Stochastic LQ and Associated Riccati equation of PDEs Driven by State-and Control-Dependent White Noise

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Abstract

The optimal stochastic control problem with a quadratic cost functional for linear partial differential equations (PDEs) driven by a state- and control-dependent white noise is formulated and studied. Both finite- and infinite-time horizons are considered. The multiplicative white noise dynamics of the system give rise to a new phenomenon of singularity to the associated Riccati equation and even to the Lyapunov equation. Well-posedness of both Riccati equation and Lyapunov equation are obtained for the first time. The linear feedback coefficient of the optimal control turns out to be singular and expressed in terms of the solution of the associated Riccati equation. The null controllability is shown to be equivalent to the existence of the solution to Riccati equation with the singular terminal value. Finally, the controlled Anderson model is addressed as an illustrating example.
Keywords: linear quadratic optimal stochastic control, multiplicative space-time white noise, stochastic partial differential equation, Riccati equation, null controllability, singular terminal condition.

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1 Introduction

In this paper, we consider the following stochastic evolutionary equation driven by both state- and control-dependent white noise:

$$dX_t = (AX_t + B_t u_t)\, dt + \sum_{j=1}^{\infty}(C_j(t)X_t + D_j(t)u_t)\, d\beta_t^j, \quad X_0 = x \in H$$

where A is the infinitesimal generator of a strongly continuous semigroup e^{tA} of linear operators, B, C_j, and D_j are some bounded operators, and W is a cylindrical Wiener process in a Hilbert space H, with $\{\beta^j(t) := \langle W(t), e_j\rangle, j = 1, 2, \ldots\}$ being independent Brownian motions for an orthonormal basis $\{e_j, j = 1, 2, \ldots\}$ of H. The cost functional is

$$J(x, u) = \mathbb{E}\int_0^T [\langle Q_t X_t, X_t\rangle + \langle R_t u_t, u_t\rangle]\, dt + \mathbb{E}[\langle GX_T, X_T\rangle],$$

where Q, G, and R are some bounded operators. The optimal control problem is to find a U-valued adapted square-integrable process $\overline{u}$ in a feedback form (via the associated Riccati equation) such that $J(x, \overline{u})$ is the minimal value of the cost functional $J(x, \cdot)$. More precise formulation will be given in the next section.

The general theory of linear quadratic optimal control (the so-called LQ theory) of Kalman [15] paved one mile stone in the deterministic optimal control theory. The general stochastic extension in a Euclidean space was given by Wonham [21] for the deterministic coefficients, and was further developed by Bismut [1] for the random coefficients. Subsequently, it was further studied by Peng [17] and Tang [19], and its theory is now rather complete.

Ichikawa [13, 14] considered the infinite-dimensional extension of Kalman's LQ theory under the following setting: H is an infinite-dimensional Hilbert space and C is a bounded linear operator. Da Prato and Ichikawa [6] studied the infinite-dimensional LQ problem for the case of $D = 0$, self-adjoint A, and unbounded coefficient B. The infinite dimensional case with stochastic coefficients driven by the so-called colored noise (where C is a Hilbert-Schmidt operator) is referred to Guatteri and Tessitore [10]. To our best knowledge, all the above-mentioned papers are restricted within the case when the linear SPDEs are driven by the so-called colored noise, which excludes the celebrated Anderson model. In this paper, we address the infinite dimensional stochastic LQ problem driven by an infinite number of Brownian motions (the so-called space-time white noise).

The introduction of the space-time white noise leads to the difficulty that the infinite sum $\sum_{i=1}^{\infty} C_i^*(s)P_s C_i(s)$ appears in both associated Lyapunov equation (3.9) and Riccati equation (4.3), and thus challenges the solvability of both equations. To overcome this difficulty for Lyapunov equation (3.9), we introduce the representation via the solution of forward SPDE to establish an estimate of the sum, and for more details, see our Proposition 3.3 and its proof. It is conventional to study the Riccati equation via the quasi-linearization method. While in our context of the space-time white noise, the coefficients of these quasi-linearized equations become singular in the sense that these coefficients explode at both ends (time 0 and time T). Some fine estimates are applied to deduce the monotonicity and convergence of solutions of quasi-linearized equations. For more details, see our Theorem 4.4 and its proof. Finally, due to the space-time white noise in our context, the conventional Yosida's approximation could not be applied to get the energy equality, and to attack the new difficulty, a new truncation is carefully constructed to deduce the energy equality and thus the feedback law of the optimal control. For more details, see our Theorem 5.2 and its proof.

We note that Anderson model has been widely studied in the litterature, and for more details, see Carmona and Molchanov [3], Conus, Joseph, and Khoshnevisan [4], and the references therein. We also emphasize that our results succeed at inclusion of the controlled Anderson SPDE. See Section 8.

The paper is organized as follows. In Section 2, we give the precise formulation of our quadratic optimal stochastic control problem for linear partial differential equations driven by a white noise. In Section 3, we study well-posedness of Lyapunov equations. In Section 4, we study the associated Riccati equation. In Section 5, we characterize the optimal control as a feedback form via the solution of Riccati equation. In Section 6, we address the infinite-horizon LQ control problem for the case of time-invariant coefficients. We show that when the system is stabilizable, the associated algebraic Riccati equation has a unique solution, and is again used to synthesize the optimal control into a feedback form. In Section 7, the null controllability is proved to be equivalent to the existence of solution of Riccati equation with the singular terminal condition. Finally in Section 8, we give examples for the controlled Anderson model.

2 Formulation of the linear quadratic optimal control

Let H, U be two separable Hilbert spaces. By $\mathcal{S}(H)$, we denote the space of all self-adjoint and bounded linear operators on Hand by $\mathcal{S}^+(H)$ we denote the set of all non-negative operators in $\mathcal{S}(H)$. Moreover, if $I \subset \mathbb{R}^+$ is an interval (bounded or unbounded), we denote by $C_s(I;\mathcal{S}(H))$ (resp. $C_s(I;\mathcal{S}^+(H))$) the set of all maps $f : I \to \mathcal{S}(H)$ (resp. $f : I \to \mathcal{S}^+(H)$) such that $f(\cdot)$ is strongly continuous in H.

Consider the following stochastic evolutionary equation driven by both state- and control-dependent white noise:

$$dX_t = (AX_t + B_t u_t)\, dt + \sum_{j=1}^{\infty} (C_j(t)X_t + D_j(t)u_t)\, d\beta_t^j, \quad X_0 = x \in H, \tag{2.1}$$

which has the following mild form:

$$X_t = e^{At}x + \int_0^t e^{A(t-s)} B_s u_s\, ds + \int_0^t \sum_{j=1}^{\infty} e^{A(t-s)} (C_j(s)X_s + D_j(s)u_s)\, d\beta_s^j. \tag{2.2}$$

Here, A is the infinitesimal generator of a strongly continuous semigroup e^{tA} of linear operators, $B \in L^\infty(0,T;\mathcal{L}(U,H)), C_j \in L^\infty(0,T;\mathcal{L}(H)), D_j \in L^\infty(0,T;\mathcal{L}(U,H))$ with the standard assumption that for some $\alpha \in (0,\frac{1}{2})$ and $c > 0$,

$$\sum_{j=1}^\infty |e^{At}C_j(s)x|_H^2 \le ct^{-2\alpha}|x|_H^2, \quad t>0. \tag{2.3}$$

W is a cylindrical Wiener process in H, $\{\beta^j(t) := \langle W(t), e_j\rangle, j = 1,2,\dots\}$ are independent Brownian motions, with $\{e_j, j = 1,2,\dots\}$ being an orthonormal basis of H. The cost functional is

$$J(x,u) = \mathbb{E}\int_0^T [\langle Q_t X_t, X_t\rangle + \langle R_t u_t, u_t\rangle]\, dt + \mathbb{E}[\langle GX_T, X_T\rangle], \quad u \in L^2_{\mathcal{F}}(0,T;U) \tag{2.4}$$

where $Q \in L^\infty(0,T;\mathcal{S}^+(H)), G \in \mathcal{S}^+(H)$, and $R \in L^\infty(0,T;\mathcal{S}^+(U))$ is strictly positive in the following sense: there is a positive number δ such that $R \ge \delta I_U$. Throughout the paper, we assume that for any $v \in U$, there is a constant $c > 0$ such that

$$\sum_{j=1}^\infty |D_j v|_H^2 \le c|v|_U^2. \tag{2.5}$$

The optimal control problem is to minimize $J(x,\cdot)$ among all the controls in $L^2_{\mathcal{F}}(0,T;U)$.

Remark 2.1 *Condition* (2.5) *means that* $D = (D_1, D_2, \dots)$ *is a Hilbert-Schmidt operator. It is still open how to replace this condition with a condition like* (2.3)*.*

Lemma 2.2 *For* $u \in \mathcal{L}^2_{\mathcal{F}}(0,T;U)$*, the system* (2.1) *has a unique mild solution* X *in the space* $C_{\mathcal{F}}([0,T];L^2(\Omega,H))$ *such that for some* $C > 0$,

$$\sup_{0\le t\le T} \mathbb{E}[|X_t|^2] \;\le\; C\left(|x|^2 + \mathbb{E}\int_0^T |u_s|^2 ds\right). \tag{2.6}$$

Proof. The existence and uniqueness of the mild solution can be found in [8]. We now derive the desired estimate for the solution. From (2.2), (2.3), and (2.5), we have

$$\begin{aligned}
\mathbb{E}[|X_t|^2] \;&\le\; C\left(|x|^2 + \mathbb{E}\int_0^t |u_s|^2 ds\right) \\
&\quad + C\mathbb{E}\int_0^t \left(\sum_{j=1}^\infty |e^{A(t-s)}C_j(s)X_s|^2 + \sum_{j=1}^\infty |e^{A(t-s)}D_j(s)u_s|^2\right) ds \\
&\le\; C\left(|x|^2 + \mathbb{E}\int_0^t |u_s|^2 ds\right) + C\mathbb{E}\int_0^t \sum_{j=1}^\infty (t-s)^{-2\alpha}|X_s|_H^2\, ds.
\end{aligned}$$

Using an extended Gronwall's inequality (see, e.g. [12]), we have the desired estimate. □

3 Lyapunov equation: existence and uniqueness of solutions

We first give results on Lyapunov equation, which will be needed in the study of Riccati equation.

3.1 Forward SDE

Let $A_0, \widehat{C}_j \in L^\infty(0,T;\mathcal{L}(H))$ with $j = 1, 2, \ldots$.

Assume that for some number $c > 0$,

$$|A_0(s)|_{\mathcal{L}(H)} \le c(T-s)^{-\alpha}, \quad s \in [0,T), \tag{3.1}$$

and

$$\sum_{j=1}^{\infty} |e^{At}\widehat{C}_j(s)x|_H^2 \le c\left(t^{-2\alpha} + (T-s)^{-2\alpha}\right)|x|_H^2, \quad (t,s) \in (0,\infty)\times[0,T). \tag{3.2}$$

Remark 3.1 *Assumptions* (3.1) *and* (3.2) *are introduced to study the quasi-linearized sequence of Lyapunov equations for the original nonlinear Riccati equation. Note that both assumptions admit explosion at time T.*

Consider the following forward evolution equation: given the initial data (t,x),

$$dY_s = (A + A_0(s))Y_s ds + \sum_{i=1}^{\infty} \widehat{C}_i(s)Y_s d\beta_s^i, \quad s \in (t,T]. \tag{3.3}$$

Lemma 3.2 *Let Assumptions* (3.1) *and* (3.2) *hold true. There is a unique mild solution to* (3.3) *satisfying*

$$\sup_{t\le s\le T} \mathbb{E}|Y_s|_H^2 \le C|x|^2$$

for a positive constant C.

Proof. First we prove the uniqueness. Consider two solutions Y^1 and Y^2. Define $\Delta Y := Y^1 - Y^2$. We have

$$\Delta Y_s = \int_t^s e^{A(s-r)}A_0(r)\Delta Y_r dr + \sum_{i=1}^{\infty}\int_t^s e^{A(s-r)}\widehat{C}_i(r)\Delta Y_r d\beta_r^i, \quad s \in [t,T]$$

and

$$\begin{aligned}
\mathbb{E}[|\Delta Y_s|^2] &\le 2\mathbb{E}\left|\int_t^s e^{A(s-r)}A_0(r)\Delta Y_r dr\right|^2 + 2\mathbb{E}\left|\sum_{i=1}^{\infty}\int_t^s e^{A(s-r)}\widehat{C}_i(r)\Delta Y_r d\beta_r^i\right|^2 \\
&\le C\int_t^s \left((s-r)^{-2\alpha} + (T-r)^{-2\alpha}\right)\mathbb{E}[|\Delta Y_r|^2]\, dr \\
&\le 2C\int_t^s (s-r)^{-2\alpha}\mathbb{E}[|\Delta Y_r|^2]\, dr.
\end{aligned} \tag{3.4}$$

Thus, $\mathbb{E}[|\Delta Y_r|^2] = 0$, and the uniqueness is proved.

Then we prove the existence. Define by Picard's iteration: $Y^0 \equiv 0$, and for $n \geq 0$,

$$Y_s^{n+1} = e^{A(s-t)}x + \int_t^s e^{A(s-r)} A_0(r) Y_r^n \, dr + \sum_{i=1}^{\infty} \int_t^s e^{A(s-r)} \widehat{C}_i(r) Y_r^n \, d\beta_r^i.$$

Thus, we have

$$\mathbb{E}\left[|Y_s^{n+1}|^2\right] \leq 3|e^{A(s-t)}x|^2 + 2C \int_t^s (s-r)^{-2\alpha} E\left[|Y_r^n|^2\right] \, dr.$$

Denote by γ the solution of the following integral equation:

$$\gamma_s = 3|e^{A(s-t)}x|^2 + 2C \int_t^s (s-r)^{-2\alpha} \gamma_r \, dr, \quad s \in (t, T]. \tag{3.5}$$

By recurrence, we have $\mathbb{E}\left[|Y_s^n|^2\right] \leq \gamma_s$, for $s \in [t, T]$.

Now we show that $\{Y^n, n \geq 0\}$ is a Cauchy sequence in $C_{\mathcal{F}}([t, T]; L^2(\Omega, H))$. We have

$$\begin{aligned} Y_s^{n+k+1} - Y_s^{n+1} &= \int_t^s e^{A(s-r)} A_0(r) \left(Y_r^{n+k} - Y_r^n\right) dr \\ &\quad + \sum_{i=1}^{\infty} \int_t^s e^{A(s-r)} \widehat{C}_i(r) \left(Y_r^{n+k} - Y_r^n\right) d\beta_r^i, \\ \mathbb{E}\left[|Y_s^{n+k+1} - Y_s^{n+1}|^2\right] &\leq 2C \int_t^s (s-r)^{-2\alpha} \mathbb{E}\left[|Y_r^{n+k} - Y_r^n|^2\right] dr. \end{aligned}$$

Define

$$\phi_s = \limsup_n \sup_k \sup_{t \leq r \leq s} \mathbb{E}\left[|Y_r^{n+k+1} - Y_r^{n+1}|^2\right].$$

We have

$$\sup_{t \leq r \leq s} \mathbb{E}\left[|Y_r^{n+k+1} - Y_r^{n+1}|^2\right] \leq 2C \int_t^s (s-r)^{-2\alpha} \mathbb{E}\left[|Y_r^{n+k} - Y_r^n|^2\right] dr, \tag{3.6}$$

$$\phi_s \leq 2C \int_t^s (s-r)^{-2\alpha} \phi_r \, dr. \tag{3.7}$$

This shows that $\phi = 0$ and $\{Y^n\}$ is a Cauchy sequence in $C_{\mathcal{F}}([0, T]; L^2(\Omega, H))$, and the existence of solution is proved. □

3.2 Lyapunov equation

Let $G \in \mathcal{S}(H)$ and $f \in L^1(0,T;\mathcal{S}(H))$. Assume that for $\alpha \in (0,\frac{1}{2})$,

$$|f(s)|_{\mathcal{L}(H)} \leq c(T-s)^{-2\alpha}. \tag{3.8}$$

Consider the following form of Lyapunov equation

$$\begin{cases} P_t' + A^*P_t + P_tA + A_0^*(t)P_t + P_tA_0(t) + \sum_{i=1}^{\infty} \widehat{C}_i^*(t)P_t\widehat{C}_i(t) + f_t = 0, \quad t \in [0,T); \\ P_T = G. \end{cases} \tag{3.9}$$

We look for a mild solution:

$$\begin{aligned} P_t &= e^{A^*(T-t)}Ge^{A(T-t)} + \int_t^T e^{A^*(s-t)}f_se^{A(s-t)}ds \\ &\quad + \int_t^T e^{A^*(s-t)}\left[A_0^*(s)P_s + P_sA_0(s) + \sum_{i=1}^{\infty}\widehat{C}_i^*(s)P_s\widehat{C}_i(s)\right]e^{A(s-t)}ds. \end{aligned} \tag{3.10}$$

Using Yosida's approximation, we can prove that the following Lyapunov equation (associated to a finite number of Brownian motions)

$$\begin{aligned} P_t^n &= e^{A^*(T-t)}Ge^{A(T-t)} + \int_t^T e^{A^*(s-t)}f_se^{A(s-t)}ds \\ &\quad + \int_t^T e^{A^*(s-t)}\left[A_0^*(s)P_s^n + P_s^nA_0(s) + \sum_{i=1}^{n}\widehat{C}_i^*(s)P_s^n\widehat{C}_i(s)\right]e^{A(s-t)}ds, \end{aligned} \tag{3.11}$$

has a unique solution $P^n \in C_s([0,T],\mathcal{S}(H))$ (see, e.g. Da Prato [5]).

Proposition 3.3 *Let Assumptions* (3.1), (3.2) *and* (3.8) *hold true. Then, P^n converges weakly to a bounded solution $P \in C_s([0,T];\mathcal{S}(H))$ of* (3.10) *satisfying the estimate for some positive constant C,*

$$\sum_{i=1}^{\infty}\left|\langle\widehat{C}_i(s)^*P_s\widehat{C}_i(s)x,x\rangle\right| \leq C(T-s)^{-2\alpha}|x|^2, \quad s \in [0,T).$$

Moreover, we have the following representation of P:

$$\langle P_tx,x\rangle = \mathbb{E}\left[\langle GY_T^{t,x},Y_T^{t,x}\rangle + \int_t^T\langle f_sY_s^{t,x},Y_s^{t,x}\rangle ds\right], \quad t \in [0,T] \tag{3.12}$$

where $Y^{t,x}$ is the mild solution to (3.3).

Proof. For each interger n, let $Y^{n,t,x}$ be the mild solution of

$$dY_s = [A + A_0(s)]\, Y_s ds + \sum_{i=1}^{n} \widehat{C}_i(s) Y_s d\beta_s^i, \quad s \in (t, T]; \quad Y_t = x.$$

We have the following representation:

$$\langle P_t^n x, x\rangle = \mathbb{E}\left[\langle G Y_T^{n,t,x}, Y_T^{n,t,x}\rangle + \int_t^T \langle f_s Y_s^{n,t,x}, Y_s^{n,t,x}\rangle ds\right]. \tag{3.13}$$

Since

$$\lim_{n\to+\infty} \sup_{t\le s\le T} \mathbb{E}[|Y_s^{n,t,x} - Y_s^{t,x}|^2] = 0$$

where $Y^{t,x}$ is the mild solution to (3.3), there exists $P_t \in \mathcal{S}(H)$ such that P_t^n converges to P_t weakly and we have by passing to the limit in (3.13) the desired representation (3.12).

Set $z_i = \widehat{C}_i(t)x$.

Let us estimate $\sum_{i=1}^{\infty} |\langle P_t z_i, z_i\rangle|$. We have

$$\begin{aligned} |\langle P_t z_i, z_i\rangle| &= \left|\mathbb{E}\left[\langle G Y_T^{t,z_i}, Y_T^{t,z_i}\rangle + \int_t^T \langle f_s Y_s^{t,z_i}, Y_s^{t,z_i}\rangle ds\right]\right| \\ &\le C\mathbb{E}[||Y_T^{t,z_i}||^2] + C\int_t^T (T-s)^{-2\alpha}\mathbb{E}[||Y_s^{t,z_i}||^2] ds. \end{aligned} \tag{3.14}$$

As Y^{t,z_i} is the mild solution of the following equation

$$Y_s^{t,z_i} = e^{A(s-t)} z_i + \int_t^s e^{A(s-r)} A_0(r) Y_r^{t,z_i} dr + \int_t^s e^{A(s-r)} \sum_{i=1}^{\infty} \widehat{C}_i(r) Y_r^{t,z_i} d\beta_r^i,$$

we have

$$\begin{aligned} \mathbb{E}[||Y_s^{t,z_i}||^2] &\le C||e^{A(s-t)} z_i||^2 + C\int_t^s \left((s-r)^{-2\alpha} + (T-r)^{-2\alpha}\right) \mathbb{E}[||Y_r^{t,z_i}||^2] dr \\ &\le C||e^{A(s-t)} z_i||^2 + 2C\int_t^s (s-r)^{-2\alpha}\mathbb{E}[||Y_r^{t,z_i}||^2] dr. \end{aligned} \tag{3.15}$$

Note that

$$\sum_{i=1}^{n} ||e^{A(s-t)} z_i||^2 = \sum_{i=1}^{n} ||e^{A(s-t)} \widehat{C}_i(t) x||^2 \le C\left((s-t)^{-2\alpha} + (T-t)^{-2\alpha}\right) ||x||^2 \le 2C(s-t)^{-2\alpha}||x||^2.$$

Finally, we get from (3.15) that

$$\sum_{i=1}^{n} \mathbb{E}[||Y_s^{t,z_i}||^2] \le C(s-t)^{-2\alpha}||x||^2 + C\int_t^s (s-r)^{-2\alpha} \sum_{i=1}^{n} \mathbb{E}[||Y_r^{t,z_i}||^2] dr.$$

By the generalized Gronwall's inequality (see Henry [12]), we have

$$\sum_{i=1}^{n}\mathbb{E}[||Y_s^{t,z_i}||^2] \;\leq\; C(s-t)^{-2\alpha}||x||^2,$$

and then letting $n\to\infty$, we have

$$\sum_{i=1}^{\infty}\mathbb{E}[||Y_s^{t,z_i}||^2] \leq C(s-t)^{-2\alpha}||x||^2. \tag{3.16}$$

Furthermore from (3.14) and (3.16), we have

$$\begin{aligned}
\sum_{i=1}^{\infty}\left|\langle \widehat{C}_i(t)^*P_t\widehat{C}_i(t)x,x\rangle\right| &= \sum_{i=1}^{\infty}|\langle P_tz_i,z_i\rangle| \\
&\leq C(T-t)^{-2\alpha}+C\int_t^T(T-s)^{-2\alpha}(s-t)^{-2\alpha}|x|_H^2\,ds \\
&= C(T-t)^{-2\alpha}+C\int_0^1(T-t)^{-2\alpha}(1-r)^{-2\alpha}(T-t)^{-2\alpha}r^{-2\alpha}(T-t)dr \\
&= C(T-t)^{-2\alpha}+C(T-t)^{1-4\alpha}\leq C(1+T^{1-2\alpha})(T-t)^{-2\alpha}.
\end{aligned}$$

Passing to the limit in (3.11) by letting $n\to\infty$, we prove that P is the solution to (3.10). □

Theorem 3.4 *There exists a unique solution $P\in C_s([0,T],\mathcal{S}(H))$ for (3.9) such that*

$$\sum_{i=1}^{\infty}\left|\langle \widehat{C}_i(s)^*P_s\widehat{C}_i(s)x,x\rangle\right| \leq C(T-s)^{-2\alpha}|x|^2.$$

Proof. The existence of solution is already proved in the preceding proposition. Now we prove the uniqueness.

Let $\tilde{P}$ be a solution, then it satisfies the following truncated Riccati equation:

$$\begin{aligned}
\tilde{P}_t &= e^{A^*(T-t)}Ge^{A(T-t)}+\int_t^T e^{A^*(s-t)}\sum_{i=1}^{n}\widehat{C}_i^*(s)\tilde{P}_s\widehat{C}_i(s)e^{A(s-t)}\,ds \\
&\quad +\int_t^T e^{A^*(s-t)}(f_s+\sum_{i=n+1}^{\infty}\widehat{C}_i^*(s)\tilde{P}_s\widehat{C}_i(s))e^{A(s-t)}\,ds.
\end{aligned} \tag{3.17}$$

We have from (3.13) the following representation

$$\langle \tilde{P}_t x, x\rangle = \mathbb{E}\left[\langle GY_T^{n,t,x}, Y_T^{n,t,x}\rangle + \int_t^T \langle (f_s + \sum_{i=n+1}^{\infty} \widehat{C}_i^*(s)\tilde{P}_s\widehat{C}_i(s))Y_s^{n,t,x}, Y_s^{n,t,x}\rangle ds\right].$$

By passing to the limit, we deduce

$$\langle \tilde{P}_t x, x\rangle = \mathbb{E}\left[\langle GY_T^{t,x}, Y_T^{t,x}\rangle + \int_t^T \langle f_s Y_s^{t,x}, Y_s^{t,x}\rangle ds\right],$$

from which we deduce the uniqueness. □

From (3.12), we deduce also the following a priori estimate.

Proposition 3.5 *Let $P \in C_s([0,T], \mathcal{S}(H))$ be the unique solution, then the following a priori estimate holds:*

$$|P_t| \le C(|G|_{\mathcal{L}(H)} + \int_t^T |f_s|_{\mathcal{L}(H)} ds).$$

4 Riccati equation: existence and uniqueness of solutions

In this section, we study the Riccati equation associated to the linear-quadratic optimal control problem (2.2) and (2.4). Let us first state a lemma which will be used later.

Lemma 4.1 *Let assumption* (2.5) *hold true. For $P \in \mathcal{S}^+(H)$ such that for any $x \in H$,*

$$\sum_{j=1}^{\infty} \langle C_j^*(t)PC_j(t)x, x\rangle < \infty.$$

Then, $\sum_{j=1}^{N} D_j^*(t)PC_j(t)$ converges strongly, whose limit is denoted by $\sum_{j=1}^{\infty} D_j^*(t)PC_j(t)$ and satisfies the following estimate:

$$\left|\sum_{j=1}^{\infty} D_j^*(t)PC_j(t)x\right|_U \le C\left(\sum_{j=1}^{\infty}\langle C_j^*(t)PC_j(t)x, x\rangle\right)^{\frac{1}{2}}, \quad x \in H$$

for some constant $C > 0$.

Proof. In view of Assumption (2.5),

$$\begin{aligned}\left|\sum_{j=M+1}^{N} D_j^*(t)PC_j(t)x\right|_U &= \sup_{|y|_U\le 1}\left\langle \sum_{j=M+1}^{N} D_j^*(t)PC_j(t)x, y\right\rangle\\ &\le \sup_{|y|_U\le 1}\left(\sum_{j=M+1}^{N}|P^{\frac12}C_j(t)x|_H^2\right)^{\frac12}\left(\sum_{j=M+1}^{N}|P^{\frac12}D_j(t)y|_H^2\right)^{\frac12}\\ &\le C\left(\sum_{j=M+1}^{N}|P^{\frac12}C_j(t)x|_H^2\right)^{\frac12}\\ &= C\left(\sum_{j=M+1}^{N}\langle C_j^*(t)PC_j(t)x, x\rangle\right)^{\frac12}.\end{aligned}$$

Hence the sequence $\sum_{j=1}^{N} D_j^*(t)PC_j(t)x$ is a Cauchy one, and we have the desired result. □

Define for $P \in \mathcal{S}^+(H)$,

$$\Lambda(t,P) := R_t + \sum_{i=1}^{\infty} D_i^*(t)PD_i(t).$$

Since $\Lambda(t,P) \ge \delta I_U$, we see that $\Lambda(t,P)$ has an inverse $\Lambda(t,P)^{-1} \le \frac{1}{\delta}I_U$.

Define for $s \in [0,T]$ and $P \in \mathcal{S}^+(H)$,

$$\lambda(s,P) := -\Lambda(s,P)^{-1}\left(B_s^*P + \sum_i D_i^*(s)PC_i(s)\right),$$

and for $P \in \mathcal{S}^+(H)$ such that $\sum_{j=1}^{\infty}\langle C_j^*(t)PC_j(t)x, x\rangle < \infty$ for each $x \in H$,

$$\begin{aligned}\hat{B}(t,P) &:= -B_t\Lambda(t,P)^{-1}\left(B_t^*P + \sum_j D_j^*(t)PC_j(t)\right) = B_t\lambda(t,P),\\ \hat{C}_i(t,P) &:= C_i(t) - D_i(t)\Lambda(t,P)^{-1}\left(B_t^*P + \sum_j D_j^*(t)PC_j(t)\right) = C_i(t) + D_i(t)\lambda(t,P).\end{aligned}$$

We have

Lemma 4.2 *For $P \in C_s([0,T], \mathcal{S}^+(H))$ such that*

$$|\sum_{i=1}^{\infty} C_i^*(s)P_sC_i(s)|_{\mathcal{L}(H)} \le C(T-s)^{-2\alpha},$$

we have

$$\sum_i |e^{At}\widehat{C}_i(s,P_s)x|^2 \le c\left(t^{-2\alpha} + (T-s)^{-2\alpha}\right)|x|^2,$$

$$|Q_s + \lambda^*(s,P_s)R_s\lambda(s,P_s)|_{\mathcal{L}(H)} \le c(T-s)^{-2\alpha},$$

$$|\widehat{B}(s,P_s)|_{\mathcal{L}(H)} \le c(T-s)^{-\alpha}.$$

Proof. The third inequality is obvious. We now prove the first inequality.

$$\begin{aligned}
& \sum_i |e^{At}\widehat{C}_i(s,P_s)x|^2 \\
\le\ & 2\sum_i |e^{At}C_i(s)x|^2 + 2\sum_i \left| e^{At}D_i(s)\Lambda(s,P_s)^{-1}\left(B_s^*P_s + \sum_j D_j^*(s)P_sC_j(s)\right)x\right|^2 \\
\le\ & 2ct^{-2\alpha}|x|^2 + 2c|e^{At}|_{\mathcal{L}(H)}\left|\Lambda(s,P_s)^{-1}\left(B_s^*P_s + \sum_j D_j^*(s)P_sC_j(s)\right)x\right|^2 \\
\le\ & 2ct^{-2\alpha}|x|^2 + C|x|^2 + C\left\langle \sum_j C_j^*(s)P_sC_j(s)x, x\right\rangle \\
\le\ & c\left(t^{-2\alpha} + (T-s)^{-2\alpha}\right)|x|^2. \qquad (4.1)
\end{aligned}$$

It remains to prove the second inequality. We have for each $x \in H$, since $R_s \le \Lambda(s,P_s)$,

$$\begin{aligned}
& \langle \lambda^*(s,P_s)R_s\lambda(s,P_s)x, x\rangle \\
\le\ & \left\langle (P_sB_s + \sum_i C_i^*(s)P_sD_i(s))\Lambda(s,P_s)^{-1}(B_s^*P_s + \sum_i D_i^*(s)P_sC_i(s))x, x\right\rangle \\
\le\ & 2\left\langle P_sB_s\Lambda(s,P_s)^{-1}B_s^*P_sx, x\right\rangle + 2\left\langle \sum_j C_j^*(s)P_sD_j(s)\Lambda(s,P_s)^{-1}\sum_i D_i^*(s)P_sC_i(s)x, x\right\rangle \\
\le\ & 2\left\langle P_sB_s\Lambda(s,P_s)^{-1}B_s^*P_sx, x\right\rangle + 2\left\langle \Lambda(s,P_s)^{-1}\sum_i D_i^*(s)P_sC_i(s)x, \sum_i D_i^*(s)P_sC_i(s)x\right\rangle \\
\le\ & c(T-s)^{-2\alpha}|x|_H^2.
\end{aligned}$$

□

Let us consider the general Riccati equation:

$$\begin{aligned}
P_t \ =\ & e^{A^*(T-t)}Ge^{A(T-t)} + \int_t^T e^{A^*(s-t)}\sum_{i=1}^{\infty} C_i^*(s)P_sC_i(s)e^{A(s-t)}ds \\
& + \int_t^T e^{A^*(s-t)}\left(Q_s - \lambda^*(s,P_s)\Lambda(s,P_s)\lambda(s,P_s)\right)e^{A(s-t)}ds, \quad t \in [0,T]. \qquad (4.2)
\end{aligned}$$

It is equivalent to the following form:

$$
\begin{aligned}
P_t &= e^{A^*(T-t)}Ge^{A(T-t)} + \int_t^T e^{A^*(s-t)} \left(\hat{B}^*(s,P_s)P_s + P_s\hat{B}(s,P_s)\right) e^{A(s-t)}ds \\
&\quad + \int_t^T e^{A^*(s-t)} \sum_{i=1}^{\infty} \hat{C}_i^*(s,P_s)P_s\hat{C}_i(s,P_s)e^{A(s-t)}ds \\
&\quad + \int_t^T e^{A^*(s-t)} \left(Q_s + \lambda^*(s,P_s)R_s\lambda(s,P_s)\right) e^{A(s-t)}\, ds, \quad t \in [0,T]. \qquad (4.3)
\end{aligned}
$$

Our existence proof will make use of the following quasi-linearized sequence $\{P^N\}$ defined by the following Lyapunov equations: $P^0 \equiv 0$, and

$$
\begin{aligned}
P_t^{N+1} &= e^{A^*(T-t)}Ge^{A(T-t)} + \int_t^T e^{A^*(s-t)} \left(\hat{B}^*(s,P_s^N)P_s^{N+1} + P_s^{N+1}\hat{B}(s,P_s^N)\right) e^{A(s-t)}ds \\
&\quad + \int_t^T e^{A^*(s-t)} \sum_{i=1}^{\infty} \hat{C}_i^*(s,P_s^N)P_s^{N+1}\hat{C}_i(s,P_s^N)e^{A(s-t)}ds \\
&\quad + \int_t^T e^{A^*(s-t)} \left(Q_s + \lambda^*(s,P_s^N)R_s\lambda(s,P_s^N)\right) e^{A(s-t)}\, ds, \quad N = 0,1,\ldots. \qquad (4.4)
\end{aligned}
$$

Note that if $P^N \in C_s([0,T],\mathcal{S}^+(H))$ and satisfies the inequality for a positive constant c which might depend on N:

$$
|\sum_i C_i^* P_s^N C_i(s)|_{\mathcal{L}(H)} \leq c(T-s)^{-2\alpha},
$$

then we see from Lemma 4.2 and Theorem 3.4 that the preceding Lyapunov equation (4.4) has a unique solution $P^{N+1} \in C_s([0,T],\mathcal{S}^+(H))$ satisfying also the last inequality. Since obviously P^0 satisfies the last inequality, we can define by induction a sequence P^{N+1} satisfying Lyapunov equation (4.4) for $N \geq 0$.

Lemma 4.3 *The sequence $\{P_t^N, N \geq 1\}$ is a non-increasing sequence of self-adjoint operators for each $t \in [0,T]$.*

Proof. Now we show that $P_t^N \geq P_t^{N+1}$ for $N \geq 1$.

Define $\Delta P_t^N := P_t^N - P_t^{N+1}, \quad t \in [0,T]$. We have

$$\begin{aligned}
\Delta P_t^N &= \int_t^T e^{A^*(s-t)} \left(\hat{B}^*(s, P_s^{N-1}) P_s^N + P_s^N \hat{B}(s, P_s^{N-1})\right) e^{A(s-t)} ds \\
&+ \int_t^T e^{A^*(s-t)} \sum_{i=1}^{\infty} \hat{C}_i^*(s, P_s^{N-1}) P_s^N \hat{C}_i(s, P_s^{N-1}) e^{A(s-t)} ds \\
&+ \int_t^T e^{A^*(s-t)} \lambda^*(s, P_s^{N-1}) R_s \lambda(s, P_s^{N-1}) e^{A(s-t)}\, ds \\
&- \int_t^T e^{A^*(s-t)} \left(\hat{B}^*(s, P_s^N) P_s^{N+1} + P_s^{N+1} \hat{B}(s, P_s^N)\right) e^{A(s-t)} ds \\
&- \int_t^T e^{A^*(s-t)} \sum_{i=1}^{\infty} \hat{C}_i^*(s, P_s^N) P_s^{N+1} \hat{C}_i(s, P_s^N) e^{A(s-t)} ds \\
&- \int_t^T e^{A^*(s-t)} \lambda^*(s, P_s^N) R_s \lambda(s, P_s^N) e^{A(s-t)}\, ds. \qquad (4.5)
\end{aligned}$$

Define for $K \in \mathcal{L}(H,U)$ and $P \in \mathcal{S}^+(H)$ such that $\sum_{i=1}^{\infty} \langle C_i^*(s) P C_i(s) x, x\rangle < \infty$,

$$F(s,K,P) := (B_s K)^* P + P B_s K + \sum_{i=1}^{\infty} \left[C_i(s) + D_i(s) K\right]^* P \left[C_i(s) + D_i(s) K\right] + K^* R_s K. \quad (4.6)$$

We have for $K \in L(H,U)$,

$$F(s,K,P) = F(s, \lambda(s,P), P) + \left[K - \lambda(s,P)\right]^* \Lambda(s,P) \left[K - \lambda(s,P)\right] \geq F(s, \lambda(s,P), P).$$

Equality (4.5) can be written into the following form:

$$\begin{aligned}
\Delta P_t^N &= \int_t^T e^{A^*(s-t)} \left(\hat{B}^*(s, P_s^N) \Delta P_s^N + \Delta P_s^N \hat{B}(s, P_s^N)\right) e^{A(s-t)} ds \\
&+ \int_t^T e^{A^*(s-t)} \sum_{i=1}^{\infty} \hat{C}_i(s, P_s^N) \Delta P_s^N \hat{C}_i(s, P_s^N) e^{A(s-t)} ds \\
&+ \int_t^T e^{A^*(s-t)} \left[F(s, \lambda(s, P_s^{N-1}), P_s^N) - F(s, \lambda(s, P_s^N), P_s^N)\right] e^{A(s-t)} ds. \quad (4.7)
\end{aligned}$$

Note that $F(s, \lambda(s, P_s^{N-1}), P_s^N) - F(s, \lambda(s, P_s^N), P_s^N) \in \mathcal{S}^+(H)$ for each $s \in [0,T]$. Therefore, we have from the representation theorem that $\Delta P_s^N \geq 0$. □

We have the following theorem.

Theorem 4.4 *The Riccati equation* (4.2) *has a unique solution* $P \in C_s([0,T]; \mathcal{S}^+(H))$ *such that*

$$|\sum_i C_i^* P_s C_i(s)|_{\mathcal{L}(H)} \leq C(T-s)^{-2\alpha}, \quad s \in [0,T].$$

Proof. First, we see from the last lemma that P^N is a nondecreasing sequence of self-adjoint operators. Moreover, we see from (3.12) that each P^N is non-negative. Using the monotone sequence theorem (see Kantorovich and Akilov [16, Theorem 1, p. 169]), we see that P_t^N converges strongly to a non-negative self-adjoint operator, denoted by P_t, which also satisfies the last inequality.

As P_t^N converges strongly to P_t, noting the following

$$\begin{aligned}\Lambda(t,P_t^N)^{-1}-\Lambda(t,P_t)^{-1} &= \Lambda(t,P_t^N)^{-1}\left(\Lambda(t,P_t)-\Lambda(t,P_t^N)\right)\Lambda(t,P_t)^{-1}\\ &= \Lambda(t,P_t^N)^{-1}\left(\sum_{i=1}^{\infty} D_i^*(t)(P_t-P_t^N)D_i(t)\right)\Lambda(t,P_t)^{-1},\end{aligned}$$

we see that $\Lambda(t,P_t^N)^{-1}$ converges strongly to $\Lambda(t,P_t)^{-1}$.

In view of our assumption (2.5), we have

$$\begin{aligned}&\left|\sum_{j=1}^{\infty} D_j^*(t)P_t^N C_j(t)x-\sum_{j=1}^{\infty} D_j^*(t)P_t C_j(t)x\right|_U\\ =& \sup_{|y|_U\le 1}\left\langle \sum_{j=1}^{\infty} D_j^*(t)P_t^N C_j(t)x-\sum_{j=1}^{\infty} D_j^*(t)P_t C_j(t)x, y\right\rangle\\ =& \sup_{|y|_U\le 1}\sum_{j=1}^{\infty}\left\langle (P_t^N-P_t)^{\frac12}C_j(t)x, (P_t^N-P_t)^{\frac12}D_j(t)y\right\rangle\\ \le& C\left(\sum_{j=1}^{\infty}\left|(P_t^N-P_t)^{\frac12}C_j(t)x\right|^2\right)^{\frac12}\\ =& C\left(\sum_{j=1}^{\infty}\langle C_j^*(t)(P_t^N-P_t)C_j(t)x,x\rangle\right)^{\frac12}.\end{aligned}$$

Since $\langle C_j^*(t)(P_t^N-P_t)C_j(t)x,x\rangle \le \langle C_j^*(t)(P_t^1-P_t)C_j(t)x,x\rangle$ and

$$\sum_{j=1}^{\infty}\langle (C_j^*(t)(P_t^1-P_t)C_j(t)x,x\rangle<\infty,$$

by the Dominated Convergence Theorem, we see that $\sum_{j=1}^{\infty} D_j^*(t)P_t^N C_j(t)$ converges strongly to $\sum_{j=1}^{\infty} D_j^*(t)P_t C_j(t)$. Therefore, the non-homogeneous term in the Lyapunov equation of P^{N+1} converges strongly.

By passing to the strong limit in the Lyapunov equation (4.4), we conclude that P is a solution.

Finally, we show the uniqueness. Let $\widetilde{P}$ be another solution of Riccati equation (4.2) such that

$$|\sum_i C_i^*(s)\widetilde{P}_s C_i(s)|_{\mathcal{L}(H)} \le C(T-s)^{-2\alpha}, \quad s \in [0,T).$$

Define $\delta P := P - \widetilde{P}$. Then proceeding identically as in the last lemma, we have

$$\begin{aligned}
\delta P_t &= \int_t^T e^{A^*(s-t)} \left(\hat{B}^*(s,P_s)\delta P_s + \delta P_s \hat{B}(s,P_s)\right) e^{A(s-t)} ds \\
&\quad + \int_t^T e^{A^*(s-t)} \sum_{i=1}^{\infty} \hat{C}_i(s,P_s)\delta P_s \hat{C}_i(s,P_s) e^{A(s-t)} ds \\
&\quad + \int_t^T e^{A^*(s-t)} \left[F(s,\lambda(s,P_s),\widetilde{P}_s) - F(s,\lambda(s,\widetilde{P}_s),\widetilde{P}_s)\right] e^{A(s-t)} ds. \qquad (4.8)
\end{aligned}$$

Since $F(s,\lambda(s,P_s),P_s) - F(s,\lambda(s,\widetilde{P}_s),P_s)$ is non-negative, we have $\delta P \ge 0$. By symmetry, we also have $\delta P \le 0$. Hence, we have $\delta P \equiv 0$. □

5 Optimal feedback control

In this section, we study the linear quadratic optimal control problem (2.1) and (2.4).

Note that Itô's formula could not be applied to systems driven by a white noise. To overcome the difficulty, we truncate the white noise by a finite number of Brownian motions.

Define X^N to be the unique solution of the following truncated state equation:

$$\begin{aligned}
dX_t^N &= (AX_t^N + B_t u_t)\, dt + \sum_{j=1}^{N} \left[C_j(t)X_t^N + D_j(t)u_t\right] d\beta_t^j, \qquad (5.1)\\
X_0^N &= x \in H.
\end{aligned}$$

We denote by P^N the solution of the following truncated Lyapunov equation:

$$\begin{aligned}
P_t^N &= e^{A^*(T-t)} G e^{A(T-t)} + \int_t^T e^{A^*(s-t)} \left(\hat{B}^*(s,P_s)P_s^N + P_s^N \hat{B}(s,P_s)\right) e^{A(s-t)} ds \\
&\quad + \int_t^T e^{A^*(s-t)} \sum_{i=1}^{N} \hat{C}_i^*(s,P_s)P_s^N \hat{C}_i(s,P_s) e^{A(s-t)} ds \qquad (5.2)\\
&\quad + \int_t^T e^{A^*(s-t)} \left(Q_s + \lambda^*(s,P_s)R_s\lambda(s,P_s)\right) e^{A(s-t)}\, ds, \quad t \in [0,T],
\end{aligned}$$

where P is the unique solution of the Riccati equation (4.2).

We have

Lemma 5.1 *For $t \in [0,T]$, P_t^N is non-decreasing, and is bounded from above, and strongly converges to P_t.*

Proof. In view of (5.2), we see from the representation that P_t^N is nonnegative for each $N \geq 1$. Furthermore, we have

$$\begin{aligned} P_t^{N+1} - P_t^N &= \int_t^T e^{A^*(s-t)} \left(\hat{B}^*(s,P_s)(P_s^{N+1}-P_s^N) + (P_s^{N+1}-P_s^N)\hat{B}(s,P_s)\right) e^{A(s-t)} ds \\ &\quad + \int_t^T e^{A^*(s-t)} \sum_{j=1}^N C_j^*(s)(P_s^{N+1}-P_s^N)C_j(s)e^{A(s-t)} ds \\ &\quad + \int_t^T e^{A^*(s-t)} \widehat{C}_{N+1}^*(s,P_s)P_s^{N+1}\widehat{C}_{N+1}(s,P_s)e^{A(s-t)} ds. \end{aligned}$$

From the representation of the solution of Lyapunov equation, it is clear that P^N is non-decreasing. Using the same argument to consider the equation of $P - P^N$, we see that $P_t^N \leq P_t$. Hence there is a bounded $\overline{P} \leq P$ such that P^N strongly converges to $\overline{P}$ which satisfies the Lyapunov equation:

$$\begin{aligned} \overline{P}_t &= e^{A^*(T-t)}Ge^{A(T-t)} + \int_t^T e^{A^*(s-t)} \left(\hat{B}^*(s,P_s)\overline{P}_s + \overline{P}_s\hat{B}(s,P_s)\right) e^{A(s-t)} ds \\ &\quad + \int_t^T e^{A^*(s-t)} \sum_{i=1}^\infty \hat{C}_i^*(s,P_s)\overline{P}_s\hat{C}_i(s,P_s)e^{A(s-t)} ds \\ &\quad + \int_t^T e^{A^*(s-t)} \left(Q_s + \lambda^*(s,P_s)R_s\lambda(s,P_s)\right) e^{A(s-t)}\, ds, \quad t \in [0,T], \end{aligned} \tag{5.3}$$

and

$$|\sum_i C_i^*\overline{P}_sC_i(s)|_{\mathcal{L}(H)} \leq C(T-s)^{-2\alpha}, \quad s \in [0,T].$$

Since P (as a solution to the Riccati equation) is also a solution to the preceding Lyapunov equation with the non-homogeneous term being $f_s := Q_s + \lambda^*(s,P_s)R_s\lambda(s,P_s)$, we conclude from the uniqueness of the solution to the Lyapunov equation that $\overline{P} = P$. □

Theorem 5.2 *The cost functional has the following representation :*

$$J(x,u) = \langle P_0x, x\rangle + \mathbb{E}\left[\int_0^T \langle \Lambda(s,P_s)(u_s - \lambda(s,P_s)X_s), u_s - \lambda(s,P_s)X_s\rangle ds\right].$$

The following feedback form:

$$\overline{u}_t = \lambda(t,P_t)\overline{X}_t, \quad t \in [0,T], \tag{5.4}$$

with $\overline{X}$ being the solution of the associated feedback system, is admissible and optimal.

Proof. We have the duality between the truncated state equation and the truncated Lyapunov equation by Yosida approximation of A:

$$
\begin{aligned}
& \mathbb{E}\left[\langle GX_T^N, X_T^N\rangle + \int_0^T \langle (Q_s + \lambda^*(s,P_s)R_s\lambda(s,P_s))X_s^N, X_s^N\rangle\right] ds \\
= \; & \langle P_0^N x, x\rangle + 2\mathbb{E}\int_0^T \langle P_s^N X_s^N, B_s u_s\rangle ds \\
& -\mathbb{E}\int_0^T \left\langle \left[\widehat{B}^*(s,P_s)P_s^N + P_s^N\widehat{B}(s,P_s) + \sum_{j=1}^N \widehat{C}_j^*(s,P_s)P_s^N\widehat{C}_j(s,P_s)\right] X_s^N, X_s^N \right\rangle ds \\
& +\mathbb{E}\int_0^T \sum_{j=1}^N \left\langle P_s^N(C_j(s)X_s^N + D_j(s)u_s), C_j(s)X_s^N + D_j(s)u_s\right\rangle ds.
\end{aligned}
$$

Setting $N \to \infty$, noting the following limit

$$
\lim_{N\to\infty} \sup_{0\le t\le T} \mathbb{E}\left[|X_t^N - X_t|^2\right] = 0,
$$

and Lemma 5.1, we have

$$
\begin{aligned}
& \mathbb{E}\left[\langle GX_T, X_T\rangle + \int_0^T \langle (Q_s + \lambda^*(s,P_s)R_s\lambda(s,P_s))X_s, X_s\rangle\right] ds \\
= \; & \langle P_0 x, x\rangle + 2\mathbb{E}\int_0^T \langle P_s X_s, B_s u_s\rangle ds \\
& -\mathbb{E}\int_0^T \left\langle \left[\widehat{B}^*(s,P_s)P_s + P_s\widehat{B}(s,P_s) + \sum_{j=1}^\infty \widehat{C}_j^*(s,P_s)P_s\widehat{C}_j(s,P_s)\right] X_s, X_s \right\rangle ds \\
& +\mathbb{E}\int_0^T \sum_{j=1}^\infty \langle P_s(C_j(s)X_s + D_j(s)u_s), C_j(s)X_s + D_j(s)u_s\rangle ds.
\end{aligned}
$$

Then, we have for any admissible control u,

$$
\begin{aligned}
J(x,u) &= \mathbb{E}\left[\langle GX_T, X_T\rangle + \int_0^T \langle Q_s X_s, X_s\rangle ds + \int_0^T \langle R_s u_s, u_s\rangle ds\right] \\
&= \langle P_0 x, x\rangle + 2\mathbb{E}\int_0^T \langle P_s X_s, B_s u_s\rangle ds + \mathbb{E}\int_0^T \langle R_s u_s, u_s\rangle ds \\
&\quad - \int_0^T \langle \lambda^*(s,P_s) R_s \lambda(s,P_s) X_s, X_s\rangle ds \\
&\quad - \mathbb{E}\int_0^T \left\langle \left[\widehat{B}^*(s,P_s)P_s + P_s\widehat{B}(s,P_s) + \sum_{j=1}^\infty \widehat{C}_j^*(s,P_s)P_s\widehat{C}_j(s,P_s)\right] X_s, X_s \right\rangle ds \\
&\quad + \mathbb{E}\int_0^T \sum_{j=1}^\infty \langle P_s(C_j(s)X_s + D_j(s)u_s), C_j(s)X_s + D_j(s)u_s\rangle \, ds.
\end{aligned}
$$

Noting that

$$
\begin{aligned}
& \sum_{j=1}^\infty C_j^*(s)PC_j(s) - \lambda^*(s,P)R_s\lambda(s,P) \\
& -[\widehat{B}^*(s,P)P + P\widehat{B}(s,P) + \sum_{j=1}^\infty \widehat{C}_j^*(s,P)P\widehat{C}_j(s,P)] \\
= \;& \lambda^*(s,P)\Lambda(s,P)\lambda(s,P)
\end{aligned}
$$

and

$$
2B_s^*P_s + 2\sum_{j=1}^\infty D_j^*(s)P_sC_j(s) = -2\Lambda(s,P_s)\lambda(s,P_s),
$$

we have for any admissible control u,

$$
\begin{aligned}
J(x,u) &= \langle P_0 x, x\rangle + \mathbb{E}\left[\int_0^T \langle \Lambda(s,P_s)(u_s - \lambda(s,P_s)X_s), u_s - \lambda(s,P_s)X_s\rangle ds\right] \\
&\geq \langle P_0 x, x\rangle.
\end{aligned}
$$

In view of Lemmas 4.2 and 3.2, we see that the closed-loop state equation has a unique solution $\overline{X}$, satisfying the following estimate

$$
\sup_{0\leq t\leq T} \mathbb{E}[|\overline{X}_t|_H^2] < \infty.
$$

Then

$$\begin{aligned}\mathbb{E}\int_0^T |\overline{u}_s|_U^2 ds &= \mathbb{E}\int_0^T |\lambda(s,P_s)\overline{X}_s|_U^2 ds \\ &\le C\int_0^T (T-s)^{-2\alpha} ds \sup_{0\le s\le T}\mathbb{E}|\overline{X}_s|^2 < \infty.\end{aligned}$$

This shows that $\overline{u}$ is admissible and $J(x,\overline{u}) = \langle P_0 x, x\rangle$. Therefore, $\overline{u}$ is optimal. □

6 Algebraic Riccati equation

In this section, we discuss the solvability of algebraic Riccati equation. For this, we need the following notion of stabilizability. Now we suppose that all the coefficients B, C, D, Q, R are time-invariant.

Definition 6.1 *We say that the system (A, B, C, D) is feedback stabilizable if there is an operator $K \in \mathcal{L}(H, U)$ such that the system corresponding to the feedback control $u = KX$ is stable, i.e. for any initial state $x \in H$,*

$$\mathbb{E}\int_0^\infty |X_t^{0,x}|^2 dt < \infty.$$

Theorem 6.1 *Assume that the system (A, B, C, D) is feedback stabilizable. Then there is a non-negative operator $P \in \mathcal{S}^+(H)$ such that $\sum_{i=1}^\infty C_i^* P C_i \in \mathcal{S}^+(H)$ and for any $T > 0$,*

$$\begin{aligned}P &= e^{A^*(T-t)} P e^{A(T-t)} + \int_t^T e^{A^*(s-t)} \left(B^*P + PB + \sum_{i=1}^\infty C_i^* P C_i\right) e^{A(s-t)} ds \\ &\quad + \int_t^T e^{A^*(s-t)} \left[Q - \lambda^*(P)\left(R + \sum_{i=1}^\infty D_i^* P D_i\right)^{-1} \lambda(P)\right] e^{A(s-t)}\, ds, \quad t \in [0,T].\end{aligned} \tag{6.1}$$

Proof. Let $N \ge 1$. Consider the following Riccati equation on $[0, N]$:

$$\begin{aligned}P_t^N &= \int_t^N e^{A^*(s-t)} \left(B^* P_s^N + P_s^N B + \sum_{i=1}^\infty C_i^* P_s^N C_i\right) e^{A(s-t)} ds \\ &\quad + \int_t^N e^{A^*(s-t)} \left[Q - \lambda^*(P_s^N)\left(R + \sum_{i=1}^\infty D_i^* P_s^N D_i\right)^{-1} \lambda(P_s^N)\right] e^{A(s-t)}\, ds, \quad t \in [0,N]\end{aligned}$$

with the following estimate

$$|\sum_{i=1}^\infty C_i^* P_s^N C_i|_{\mathcal{L}(H)} \le C_N (N-s)^{-2\alpha}.$$

It is easy to see that P^N is non-decreasing in N. By the stabilizability assumption, there is a feedback control $u = KX$ such that $\mathbb{E}\int_0^\infty |X_s|^2\,ds < \infty$ and

$$\begin{aligned}\langle P_0^N x, x\rangle &\leq \mathbb{E}\langle P_N^N X_N^{0,x}, X_N^{0,x}\rangle + \mathbb{E}\int_0^N \left(\langle QX_s^{0,x}, X_s^{0,x}\rangle + \langle Ru_s, u_s\rangle\right) ds \\ &\leq \mathbb{E}\int_0^\infty \left(\langle QX_s^{0,x}, X_s^{0,x}\rangle + \langle Ru_s, u_s\rangle\right) ds =: C|x|^2 \qquad (6.2)\end{aligned}$$

with the number C not depending on N. Using the time-invariance of the underlying coefficients, we also have for each $t \in [0,\infty)$, $\langle P_t^N x, x\rangle \leq C_t|x|^2$ with the number C_t not depending on N. Thus there exists P_t such that P_t^N converges to P_t in a strong way.

For $t \leq T \leq N$, P^N is the solution of the following Riccati equation

$$\begin{aligned}P_t^N &= e^{A^*(T-t)}P_T^N e^{A(T-t)} + \int_t^T e^{A^*(s-t)}\left(B^*P_s^N + P_s^N B + \sum_{i=1}^\infty C_i^* P_s^N C_i\right) e^{A(s-t)}ds \qquad (6.3)\\ &\quad + \int_t^T e^{A^*(s-t)}\left[Q - \lambda^*(P_s^N)\left(R + \sum_{i=1}^\infty D_i^* P_s^N D_i\right)^{-1}\lambda(P_s^N)\right] e^{A(s-t)}\,ds, \quad t \in [0,T].\end{aligned}$$

As $|P_T^N|_{\mathcal{L}(H)} \leq C_T$, from Theorem 3.4, there exists a constant C'_T such that

$$|\sum_{i=1}^\infty C_i^* P_s^N C_i|_{\mathcal{L}(H)} \leq C'_T(T-s)^{-2\alpha}, \quad s \in [0,T),$$

or equivalently

$$\sum_{i=1}^\infty \langle C_i^* P_s^N C_i x, x\rangle \leq C'_T(T-s)^{-2\alpha}|x|^2, \quad (s,x) \in [0,T) \times H.$$

Passing to the limit in N, using Fatou's lemma, we derive

$$|\sum_{i=1}^\infty C_i^* P_s C_i|_{\mathcal{L}(H)} \leq C'_T(T-s)^{-2\alpha}, \quad s \in [0,T). \qquad (6.4)$$

Taking the strong limit in (6.3), we deduce that P_t satisfies the following equation:

$$\begin{aligned}P_t &= e^{A^*(T-t)}P_T e^{A(T-t)} + \int_t^T e^{A^*(s-t)}\left(B^*P_s + P_s B + \sum_{i=1}^\infty C_i^* P_s C_i\right) e^{A(s-t)}ds \\ &\quad + \int_t^T e^{A^*(s-t)}\left[Q - \lambda^*(P_s)\left(R + \sum_{i=1}^\infty D_i^* P_s D_i\right)^{-1}\lambda(P_s)\right] e^{A(s-t)}\,ds, \quad t \in [0,T]\end{aligned}$$

Due to the time invariance of the coefficients, we prove that P_t does not depend on t. From (6.4), we see that $\sum_{i=1}^{\infty} C_i^* P C_i \in \mathcal{L}(H)$, and that P is the mild solution of (6.1). □

Theorem 6.2 *Let $Q \in \mathcal{S}^+(H)$ be positive. Assume that the non-negative operator $P \in \mathcal{S}^+(H)$ satisfies $\sum_{i=1}^{\infty} C_i^* P C_i \in \mathcal{S}^+(H)$ and algebraic Riccati equation* (6.1)*. Then, the feedback law $\overline{u} = \lambda(P)\overline{X}$ is admissible and optimal, and the value function $= J(x, \overline{u}) = \langle Px, x\rangle, x \in H$. Consequently, the non-negative solution P of algebraic Riccati equation* (6.1) *such that $\sum_{i=1}^{\infty} C_i^* P C_i \in \mathcal{S}^+(H)$ is unique.*

Proof. For any admissible u, there is a sequence $T_i \to \infty$ such that

$$\lim_{i\to\infty} \mathbb{E}[|X_{T_i}|^2] = 0.$$

Since $P_t \equiv P$ solves the Riccati equation on the finite time interval $[0, T_i]$ with the terminal condition P, we have

$$\mathbb{E}\langle PX_{T_i}, X_{T_i}\rangle + \mathbb{E}\int_0^{T_i} (\langle QX_s, X_s\rangle + \langle Ru_s, u_s\rangle)\, ds \geq \langle Px, x\rangle.$$

Letting $i \to \infty$, we have $J(x, u) \geq \langle Px, x\rangle$.

Now we prove that $\overline{u}$ is admissible and $J(x, \overline{u}) = \langle Px, x\rangle$. Again since $P_t \equiv P$ solves the Riccati equation on the finite time interval $[0, T]$ with the terminal condition P, we have

$$\mathbb{E}\langle P\overline{X}_T, \overline{X}_T\rangle + \mathbb{E}\int_0^{T} \left(\langle Q\overline{X}_s, \overline{X}_s\rangle + \langle R\overline{u}_s, \overline{u}_s\rangle\right) ds = \langle Px, x\rangle.$$

By the monotone convergence theorem, we have

$$\mathbb{E}\int_0^{\infty} \left(\langle Q\overline{X}_s, \overline{X}_s\rangle + \langle R\overline{u}_s, \overline{u}_s\rangle\right) ds \leq \langle Px, x\rangle.$$

As Q is positive, $\overline{u}$ is admissible and thus $J(x, \overline{u}) = \langle Px, x\rangle$. □

7 Null controllability of SPDEs via Riccati equation

In this section, we characterize the null controllability of the system (2.1) via the existence of Riccati equation with the singular terminal value in the spirit of Sîrbu and Tessitore [18].

Definition 7.1 *The system* (2.1) *is T-null (exact) controllable if for any $(t, x) \in [0, T) \times H$, there is $u \in L^2_{\mathcal{F}}(t, T; U)$ such that $X_T^{t,x,u} = 0$, $\mathbb{P}$-almost surely; and it is null controllable if it is T-null controllable for each $T > 0$.*

For each $T > 0$ and $x \in H$, we consider the following optimal null-controllability control problem:

$$V(t,x) := \min_{u \in L^2_{\mathcal{F}}(t,T;U)} J(t,x;u) := \mathbb{E}\int_t^T (|X_s^{t,x;u}|^2 + |u_s|^2)\, ds \tag{7.1}$$

subject to $X_T^{t,x;u} = 0$. If there is no u satisfying $X_T^{t,x;u} = 0$, we set $V(t,x) = +\infty$.

Let Id denote the identity operator in H. We introduce the following Riccati equation:

$$\begin{aligned} P_t &= e^{A^*(T'-t)} P_{T'} e^{A(T'-t)} + \int_t^{T'} e^{A^*(s-t)} \sum_{i=1}^{\infty} C_i^*(s) P_s C_i(s) e^{A(s-t)} ds \\ &\quad + \int_t^{T'} e^{A^*(s-t)} \left(Id - \lambda^*(s,P_s)\Lambda(s,P_s)\lambda(s,P_s)\right) e^{A(s-t)} ds, \quad 0 \le t \le T' < T; \end{aligned} \tag{7.2}$$

with the following singular terminal condition $P_T = +\infty$ in the following sense: for any $x \in H$ such that $x \neq 0$,

$$\lim_{(t,y)\to(T,x)} \langle P_t y, y\rangle = +\infty. \tag{7.3}$$

We then have

Theorem 7.1 *For given $T > 0$, the following conditions are equivalent:*

(i) the Riccati equation (7.2) *has a mild solution P satisfying the singular terminal condition $P_T = +\infty$ and the map $s \mapsto \sum_{i=1}^{\infty} C_i^*(s)P_sC_i(s) \in \mathcal{L}(H)$ is bounded in any compact interval of $[0,T)$;*

(ii) the state system (2.1) *is T-null controllable.*

If the system (2.1) *is T-null controllable, then the associated optimal null-control problem with the cost* (7.1) *has the optimal control of the following feedback form:*

$$\overline{u}_s := \lambda(s,P_s)\overline{X}_s, \quad s \in [0,T]$$

where P is the solution of the Riccati equation (7.2) *with the singular terminal condition $P_T = +\infty$.*

Proof. First we prove that Assertion (i) implies (ii). In fact, if P is the solution of the Riccati equation (7.2) with the singular terminal condition $P_T = +\infty$, then its restriction on $[t,s]$ for $s \in (t,T)$ can be regarded as the solution of the Riccati equation (7.2) with the terminal condition P_s. Set $\overline{u}_s := \lambda(s,P_s)\overline{X}_s$ for $s \in [0,T]$. From Theorem 5.2, we have

$$\begin{aligned} \langle P_t x, x\rangle &= \mathbb{E}\langle P_s \overline{X}_s^{t,x;\overline{u}}, \overline{X}_s^{t,x;\overline{u}}\rangle + \mathbb{E}\int_t^s (|\overline{X}_r^{t,x;\overline{u}}|^2 + |\overline{u}_r|^2)\, dr \\ &\ge \mathbb{E}\langle P_s \overline{X}_s^{t,x;\overline{u}}, \overline{X}_s^{t,x;\overline{u}}\rangle. \end{aligned} \tag{7.4}$$

Therefore, we have $\overline{u} \in L^2_{\mathcal{F}}([t,T];U)$ and we can extend $\overline{X}^{t,x;\overline{u}}$ to $[t,T]$ lying in $C_{\mathcal{F}}([t,T];L^2(\Omega,H))$. From the inequality (7.4) and Fatou's lemma, we have

$$\langle P_t x, x\rangle \geq \mathbb{E}\liminf_{s\to T-}\left[\langle P_s\overline{X}^{t,x;\overline{u}}_s, \overline{X}^{t,x;\overline{u}}_s\rangle\right] \geq \mathbb{E}\left[(+\infty)I_{\overline{X}^{t,x;\overline{u}}_T\neq 0}\right].$$

Hence, $\overline{X}^{t,x;\overline{u}}_T = 0$, $\mathbb{P}$-almost surely. Assertion (ii) is proved.

Now we show that Assertion (ii) implies (i). For any integer $n \geq 1$,

$$V^n(t,x) := \min_{u\in L^2_{\mathcal{F}}(t,T;U)} J^n(t,x;u) := n\mathbb{E}|X^{t,x;u}_T|^2 + \mathbb{E}\int_t^T (|X^{t,x;u}_s|^2 + |u_s|^2)\,ds. \tag{7.5}$$

It is associated to the following Riccati equation

$$\begin{aligned} P_t &= ne^{A^*(T-t)}e^{A(T-t)} + \int_t^T e^{A^*(s-t)}\sum_{i=1}^{\infty} C_i^*(s)P_sC_i(s)e^{A(s-t)}ds \\ &\quad + \int_t^T e^{A^*(s-t)}\left(Id - \lambda^*(s,P_s)\Lambda(s,P_s)\lambda(s,P_s)\right)e^{A(s-t)}ds, \quad 0\leq t\leq T. \end{aligned} \tag{7.6}$$

Denoting by P^n its unique solution, we have for $s<T$,

$$\begin{aligned} P^n_t &= e^{A^*(s-t)}P^n_s e^{A(s-t)} + \int_t^s e^{A^*(r-t)}\sum_{i=1}^{\infty} C_i^*(r)P^n_rC_i(r)e^{A(r-t)}dr \\ &\quad + \int_t^s e^{A^*(r-t)}\left(Id - \lambda^*(r,P^n_r)\Lambda(r,P^n_r)\lambda(r,P^n_r)\right)e^{A(r-t)}dr, \quad 0\leq t\leq s. \end{aligned} \tag{7.7}$$

From Theorem 5.2, we see that P^n is non-decreasing in n. Moreover, from the T-null controllability, there is $u^0 \in \mathcal{L}^2_{\mathcal{F}}(t,T;U)$ such that $X^{t,x;u^0}_T = 0$. Hence,

$$\langle P^n_t x, x\rangle \leq \mathbb{E}\int_t^T (|X^{t,x;u^0}_s|^2 + |u^0_s|^2)\,ds.$$

Consequently, the sequence P^n_t has a strong limit in $\mathcal{S}^+(H)$, which is denoted by P_t.

For $s' \in (s,T)$, we have

$$\left|\sum_{i=1}^{\infty} C_i^*(t)P^n_tC_i(t)\right|_{\mathcal{L}(H)} \leq \frac{C_{s'}}{(s'-t)^{2\alpha}} \leq \frac{C_{s'}}{(s'-s)^{2\alpha}}. \tag{7.8}$$

Letting $n\to+\infty$, we have

$$\left|\sum_{i=1}^{\infty} C_i^*(t)P_tC_i(t)\right|_{\mathcal{L}(H)} \leq \frac{C_{s'}}{(s'-s)^{2\alpha}},$$

meaning that the sum is bounded in $\mathcal{L}(H)$.

Taking the strong limit in (7.7), we see that P is a mild solution of Riccati equation (7.2) on the time interval $[0,T)$.

Furthermore, we have for any integer n,

$$\liminf_{s\to T-,\ y\to x}\langle P_s y, y\rangle \geq \liminf_{s\to T-,\ y\to x}\langle P_s^n y, y\rangle = n|x|^2.$$

This shows that P satisfies the singular terminal condition at time T. □

8 Examples: LQ optimal control of the Anderson model

Example 8.1 *Consider the following controlled Anderson model, that is, the following controlled stochastic heat equation in* $[0,1]$*:*

$$\begin{aligned} dX_t(y) &= \frac{\partial^2}{\partial^2 y}X_t(y)\,dt + b(t,y)u(t,y)\,dt + X_t(y)dW(t,y); &(8.1)\\ X_t(0) &= X_t(1) = 0, \quad t\in[0,T]; &(8.2)\\ X_0(y) &= x(y), \quad y\in[0,1]. &(8.3)\end{aligned}$$

The cost functional reads:

$$J(x,u) = \mathbb{E}\int_0^T\int_0^1 \left[q(t,y)X_t^2(y) + r(t,y)u^2(t,y)\right]\,dydt + \mathbb{E}\int_0^1 g(y)X_T^2(y)\,dy. \tag{8.4}$$

In the above example, $H = L^2(0,1)$*, and* W *is an* H*-valued cylindrical Wiener process. We choose an orthonormal basis* $\{e_i, i = 1,2,\ldots\}$ *in the space* H *such that*

$$\sup_i \sup_{y\in[0,1]} |e_i(y)| < \infty.$$

A *is the realization of the second derivative operator with the zero Dirichlet boundary conditions, and all the functions* b,q,r,g *are measurable and bounded . So* $\mathcal{D}(A) = H^2([0,1])\cap H_0^1([0,1])$ *and* $A\psi = \psi''$ *for all* $\psi\in\mathcal{D}(A)$*,* $C_i\phi(y) := e_i(y)\phi(y)$*, and* $(B_t\phi)(y) = b(t,y)\phi(y)$ *for* $\phi\in H$*. Then the pair* (A,C) *satisfies (see Da Prato and Zabczyk [9]) the inequality* (2.3)*.*

Finally, $(Q_t\phi)(y) := q(t,y)\phi(y), (R_t\phi)(y) := r(t,y)\phi(y)$*, and* $(G\phi)(y) := g(y)\phi(y)$*. Theorems 4.4 and 5.2 can be applied to solve the above quadratic optimal control of the Anderson model.*

Example 8.2 *Consider the controlled Anderson system with the coefficient b being time-invariant and* b^{-1} *existing and being bounded. Then the system* (2.1) *is stablizable by the feedback control* $u = -\lambda b^{-1}(y)X$ *for sufficiently large* λ*. To show this, we have for* $\widetilde{X}_t := e^{\lambda t}X_t$*,*

$$\widetilde{X}_t = e^{At}x + \int_0^t \sum_{j=1}^\infty e^{A(t-s)}C_j\widetilde{X}_s\,d\beta_s^j. \tag{8.5}$$

Therefore, we have

$$\begin{aligned}\mathbb{E}[|\widetilde{X}_t|^2] &\leq C|x|^2 + C\mathbb{E}\int_0^t \sum_{j=1}^{\infty} |e^{A(t-s)}C_j\widetilde{X}_s|^2 ds \\ &\leq C|x|^2 + C\int_0^t (t-s)^{-2\alpha}\mathbb{E}[|\widetilde{X}_s|^2]ds. \end{aligned} \tag{8.6}$$

From Gronwall's inequality, we have

$$\mathbb{E}[|\widetilde{X}_t|^2] \leq C|x|^2 e^{Ct}, \quad \mathbb{E}[|X_t|^2] \leq C|x|^2 e^{(C-\lambda)t}.$$

Therefore, Theorems 6.1 and 6.2 can be applied to the Anderson model.

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