UNIFORM SYNCHRONIZATION OF AN ABSTRACT LINEAR SECOND ORDER EVOLUTION SYSTEM

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Abstract. Although the mathematical study on the synchronization of wave equations at finite horizon has been well developed, there was few results on the synchronization of wave equations for long-time horizon. The aim of the paper is to investigate the uniform synchronization at the infinite horizon for one abstract linear second order evolution system in a Hilbert space.

First, using the classical compact perturbation theory on the uniform stability of semigroups of contractions, we will establish a lower bound on the number of damping, necessary for the uniform synchronization of the considered system. Then, under the minimum number of damping, we clarify the algebraic structure of the system as well as the necessity of the conditions of compatibility on the coupling matrices. We then establish the uniform synchronization by the compact perturbation method and then give the dynamics of the asymptotic orbit. Various applications are given for the system of wave equations with boundary feedback or (and) locally distributed feedback, and for the system of Kirchhoff plate with distributed feedback. Some open questions are raised at the end of the paper for future development.

The study is based on the synchronization theory and the compact perturbation of semigroups.

Keywords: uniform synchronization, condition of compatibility, second order evolution system.

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1. Introduction

Synchronization is a widespread natural phenomenon. It was first observed by Huygens in 1665 [7]. The theoretical research on synchronization from the mathematical point of view dates back to Wiener in 1950s in [29] (Chapter 10). Since 2012, Li and Rao started the research on the synchronization in a finite time for a coupled system of wave equations with Dirichele boundary controls [13, 14]). Later, the synchronization has been carried out for a coupled system of wave equations with various boundary controls, the most part of their results was recently collected in the monograph [16]. The optimal control for the exact synchronization of parabolic system was recently investigated by Wang and Yan in [28]. Consequently, the study of synchronization becomes a part of research in control theory.

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In a recent work [17, 18], we showed that under Kalman's rank condition, the observability of a scalar equation implies the uniqueness of solution to a system of elliptic operators. Using this result, we have established the asymptotic synchronization by groups for second order evolution systems.

The objective of this work is to investigate the uniform synchronization for second order evolution systems. Let us briefly describe the formulation and the main ideas.

Let H and V be two separated Hilbert spaces such that $V \subset H \subset V'$, V' being the dual of V, with dense and compact imbeddings. Let L be the duality mapping from V onto V', and g be a linear continuous symmetric operator from V into V'. Let I denote the identity of $\mathbb{R}^N$. We define the diagonal operators

$$\mathcal{L} = LI \quad \text{and} \quad \mathcal{G} = gI. \tag{1.1}$$

Let A and D be symmetric and semi-positive definite matrices with constant elements. Consider the following second order evolution system for the state variable $U = (u^{(1)}, \cdots, u^{(N)})^T$:

$$U'' + \mathcal{L}U + AU + D\mathcal{G}U' = 0, \tag{1.2}$$

where " $'$ " stands for the time derivative.

We first show that if system (1.2) is uniformly stable in the space $(V \times H)^N$, then $\text{rank}(D) = N$ (see Corollary 2.5 below). When $\text{rank}(D) < N$, system (1.2) is not uniformly stable, we then turn to consider the synchronization.

For any given integer $p \geqslant 1$, let

$$0 = n_0 < n_1 < n_2 < \cdots < n_p = N \tag{1.3}$$

be integers such that $n_r - n_{r-1} \geqslant 2$ for all r with $1 \leqslant r \leqslant p$. We re-arrange the components of the state variable U into p groups

$$(u^{(1)}, \cdots, u^{(n_1)}),\ (u^{(n_1+1)}, \cdots, u^{(n_2)}), \cdots\cdots, (u^{(n_{p-1}+1)}, \cdots, u^{(n_p)}). \tag{1.4}$$

Definition 1.1. *System* (1.2) *is uniformly (exponentially) synchronizable by* p*-groups, if there exist constants* $M \geqslant 1$ *and* $\omega > 0$*, such that for any given initial data* $(U_0, U_1) \in (V \times H)^N$*, the corresponding solution* U *to system* (1.2) *satisfies*

$$\begin{aligned} &\|(u^{(k)}(t) - u^{(l)}(t), {u^{(k)}}'(t) - {u^{(l)}}'(t))\|_{V\times H} \\ \leqslant & M e^{-\omega t} \|(u_0^{(k)} - u_0^{(l)}, u_1^{(k)} - u_1^{(l)})\|_{V\times H}, \quad t \geqslant 0 \end{aligned} \tag{1.5}$$

for all k, l *with* $n_{r-1} + 1 \leqslant k, l \leqslant n_r$ *and all* r *with* $1 \leqslant r \leqslant p$.

Now let us outline the main ideas in the study of the uniform synchronization by p-groups.

Let C_p be the matrix given by (3.2) below. Then (1.5) can be equivalently rewritten as

$$\|C_p(U(t), U'(t))\|_{(V\times H)^{N-p}} \leqslant M e^{-\omega t} \|C_p(U_0, U_1)\|_{(V\times H)^{N-p}} \quad t \geqslant 0. \tag{1.6}$$

The matrix A satisfies the condition of C_p-compatibility, if there exists a symmetric and semi-positive definite matrix $\overline{A}_p$ such that

$$(C_p C_p^T)^{-1/2} C_p A = \overline{A}_p (C_p C_p^T)^{-1/2} C_p. \tag{1.7}$$

Correspondingly, the reduced matrix $\overline{D}_p$ can be introduced for D (see Proposition 3.2). Applying $(C_p C_p^T)^{-1/2} C_p$ to (1.2) and setting $W = (C_p C_p^T)^{-1/2} C_p U$, we get a self-closed reduced system

$$W'' + \mathcal{L}W + \overline{A}_p W + \overline{D}_p \mathcal{G} W' = 0. \tag{1.8}$$

It is clear that the uniform synchronization by p-groups of system (1.2) is equivalent to the uniform stability of the reduced system (1.8).

In Theorem 3.7, we will show that under the condition $\operatorname{rank}(D) = N - p$, if the scalar equation

$$u'' + Lu + gu' = 0 \tag{1.9}$$

is uniformly stable in the space $V \times H$, then system (1.2) is uniformly synchronizable by p-groups.

Furthermore (see Theorem 3.9), there exist some functions $u_1, \cdots, u_p$, such that

$$\begin{aligned} &\|(u^{(k)}(t) - u_r(t), {u^{(k)}}'(t) - u'_r(t))\|_{V\times H} \\ \leqslant &Me^{-\omega t}\|(u_0^{(k)} - u_0^{(l)}, u_1^{(k)} - u_1^{(l)})\|_{V\times H}, \quad t \geqslant 0 \end{aligned} \tag{1.10}$$

for all k, l with $n_{r-1} + 1 \leqslant k, l \leqslant n_r$ and all r with $1 \leqslant r \leqslant p$.

Moreover, the functions $u_1, \cdots, u_p$ satisfy a homogeneous system, then, the solution U to system (1.2) follows a conservative orbit. This is quite different from the approximate boundary synchronization by p-groups, since the approximate boundary synchronization by p-groups in the consensus sense does not imply that in the pinning sense in general (see Chapter 11 in [16]).

The above approach is direct and efficient. The difficult part of the problem is to show the necessity of the conditions of C_p-compatibility which are imposed as physically reasonable hypotheses even for the systems of ordinary differential equations. So, we have to first justify the necessity of the conditions of compatibility, then, the uniform synchronization will be studied by means of a serious mathematical consideration.

The necessity of the condition of C_p-compatibility for A, respectively D is intrinsically linked with the rank of the matrix D. We will show (see Proposition 3.5) that $\operatorname{rank}(D) \geqslant N - p$ is a necessary condition for the uniform synchronization by p-groups. Then under the minimum rank condition $\operatorname{rank}(D) = N - p$, we establish the necessity of the condition of C_p-compatibility for the matrix A, respectively D (see Theorem 3.7).

Now we give some related literatures. One of the motivation of studying the synchronization consists of establishing the controllability for fewer boundary controls. When the number of boundary controls is fewer than the number of state variables, the non-exact boundary controllability for a coupled system of wave equations with various boundary controls in the usual energy space was established in Li and Rao [16]. However, if the components of initial data are allowed to have different levels of energy, then the exact boundary controllability for a system of two wave equations was established by means of only one boundary control in Alabau-Boussouira [1, 2], Liu and Rao [20], Rosier and de Teresa [25]. In [4], Dehman established the controllability of two coupled wave equations on a compact manifold with only one local distributed control. In [21, 31], Zuazua proposed the average controllability as another way to deal with the controllability with fewer controls. The observability inequality is particularly interesting for a trial on the decay rate of approximate controllability.

The paper is organized as follows. In §2, we consider the uniform stability and establish a lower bound on the rank of the control matrix, which is necessary for the study of the uniform synchronization. §3 is devoted to the uniform synchronization by p-groups. Under the minimum rank condition, we show the necessity of the conditions of C_p-compatibility for the coupling matrices in the considered system. In §4, we give some examples of applications such as the system of wave equations with boundary feedback or (and) locally distributed feedback, and the system of Kirchhoff plate with distributed feedback. In §5, we give some comments on the obtained results and propose some open questions for future development.

2. Uniform stability

We first recall the following well-posedness result (see Proposition 3.1 in [18]).

Proposition 2.1. *System* (1.2) *generates a semi-group of contractions with a compact resolvent in the space* $(V \times H)^N$. *More precisely, for any given initial data* $(U_0, U_1) \in (V \times H)^N$, *the corresponding weak solution* U *to system* (1.2) *satisfies*

$$U \in C^0(\mathbb{R}^+, V^N) \cap C^1(\mathbb{R}^+, H^N) \tag{2.1}$$

and

$$\|(U(t), U'(t))\|_{(V\times H)^N} \leqslant \|(U_0, U_1)\|_{(V\times H)^N}, \quad t \geqslant 0. \tag{2.2}$$

Definition 2.2. *System* (1.2) *is uniformly (exponentially) stable in the space* $(V \times H)^N$, *if there exist constants* $M \geqslant 1$ *and* $\omega > 0$, *such that for any given initial data* $(U_0, U_1) \in (V \times H)^N$, *the corresponding solution* U *to system* (1.2) *satisfies*

$$\|(U(t), U'(t))\|_{(V\times H)^N} \leqslant Me^{-\omega t}\|(U_0, U_1)\|_{(V\times H)^N}, \quad t \geqslant 0. \tag{2.3}$$

Proposition 2.3. *Let* $\mathcal{R}$ *be a linear compact mapping from* V *to* $L^2(0, T; H)$. *Then we can not find positive constants* $M \geqslant 1$ *and* $\omega > 0$, *such that for all* $\theta \in V$, *the solution to the following problem*

$$\begin{cases} u'' + Lu = \mathcal{R}\theta, \\ t = 0: \quad u = \theta, \ u' = 0 \end{cases} \tag{2.4}$$

satisfies

$$\|(u(t), u'(t))\|_{V\times H} \leqslant Me^{-\omega t}\|\theta\|_V, \quad t \geqslant 0. \tag{2.5}$$

Proof. Noting that problem (2.4) is time invertible, by well-posedness we have

$$\|\theta\|_V \leqslant \|u(T)\|_V + \|u'(T)\|_H + \int_0^T \|\mathcal{R}\theta\|_H dt. \tag{2.6}$$

Assume by contradiction that (2.5) holds for all $\theta \in V$, then we have

$$\|\theta\|_V \leqslant Me^{-\omega T}\|\theta\|_V + \int_0^T \|\mathcal{R}\theta\|_H dt. \tag{2.7}$$

When T is large enough, it follows that for all $\theta \in V$, we have

$$\|\theta\|_V \leqslant \frac{\sqrt{T}}{1 - Me^{-\omega T}}\|\mathcal{R}\theta\|_{L^2(0,T;H)}. \tag{2.8}$$

This contradicts the compactness of $\mathcal{R}$. The proof is complete. □

Theorem 2.4. *Let* $\widetilde{C}_q$ *be a full row-rank matrix of order* $(N - q) \times N$ *with* $0 \leqslant q < N$. *Assume that there exist constants* $M \geqslant 1$ *and* $\omega > 0$, *such that for any* *given initial data* $(U_0, U_1) \in (V \times H)^N$, *the corresponding solution* U *to system* (1.2) *satisfies*

$$\|\widetilde{C}_q(U(t), U'(t))\|_{(V\times H)^{N-q}} \leqslant Me^{-\omega t}\|(U_0, U_1)\|_{(V\times H)^N}, \quad t \geqslant 0. \tag{2.9}$$

Then

$$rank(\widetilde{C}_q D) \geqslant N - q. \tag{2.10}$$

Proof. Assume by contradiction that the rank condition (2.10) fails. Then, we have

$$rank(\widetilde{C}_q D) = rank(D\widetilde{C}_q^T) < N - q = rank(\widetilde{C}_q^T). \tag{2.11}$$

By Proposition 2.11 in [16], we have

$$Im(\widetilde{C}_q^T) \cap Ker(D) \neq \{0\}. \tag{2.12}$$

Let $E \in \mathrm{Im}(\widetilde{C}_q^T)$ be a unit vector such that $DE = 0$. Applying E to system (1.2) associated with the initial data

$$t = 0: \quad U = \theta E, \quad U' = 0 \tag{2.13}$$

with $\theta \in V$, and setting $u = (\!(E, U)\!)$, we get

$$\begin{cases} u'' + Lu = -(\!(E, AU)\!), \\ t = 0: \quad u = \theta, \ u' = 0, \end{cases} \tag{2.14}$$

here and hereafter $(\!(\cdot,\cdot)\!)$ denotes the inner product with the associated norm $\|\cdot\|$ in the euclidian space $\mathbb{R}^N$.

Now, we define the linear mapping

$$\mathcal{R}: \quad \theta \to (\!(E, AU)\!). \tag{2.15}$$

Since the matrices A and D are symmetric and semi-positive definite, by the dissipation of system (1.2) with the initial data (2.13), we have

$$\|\mathcal{R}\theta\|_{L^2(0,T;V)} + \|\mathcal{R}\theta\|_{H^1(0,T;H)} \leqslant c_T\|\theta\|_V, \tag{2.16}$$

where c_T is a positive constant depending only on T.

Noting that the imbedding from $L^2(0,T;V) \cap H^1(0,T;H)$ into $L^2(0,T;H)$ is compact (see Theorem 5.1 in [19]), the mapping $\mathcal{R}$ is compact from V into $L^2(0,T;H)$.

On the other hand, noting $E = \widetilde{C}_q^T x$, we have

$$u = (\!(E, U)\!) = (\!(x, \widetilde{C}_q U)\!). \tag{2.17}$$

Then, it follows from (2.9) that

$$\|(u(t), u'(t))\|_{V\times H} \leqslant c\|\widetilde{C}_q(U(t), U'(t))\|_{V\times H} \leqslant cMe^{-\omega t}\|\theta\|_V, \quad t \geqslant 0 \tag{2.18}$$

for all $\theta \in V$. This contradicts Proposition 2.3. □

In particular, taking $\widetilde{C}_q = I$ in Theorem 2.4, we get immediately

Corollary 2.5. *If system (1.2) is uniformly stable, then we have rank*$(D) = N$.

Conversely, we have

Theorem 2.6. *Assume that the scalar equation*

$$u'' + Lu + gu' = 0 \tag{2.19}$$

is uniformly stable in the space $V \times H$. *If rank*$(D) = N$, *then system (1.2) is uniformly stable in the space* $(V \times H)^N$.

Proof. Following the classical theory (see [26, 27]), the uniform stability of a semigroup is robust by compact perturbation. This property was served in [8, 22] for obtaining the uniform stability.

More precisely, since the mapping $U \to AU$ is compact from V into H, the asymptotic stability of the coupled system (1.2) and the uniform stability of the following decoupled system

$$U'' + \mathcal{L}U + D\mathcal{G}U' = 0 \tag{2.20}$$

yield the uniform stability of the coupled system (1.2).

Since $\text{rank}(D) = N$, system (2.20) can be decomposed into N scalar equations of the same type as those in (2.19), therefore, it is uniformly stable.

On the other hand, by Proposition 2.1, the resolvent of system (1.2) is compact. Then by the classical theory of semigroups, the asymptotic stability of system (1.2) is equivalent to the uniqueness of the following over-determined system:

$$(\mathcal{L} + A)\Phi = \beta^2\Phi \quad \text{and} \quad \mathcal{G}\Phi = 0. \tag{2.21}$$

Let $AE = \lambda E$ with $\lambda \geqslant 0$. Then, setting $\phi = (E, \Phi)$, it follows that

$$L\phi = (\beta^2 - \lambda)\phi \quad \text{and} \quad g\phi = 0. \tag{2.22}$$

By the definition of the dual mapping, we have

$$\langle L\phi, \phi\rangle_{V',V} = (L\phi, \phi)_H = \|\phi\|_V^2.$$

It follows that $\beta^2 - \lambda > 0$. Then, we check easily that

$$U = e^{it\sqrt{\beta^2-\lambda}}\phi E \tag{2.23}$$

satisfies system (2.20), which is uniformly stable. We get $(\!(E, \Phi)\!) = \phi = 0$ for each eigenvector E of A, then $\Phi = 0$. □

Remark 2.7. *Roughly speaking, Theorem 2.6 indicates that the uniform stability of system* (1.2) *can be obtained by means of the scalar equation* (2.19)*. It provides thus a direct and efficient approach to solve a seemingly difficult problem of uniform stability of a complex system.*

3. Uniform synchronization by p-groups

By Corollary 2.5, when $\text{rank}(D) < N$, system (1.2) is not uniformly stable. Instead of the stability, we turn to consider its synchronization by p-groups.

Let S_r be the full row-rank matrix of order $(n_r - n_{r-1} - 1) \times (n_r - n_{r-1})$:

$$S_r = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{pmatrix}, \quad 1 \leqslant r \leqslant p. \tag{3.1}$$

Define the $(N - p) \times N$ matrix C_p of synchronization by p-groups as

$$C_p = \begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_p \end{pmatrix}. \tag{3.2}$$

The uniform synchronization by p-groups (1.5) can be equivalently rewritten by (1.6), which is easy to be analyzed.

Let $\epsilon_1, \cdots, \epsilon_N$ be the vectors of the canonical basis of $\mathbb{R}^N$. Defining

$$e_r = \sum_{i=n_{r-1}+1}^{n_r} \epsilon_i, \quad 1 \leqslant r \leqslant p, \tag{3.3}$$

we have

$$Ker(C_p) = Span\{e_1, \cdots, e_p\}. \tag{3.4}$$

Proposition 3.1. *(see Proposition 4.2 in [18]) The matrix A satisfies the condition of C_p-compatibility:*

$$AKer(C_p) \subseteq Ker(C_p) \tag{3.5}$$

if and only if there exists a symmetric and semi-positive definite matrix $\overline{A}_p$ of order $(N-p)$, such that (1.7) holds.

Proposition 3.2. *(see Proposition 4.4 in [18]) The matrix D satisfies the condition of strong C_p-compatibility:*

$$Ker(C_p) \subseteq Ker(D) \tag{3.6}$$

if and only if there exists a symmetric and semi-positive definite matrix R of order $(N-p)$, such that

$$D = C_p^T R C_p. \tag{3.7}$$

Moreover, setting

$$\overline{D}_p = (C_p C_p^T)^{1/2} R (C_p C_p^T)^{1/2}, \tag{3.8}$$

we have

$$(C_p C_p^T)^{-1/2} C_p D = \overline{D}_p (C_p C_p^T)^{-1/2} C_p. \tag{3.9}$$

Remark 3.3. *By the expression (3.3), it is easy to check that the condition of C_p-compatibility (3.5) is equivalent to the row-sum condition by blocks*

$$\sum_{j=n_{s-1}+1}^{n_s} a_{ij} = \alpha_{rs}, \quad n_{r-1}+1 \leqslant i \leqslant n_r, \quad 1 \leqslant r, s \leqslant p, \tag{3.10}$$

where α_{rs} are some constants. In particular, when $p=1$, A satisfies the row-sum condition:

$$\sum_{p=1}^{N} a_{kp} = \alpha, \quad k = 1, \cdots, N. \tag{3.11}$$

The condition of strong C_p-compatibility (3.6) is equivalen to

$$DKer(C_p) = \{0\}. \tag{3.12}$$

That means that $D = (d_{ij})$ satisfies the null row-sum condition by blocks

$$\sum_{j=n_{s-1}+1}^{n_s} d_{ij} = 0, \quad n_{r-1}+1 \leqslant i \leqslant n_r, \quad 1 \leqslant r, s \leqslant p. \tag{3.13}$$

Now applying C_p to system (1.2), and setting $W = C_pU$, we get a self-closed reduced system

$$W'' + \mathcal{L}W + \overline{A}_pW + \overline{D}_pGW' = 0. \tag{3.14}$$

Moreover, it is easy to check the following basic result.

Proposition 3.4. *Assume that the matrices A and D satisfy the condition of C_p-compatibility* (3.5) *and the condition of strong C_p-compatibility* (3.6)*, respectively. The uniform synchronization by p-groups of system* (1.2) *in the space $(V \times H)^N$ is equivalent to the uniform stability of the reduced system* (3.14) *in the space $(V \times H)^{N-p}$.*

Since the reduced matrices $\overline{A}_p$ and $\overline{D}_p$ are still symmetric and semi-positive definite, the uniform stability of the reduced system (3.14) can be treated by Theorem 2.6. So, the uniform synchronization by p-groups is reduced to the uniform stability. However, the necessity of the condition of C_p-compatibility for A and that of the condition of strong C_p-compatibility for D are intrinsically linked with the rank of the matrix D.

Proposition 3.5. *If system* (1.2) *is uniformly synchronizable by p-groups, then we necessarily have*

$$rank(C_pD) \geqslant N - p. \tag{3.15}$$

Proof. It is sufficient to take $\widetilde{C}_q = C_p$ in Theorem 2.4. □

Proposition 3.6. *The following rank condition*

$$rank(D) = rank(C_pD) = N - p \tag{3.16}$$

holds, if and only if $Ker(D)$ and $Ker(C_p)$ are bi-orthonormal.

Proof. By Proposition 2.11 in [16], the rank condition (3.16) is equivalent to

$$Ker(D) \cap Im(C_p^T) = Ker(C_p) \cap Im(D) = \{0\}, \tag{3.17}$$

namely,

$$Ker(D) \cap \{Ker(C_p)\}^{\perp} = Ker(C_p) \cap \{Ker(D^T)\}^{\perp} = \{0\}. \tag{3.18}$$

Hence by Proposition 2.5 in [16], Ker(D) and Ker(C_p) are bi-orthogonal. □

Theorem 3.7. *Assume that system (1.2) is uniformly synchronizable by p-groups under the minimal rank conditions (3.16). Then A satisfies the condition of C_p-compatibility* (3.5) *and D satisfies the condition of strong C_p-compatibility* (3.6).

Proof. Let U be the solution to system (1.2) with the following initial data:

$$U_0 = \sum_{r=1}^{p} u_{0r}e_r, \quad U_1 = \sum_{r=1}^{p} u_{1r}e_r, \tag{3.19}$$

where $u_{0r} \in V$ and $u_{1r} \in H$ for $r = 1, \cdots, p$. Then by (1.6) we have

$$t \geqslant 0: \quad \|C_p(U(t), U'(t))\|_{(V\times H)^{N-p}} \leqslant Me^{-\omega t}\|C_p(U_0, U_1)\|_{(V\times H)^{N-p}} = 0. \tag{3.20}$$

There exist some functions $u_1, \cdots, u_p$ in $C^0(\mathbb{R}^+, V) \cap C^1(\mathbb{R}^+, H)$, such that

$$U = \sum_{s=1}^{p} u_se_s. \tag{3.21}$$

Then

$$\sum_{s=1}^{p} u_s'' e_s + \sum_{s=1}^{p} L u_s e_s + \sum_{s=1}^{p} g u_s' D e_s + \sum_{s=1}^{p} u_s A e_s = 0. \tag{3.22}$$

Applying C_p to both sides of the above system, it follows that

$$\sum_{s=1}^{p} g u_s' C_p D e_s + \sum_{s=1}^{p} u_s C_p A e_s = 0. \tag{3.23}$$

In particular, by the continuity at $t = 0$, we have

$$\sum_{s=1}^{p} g u_{1s} C_p D e_s + \sum_{s=1}^{p} u_{0s} C_p A e_s = 0, \tag{3.24}$$

then

$$C_p A e_s = 0, \quad C_p D e_s = 0, \quad s = 1, \cdots, p. \tag{3.25}$$

Thus A satisfies the condition of C_p-compatibility (3.5), and D satisfies a similar condition of C_p-compatibility as in (3.5).

We next show that D satisfies the condition of strong C_p-compatibility (3.6). In fact, for $s = 1, \cdots, p$, we have

$$(\!(De_s, d)\!) = (\!(e_s, Dd)\!) = 0, \quad d \in Ker(D), \tag{3.26}$$

then $De_s \in \mathrm{Ker}(D)^\perp \cap \mathrm{Ker}(C_p)$. By Proposition 3.6, $\mathrm{Ker}(D)$ is bi-orthogonal to $\mathrm{Ker}(C_p)$, so $\mathrm{Ker}(D)^\perp \cap \mathrm{Ker}(C_p) = \{0\}$. Then

$$De_s = 0, \quad s = 1, \cdots, p. \tag{3.27}$$

We get thus the condition of strong C_p-compatibility (3.6) for the matrix D. □

Theorem 3.8. *Assume that A satisfies the condition of C_p-compatibility* (3.5) *and D satisfies the condition of strong C_p-compatibility* (3.7) *with rank$(R) = N - p$. Assume furthermore that the scalar equation* (2.19) *is uniformly stable in the space $V \times H$. Then system (1.2) is uniformly synchronizable by p-groups in $(V \times H)^N$.*

Proof. By Proposition 3.4, it is sufficient to show the uniform stability of the reduced system (3.14). By (3.8), $\mathrm{rank}(\overline{D}_p) = \mathrm{rank}(R) = N - p$. Then by Theorem 2.6, the reduced system (3.14) is uniformly stable. □

Theorem 3.9. *Assume that system (1.2) is uniformly synchronizable by p-groups in $(V \times H)^N$, then there exist some functions $u_1, \cdots, u_p$ in $C^0(\mathbb{R}^+, V) \cap C^1(\mathbb{R}^+, H)$ and some positive constants $M \geqslant 1$ and $\omega > 0$, such that setting*

$$u = \sum_{r=1}^{p} u_r e_r / \|e_r\|, \tag{3.28}$$

we have for all $t \geqslant 0$,

$$\|(U(t) - u(t), U'(t) - u'(t))\|_{(V \times H)^N} \leqslant M e^{-\omega t} \|C_p(U_0, U_1)\|_{(V \times H)^{N-p}}. \tag{3.29}$$

Assume furthermore that A satisfies the condition of C_p-compatibility (3.5) *and D satisfies the condition of strong C_p-compatibility* (3.6)*. Then u obeys a conservative system.*

Proof. Let U be the solution to system (1.2) with any given initial data $(U_0, U_1) \in (V \times H)^N$. For $r = 1, \cdots, p$, let $u_r = ((U, e_r))/\|e_r\|$. Noting that $\mathbb{R}^N = \mathrm{Ker}(C_p) \oplus \mathrm{Im}(C_p^T)$, we have

$$U = \sum_{r=1}^{p} u_r e_r/\|e_r\| + C_p^T(C_pC_p^T)^{-1}C_pU = u + C_p^T(C_pC_p^T)^{-1}C_pU. \tag{3.30}$$

By (1.6), we get

$$\begin{aligned} &\|(U(t) - u(t), U'(t) - u'(t))\|_{(V\times H)^N} \\ \leqslant &\|C_p^T(C_pC_p^T)^{-1}\|\|C_p(U(t), U'(t))\|_{(V\times H)^{N-p}} \\ \leqslant &M'e^{-\omega t}\|C_p(U_0, U_1)\|_{(V\times H)^{N-p}}, \quad t \geqslant 0 \end{aligned} \tag{3.31}$$

for some constant $M' \geqslant 1$.

Now we will precisely show the dynamics of the functions $u_1, \cdots, u_p$. First, recall that the condition of C_p-compatibility (3.5) implies

$$Ae_r = \sum_{s=1}^{p} \beta_{rs}\frac{\|e_r\|}{\|e_s\|}e_s, \quad r = 1, \cdots, p. \tag{3.32}$$

Moreover, since A is symmetric, a straightforward computation shows that

$$(Ae_r, e_s) = \sum_{q=1}^{p} \beta_{rq}\frac{\|e_r\|}{\|e_q\|}((e_q, e_s)) = \beta_{rs}\|e_r\|\|e_s\| \tag{3.33}$$

and

$$((e_r, Ae_s)) = \sum_{q=1}^{p} \beta_{sq}\frac{\|e_s\|}{\|e_q\|}((e_r, e_q)) = \beta_{sr}\|e_s\|\|e_r\|. \tag{3.34}$$

It follows that

$$\beta_{rs} = \beta_{sr}, \quad 1 \leqslant r, s \leqslant p.$$

On the other hand, the condition of strong C_p-compatibility (3.6) implies

$$De_r = 0, \quad r = 1, \cdots, p. \tag{3.35}$$

Then, applying e_r to system (1.2), we get the following conservative system

$$\begin{cases} u_r'' + Lu_r + \sum_{s=1}^{p} \beta_{rs}u_s = 0, \\ t = 0: \quad u_r = ((U_0, e_r))/\|e_r\|, \quad u_r' = ((U_1, e_r))/\|e_r\| \end{cases} \tag{3.36}$$

for $r = 1, \cdots, p$. □

Remark 3.10. *Classically, the convergence* (1.5) *or equivalently* (1.6) *is called uniform synchronization by p-groups in the consensus sense, while the convergence* (1.5) *is in the pinning sense. Moreover, the p-tuple* $u = (u_1, \cdots, u_p)$ *is called the uniformly synchronizable state by p-groups. Theorem 3.9 indicates that two notions are simply the same.*

Moreover, setting the matrix $B = (\beta_{rs})$*, we define the energy by*

$$E(t) = \|u(t)\|^2_{V^p} + (Bu(t), u(t))_{H^p} + \|u'(t)\|^2_{H^p}.$$

Since B is symmetric, we have

$$E(t) = E(0), \quad t \geqslant 0. \tag{3.37}$$

Then the orbit of u is lacalized on the sphere (3.37) *which is uniquely determined by the projection of the initial data (U_0, U_1) to $Ker(C_p)$.*

Remark 3.11. *The condition of strong C_p-compatibility* (3.6) *implies that (see Proposition 2.13 in* [16]*)*

$$rank(D, AD, \cdots, A^{N-1}D) = N - p.$$

Following Theorem 4.7 in [18]*, there does not exist an extended matrix $\widetilde{C}_q$ $(q < p)$, such that*

$$\widetilde{C}_q(U(t), U'(t)) \to (0, 0) \quad in\ (V \times H)^N\ as\ t \to +\infty.$$

Unlike in the case of the approximate boundary synchronization by p-groups, there is no possibility to get any induced synchronization in the present situation (see Chapter 11 in [16]*).*

4. Applications

4.1. **Wave equations with boundary feedback.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary $\Gamma = \Gamma_1 \cup \Gamma_0$ such that $\overline{\Gamma}_1 \cap \overline{\Gamma}_0 = \emptyset$ and $\text{mes}(\Gamma_1) > 0$. For fixing idea, we assume that $\text{mes}(\Gamma_0) > 0$.

Consider the following wave equation

$$\begin{cases} u'' - \Delta u = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ u = 0 & \text{on } \mathbb{R}^+ \times \Gamma_0, \\ \partial_\nu u + u' = 0 & \text{on } \mathbb{R}^+ \times \Gamma_1, \end{cases} \tag{4.1}$$

where ∂_ν denotes the outward normal derivative on the boundary. The uniform stability of (4.1) was abundantly studied by different approaches in the literature, we only quote [3, 10, 11] and the references therein.

Now, let A and D be symmetric and semi-positive definite matrices of order N. We consider the following system of wave equations:

$$\begin{cases} U'' - \Delta U + AU = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ U = 0 & \text{on } \mathbb{R}^+ \times \Gamma_0, \\ \partial_\nu U + DU' = 0 & \text{on } \mathbb{R}^+ \times \Gamma_1. \end{cases} \tag{4.2}$$

Let $H^1_{\Gamma_0}(\Omega)$ denote the subspace of $H^1(\Omega)$, composed of functions with vanishing trace on Γ_0. Multiplying (4.2) by $\Phi \in H^1_{\Gamma_0}(\Omega)$ and integrating by parts, we get the following variational formulation:

$$\int_\Omega ((U'', \Phi))dx + \int_\Omega ((\nabla U, \nabla \Phi))dx + \int_\Omega ((AU, \Phi))dx + \int_{\Gamma_1} ((DU', \Phi))d\Gamma = 0. \tag{4.3}$$

Define

$$\langle Lu, \phi \rangle = \int_\Omega \nabla u \cdot \nabla \phi dx, \quad \langle gv, \phi \rangle = \int_{\Gamma_1} v\phi d\Gamma. \tag{4.4}$$

Then (4.3) can be rewritten as

$$U'' + \mathcal{L}U + AU + D\mathcal{G}U' = 0. \tag{4.5}$$

Moreover, since the scalar equation (4.1) is uniformly stable in $H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$, applying Theorem 3.8 and Theorem 3.9, we immediately obtain the following

Theorem 4.1. *Assume that A satisfies the condition of C_p-compatibility* (3.5) *and D the condition of strong C_p-compatibility* (3.7) *with rank$(R) = N - p$. Then the system of wave equations (4.2) is uniformly synchronizable by p-groups in the space $(H^1_{\Gamma_0}(\Omega) \times L^2(\Omega))^N$.*

Moreover, for any given initial data $(U_0, U_1) \in (H^1_{\Gamma_0}(\Omega) \times L^2(\Omega))^N$, consider the problem

$$
\begin{cases} u_r'' - \Delta u_r + \sum_{s=1}^p \beta_{rs} u_s = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ u_r = 0 & \text{on } \mathbb{R}^+ \times \Gamma_0, \\ \partial_\nu u_r = 0 & \text{on } \mathbb{R}^+ \times \Gamma_1, \\ t = 0: \quad u_r = ((U_0, e_r))/\|e_r\|, \quad u_r' = ((U_0, e_r))/\|e_r\| & \text{in } \Omega \end{cases} \tag{4.6}
$$

for $r = 1, \cdots, p$, and the coefficients β_{rs} are given by (3.32). *Then, setting $u = \sum_{r=1}^p u_r e_r/\|e_r\|$, the corresponding solution U to system (4.2) satisfies*

$$
\begin{aligned} &\|(U(t) - u(t), U'(t) - u'(t))\|_{(H^1_{\Gamma_0}(\Omega) \times L^2(\Omega))^N} \\ \leqslant & M e^{-\omega t} \|C_p(U_0, U_1)\|_{(H^1_{\Gamma_0}(\Omega) \times L^2(\Omega))^{N-p}}, \quad t \geqslant 0. \end{aligned} \tag{4.7}
$$

4.2. **Wave equations with locally distributed feedback.** Let $\Omega \subset \mathbb{R}^n$ denote a bounded domain with smooth boundary Γ. Let $\omega \subset \Omega$ denote the damped domain.

Let a be a smooth function such that

$$
a(x) \geqslant 0, \quad \forall x \in \Omega \quad \text{and} \quad a(x) \geqslant a_0 > 0, \quad \forall x \in \omega. \tag{4.8}
$$

Consider the uniform stability of the following locally damped scalar wave system

$$
\begin{cases} u'' - \Delta u + a u' = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ u = 0 & \text{on } \mathbb{R}^+ \times \Gamma. \end{cases} \tag{4.9}
$$

This is a very challenge and promising issue. There is a large amount of literatures that we will comment briefly. The uniform decay was first established by multipliers in [6] as ω is a neighbourhood of the boundary. Later, the result was generalized in [30] to semi-linear case. When Ω is a compact Riemann manifold without boundary and ω satisfies the geometric optic condition, the uniform stability was established by a micro-local approach in [24].

Now, consider the following system of locally damped wave equations:

$$
\begin{cases} U'' - \Delta U + AU + aDU' = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ U = 0 & \text{on } \mathbb{R}^+ \times \Gamma, \end{cases} \tag{4.10}
$$

where A and D are symmetric and semi-positive definite matrices with constant elements. Multiplying system (4.10) by $\Phi \in H^1_0(\Omega)$ and integrating by parts, we get the following variational formulation:

$$
\int_\Omega ((U'', \Phi))dx + \int_\Omega ((\nabla U, \nabla \Phi))dx + \int_\Omega ((AU, \Phi))dx + \int_\Omega a((DU', \Phi))d\Gamma = 0. \tag{4.11}
$$

Let L and g be the linear continuous mappings from $H^1_0(\Omega)$ into $H^{-1}(\Omega)$, defined by

$$
\langle Lu, \phi \rangle = \int_\Omega \nabla u \cdot \nabla \phi dx \quad \text{and} \quad \langle gv, \phi \rangle = \int_\Omega a v \phi dx, \tag{4.12}
$$

respectively. Then the variational problem (4.11) can be rewritten as

$$
U'' + \mathcal{L}U + AU + D\mathcal{G}U' = 0. \tag{4.13}
$$

Then, applying Theorem 3.8 and Theorem 3.9, we have

Theorem 4.2. *Assume that the damped domain $\omega \subset \Omega$ contains a neighbourhood of the whole boundary Γ. Assume furthermore that A satisfies the condition of C_p-compatibility* (3.5) *and D the condition of strong C_p-compatibility* (3.7) *with rank$(R) = N - p$. Then system (4.13) is uniformly synchronizable by p-groups in the space $(H_0^1(\Omega) \times L^2(\Omega))^N$.*

Moreover, for any given initial data $(U_0, U_1) \in (H_0^1(\Omega) \times L^2(\Omega))^N$, consider the problem

$$(4.14) \qquad \begin{cases} u_r'' - \Delta u_r + \sum_{s=1}^p \beta_{rs} u_s = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ u_r = 0 & \text{on } \mathbb{R}^+ \times \Gamma, \\ t = 0: \quad u_r = ((U_0, e_r))/\|e_r\|, \quad u_r' = ((U_0, e_r))/\|e_r\| & \text{in } \Omega \end{cases}$$

for $r = 1, \cdots, p$, and the coefficients β_{rs} are given by (3.32). *Then, setting $u = \sum_{r=1}^p u_r e_r/\|e_r\|$, the corresponding solution U to system (4.13) satisfies*

$$(4.15) \qquad \begin{aligned} &\|(U(t) - u(t), U'(t) - u'(t))\|_{(H^1(\Omega) \times L^2(\Omega))^N} \\ \leqslant &M e^{-\omega t} \|C_p(U_0, U_1)\|_{(H^1(\Omega) \times L^2(\Omega))^{N-p}}, \quad t \geqslant 0. \end{aligned}$$

4.3. **Kirchhoff plate equations with locally distributed feedback.** In this sub-section Ω is a bounded domain in $\mathbb{R}^2$, occupied by an elastic thin plate. We refer to [9] for the stabilization of linear models.

Let a be a smooth and non-negative function such that (4.8) holds. Assume that ω contains a neighbourhood of the whole boundary Γ. Then, the following system of plate equation

$$(4.16) \qquad \begin{cases} u'' + \Delta^2 u + a u' = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ u = \partial_\nu u = 0 & \text{on } \mathbb{R}^+ \times \Gamma \end{cases}$$

is uniformly stable in $H_0^2(\Omega) \times L^2(\Omega)$ (see [9] for details).

Consider the following system :

$$(4.17) \qquad \begin{cases} U'' + \Delta^2 U + AU + aDU' = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ U = \partial_\nu U = 0 & \text{on } \mathbb{R}^+ \times \Gamma, \end{cases}$$

where A and D are symmetric and semi-positive definite matrices with constant elements. Multiplying system (4.17) by $\Phi \in H_0^2(\Omega)$ and integrating by parts, we get the following variational formulation:

$$(4.18) \qquad \int_\Omega ((U'', \Phi))dx + \int_\Omega ((\Delta U, \Delta \Phi))dx + \int_\Omega ((AU, \Phi))dx + \int_\Omega a((DU', \Phi))dx = 0.$$

Let L and g be defined by

$$(4.19) \qquad \langle Lu, \phi \rangle = \int_\Omega \Delta u \Delta \phi dx \quad \text{and} \quad \langle gv, \phi \rangle = \int_\Omega a v \phi dx,$$

respectively. (4.18) can be interpreted as

$$(4.20) \qquad U'' + \mathcal{L}U + AU + D\mathcal{G}U' = 0.$$

Then, applying Theorem 3.8 and Theorem 3.9, we have

Theorem 4.3. *Assume that A satisfies the condition of C_p-compatibility* (3.5) *and D the condition of strong C_p-compatibility* (3.7) *with rank$(R) = N - p$. Then system (4.17) is uniformly synchronizable by p-groups in $(H_0^2(\Omega) \times L^2(\Omega))^N$.*

Moreover, for any given initial data $(U_0, U_1) \in (H_0^2(\Omega) \times L^2(\Omega))^N$*, consider the problem*

$$\begin{cases} u_r'' + \Delta^2 u_r + \sum_{s=1}^p \beta_{rs} u_s = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ u_r = \partial_\nu u_r = 0 & \text{on } \mathbb{R}^+ \times \Gamma, \\ t = 0: \quad u_r = ((U_0, e_r))/\|e_r\|, \quad u_r' = ((U_0, e_r))/\|e_r\| & \text{in } \Omega \end{cases} \tag{4.21}$$

for $r = 1, \cdots, p$*, and the coefficients* β_{rs} *are given by* (3.32)*. Then, setting* $u = \sum_{r=1}^p u_r e_r/\|e_r\|$*, the corresponding solution* U *to system (4.17) satisfies*

$$\begin{aligned} &\|(U(t) - u(t), U'(t) - u'(t))\|_{(H^2(\Omega)\times L^2(\Omega))^N} \\ \leqslant &Me^{-\omega t}\|C_p(U_0, U_1)\|_{(H^2(\Omega)\times L^2(\Omega))^{N-p}}, \quad t \geqslant 0. \end{aligned} \tag{4.22}$$

Remark 4.4. *The above examples are classic and illustrate the applications of the abstract theory. In fact, Theorems 3.8 and 3.9 are also applicable for many other models, such as system of wave equations with viscoelastic (Kelvin-Voigt) damping, system of Kirchhoff plate equations with boundary shear force and bending moment damping etc.*

5. Perspective comments.

Up to now, we have started the work on a simplified model with only one damping. Many related problems can be considered later.

(i) By the definition of uniform synchronization by p-groups:

$$\|C_p(U(t), U'(t))\|_{(V\times H)^{N-p}} \leqslant Me^{-\omega t}\|C_p(U_0, U_1)\|_{(V\times H)^{N-p}}, \quad t \geqslant 0, \tag{5.1}$$

if $C_p(U_0, U_1) = (0, 0)$, then

$$C_p U(t) \equiv 0, \quad t \geqslant 0. \tag{5.2}$$

Thus, for any given synchronized initial data, the solution is always synchronized. This simplifies much the study on the necessity of the conditions of C_p-compatibility given in Theorem 3.7.

A more natural definition of uniform synchronization by p-groups should be given by

$$\|C_p(U(t), U'(t))\|_{(V\times H)^{N-p}} \leqslant Me^{-\omega t}\|(U_0, U_1)\|_{(V\times H)^N}, \quad t \geqslant 0. \tag{5.3}$$

In this case, the solution is not automatically synchronized even for the synchronized initial data. The situation will be chaotic and presents certainly many interesting questions.

(ii) Instead of the uniform decay rate given by (5.1), we can consider the polynomial decay rate as

$$\|C_p(U(t), U'(t))\|_{(V\times H)^{N-p}} = O((1+t)^{-\delta}), \quad t \geqslant 0, \tag{5.4}$$

with a positive power δ. We refer to [5, 23] and the references therein for the recent progress on the polynomial stability of indirectly damped wave equations.

(iii) We may consider a system with several damping of different types:

$$U'' + \mathcal{L}U + AU + D_1\mathcal{G}_1U' + D_2\mathcal{G}_2U' = 0, \tag{5.5}$$

where $\mathcal{G}_1$ and $\mathcal{G}_2$ can be internal and boundary damping for wave equations, and bending moment and shear force damping for plate equations, respectively. Many related questions can be asked, for example:

(a) Let $D = (D_1, D_2)$ be the composite damping matrix. Is Kalman's rank condition on (A, D) still sufficient for the asymptotic stability as what has been done in [18]?

(b) Is the condition $\text{rank}(D) = N$ still sufficient for the uniform stability as we have done in the present work?

The main difficulty comes from the interaction of the numerous matrices A, D_1, D_2, somewhat like for coupled Robin problem in [12]. The key idea is to separate them as the coupling terms are compact, so more regularity seems to be necessary.

We do not have any answer yet for each question, but the first attempt already shows some interesting results for developing the research in these directions.

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