $\frac{\nu \sqcup \{0,\varepsilon\} \sqcup \cdots}{\varepsilon^{d-1}}$ and $E_{\nu} = \{x \cdot \nu \leq 0\}.$ Thanks to the results obtained in [\[4\]](#page-30-9), if one shows that, for all sets E with $\partial E \in C^2$

$$
\frac{1}{C}(1 + \mathcal{H}^{d-1}(E)) \le \mathcal{P}(E) \le C(1 + \mathcal{H}^{d-1}(E))
$$
\n(6.13)

then the representation Lemma [6.2](#page-27-2) holds.

Let us then prove [\(6.13\)](#page-28-0). First of all, consider the upper bound. Being the boundary of $E \cap [0, L)^d$ compact and of class C^2 , there exists a covering of it with finitely many cubes $\{C(x_j, \nu_j, h_j)\}_{j=1}^N$, where $x_j \in \mathbb{R}^d$, $\nu_j \in \mathcal{S}^{d-1}$, $h_j > 0$, of the form

$$
C(x_j, \nu_j, h_j) = \{ y \in \mathbb{R}^d : y_j := (y - x_j) \cdot \nu_j \le h_j, \quad |y - y_j \nu_j - x_j|_{\infty} \le h_j \}
$$
(6.14)

and such that in each of the cubes $4(C(x_j, \nu_j, h_j) - x_j)$ the boundary of $E - x_j$ is the graph of a Lipschitz function of the plane $\{y \cdot \nu_j = 0\}$, with Lipschitz constant bounded by some constant independent of j.

Let $C_j := C(x_j, \nu_j, h_j), E_j := E \cap C_j.$ Then,

$$
C_{\beta,L} \int_{[0,L)^d} \int_{\mathbb{R}^d} |\chi_E(x+\zeta) - \chi_E(x)| \frac{e^{-\beta|\zeta|_1}}{|\zeta|_1^{d-2}} d\zeta dx \le
$$

$$
\leq C + \sum_{j=1}^N C_{\beta,L} \int_{C_j} \int_{C_{j-1} \cup C_j \cup C_{j+1} - x} |\chi_E(x+\zeta) - \chi_E(x)| \frac{e^{-\beta|\zeta|_1}}{|\zeta|_1^{d-2}} d\zeta dx \qquad (6.15)
$$

where the sets C_0 and C_{N+1} are defined by periodicity and C is independent of β . This is due to the decay of the kernel, which gives finite weight to interactions at distances bigger than $\min_i h_i$. Now, let us consider one of the contributions above where we assume for simplicity that $\nu_j = e_1$, $x_j = 0, \varepsilon = \max\{h_{j-1}, h_j, h_{j+1}\}\$ and we denote by Φ the Lipschitz map that maps $H_{\varepsilon}^- = [-\varepsilon/2, 0] \times$ $[-\varepsilon/2, \varepsilon/2]^{d-1}$ in E_j and $H_{\varepsilon}^+ = [0, \varepsilon/2] \times [-3/2\varepsilon, 3/2\varepsilon]^{d-1}$ in $(2C_j) \setminus (E_j \cup E_{j-1} \cup E_{j+1})$:

$$
C_{\beta,L} \int_{C_j} \int_{(C_{j-1} \cup C_j \cup C_{j+1})-x} |\chi_E(x+\zeta) - \chi_E(x)| \frac{e^{-\beta|\zeta|_1}}{|\zeta|_1^{d-2}} d\zeta dx \sim
$$

$$
\sim C_{\beta,L} \int_{H_{\varepsilon}^-} \int_{H_{\varepsilon}^+} |\chi_E(\Phi(x)) - \chi_E(\Phi(y))| \frac{e^{-\beta|\Phi(x) - \Phi(y)|_1}}{|\Phi(x) - \Phi(y)|_1^{d-2}} |D\Phi|(x) |D\Phi|(y) dx dy. \quad (6.16)
$$

Carrying on analogous calculations to those of Section [6.1](#page-24-3) one obtains that the above limit as $\beta \to +\infty$ is less or equal than a constant times ε^{d-1} , namely comparable to the measure of $\partial(E \cap C_i)$. Therefore, the estimate from above is proved.

Now, let us prove the estimate from below in [\(6.13\)](#page-28-0). To this aim, take a Besicovitch covering of ∂E with cylinders $\{C_j^{\alpha}\}_{j=1,\dots,N_{\alpha}}$ such that for all $\alpha \in \{1,\dots,N_0\}$ the sets $\{C_j^{\alpha}\}_{j=1,\dots,N_{\alpha}}$ are disjoint and such that in each of the cubes $4(C_j^{\alpha} - x_j^{\alpha})$ the boundary of $E - x_j^{\alpha}$ is the graph of a Lipschitz function of the plane $\{y \cdot \nu_j^{\alpha} = 0\}$ with Lipschitz constant uniformly bounded in j. Then

$$
C_{\beta,L} \int_{[0,L)^d} \int_{\mathbb{R}^d} |\chi_E(x+\zeta) - \chi_E(x)| \frac{e^{-\beta|\zeta|_1}}{|\zeta|_1^{d-2}} d\zeta dx \ge
$$

$$
\geq -C + \sum_{j=1}^{N_\alpha} C_{\beta,L} \int_{C_j^\alpha} \int_{C_{j-1}^\alpha \cup C_j^\alpha \cup C_{j+1}^\alpha - x} |\chi_E(x+\zeta) - \chi_E(x)| \frac{e^{-\beta|\zeta|_1}}{|\zeta|_1^{d-2}} d\zeta dx \quad (6.17)
$$

where the sets C_0^{α} and C_{N+1}^{α} are defined by periodicity and C is independent of β . This is due again to the decay of the kernel and the fact that the sets $\{C_j^{\alpha}\}_{j=1,\dots,N_{\alpha}}$ are disjoint. After making a change of variables with the map Φ^{α} that maps part of the sets $E \cap C_j^{\alpha}$, $E \cap C_{j-1}^{\alpha}$, $E \cap C_{j+1}^{\alpha}$ into adjacent half squares of side length $\min\{h_j, h_{j-1}, h_{j+1}\}$ The single contributions of the sets C_j^{α} can be estimated in the same way as in the estimates from below in Section [6.1,](#page-24-3) leading to something of the order of $\partial (E \cap C_j^{\alpha})$. Applying the same reasoning to the other families $\{C_j^{\beta}\}$ $\{f_j^{\beta}\}\,\,\text{with}\,\,\beta\neq\alpha,$ $\beta \in \{1, \ldots, N_0\}$ one obtains the estimate from below as well.

The second step to prove Theorem [6.1](#page-25-2) after Lemma [6.2](#page-27-2) is to characterize the function ϕ in [\(6.12\)](#page-27-3). W.l.o.g. we consider $d = 2$. Then the kernel is given by $K_{\beta}(\zeta) := -e^{-\beta |\zeta|_1} \ln(|\zeta|_1)$. Let us recall

$$
|\chi_E(x) - \chi_E(x+\zeta)| = |\chi_E(x) - \chi_E(x+\zeta_1)| + |\chi_E(x+\zeta_1) - \chi_E(x+\zeta)|
$$

- 2 $|\chi_E(x) - \chi_E(x+\zeta_1)||\chi_E(x+\zeta_1) - \chi_E(x+\zeta)|$.

Integrating and using the $[0, L)^2$ -periodicity of E

$$
\int_{[0,L)^2} \int_{\mathbb{R}^2} |\chi_E(x) - \chi_E(x+\zeta)| K_{\beta}(\zeta) d\zeta dx = \int_{[0,L)^2} \int_{\mathbb{R}^2} |\chi_E(x) - \chi_E(x+\zeta_1)| K_{\beta}(\zeta) d\zeta dx
$$

+
$$
\int_{[0,L)^2} \int_{\mathbb{R}^2} |\chi_E(x) - \chi_E(x+\zeta_2)| K_{\beta}(\zeta) d\zeta dx
$$

-
$$
-2 \int_{[0,L)^2} \int_{\mathbb{R}^2} |\chi_E(x) - \chi_E(x+\zeta_1)| |\chi_E(x) - \chi_E(x+\zeta_2)| K_{\beta}(\zeta) d\zeta dx \qquad (6.18)
$$

Let now E be given, up to translations, of the form $\{x \cdot \nu \leq 0\} \cap [-2\varepsilon, 2\varepsilon]^2$. Then,

$$
C_{\beta,L} \int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} \int_{\mathbb{R}} \int_{\mathbb{R}} |\chi_E(x) - \chi_E(x + \zeta_1)| K_{\beta}(\zeta_1, \zeta_2) d\zeta_2 d\zeta_1 dx_1 dx_2 =
$$

$$
C_{\beta,L} \int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} \int_{\mathbb{R}} |\chi_{E_{x_2}}(x_1) - \chi_{E_{x_2}}(x_1 + \zeta_1)| \widehat{K}_{\beta}(\zeta_1) d\zeta_1 dx_1 dx_2 \quad (6.19)
$$

which converges, as $\beta \to +\infty$, to

$$
\int_{-\varepsilon}^{\varepsilon} \int_{\partial E_{x_2}} d\mathcal{H}^0(x_1) dx_2 = \text{Per}_{11}(E, [\varepsilon, \varepsilon)^2). \tag{6.20}
$$

Analogously, the second term in [\(6.18\)](#page-29-0) converges to

$$
\int_{-\varepsilon}^{\varepsilon} \int_{\partial E_{x_1}} d\mathcal{H}^0(x_2) dx_1 = \text{Per}_{12}(E, [\varepsilon, \varepsilon)^2). \tag{6.21}
$$

We claim that the third term in [\(6.18\)](#page-29-0) is of lower order and therefore converges to 0 as $\beta \to +\infty$. We have that

$$
C_{\beta,L} \int_{[-\varepsilon,\varepsilon)^2} \int_{\mathbb{R}^2} |\chi_E(x) - \chi_E(x + \zeta_1)| |\chi_E(x) - \chi_E(x + \zeta_2)| K_\beta(\zeta) d\zeta dx \sim
$$

$$
\sim C_{\beta,L} \int_0^{\varepsilon} \int_{-\varepsilon}^{x_2 \tan \theta} \int_{-x_1 + x_2 \tan \theta}^{+\infty} \int_{-x_2 + \frac{x_1}{\tan \theta}} e^{-\beta |\zeta_1|} e^{-\beta |\zeta_2|} (-\ln(|\zeta_1| + |\zeta_2|)) d\zeta_2 d\zeta_1 dx_1 dx_2,
$$
(6.22)

where $\zeta_1 + \zeta_2 \ge -x_2 + \frac{x_1}{\tan \theta} - x_1 + x_2 \tan \theta$, θ is the minus the angle between e_1 and ν and, w.l.o.g. is assumed to be between 0 and $-\pi/4$.

Since now in comparison to Section [6.1](#page-24-3) the variable x_2 appears as well in the independent variable of integration, simple estimates show that such term goes like $\frac{C_{\beta,L}}{\beta^4}$ and therefore vanishes as $\beta \to +\infty$.

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