Characterizations of complete stabilizability*

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Abstract

We present several characterizations, via some weak observability inequalities, of the complete stabilizability for a control system $[A, B]$, i.e., $y'(t) = Ay(t) + Bu(t)$, $t \geq 0$, where A generates a C_0-semigroup on a Hilbert space X and B is a linear and bounded operator from another Hilbert space U to X. We then extend these characterizations in two directions: first, the control operator B is unbounded; second, the control system is time-periodic. We also give some sufficient conditions, from the perspective of the spectral projections, to ensure the weak observability inequalities. As applications, we provide several examples, which are not null controllable, but can be verified, via the weak observability inequalities, to be completely stabilizable.

Keywords. complete stabilizability, weak observability inequality, infinite-dimensional system

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1 Introduction

1.1 Control system and notation

In the literature on infinite-dimensional linear systems, several concepts of stabilization appear, such as complete stabilization, exponential stabilization, strong stabilization, polynomial stabilization, and logarithmic stabilization. This paper mainly studies the complete stabilization for the control system $[A, B]$, i.e.,

$$y'(t) = Ay(t) + Bu(t), \quad t \geq 0, \tag{1.1}$$

under the assumptions:

(H_1) The operator A, with its domain $D(A) \subset X$, generates a C_0-semigroup $S(t)$ $(t \geq 0)$ on a Hilbert space X.

(H_2) The operator B is a linear and bounded operator from another Hilbert space U to X. The Hilbert spaces X and U are identified with their dual spaces respectively.

We further study the complete stabilization for both a system $[A, B]$ (where B is unbounded) and a periodic system $[A(\cdot), B(\cdot)]$. To avoid complex definitions in the introduction, we treat them as extensions in Section 3 of this paper.

Throughout the paper, the following notations will be used: Given $u \in L^2(\mathbb{R}^+; U)$ and $y_0 \in X$, we write $y(\cdot; u, y_0)$ for the solution to the system (1.1) with the initial condition $y(0) = y_0$; $\mathbb{R}^+ := [0, +\infty)$, $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbf{N} := \mathbb{N} \cup \{0\}$; Given a Hilbert space X_1, we write $\|\cdot\|_{X_1}$ and $\langle \cdot, \cdot \rangle_{X_1}$ for the norm and

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the inner product of X_1 respectively; Given Banach spaces X_1 and X_2, we write $\mathcal{L}(X_1; X_2)$ for the space of all linear and bounded operators from X_1 to X_2 and $\mathcal{L}(X_1) := \mathcal{L}(X_1; X_1)$; Given a linear operator F, we use F^* to denote its adjoint operator; We denote by I the identity operator on any space; Write $\rho(A)$ for the resolvent set of the operator A.

1.2 Aim and motivation

Let us first review several concepts related to the control system (1.1):

(a_1) The system (1.1) is said to be exponentially stabilizable, if there exists $K \in \mathcal{L}(X; U)$, $\mu > 0$ and $C > 0$ such that $\|S_K(t)\|_{\mathcal{L}(X)} \le Ce^{-\mu t}$ for all $t \in \mathbb{R}^+$. Here, $S_K(t)$ $(t \ge 0)$ denotes the semigroup generated by $A + BK$.

(a_2) The system (1.1) is said to be completely (or rapidly) stabilizable, if for any $\mu > 0$, there exists $K := K(\mu) \in \mathcal{L}(X; U)$ and $C := C(\mu) > 0$ such that $\|S_K(t)\|_{\mathcal{L}(X)} \le Ce^{-\mu t}$ for all $t \in \mathbb{R}^+$.

(a_3) The system (1.1) is said to be null controllable over $[0, T]$ for some $T > 0$, if for any $y_0 \in X$, there exists $u \in L^2(0, T; U)$ such that $y(T; u, y_0) = 0$.

For these concepts, we have the following known facts:

(b_1) In finite-dimensional settings where A, B are matrices, $(a_2) \Leftrightarrow (a_3)$. However, in infinite-dimensional settings, $(a_3) \Rightarrow (a_2) \Rightarrow (a_1)$ (see [39, Proposition 21]), but the reverse may be not true.

(b_2) The null controllability over $[0, T]$ is equivalent to the following observability inequality: there exists $C := C(T) > 0$ such that $\|S(T)^*\varphi\|_X \le C\|B^*S(T - \cdot)^*\varphi\|_{L^2(0,T;U)}$ for all $\varphi \in X$ (This inequality can be equivalently written as the "initial time" observability inequality for the adjoint equation of (1.1), that is, $\|z(0)\|_X \le C\|B^*z(\cdot)\|_{L^2(0,T;U)}$, where $z(\cdot)$ is the solution to the adjoint equation $z'(t) = -A^*z(t), z(T) = \varphi \in X$ (see [27, Chapter 7, Section 2.2])).

(b_3) The exponential stabilizability is equivalent to the weak observability of the dual system: there is $\alpha \in (0, 1)$, $T > 0$ and $C > 0$ such that $\|S(T)^*\varphi\|_X \le C\|B^*S(T - \cdot)^*\varphi\|_{L^2(0,T;U)} + \alpha\|\varphi\|_X$ for all $\varphi \in X$. (This was proved in [39, Theorem 1].)

According to the above facts (b_1)-(b_3), the following question is natural and interesting:

- How to characterize the complete stabilizability by some kind of observability inequalities?

The aim of this paper is to answer the above question.

1.3 Main results

The main theorem of this paper is as follows:

Theorem 1.1. *The following statements are equivalent:*

(i) The control system (1.1) is completely stabilizable.

(ii) For any $\alpha > 0$, there are positive constants $C(\alpha)$ and $D(\alpha)$ such that

$$\|S(T)^*\varphi\|_X \le D(\alpha)\|B^*S(T - \cdot)^*\varphi\|_{L^2(0,T;U)} + C(\alpha)e^{-\alpha T}\|\varphi\|_X, \quad when \ \varphi \in X \ and \ T > 0. \quad (1.2)$$

(iii) There exists $T_0 \ge 0$ such that for any $T > T_0$ and $\alpha > 0$, there are positive constants $C(\alpha)$ (which is independent of T) and $D(\alpha, T)$ such that

$$\|S(T)^*\varphi\|_X \le D(\alpha, T)\|B^*S(T - \cdot)^*\varphi\|_{L^2(0,T;U)} + C(\alpha)e^{-\alpha T}\|\varphi\|_X, \quad when \ \varphi \in X. \quad (1.3)$$

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(iv) For each $k \in \mathbb{N}$, there are positive constants T_k and $D(k)$ such that

$$\|S(T_k)^*\varphi\|_X \le D(k)\|B^*S(T_k - \cdot)^*\varphi\|_{L^2(0,T_k;U)} + e^{-kT_k}\|\varphi\|_X, \quad when \ \varphi \in X. \tag{1.4}$$

Some comments on Theorem 1.1 are given.

(c_1) For each $\alpha > 0$, inequality (1.2) is a weak observability inequality. Hence, the complete stabilizability is characterized by a family of weak observability inequalities. This essentially differs from the exponential stabilizability which corresponds to only one weak observability inequality (see the note (b_3)). The reason that an exponential function appears in (1.2) (as well as (1.3) and (1.4)) is that the system $[A, B]$ is completely stabilizable if and only if for each $\mu > 0$, the system $[A + \mu I, B]$ is exponentially stabilizable. This can be seen from the proof of Theorem 1.1.

(c_2) The statement (iv) can be understood as a kind of discretization of the statement (ii).

(c_3) In this paper, we will further extend Theorem 1.1 in two directions: First, the control operator B is unbounded; Second, the control system is time-periodic. We give these extensions in Section 3.

(c_4) As applications of Theorem 1.1, as well as its extensions, some examples will be given in Subsection 4.2. These examples present several concrete control systems, which are not null controllable, but can be proved to be completely stabilizable, via the weak observability inequalities in Theorem 1.1, as well as its extensions.

(c_5) We provide some sufficient conditions, from the perspective of the spectral projections, to ensure the weak observability inequalities in Theorem 1.1 (see Subsection 4.1).

1.4 Related works and the novelty of this paper

There is a lot of literature on the stabilization of infinite-dimensional systems. We mention [13, 25, 30, 35, 40] for time-invariant linear systems with bounded control operators; [1, 4, 12, 15, 17, 21, 22, 23, 26, 28, 37, 42, 43, 46] for time-invariant linear systems with unbounded control operators; [2, 19, 24] for time-varying linear systems with bounded or unbounded control operators; [3, 31, 45] for time-periodic systems; [6, 7, 8, 10, 11] for nonlinear systems.

About the characterizations of the exponential stabilization for infinite-dimensional linear time-invariant systems, we would like to mention works [4, 30, 39]: A frequency domain criterion on the stabilizability is built up for conservative systems with distributed control in [30]; A unique continuation type criterion on the stabilizability (which is also called Fattorini's criterion) is established in [4] for parabolic systems; A characterization, via a weak observability inequality, of the stabilizability is given for some infinite dimensional systems in [39]. About the characterizations of the periodically exponential stabilization for infinite-dimensional linear time-periodic systems, we mention works [3, 5, 44, 45, 47]: Some unique continuation type criterions on the periodic stabilizability, as well as a characterization, via a weak observability inequality, are presented for some parabolic-like time periodic evolution equations in [3, 5]; A characterization, via a detectability inequality, is given for some linear time-periodic evolution systems in [47]. Certain geometric and analytic characterizations of the periodic stabilizability are provided for some linear time-periodic evolution systems in [44, 45].

We emphasize here the works [3], [4] and [5], where some characterizations of the stabilizability/ the periodic stabilizability, with an arbitrarily given decay rate, were obtained for some parabolic-like evolution equations. It seems for us that some characterizations of the complete stabilizability/the periodically complete stabilizability for those equations can be derived from these works. Compared these works with ours, we would like to emphasize what follows: First, the results obtained there need the assumption that the generator of the control system has compact resolvent, while such assumption is not necessary in our work. (This assumption allows one to decompose the control system into two sub-systems, one is unstable and of finite-dimension and another is stable and of infinite-dimension.) Second, the works [3], [4] and [5] concern stabilizability, while ours deals with complete stabilizability.

We now explain the novelty of this work:

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- We have not found any characterization via observability inequalities on the complete/periodic complete stabilizability for time-invariant/time-periodic evolution equations in the literature. Hence, Theorem 1.1, as well as its extensions obtained in this work, seem to be new results. These results may help us to understand the connections and the differences between stabilizability, complete stabilizability and null controllability, from the perspective of observability inequalities.

- We are working in a general framework where the generator of the system does not need to have compact resolvents. Our Example 1 in Subsection 4.2 is under such framework. Indeed, the generator of the system in that example has only continuous spectrum, and consequently does not have compact resolvents.

- Though the generators of the systems studied in [39, 47] also do not need to have compact resolvents, the authors there did not obtain the characterizations on the stabilizability/the periodic stabilizability for an arbitrarily given decay rate, like [3, 4, 5]. So the approach to the characterizations on the complete stabilizability obtained in this work does not follow from [39, 47].

1.5   The plan of this paper

The rest of the paper is organized as follows: Section 2 gives the proof of Theorem 1.1; Section 3 presents two extensions of Theorem 1.1; Section 4 provides several examples and shows some sufficient conditions ensuring the weak observability inequalities.

2   The proof of Theorem 1.1

The next lemma is quoted from [39] and will play an important role in the proof of Theorem 1.1.

Lemma 2.1. *([39, Theorem 1]) Let $\mu \geq 0$. Let $S_\mu(t)$ ($t \geq 0$) be the semigroup generated by $A + \mu I$. Then the following statements are equivalent:*

(i) *The following system is exponentially stabilizable:*

$$z'(t) = (A + \mu I)z(t) + Bu(t), \quad t \in \mathbb{R}^+. \tag{2.1}$$

(ii) *There exists $\alpha \in (0,1), T > 0$ and $C \geq 0$ such that*

$$\|S_\mu(T)^* \varphi\|_X \leq C\|B^* S_\mu(T - \cdot)^* \varphi\|_{L^2(0,T;U)} + \alpha\|\varphi\|_X \ \text{ for any } \ \varphi \in X. \tag{2.2}$$

Now, we are in position to prove Theorem 1.1.

The proof of Theorem 1.1. We organize the proof in several steps.

Step 1. We show $(i) \Rightarrow (ii)$.

Arbitrarily fix $\alpha > 0$. By (i), there exists $K := K(\alpha) \in \mathcal{L}(X;U)$ and $C := C(\alpha) \geq 1$ such that

$$\|S_K(t)\|_{\mathcal{L}(X)} \leq Ce^{-\alpha t} \ \text{ for all } \ t \in \mathbb{R}^+. \tag{2.3}$$

Meanwhile, we arbitrarily fix $y_0 \in X$ and set

$$u_{y_0}(t) := KS_K(t)y_0, \ \ t \in \mathbb{R}^+. \tag{2.4}$$

Then by (2.3) and (2.4), we have

$$\|u_{y_0}\|_{L^2(0,T;U)} \leq C(2\alpha)^{-\frac{1}{2}}\|K\|_{\mathcal{L}(X;U)}\|y_0\|_X \ \text{ for any } \ T > 0. \tag{2.5}$$

Next, we arbitrarily fix $T > 0$. By the definitions of $S_K(\cdot)$ and $S(\cdot)$ and by (2.4), we see

$$S_K(T)y_0 = S(T)y_0 + \int_0^T S(T-t)Bu_{y_0}(t)dt,$$

which implies

$$-\langle y_0, S(T)^*\varphi\rangle_X = -\langle S_K(T)y_0, \varphi\rangle_X + \int_0^T \langle u_{y_0}(t), B^*S(T-t)^*\varphi\rangle_X dt \ \text{ for any } \ \varphi \in X.$$

The above, along with (2.3) and (2.5), yields that for any $\varphi \in X$,

$$|\langle y_0, S(T)^*\varphi\rangle_X| \le C\left(e^{-\alpha T}\|\varphi\|_X + \|K\|_{\mathcal{L}(X;U)}(2\alpha)^{-\frac{1}{2}}\|B^*S(T-\cdot)^*\varphi\|_{L^2(0,T;U)}\right)\|y_0\|_X.$$

Since $y_0 \in X$ and $T > 0$ were arbitrarily taken, the above implies that for any $\varphi \in X$ and $T > 0$,

$$\|S(T)^*\varphi\|_X \le C\left(e^{-\alpha T}\|\varphi\|_X + \|K\|_{\mathcal{L}(X;U)}(2\alpha)^{-\frac{1}{2}}\|B^*S(T-\cdot)^*\varphi\|_{L^2(0,T;U)}\right). \tag{2.6}$$

Now, (1.2), with $D(\alpha) := C\|K\|_{\mathcal{L}(X;U)}(2\alpha)^{-\frac{1}{2}}$, follows from (2.6) at once. Since $\alpha > 0$ was arbitrarily taken, we obtain (ii).

Step 2. It is trivial that $(ii) \Rightarrow (iii)$.

Step 3. We show $(iii) \Rightarrow (iv)$.

Let T_0, $C(\alpha)$ and $D(\alpha, T)$ be given by (iii). Arbitrarily fix $k \in \mathbb{N}$. Let T_k be such that $T_k > T_0$ and $C(k+1) < e^{T_k}$. Write $D(k) := D(k+1, T_k)$. Then, by (1.3) (where $\alpha = k+1$ and $T = T_k$), after some direct computations, we get (1.4) with the above T_k and $D(k)$. Since k was arbitrarily taken from $\mathbb{N}$, we get (iv).

Step 4. We show $(iv) \Rightarrow (i)$.

Arbitrarily fix $\mu > 0$. We first show that the system (2.1) is exponentially stabilizable. To this end, we take $k_\mu \in \mathbb{N}$ so that $k_\mu - 1 \le \mu < k_\mu$. Then by (iv), we can find $D(k_\mu) > 0$ and $T_{k_\mu} > 0$ such that

$$\|S(T_{k_\mu})^*\varphi\|_X \le D(k_\mu)\|B^*S(T_{k_\mu} - \cdot)^*\varphi\|_{L^2(0,T_{k_\mu};U)} + e^{-k_\mu T_{k_\mu}}\|\varphi\|_X \ \text{ for all } \ \varphi \in X. \tag{2.7}$$

Meanwhile, we write $S_\mu(t)$ $(t \ge 0)$ for the C_0-semigroup generated by $A + \mu I$. Then it is clear that

$$S_\mu(t)^* = e^{\mu t}S(t)^* \ \text{ for all } \ t \ge 0. \tag{2.8}$$

Now, multiplying (2.7) by $e^{\mu T_{k_\mu}}$, using (2.8), we have

$$\|S_\mu(T_{k_\mu})^*\varphi\|_X \le D(k_\mu)e^{\mu T_{k_\mu}}\|B^*S(T_{k_\mu} - \cdot)^*\varphi\|_{L^2(0,T_{k_\mu};U)} + e^{-(k_\mu - \mu)T_{k_\mu}}\|\varphi\|_X.$$

Since the first term in the righthand side of above inequality can be written as

$$\begin{aligned}
D(k_\mu)e^{\mu T_{k_\mu}}\|B^*S(T_{k_\mu} - \cdot)^*\varphi\|_{L^2(0,T_{k_\mu};U)} &= D(k_\mu)e^{\mu T_{k_\mu}}\|e^{-\mu(T_{k_\mu} - \cdot)}B^*S_\mu(T_{k_\mu} - \cdot)^*\varphi\|_{L^2(0,T_{k_\mu};U)} \\
&= D(k_\mu)\|e^{\mu \cdot}B^*S_\mu(T_{k_\mu} - \cdot)^*\varphi\|_{L^2(0,T_{k_\mu};U)},
\end{aligned}$$

and the function $e^{\mu t}, t \in [0, T_{k_\mu}]$ can be dominated by $e^{\mu T_{k_\mu}}$, we get

$$\|S_\mu(T_{k_\mu})^*\varphi\|_X \le D(k_\mu)e^{\mu T_{k_\mu}}\|B^*S_\mu(T_{k_\mu} - \cdot)^*\varphi\|_{L^2(0,T_{k_\mu};U)} + e^{-(k_\mu - \mu)T_{k_\mu}}\|\varphi\|_X \ \text{ for all } \ \varphi \in X. \tag{2.9}$$

Since $e^{-(k_\mu - \mu)T_{k_\mu}} < 1$, the above (2.9) leads to (2.2) with

$$T = T_{k_\mu} > 0, \ \alpha = e^{-(k_\mu - \mu)T_{k_\mu}} \in (0,1), \ C = D(k_\mu)e^{\mu T_{k_\mu}} > 0.$$

Then according to Lemma 2.1, the system (2.1) is exponentially stabilizable.

We next claim that the system (1.1) is completely stabilizable. Indeed, since the system (2.1) is exponentially stabilizable, there exists $K := K(\mu) \in \mathcal{L}(X; U)$ and $C(\mu) > 0$ such that

$$\|S_{\mu,K}(t)\|_{\mathcal{L}(X)} \leq C(\mu) \text{ for all } t \in \mathbb{R}^+,$$

where $S_{\mu,K}(t)$ $(t \geq 0)$ is the semigroup generated by $A + \mu I + BK$. This, together with the fact:

$$S_{\mu,K}(t) = e^{\mu t} S_K(t) \text{ for all } t \in \mathbb{R}^+,$$

yields that

$$\|S_K(t)\|_{\mathcal{L}(X)} \leq C(\mu) e^{-\mu t} \text{ for all } t \in \mathbb{R}^+.$$

Since $\mu > 0$ was arbitrarily taken, the above leads to the complete stabilizability for the system (1.1), i.e., (i) is true. □

3 Extensions

In this section we will extend Theorem 1.1 in two directions: The first one is the case that the control operator B is unbounded, while the second one is the case that the control system is time-periodic.

3.1 The case that the control operator is unbounded

This subsection aims to extend Theorem 1.1 to the control system $[A, B]$, i.e.,

$$y'(t) = Ay(t) + Bu(t), \quad t \geq 0, \tag{3.1}$$

under the following assumptions:

$(\widetilde{H_1})$ The operator A, with its domain[1] $D(A) \subset X$, is the generator of a C_0-semigroup $S(t)$ $(t \geq 0)$ on X.

$(\widetilde{H_2})$ The operator B belongs to $\mathcal{L}(U; X_{-1})$, where X_{-1} is the completion of X with respect to the norm $\|z\|_{-1} := \|(\rho_0 I - A)^{-1} z\|_X$, $z \in X$ (where $\rho_0 \in \rho(A)$ is arbitrarily fixed).

$(\widetilde{H_3})$ There exists a time $T > 0$ and a constant $C(T) > 0$ such that

$$\int_0^T \|B^* S(t)^* x\|_U^2 dt \leq C(T) \|x\|_X^2 \text{ for all } x \in D(A^*). \tag{3.2}$$

(This condition is called the regularity property or the admissibility condition (see, for example, [10, Chapter 2, Section 2.3] or [26]). Here, we notice that $B^* \in \mathcal{L}(D(A^*); U)$ by $(\widetilde{H_2})$ and (d_3) in Remark 3.1 below.)

Given $y_0 \in X$ and $u \in L^2(\mathbb{R}^+; U)$, we write $y(\cdot; u, y_0)$ for the solution to (3.1) with the initial condition $y(0) = y_0$.

Remark 3.1. *Several comments on the above assumptions are given.*

(d_1) *The operator A (which belongs to $\mathcal{L}(D(A); X)$) has a unique extension, denoted by $\widetilde{A}$, in the space $\mathcal{L}(X; X_{-1})$, moreover $(\rho_0 I - \widetilde{A})^{-1} \in \mathcal{L}(X_{-1}, X)$ (see [41, Chapter 2, Proposition 2.10.3]). Hence, $(\widetilde{H_2})$ can be replaced by the assumption: $(\rho_0 I - \widetilde{A})^{-1} B \in \mathcal{L}(U; X)$ (see [15, 26]).*

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[1]We define a norm on $D(A)$ by: $\|x\|_{D(A)} := \|(\rho_0 I - A)x\|_X$, $x \in D(A)$, where $\rho_0 \in \rho(A)$ is arbitrarily fixed. Then $D(A)$ with this norm is a Hilbert space since A as the generator of a C_0-semigroup is closed. It is well known that this norm is equivalent to the classical graph norm $\|x\|'_{D(A)} := (\|x\|_X^2 + \|Ax\|_X^2)^{\frac{1}{2}}$, $x \in D(A)$. The same can be said about any generator of a C_0-semigroup on X.

(d_2) *Let $\widetilde{S}(t) := (\rho_0 I - \widetilde{A})S(t)(\rho_0 I - \widetilde{A})^{-1}$ on X_{-1} for any $t \geq 0$. Then $\widetilde{S}(t)$ ($t \geq 0$) is a C_0-semigroup on X_{-1} and $\widetilde{A}$ is the generator of this semigroup (see [41, Chapter 2, Proposition 2.10.4]). We call $\widetilde{S}(t)$ ($t \geq 0$) as the extension of $S(t)$ ($t \geq 0$).*

(d_3) *The space $D(A^*)$, with the norm $\|z\|_{D(A^*)} := \|(\overline{\rho_0} I - A^*)z\|_X$, $z \in D(A^*)$, is a Hilbert space and X_{-1} is isomorphic to the dual space of $D(A^*)$ (see [41, Chapter 2, Proposition 2.10.1 and Proposition 2.10.2]). For convenience, we identify the dual space of $D(A^*)$ with X_{-1}. Thus, X_{-1} is the dual space of $D(A^*)$ with respect to the pivot space X (see [41, Chapter 2, Section 2.9]).*

(d_4) *Assumption $(\widetilde{H_3})$ is equivalent to that for any $T > 0$, there exists a constant $C(T) > 0$ such that (3.2) holds. (See [41, Chapter 4, Proposition 4.3.2].)*

(d_5) *We can easily check that when $(\widetilde{H_1})$-$(\widetilde{H_3})$ hold, B^* is an admissible observation operator and consequently B is an admissible control operator. (See [41, Chapter 4, Definition 4.3.1], [41, Chapter 4, Definition 4.2.1] and [41, Chapter 4, Theorem 4.4.3].) Hence, if $(\widetilde{H_1})$-$(\widetilde{H_3})$ are true, then, it follows by [41, Chapter 4, Proposition 4.2.5]) that when $y_0 \in X$ and $u \in L^2(\mathbb{R}^+; U)$, the equation (3.1) has a unique solution $y(\cdot; u, y_0)$ (in $C([0, +\infty); X)$) which is given by $y(t; u, y_0) = \widetilde{S}(t)y_0 + \int_0^t \widetilde{S}(t - s)Bu(s)ds$. Moreover, for each $T > 0$, there exists $C := C(T) > 0$ such that*

$$\|y(t; u, y_0)\|_X \leq C(\|y_0\|_X + \|u\|_{L^2(0,T;U)}), \quad t \in [0, T].$$

(d_6) *When $u \in L^2(\mathbb{R}^+; U)$ and $t \geq 0$, we only have $\int_0^t \widetilde{S}(t - s)Bu(s)ds \in X_{-1}$ under the assumptions $(\widetilde{H_1})$-$(\widetilde{H_2})$; but it holds that $\int_0^t \widetilde{S}(t - s)Bu(s)ds \in X$, if we further assume $(\widetilde{H_3})$ (see [41, Chapter 4, Proposition 4.2.2]).*

(d_7) *Some examples satisfying $(\widetilde{H_1})$-$(\widetilde{H_3})$ are given in [15, 26, 41].*

Throughout this subsection, $\widetilde{A}$ and $\widetilde{S}(t)$ ($t \geq 0$) denote respectively the extensions of A and $S(t)$ ($t \geq 0$), which are given in (d_1) and (d_2) of Remark 3.1.

To state the main results of this subsection, we need the following definitions on the stabilization for the system (3.1):

Definition 3.2. *(i) The system (3.1) is said to be exponentially stabilizable, if there exists a C_0-semigroup $\Phi(t)$ ($t \geq 0$) on X, with its generator $\Lambda : D(\Lambda) \subset X \to X$, and $K \in \mathcal{L}(D(\Lambda); U)$ such that*

(a) $\Lambda x = (\widetilde{A} + BK)x$ for all $x \in D(\Lambda)$;

(b) there exists $\alpha > 0$ and $C > 0$ such that $\|\Phi(t)\|_{\mathcal{L}(X)} \leq Ce^{-\alpha t}$ for any $t \in \mathbb{R}^+$;

(c) there exists $D > 0$ such that $\|K\Phi(\cdot)x\|_{L^2(\mathbb{R}^+;U)} \leq D\|x\|_X$ for all $x \in D(\Lambda)$.

(ii) The system (3.1) is said to be completely stabilizable, if for any $\alpha > 0$, there exists a C_0-semigroup $\Phi_\alpha(t)$ ($t \geq 0$) on X, with its generator $\Lambda_\alpha : D(\Lambda_\alpha) \subset X \to X$, and $K_\alpha \in \mathcal{L}(D(\Lambda_\alpha); U)$ such that

(a') $\Lambda_\alpha x = (\widetilde{A} + BK_\alpha)x$ for all $x \in D(\Lambda_\alpha)$;

(b') there exists $C(\alpha) > 0$ such that $\|\Phi_\alpha(t)\|_{\mathcal{L}(X)} \leq C(\alpha)e^{-\alpha t}$ for any $t \in \mathbb{R}^+$;

(c') there exists $D(\alpha) > 0$ such that $\|K_\alpha\Phi_\alpha(\cdot)x\|_{L^2(\mathbb{R}^+;U)} \leq D(\alpha)\|x\|_X$ for all $x \in D(\Lambda_\alpha)$.

Remark 3.3. *Definition 3.2 is inspired by [15, 26], where the authors proved that the solvability of the LQ problem $V(y_0) = \inf_{u \in L^2(\mathbb{R}^+;U)} \int_0^\infty [\|y(t; u, y_0)\|_X^2 + \|u(t)\|_U^2]dt$ (i.e., $V(y_0) < +\infty$ for all $y_0 \in X$) implies the exponential stabilizability of the system (3.1) in the sense of (i) in Definition 3.2. On the other hand, with the aid of Lemma 3.8 below, we obtain the reverse. Hence, the solvability of the above LQ problem is equivalent to the exponential stabilizability of the system (3.1) in the sense of (i) in Definition 3.2 (see Proposition 3.9 below).*

Besides, it deserves mentioning that Definition 3.2 can be viewed as the dual of the concept of estimatability (see [36, Definition 2.1]).

The main result in this subsection is as follows:

Theorem 3.4. *Suppose that $(\widetilde{H_1})$-$(\widetilde{H_3})$ are true. Then the following statements are equivalent:*

(i) *The system (3.1) is completely stabilizable.*

(ii) *For any $\alpha > 0$, there are positive constants $C(\alpha)$ and $D(\alpha)$ such that*

$$\|S(T)^*\varphi\|_X \leq D(\alpha)\|B^*S(T-\cdot)^*\varphi\|_{L^2(0,T;U)} + C(\alpha)e^{-\alpha T}\|\varphi\|_X, \quad \text{when } \varphi \in D(A^*), \ T > 0. \quad (3.3)$$

(iii) *There exists $T_0 \geq 0$ such that for any $T > T_0$ and $\alpha > 0$, there are positive constants $C(\alpha)$ (which is independent of T) and $D(\alpha, T)$ such that*

$$\|S(T)^*\varphi\|_X \leq D(\alpha,T)\|B^*S(T-\cdot)^*\varphi\|_{L^2(0,T;U)} + C(\alpha)e^{-\alpha T}\|\varphi\|_X, \quad \text{when } \varphi \in D(A^*). \quad (3.4)$$

(iv) *For each $k \in \mathbb{N}$, there are positive constants T_k and $D(k)$ such that*

$$\|S(T_k)^*\varphi\|_X \leq D(k)\|B^*S(T_k-\cdot)^*\varphi\|_{L^2(0,T_k;U)} + e^{-kT_k}\|\varphi\|_X, \quad \text{when } \varphi \in D(A^*). \quad (3.5)$$

Remark 3.5. *It follows from (3.2) that the operators*

$$\begin{cases} (x \in D(A^*)) \to ((t \to B^*S(t)^*x) \in L^2(0,T;U)), \\ (x \in D(A^*)) \to ((t \to B^*S(T-t)^*x) \in L^2(0,T;U)), \end{cases}$$

can be extended in a unique way as linear and bounded operators from X into $L^2(0,T;U)$. If we denote these extensions in the same manners, then (3.3) is equivalent to

$$\|S(T)^*\varphi\|_X \leq D(\alpha)\|B^*S(T-\cdot)^*\varphi\|_{L^2(0,T;U)} + C(\alpha)e^{-\alpha T}\|\varphi\|_X, \quad \text{when } \varphi \in X, \ T > 0.$$

The same can be said about (3.4) and (3.5).

Before proving Theorem 3.4, we give some preliminaries. The first one is about the LQ problem

$$(\textbf{LQ})_{y_0} : \quad \inf_{u \in L^2(\mathbb{R}^+;U)} J(u;y_0), \quad y_0 \in X, \quad (3.6)$$

where

$$J(u;y_0) := \int_0^\infty [\|y(t;u,y_0)\|_X^2 + \|u(t)\|_U^2]dt, \quad u \in L^2(\mathbb{R}^+;U). \quad (3.7)$$

Let

$$\mathcal{U}_{ad}(y_0) := \{u \in L^2(\mathbb{R}^+;U) : y(\cdot;u,y_0) \in L^2(\mathbb{R}^+;X)\}. \quad (3.8)$$

The following Lemma 3.6 can be found in [26, Theorem 5.2, Page 40] (see also [15, Theorem 2.2] and [46, Propositions 3.2-3.4]):

Lemma 3.6. *Assume that $(\widetilde{H_1})$-$(\widetilde{H_3})$ hold. Suppose that $\mathcal{U}_{ad}(y_0) \neq \emptyset$ for any $y_0 \in X$. Then for each $y_0 \in X$, the problem $(\textbf{LQ})_{y_0}$ has a unique solution $u^*_{y_0}$. Moreover, there exists a self-adjoint and non-negative operator $P \in \mathcal{L}(X)$ and a C_0-semigroup $S_P(t)$ $(t \geq 0)$ on X, with its generator $A_P : D(A_P) \subset X \to X$, such that the following conclusions are true:*

(i) *It holds that $P \in \mathcal{L}(D(A_P); D(A^*))$ and $B^*P \in \mathcal{L}(D(A_P); U)$.*

(ii) *For each $x \in D(A_P)$, $A_Px = (\widetilde{A} - BB^*P)x$.*

(iii) *If $y_0 \in D(A_P)$, then $u^*_{y_0}(t) = -B^*PS_P(t)y_0$ for a.e. $t \in \mathbb{R}^+$.*

(iv) *The semigroup $S_P(\cdot)$ is exponentially stable on X, i.e., there exists $C > 0$ and $\alpha > 0$ (depending on P) such that $\|S_P(t)\|_{\mathcal{L}(X)} \leq Ce^{-\alpha t}$ for any $t \in \mathbb{R}^+$.*

Remark 3.7. *In general, to ensure the conclusion (iv) in Lemma 3.6, one needs the* detectability con-
dition *given in [26, (D.C), Page 41]. Fortunately, this condition holds automatically in our setting, since
the operator R (given in [26, (D.C), Page 41]) is the identity operator currently.*

The next lemma is the extension of [39, Proposition 6] to the current setting. Since the proof is the
same as that of [39, Proposition 6], we omit it.

Lemma 3.8. *Suppose that $(\widetilde{H_1})$-$(\widetilde{H_3})$ hold. Let $T > 0$ and $\alpha > 0$. Then the following statements are
equivalent:*

- *(i) The system (3.1) is cost-uniformly α-null controllable in time $T > 0$, i.e., there exists $C(\alpha, T) \geq 0$
  such that for each $y_0 \in X$, there exists $u \in L^2(0, T; U)$ such that $\|y(T; u, y_0)\|_X \leq \alpha \|y_0\|_X$ and
  $\|u\|_{L^2(0,T;U)} \leq C(\alpha, T)\|y_0\|_X$.*

- *(ii) There exists $C(\alpha, T) \geq 0$ such that*

$$\|S(T)^*\varphi\|_X \leq C(\alpha, T)\|B^*S(T - \cdot)^*\varphi\|_{L^2(0,T;U)} + \alpha\|\varphi\|_X \ \text{ for any } \ \varphi \in D(A^*).$$

  Moreover, $C(\alpha, T)$ in both (i) and (ii) can be taken as the same.

Proof of Theorem 3.4. We organize the proof in several steps.

Step 1. We show $(i) \Rightarrow (ii)$.

Suppose that (i) holds, i.e., $[A, B]$ is completely stabilizable in the sense of Definition 3.2. Arbitrarily
fix $\alpha > 0$. Then by (ii) in Definition 3.2, there exists a C_0-semigroup $\Phi_\alpha(t)$ $(t \geq 0)$, with the generator
$\Lambda_\alpha : D(\Lambda_\alpha) \subset X \to X$, and $K_\alpha \in \mathcal{L}(\Lambda_\alpha); U)$ such that (a')-(c') are true. Several observations are given
in order: First, by (a') and (c'), we have that, for any $y_0 \in D(\Lambda_\alpha)$,

$$\Phi_\alpha(t)y_0 = \widetilde{S}(t)y_0 + \int_0^t \widetilde{S}(t - s)BK_\alpha\Phi_\alpha(s)y_0 ds, \quad t \in \mathbb{R}^+.$$

Here, we notice that $\Phi_\alpha(t)y_0 \in D(\Lambda_\alpha)$ for each $t \in \mathbb{R}^+$ when $y_0 \in D(\Lambda_\alpha)$. Second, by (b'), we can
find $C(\alpha) > 0$ such that $\|\Phi_\alpha(t)\|_{\mathcal{L}(X)} \leq C(\alpha)e^{-\alpha t}$ for all $t \in \mathbb{R}^+$. Third, we let, for each $y_0 \in D(\Lambda_\alpha)$,
$u_{y_0}(t) := K_\alpha\Phi_\alpha(t)y_0$, $t \in \mathbb{R}^+$. Then it follows from (c') of Definition 3.2 that there exists $D(\alpha) > 0$
(independent of y_0) such that $\|u_{y_0}(\cdot)\|_{L^2(\mathbb{R}^+;U)} \leq D(\alpha)\|y_0\|_X$, $y_0 \in D(\Lambda_\alpha)$.

From these observations and by a very similar way as that used in the proof of $(i) \Rightarrow (ii)$ of Theo-
rem 1.1, we can verify that, for each $y_0 \in D(\Lambda_\alpha)$,

$$|\langle y_0, S(T)^*\varphi\rangle_X| = |\langle S(T)y_0, \varphi\rangle_X| = |\langle \widetilde{S}(T)y_0, \varphi\rangle_X| = |\langle \widetilde{S}(T)y_0, \varphi\rangle_{X_{-1}, D(A^*)}|$$

$$= \left|\left\langle \int_0^T \widetilde{S}(T - s)Bu_{y_0}(s)ds, \varphi\right\rangle_{X_{-1}, D(A^*)} - \langle \Phi_\alpha(T)y_0, \varphi\rangle_{X_{-1}, D(A^*)}\right|$$

$$= \left|\int_0^T \langle u_{y_0}(s), B^*S^*(T - s)\varphi\rangle_U ds - \langle \Phi_\alpha(T)y_0, \varphi\rangle_X\right|$$

$$\leq \left(C(\alpha)e^{-\alpha T}\|\varphi\|_X + D(\alpha)\|B^*S(T - \cdot)^*\varphi\|_{L^2(0,T;U)}\right)\|y_0\|_X, \text{ when } \varphi \in D(A^*), T > 0.$$

The above, along with the density of $D(\Lambda_\alpha)$ in X, leads to (3.3). The reason why $D(\Lambda_\alpha)$ is dense in X
is that Λ_α is the generator of the semigroup $\Phi_\alpha(t)$ $(t \geq 0)$ (see [34, Chapter 1, Theorem 1.3]).

Step 2. It is trivial that $(ii) \Rightarrow (iii)$.

Step 3. The proof of $(iii) \Rightarrow (iv)$ is very similar to that used in the proof of Theorem 1.1. We omit it.

Step 4. We show $(iv) \Rightarrow (i)$.

Arbitrarily fix $\beta > 0$ and $y_0 \in X$. Let $\mathcal{U}_{ad}^\beta(y_0) := \{v \in L^2(\mathbb{R}^+; U) : z_\beta(\cdot; v, y_0) \in L^2(\mathbb{R}^+; X)\}$,
where $z_\beta(\cdot; v, y_0)$ (with $v \in L^2(\mathbb{R}^+; U)$) is the unique solution to the system: $z'(t) = (A + \beta I)z(t) + Bv(t)$,

$t \in \mathbb{R}^+$; $z(0) = y_0$. One can directly check that, for each $t \geq 0$, $z_\beta(t; v, y_0) = \widetilde{S}_\beta(t)y_0 + \int_0^t \widetilde{S}_\beta(t-s)Bv(s)ds$. (Here, $\widetilde{S}_\beta(t)$ ($t \geq 0$) is the C_0-semigroup on X_{-1}, generated by $\widetilde{A} + \beta I : X \to X_{-1}$. It is easy to check that $\widetilde{S}_\beta(t) = e^{\beta t}\widetilde{S}(t)$, $t \geq 0$.) This, along with the note (d_4) in Remark 3.1, yields that for each $T > 0$, $z_\beta(\cdot; v, y_0) \in C([0, T]; X)$. Consider the next LQ problem

$$(\mathbf{LQ})_{y_0}^\beta : \quad \inf_{v \in L^2(\mathbb{R}^+;U)} \left\{ J^\beta(v; y_0) := \int_0^\infty [\|z_\beta(t; v, y_0)\|_X^2 + \|v(t)\|_U^2]dt \right\}.$$

The rest of the proof of this step is divided into two sub-steps.

Sub-step 4.1. We prove $\mathcal{U}_{ad}^\beta(y_0) \neq \emptyset$.

Clearly, this will be done, if one can show the existence of $\hat{v} \in L^2(\mathbb{R}^+; U)$ such that

$$\|z_\beta(\cdot; \hat{v}, y_0)\|_{L^2(\mathbb{R}^+;X)} \leq \tilde{C}(\beta)\|y_0\|_X \quad \text{and} \quad \|\hat{v}\|_{L^2(\mathbb{R}^+;U)} \leq \tilde{D}(\beta)\|y_0\|_X, \tag{3.9}$$

for some $\tilde{C}(\beta) > 0$ and $\tilde{D}(\beta) > 0$ depending only on β.

To show (3.9), we construct a control $\hat{v}$ in the following manner: With respect to the above $\beta > 0$, there is a unique $k := k(\beta) \in \mathbb{N}$ satisfying $k - 1 \leq 2\beta < k$. Then, according to (iv), there exists $T_k > 0$ and $D(k) > 0$ such that (3.5) holds. This, together with Lemma 3.8, implies that the system (3.1) is cost-uniformly e^{-kT_k}-null controllable in time T_k. Therefore, there exists $u_0 \in L^2(0, T_k; U)$ such that

$$\|y(T_k; u_0, y_0)\|_X \leq e^{-kT_k}\|y_0\|_X \leq e^{-2\beta T_k}\|y_0\|_X \quad \text{and} \quad \|u_0\|_{L^2(0,T_k;U)} \leq D(k)\|y_0\|_X. \tag{3.10}$$

Let $y_1 := y(T_k; u_0, y_0)$. Then by making use of the above cost-uniformly e^{-kT_k}-null controllability again, we can find $u_1 \in L^2(0, T_k; U)$ such that

$$\|y(T_k; u_1, y_1)\|_X \leq e^{-2\beta T_k}\|y_1\|_X \quad \text{and} \quad \|u_1\|_{L^2(0,T_k;U)} \leq D(k)\|y_1\|_X.$$

Since the system (3.1) is time-invariant, continuing the above process leads to a sequence $\{u_i\}_{i \in \mathbb{N}} \subset L^2(0, T_k; U)$ such that

$$\|y(T_k; u_i, y_i)\|_X \leq e^{-2\beta T_k}\|y_i\|_X \quad \text{and} \quad \|u_i\|_{L^2(0,T_k;U)} \leq D(k)\|y_i\|_X \quad \text{for any} \ i \in \mathbb{N},$$

where $y_i := y(T_k; u_{i-1}, y_{i-1})$. This, together with (3.10), shows that

$$\|y(T_k; u_i, y_i)\|_X \leq (e^{-2\beta T_k})^{i+1}\|y_0\|_X \leq e^{-2\beta(i+1)T_k}\|y_0\|_X \quad \text{for all} \ i \in \mathbb{N}; \tag{3.11}$$

$$\|u_i\|_{L^2(0,T_k;U)} \leq D(k)(e^{-2\beta T_k})^i\|y_0\|_X \leq D(k)e^{-2\beta i T_k}\|y_0\|_X \quad \text{for all} \ i \in \mathbb{N}. \tag{3.12}$$

Let

$$\hat{u}(t) := \sum_{i=0}^\infty \chi_{[iT_k,(i+1)T_k)}(t)u_i(t - iT_k), \quad t \in \mathbb{R}^+ \tag{3.13}$$

and

$$z_\beta(t) := e^{\beta t}y(t; \hat{u}, y_0), \quad \hat{v}(t) := e^{\beta t}\hat{u}(t), \quad t \in \mathbb{R}^+. \tag{3.14}$$

Now, we show that $\hat{v}$, given in (3.14), satisfies the second inequality in (3.9). Indeed, by (3.12), (3.13) and the second equality in (3.14), we find

$$\|\hat{v}\|_{L^2(\mathbb{R}^+;U)} \leq \sum_{i=0}^\infty e^{\beta(i+1)T_k}\|u_i\|_{L^2(0,T_k;U)} \leq D(k)\sum_{i=0}^\infty e^{-\beta(i-1)T_k}\|y_0\|_X = \frac{D(k)e^{\beta T_k}}{1 - e^{-\beta T_k}}\|y_0\|_X,$$

which leads to the second inequality in (3.9) with $\tilde{D}(\beta) = \frac{D(k)e^{\beta T_k}}{1-e^{-\beta T_k}}$. (Here, we notice that k is uniquely determined by β, and thus the constant $\frac{D(k)e^{\beta T_k}}{1-e^{-\beta T_k}}$ depends only on β.)

Finally, we show that $\hat{v}$, given by (3.14), satisfies the first inequality in (3.9). To this end, several observations are given in order. First, it follows from (3.14) that $z_\beta(t) = z_\beta(t; \hat{v}, y_0)$ for any $t \geq 0$. Then, it follows by the equation satisfied by $z_\beta(\cdot; \hat{v}, y_0)$, (3.14) and (3.13) that when $t \in [iT_k, (i+1)T_k]$, with $i \in \mathbf{N}$ arbitrarily fixed,

$$
\begin{aligned}
|\langle z_\beta(t; \hat{v}, y_0), \varphi \rangle_X| &= |\langle z_\beta(t; \hat{v}, y_0), \varphi \rangle_{X_{-1}, D(A^*)}| \\
&\leq |\langle \widetilde{S}_\beta(t - iT_k) z_\beta(iT_k; \hat{v}, y_0), \varphi \rangle_{X_{-1}, D(A^*)}| + \left| \left\langle \int_{iT_k}^t \widetilde{S}_\beta(t - s) B\hat{v}(s) ds, \varphi \right\rangle_{X_{-1}, D(A^*)} \right| \\
&= \left| \langle e^{\beta(t - iT_k)} S(t - iT_k) z_\beta(iT_k; \hat{v}, y_0), \varphi \rangle_X \right| + \left| \int_{iT_k}^t \langle \widetilde{S}_\beta(t - s) B\hat{v}(s), \varphi \rangle_{X_{-1}, D(A^*)} ds \right| \\
&\leq \sup_{t \in [0, T_k]} \|S(t)\|_{\mathcal{L}(X)} e^{\beta T_k} \|z_\beta(iT_k; \hat{v}, y_0)\|_X \|\varphi\|_X \\
&\quad + e^{\beta(i+1)T_k} \left| \int_{iT_k}^t \langle \widetilde{S}(t - s) B\hat{u}(s), \varphi \rangle_{X_{-1}, D(A^*)} ds \right| \\
&\leq \sup_{t \in [0, T_k]} \|S(t)\|_{\mathcal{L}(X)} e^{\beta(i+1)T_k} \|y(iT_k; \hat{u}, y_0)\|_X \|\varphi\|_X \\
&\quad + e^{\beta(i+1)T_k} \left| \int_0^{t - iT_k} \langle \widetilde{S}(t - iT_k - s) Bu_i(s), \varphi \rangle_{X_{-1}, D(A^*)} ds \right| \quad \text{for any } \varphi \in D(A^*). \quad (3.15)
\end{aligned}
$$

Here, we used the fact $\widetilde{S}(t) = S(t)$ $(t \geq 0)$ on X. Second, we get from (3.13) and the construction of $\{y_i\}_{i \in \mathbf{N}}$ that for each $i \in \mathbb{N}$, $y(iT_k; \hat{u}, y_0) = y(T_k; u_{i-1}, y_{i-1})$. This, along with (3.11), leads to

$$
\|y(iT_k; \hat{u}, y_0)\|_X \leq e^{-2\beta i T_k} \|y_0\|_X, \quad \text{when } i \in \mathbb{N}. \quad (3.16)
$$

Third, it follows by (3.2) and (3.12) that for each $t \in [iT_k, (i+1)T_k]$ (with $i \in \mathbf{N}$ arbitrarily fixed) and for each $\varphi \in D(A^*)$,

$$
\begin{aligned}
\left| \int_0^{t - iT_k} \langle \widetilde{S}(t - iT_k - s) Bu_i(s), \varphi \rangle_{X_{-1}, D(A^*)} ds \right| &= \left| \int_0^{t - iT_k} \langle u_i(s), B^* S^*(t - iT_k - s) \varphi \rangle_U ds \right| \\
&\leq C(T_k) \|u_i\|_{L^2(0, T_k; U)} \|\varphi\|_X \leq C(T_k) D(k) e^{-2\beta i T_k} \|y_0\|_X \|\varphi\|_X. \quad (3.17)
\end{aligned}
$$

Since $D(A^*)$ is dense in X, it follows (3.15), (3.16) and (3.17) that

$$
\begin{aligned}
\|z_\beta(t; \hat{v}, y_0)\|_{L^2(\mathbb{R}^+; X)} &\leq \sum_{i=0}^\infty \|z_\beta(\cdot; \hat{v}, y_0)\|_{L^2(iT_k, (i+1)T_k; X)} \leq T_k \sum_{i=0}^\infty \sup_{t \in [iT_k, (i+1)T_k]} \|z_\beta(t; \hat{v}, y_0)\|_X \\
&\leq T_k e^{\beta T_k} \left(\sup_{t \in [0, T_k]} \|S(t)\|_{\mathcal{L}(X)} + C(T_k) D(k) \right) \sum_{i=0}^\infty e^{-\beta i T_k} \|y_0\|_X,
\end{aligned}
$$

which leads to the first inequality in (3.9) with $\tilde{C}(\beta) := T_k e^{\beta T_k} (\sup_{t \in [0, T_k]} \|S(t)\|_{\mathcal{L}(X)} + C(T_k) D(k)) / (1 - e^{-\beta T_k})$. The reason why $D(A^*)$ is dense in X is that A^* is the generator of the adjoint semigroup $S(t)^*$ $(t \geq 0)$ (see [34, Chapter 1, Corollary 10.6]).

Sub-step 4.2. We prove the desired complete stabilizability.

One can directly check that $[A + \beta I, B]$ still satisfies the assumptions $(\widetilde{H_1})$-$(\widetilde{H_3})$. Meanwhile, by Sub-step 4.1, we have $\mathcal{U}_{ad}^\beta(y_0) \neq \emptyset$ for each $y_0 \in X$. Thus, by Lemma 3.6 (where $(\mathbf{LQ})_{y_0}$ is replaced by $(\mathbf{LQ})_{y_0}^\beta$), there is a unique solution $v_{y_0}^*$ to $(\mathbf{LQ})_{y_0}^\beta$; a self-adjoint and non-negative definite operator $P := P(\beta) \in \mathcal{L}(X)$; a C_0-semigroup $S_P^\beta(t)$ $(t \geq 0)$ on X, with the generator $A_P^\beta : D(A_P^\beta) \subset X \to X$, such that the following conclusions are true:

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(e_1) It holds that $P \in \mathcal{L}(D(A_P^\beta); D(A^*))$ and $B^*P \in \mathcal{L}(D(A_P^\beta); U)$;

(e_2) For any $y_0 \in D(A_P^\beta)$, $A_P^\beta y_0 = (\widetilde{A} + \beta I - BB^*P)y_0$;

(e_3) If $y_0 \in D(A_P^\beta)$, then $v_{y_0}^*(t) = -B^*P S_P^\beta(t)y_0$ for a.e. $t > 0$;

(e_4) The semigroup $S_P^\beta(t)$ $(t \geq 0)$ is exponentially stable on X.

Let $K_\beta := -B^*P$ and $\Phi_\beta(t) := e^{-\beta t} S_P^\beta(t)$, $t \geq 0$. Then one can directly check that $\Phi_\beta(t)$ $(t \geq 0)$ is a C_0-semigroup on X generated by $\Lambda_\beta := A_P^\beta - \beta I$, with $D(\Lambda_\beta) = D(A_P^\beta)$. Moreover, by ($e_1$), $K_\beta \in \mathcal{L}(D(\Lambda_\beta); U)$.

Next, we will check that the above K_β and $\Phi_\beta(t)$ $(t \geq 0)$ satisfy (a')-(c') in Definition 3.2 one by one. First, it follows from (e_2) that $\Lambda_\beta y_0 = (A_P^\beta - \beta I)y_0 = (\widetilde{A} - BB^*P)y_0$ for any $y_0 \in D(\Lambda_\beta)(= D(A_P^\beta))$, which leads to (a') in Definition 3.2. Second, we use (e_4) to find $C := C(\beta) > 0$ such that $\|S_P^\beta(t)\|_{\mathcal{L}(X)} \leq C$ for all $t > 0$. Thus we have

$$\|\Phi_\beta(t)\|_{\mathcal{L}(X)} = e^{-\beta t}\|S_P^\beta(t)\|_{\mathcal{L}(X)} \leq Ce^{-\beta t}, \quad \text{when } t > 0,$$

which leads to (b') in Definition 3.2. Finally, we use (3.9) to find $\hat{D}(\beta) > 0$ such that

$$\|v_{y_0}^*\|_{L^2(\mathbb{R}^+;U)} \leq \sqrt{J^\beta(v_{y_0}^*; y_0)} = \sqrt{\inf_{v \in L^2(\mathbb{R}^+;U)} J^\beta(v; y_0)} \leq \hat{D}(\beta)\|y_0\|_X.$$

This, together with (e_3), implies that when $y_0 \in D(\Lambda_\beta)(= D(A_P^\beta))$,

$$\|K_\beta \Phi_\beta(\cdot)y_0\|_{L^2(\mathbb{R}^+;U)} = \| - e^{-\beta \cdot} B^*P S_P^\beta(\cdot)y_0\|_{L^2(\mathbb{R}^+;U)} \leq \|v_{y_0}^*\|_{L^2(\mathbb{R}^+;U)} \leq \hat{D}(\beta)\|y_0\|_X,$$

which leads to (c') in Definition 3.2.

Now, since $\beta > 0$ was arbitrarily taken, the above checked (a')-(c'), along with Definition 3.2, yields that the system (3.1) is completely stabilizable. $\qquad \square$

The following proposition may have independent interest.

Propositon 3.9. *Suppose that $(\widetilde{H_1})$-$(\widetilde{H_3})$ hold. Then the following statements are equivalent:*

(i) *The set $\mathcal{U}_{ad}(y_0)$ defined by (3.8) is nonempty for any $y_0 \in X$.*

(ii) *The system (3.1) is exponentially stabilizable (in the sense of (i) in Definition 3.2).*

(iii) *For any $y_0 \in X$, $V(y_0) := \inf_{u \in L^2(\mathbb{R}^+;U)} J(u; y_0) < +\infty$, where $J(u; y_0)$ is given by (3.7).*

Proof. First of all, it is well known that $(i) \Leftrightarrow (iii)$.

We next prove $(i) \Rightarrow (ii)$. Suppose (i) holds. Then it follows from Lemma 3.6 that for each $y_0 \in X$, the problem $\mathbf{(LQ)}_{y_0}$ (see (3.6)) has a unique solution $u_{y_0}^*$, moreover there exists a self-adjoint and non-negative operator $P \in \mathcal{L}(X)$ and a C_0-semigroup $S_P(t)$ $(t \geq 0)$ on X, with its generator $A_P : D(A_P) \subset X \to X$, such that (i)-(iv) in Lemma 3.6 are true. Let $\Phi(t) := S_P(t)$ $(t \geq 0)$. (Its the generator is $\Lambda := A_P$.) Let $K := -B^*P$. Then by (i) in Lemma 3.6, we have $K \in \mathcal{L}(D(\Lambda); U)(= \mathcal{L}(D(A_P); U))$.

Now we show that the above $\Phi(t)$ and K satisfy the conditions (a), (b) and (c) in Definition 3.2. Indeed, (a) and (b) follow from (ii) and (iv) in Lemma 3.6, respectively. While the condition (c) can be deduced from our assumptions and (iii) in Lemma 3.6. Indeed, by our assumptions and (ii) in [15, Theorem 2.2], there exists a constant $C > 0$ such that

$$\|u_x^*\|_{L^2(\mathbb{R}^+;U)} \leq \sqrt{\inf_{u \in L^2(\mathbb{R}^+;U)} J(u; x)} \leq C\|x\|_X, \quad \text{when } x \in X,$$

where $J(u; x)$ is defined by (3.7) (with $y_0 = x$). Thus, by (iii) in Lemma 3.6, we get (c). Hence (ii) is true.

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Finally, we show $(ii) \Rightarrow (i)$. We suppose (ii) holds, i.e., there exists a C_0-semigroup $\Phi(t)$ $(t \geq 0)$ on X, with its generator $\Lambda : D(\Lambda) \subset X \to X$, and $K \in \mathcal{L}(D(\Lambda); U)$ such that the conditions (a), (b) and (c) are true. Let $\alpha > 0$ be given in (b). By a very similar way used in Step 1 in the proof of Theorem 3.4, we can find positive constants $C(\alpha)$ and $D(\alpha)$ such that (3.3) holds for the aforementioned α. Thus, there exists $T_0 > 0$ and $\delta \in (0, 1)$ such that

$$\|S(T_0)^*\varphi\|_X \leq D(\alpha)\|B^*S(T_0 - \cdot)^*\varphi\|_{L^2(0,T_0;U)} + \delta\|\varphi\|_X, \quad \text{when} \ \ \varphi \in D(A^*).$$

This, together with Lemma 3.8, yields that the system (3.1) is cost-uniformly δ-null controllable at time $T_0 > 0$. Then, by the very similar way used in Sub-step 4.1 in the proof of Theorem 3.4 (or in the proof of Lemma 31 in [39]), we can conclude that $\mathcal{U}_{ad}(y_0) \neq \emptyset$ for any $y_0 \in X$, i.e., (i) is true. $\qquad \square$

3.2 Periodic feedback stabilization

This subsection studies the periodic complete stabilization for the periodic system $[A(\cdot), B(\cdot)]$, i.e.,

$$y'(t) = A(t)y(t) + B(t)u(t), \ \ t \in \mathbb{R}^+, \tag{3.18}$$

under the following hypotheses:

$(\widehat{H_1})$ For a.e. $t \in \mathbb{R}^+$, $A(t) := A + D(t)$, where the operator A generates a C_0-semigroup $S(t)$ $(t \geq 0)$ on X; the operator-valued function $D(\cdot)$ belongs to $L^1_{loc}(\mathbb{R}^+; \mathcal{L}(X))$ and is $\mathbb{T}$-periodic $(\mathbb{T} > 0)$, i.e., $D(t + \mathbb{T}) = D(t)$ for a.e. $t \in \mathbb{R}^+$.

$(\widehat{H_2})$ The operator-valued function $B(\cdot)$ belongs to $L^\infty(\mathbb{R}^+; \mathcal{L}(U; X))$ and is $\mathbb{T}$-periodic, i.e., $B(t + \mathbb{T}) = B(t)$ for a.e. $t \in \mathbb{R}^+$.

By [45, Chapter 1, Proposition 1.2], we have what follows: First, $A(\cdot)$ generates a unique $\mathbb{T}$-periodic evolution $\Phi(\cdot, \cdot)$ (i.e., $\Phi(t + \mathbb{T}, s + \mathbb{T}) = \Phi(t, s)$ for all $0 \leq s \leq t$) on X; Second, for each $\mathbb{T}$-periodic feedback operator $K \in L^\infty(\mathbb{R}^+; \mathcal{L}(X; U))$ (i.e., $K(\mathbb{T} + t) = K(t)$ for a.e. $t \in \mathbb{R}^+$), $A_K(\cdot) := A(\cdot) + B(\cdot)K(\cdot)$ generates a unique $\mathbb{T}$-periodic evolution $\Phi_K(\cdot, \cdot)$; Third, when $u \in L^2(\mathbb{R}^+; U)$ and $y_0 \in X$,

$$y(t; u, y_0) = \Phi(t, 0)y_0 + \int_0^t \Phi(t, s)Bu(s)ds \ \ \text{and} \ \ y_K(t; y_0) = \Phi_K(t, 0)y_0 \ \text{for all} \ \ t \geq 0,$$

where $y(\cdot; u, y_0)$ is the solution to (3.18) with the initial condition: $y(0) = y_0$, while $y_K(\cdot; z)$ is the solution to the equation: $y'(t) = [A(t) + B(t)K(t)]y(t)$, $t \geq 0$; $y(0) = y_0$.

To present of the main result of this subsection, we need the following definitions:

Definition 3.10. *(i) The system (3.18) is said to be periodically stabilizable, if there exists $C > 0$, $\alpha > 0$ and a $\mathbb{T}$-periodic feedback operator $K(\cdot) \in L^\infty(\mathbb{R}^+; \mathcal{L}(X; U))$ such that $\|\Phi_K(t, 0)\|_{\mathcal{L}(X)} \leq Ce^{-\alpha t}$ for all $t \geq 0$.*

(ii) The system (3.18) is said to be periodically completely stabilizable, if for any $\alpha \in \mathbb{R}^+$, there exists $C := C(\alpha) > 0$ and a $\mathbb{T}$-periodic feedback operator $K(\cdot) := K_\alpha(\cdot) \in L^\infty(\mathbb{R}^+; \mathcal{L}(X; U))$ such that $\|\Phi_K(t, 0)\|_{\mathcal{L}(X)} \leq Ce^{-\alpha t}$ for all $t \geq 0$.

The main result of this subsection is as follows:

Theorem 3.11. *Suppose that $(\widehat{H_1})$ and $(\widehat{H_2})$ are true. Then the following statements are equivalent:*

(i) The system (3.18) is periodically completely stabilizable.

(ii) For any $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ and $C(k) > 0$ such that

$$\|\Phi(n_k\mathbb{T}, 0)^*\psi\|_X \leq C(k)\|B(\cdot)^*\Phi(n_k\mathbb{T}, \cdot)^*\psi\|_{L^2(0,n_k\mathbb{T};U)} + e^{-kn_k\mathbb{T}}\|\psi\|_X \ \ \text{for any} \ \ \psi \in X.$$

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To prove Theorem 3.11, we need the next Lemma 3.12, which is quoted from [47, Theorem 1.1].

Lemma 3.12. *Suppose that $(\widehat{H_1})$ and $(\widehat{H_2})$ hold. Let $\mu \geq 0$. Let $\Phi^\mu(\cdot,\cdot)$ be the $\mathbb{T}$-periodic evolution generated by $A(\cdot) + \mu I$. Then the following statements are equivalent:*

(i) The following system is periodically stabilizable:

$$y'(t) = (A(t) + \mu I)y(t) + B(t)u(t), \ \ t \in \mathbb{R}^+. \tag{3.19}$$

(ii) There exists $\delta \in (0,1)$, $n \in \mathbb{N}$ and $C(n) > 0$ such that

$$\|\Phi^\mu(n\mathbb{T},0)^*\psi\|_X \leq C(n)\|B(\cdot)^*\Phi^\mu(n\mathbb{T},\cdot)^*\psi\|_{L^2(0,n\mathbb{T};U)} + \delta\|\psi\|_X \ \text{ for any } \ \psi \in X. \tag{3.20}$$

Remark 3.13. *Given $n \in \mathbb{N}$ and $\psi \in X$, we have that $\Phi^\mu(n\mathbb{T},t)^*\psi = \varphi_n(t;\psi)$ for each $t \in [0,n\mathbb{T}]$, where $\varphi_n(\cdot;\psi)$ is the solution to the equation: $\varphi_n'(t) = -(A+\mu I)(t)^*\varphi_n(t) \ t \in [0,n\mathbb{T}]; \ \varphi_n(n\mathbb{T}) = \psi$. In [47, Theorem 1.1], (3.20) is expressed in terms of $\varphi_n(\cdot;\psi)$.*

Now, we are in the position to prove Theorem 3.11.

The proof of Theorem 3.11. First of all, when $\mu \geq 0$ and $K(\cdot) \in L^\infty(\mathbb{R}^+;\mathcal{L}(X,U))$ is $\mathbb{T}$-periodic, we have

$$\Phi^\mu(t,s) = e^{\mu(t-s)}\Phi(t,s) \ \text{ and } \ \Phi_K^\mu(t,s) = e^{\mu(t-s)}\Phi_K(t,s) \ \text{ for all } \ 0 \leq s \leq t, \tag{3.21}$$

where $\Phi_K^\mu(\cdot,\cdot)$ is the $\mathbb{T}$-periodic evolution generated by $A(\cdot) + \mu I + B(\cdot)K(\cdot)$. We now organize the rest of the proof in two steps.

Step 1. We show $(i) \Rightarrow (ii)$.

Suppose (i) is true. Arbitrarily fix $k \in \mathbf{N}$. Then according to Definition 3.10, there exists $C_k > 0$ and a $\mathbb{T}$-periodic feedback operator $K(\cdot) := K_k(\cdot) \in L^\infty(\mathbb{R}^+;\mathcal{L}(X;U))$ such that $\|\Phi_K(t,0)\|_{\mathcal{L}(X)} \leq C_k e^{-(k+1)t}$ for all $t \geq 0$, which, along with the second equality in (3.21), implies that $\|\Phi_K^k(t,0)\|_{\mathcal{L}(X)} \leq C_k e^{-t}$ for all $t \geq 0$. This, along with Definition 3.10, leads to the periodic stabilizability of the system (3.19) (where $\mu = k$). Then according to Lemma 3.12, there exists $\delta_k \in (0,1)$, $n_k \in \mathbb{N}$ and $C(k) > 0$ such that

$$\|\Phi^k(n_k\mathbb{T},0)^*\psi\|_X \leq C(k)\|B(\cdot)^*\Phi^k(n_k\mathbb{T},\cdot)^*\psi\|_{L^2(0,n_k\mathbb{T};U)} + \delta_k\|\psi\|_X \ \text{ for any } \ \psi \in X.$$

This, together with the first equality in (3.21), implies that

$$
\begin{aligned}
\|\Phi(n_k\mathbb{T},0)^*\psi\|_X &\leq C(k)\|e^{-k\cdot}B(\cdot)^*\Phi(n_k\mathbb{T},\cdot)^*\psi\|_{L^2(0,n_k\mathbb{T};U)} + \delta_k e^{-kn_k\mathbb{T}}\|\psi\|_X \\
&\leq C(k)\|B(\cdot)^*\Phi(n_k\mathbb{T},\cdot)^*\psi\|_{L^2(0,n_k\mathbb{T};U)} + e^{-kn_k\mathbb{T}}\|\psi\|_X \ \text{ for any } \ \psi \in X,
\end{aligned}
$$

which leads to (ii).

Step 2. We show $(ii) \Rightarrow (i)$.

Suppose that (ii) is true. Arbitrarily fix $\mu > 0$. We first show that the system (3.19) is periodically stabilizable. To this end, we take $k = [\mu] + 1$, where $[\mu]$ denotes the integer part of μ. Then by (ii), we can find $n_k \in \mathbb{N}$ and $C(k) > 0$ such that

$$\|\Phi(n_k\mathbb{T},0)^*\psi\|_X \leq C(k)\|B(\cdot)^*\Phi(n_k\mathbb{T},\cdot)^*\psi\|_{L^2(0,n_k\mathbb{T};U)} + e^{-kn_k\mathbb{T}}\|\psi\|_X \ \text{ for any } \ \psi \in X.$$

Hence, by the first equality in (3.21) and the same way to prove (2.9), we have

$$
\begin{aligned}
\|\Phi^\mu(n_k\mathbb{T},0)^*\psi\|_X &\leq C(k)\|e^{\mu\cdot}B(\cdot)^*\Phi^\mu(n_k\mathbb{T};\cdot)^*\psi\|_{L^2(0,n_k\mathbb{T};U)} + e^{-(k-\mu)n_k\mathbb{T}}\|\psi\|_X \\
&\leq C(k)e^{\mu n_k\mathbb{T}}\|B(\cdot)^*\Phi^\mu(n_k\mathbb{T},\cdot)^*\psi\|_{L^2(0,n_k\mathbb{T};U)} + e^{-(k-\mu)n_k\mathbb{T}}\|\psi\|_X \ \text{ for any } \ \psi \in X.
\end{aligned}
$$

Since $e^{-(k-\mu)n_k\mathbb{T}} < 1$, the above, along with Lemma 3.12, yields that the system (3.19) is periodically stabilizable.

We next show that the system (3.19) is periodically completely stabilizable. Indeed, by the periodic stabilizability of the system (3.19) and by (3.21), one can easily check that there is $C = C(\mu) > 0$ and $\mathbb{T}$-periodic $K(\cdot) \in L^\infty(\mathbb{R}^+;\mathcal{L}(X;U))$ (which depends on μ) so that $\|\Phi_K(t,0)\|_{\mathcal{L}(X)} \leq Ce^{-\mu t}$ for all $t \geq 0$. Then, since $\mu > 0$ was arbitrarily taken, we get, from Definition 3.10, the periodic complete stabilizability of the system (3.19). $\qquad\square$

4 Applications

In this section, we present several examples of control systems, which are not null controllable, but can be shown to be completely stabilizable, through verifying the weak observability inequalities in Theorem 1.1, as well as its extensions presented in Section 3. Besides, we give some sufficient conditions ensuring the weak observability inequalities, from the perspective of the spectral projection. These conditions not only are useful in the studies of our examples, but also have independent interest.

4.1 Conditions ensuring the weak observability

This subsection presents two theorems. One is about the setting in Section 1, while another is about the setting in Subsection 3.1.

Theorem 4.1. *Let A, with its domain $D(A)$, generate a C_0-semigroup $S(t)$ $(t \geq 0)$ on X and $B \in \mathcal{L}(U; X)$. Let $\{P_k\}_{k \in \mathbb{N}}$ be a family of orthogonal projections on X. Suppose the following dissipative inequality is satisfied:*

(a) For each $k \in \mathbb{N}$, there exists $M_k > 0$ and $\alpha_k > 0$, with $\alpha_k \to +\infty$ as $k \to +\infty$, such that

$$\|(I - P_k)S(t)^*\varphi\|_X \leq M_k e^{-\alpha_k t}\|\varphi\|_X \ \text{ for any } \ t \in \mathbb{R}^+ \text{and} \ \varphi \in X. \tag{4.1}$$

Then the statement (iii) in Theorem 1.1 holds when one of the following two conditions (b_1) and (b_2) is true:

(b_1) For each $k \in \mathbb{N}$, there exists $C_k > 0$ such that the following spectral inequality is true:

$$\|P_k\varphi\|_X \leq C_k\|B^*P_k\varphi\|_U \ \text{ for any } \ \varphi \in X; \tag{4.2}$$

(b_2) There exists $T_0 > 0$ such that for each $k \in \mathbb{N}$, there exists $C(k, T_0) > 0$ such that the following truncated observability inequality holds:

$$\|P_k S(T_0)^*\varphi\|_X^2 \leq C(k, T_0)\int_0^{T_0}\|B^*P_k S(t)^*\varphi\|_U^2 dt \ \text{ for any } \ \varphi \in X. \tag{4.3}$$

Proof. Since $S(t)$ $(t \geq 0)$ is a C_0-semigroup, there exists $M > 1$ and $\delta_0 > 0$ such that

$$\|S(t)\|_{\mathcal{L}(X)} \leq Me^{\delta_0 t} \ \text{ for all } \ t \in \mathbb{R}^+. \tag{4.4}$$

Arbitrarily fix $\alpha > 0$. Then fix $k \in \mathbb{N}$ such that $\alpha_k > \alpha$. (Such k exists, since α_k tends to $+\infty$.) We shall prove statement (iii) in Theorem 1.1 by two cases.

We first consider the case when (b_1) is true. Arbitrarily fix $T > 1$ and $\varphi \in X$. By (4.4) and the Cauchy-Schwarz inequality, we find

$$
\begin{aligned}
\|S(T)^*\varphi\|_X^2 &= \left\|\int_0^1 S(t)^*S(T-t)^*\varphi dt\right\|_X^2 \leq \left(\int_0^1 \|S(t)^*\|_{\mathcal{L}(X)}\|S(T-t)^*\varphi\|_X dt\right)^2 \\
&\leq M^2 e^{2\delta_0}\left(\int_0^1 \|S(T-t)^*\varphi\|_X dt\right)^2 \leq M^2 e^{2\delta_0}\int_0^1 \|S(T-t)^*\varphi\|_X^2 dt.
\end{aligned} \tag{4.5}
$$

Meanwhile, since P_k is an orthogonal projection, it follows from (4.2) and (4.1) that for each $t \in [0, T]$,

$$
\begin{aligned}
&\|S(T-t)^*\varphi\|_X^2 = \|P_k S(T-t)^*\varphi\|_X^2 + \|(I - P_k)S(T-t)^*\varphi\|_X^2 \\
&\leq C_k^2\|B^*P_k S(T-t)^*\varphi\|_U^2 + \|(I - P_k)S(T-t)^*\varphi\|_X^2 \\
&\leq 2C_k^2\|B^*S(T-t)^*\varphi\|_U^2 + 2C_k^2\|B^*(I - P_k)S(T-t)^*\varphi\|_U^2 + \|(I - P_k)S(T-t)^*\varphi\|_X^2
\end{aligned}
$$

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$$
\begin{aligned}
&\leq\ 2C_k^2\|B^*S(T-t)^*\varphi\|_U^2 + \left[2C_k^2\|B\|_{\mathcal{L}(U;X)}^2 + 1\right]\|(I-P_k)S(T-t)^*\varphi\|_X^2 \\
&\leq\ 2C_k^2\|B^*S(T-t)^*\varphi\|_U^2 + \left[2C_k^2\|B\|_{\mathcal{L}(U;X)}^2 + 1\right]M_k^2 e^{-2\alpha_k(T-t)}\|\varphi\|_X^2.
\end{aligned}
\tag{4.6}
$$

Next, integrating both sides of (4.6) for t over $(0,1)$, using (4.5), and noting that $\alpha_k > \alpha$, we obtain

$$
\|S(T)^*\varphi\|_X^2 \leq 2M^2C_k^2 e^{2\delta_0}\int_0^1\|B^*S(T-t)^*\varphi\|_U^2\mathrm{d}t + M^2 M_k^2 e^{2(\delta_0+\alpha)}\left[2C_k^2\|B\|_{\mathcal{L}(U;X)}^2+1\right]e^{-2\alpha T}\|\varphi\|_X^2,
$$

which, along with the fact that $T > 1$, leads to

$$
\|S(T)^*\varphi\|_X \leq D(\alpha)\|B^*S(T-\cdot)^*\varphi\|_{L^2(0,T;U)} + C(\alpha)e^{-\alpha T}\|\varphi\|_X,
\tag{4.7}
$$

where

$$
D(\alpha) := \sqrt{2}MC_k e^{\delta_0}, \quad C(\alpha) := MM_k e^{\delta_0+\alpha}\sqrt{\left[2C_k^2\|B\|_{\mathcal{L}(U;X)}^2+1\right]}.
$$

Since k depends only on α, the statement (iii) in Theorem 1.1 follows from (4.7) in the current case.

We next consider the case where (b_2) holds. Arbitrarily fix $T \geq 2T_0$. Then, there exists a natural number $N \geq 2$ such that $NT_0 \leq T < (N+1)T_0$. By (4.4), (4.3) and (4.1), we see that for each $\varphi \in X$,

$$
\begin{aligned}
\|S(T)^*\varphi\|_X^2 &=\ \|S(T-NT_0)^*S(NT_0)^*\varphi\|_X^2 \leq M^2 e^{2\delta_0 T_0}\|S(NT_0)^*\varphi\|_X^2 \\
&=\ M^2 e^{2\delta_0 T_0}\left(\|P_k S(T_0)^*S((N-1)T_0)^*\varphi\|_X^2 + \|(I-P_k)S(NT_0)^*\varphi\|_X^2\right) \\
&\leq\ M^2 e^{2\delta_0 T_0}\left(C(k,T_0)\int_0^{T_0}\|B^*P_k S((N-1)T_0+t)^*\varphi\|_U^2\,dt + \|(I-P_k)S(NT_0)^*\varphi\|_X^2\right) \\
&\leq\ 2M^2 e^{2\delta_0 T_0}C(k,T_0)\left(\int_{(N-1)T_0}^{NT_0}\|B^*S(t)^*\varphi\|_U^2\,dt + \int_{(N-1)T_0}^{NT_0}\|B^*(I-P_k)S(t)^*\varphi\|_U^2\,dt\right) \\
&\quad + M^2 e^{2\delta_0 T_0}M_k^2 e^{-2\alpha_k NT_0}\|\varphi\|_X^2.
\end{aligned}
\tag{4.8}
$$

Meanwhile, it follows from (4.1) that

$$
\begin{aligned}
\int_{(N-1)T_0}^{NT_0}\|B^*(I-P_k)S(t)^*\varphi\|_U^2\,dt &\leq\ \|B\|_{\mathcal{L}(U;X)}^2 M_k^2\int_{(N-1)T_0}^{NT_0}e^{-2\alpha_k t}\|\varphi\|_X^2\,dt \\
&\leq\ \|B\|_{\mathcal{L}(U;X)}^2 M_k^2 T_0 e^{-2\alpha_k(N-1)T_0}\|\varphi\|_X^2.
\end{aligned}
\tag{4.9}
$$

Using (4.8) and (4.9), noting that $\alpha_k > \alpha$ and $NT_0 \leq T < (N+1)T_0$, we get

$$
\|S(T)^*\varphi\|_X^2 \leq D(\alpha)^2\int_0^T\|B^*S(t)^*\varphi\|_U^2\,dt + C(\alpha)^2 e^{-2\alpha T}\|\varphi\|_X^2,
\tag{4.10}
$$

where

$$
D(\alpha) := Me^{\delta_0 T_0}\sqrt{2C(k,T_0)}, \quad C(\alpha) := MM_k e^{(\delta_0+\alpha)T_0}\sqrt{2C(k,T_0)\|B^*\|_{\mathcal{L}(X;U)}^2 T_0 e^{2\alpha T_0}+1}.
$$

Now, the statement (iii) in Theorem 1.1 follows from (4.10) in the current case. $\qquad\square$

Remark 4.2. *(i) The Lebeau-Robbiano strategy says in plain language that the null controllability can be implied by a spectral inequality and a dissipative inequality. In this strategy, the following compatibility condition on the decay rate in the dissipative inequality and the growth rate in the spectral inequality is necessary: the former is greater than the latter (see e.g. [32, Theorem 2.2]). In the studies of the exponential stabilizability, such compatibility condition is relaxed (see [18, Lemma 2.2]). Our Theorem 4.1, together with Theorem 1.1, improves [18, Lemma 2.2] from two perspectives: First, it serves for the complete stabilizability; Second, there is no any compatibility condition on the constants in (4.1) and (4.2).*

(ii) When $P_k S(\cdot) = S(\cdot) P_k$, the condition (b_1) implies condition (b_2). This can be checked directly.

(iii) We borrowed the name "truncated observability inequality" from [7], where such kind of observability inequality is applied to construct an internal feedback control stabilizing the Navier-Stokes equations.

To show the similar result to Theorem 4.1 in the setting where B is unbounded, we need the next lemma.

Lemma 4.3. *Suppose that $[A, B]$ satisfies the following conditions:*

(a) The operator A, with its domain $D(A)$, generates an analytic semigroup $S(t)$ $(t \geq 0)$ on X.

(b) There exists $\gamma \in \left(0, \frac{1}{2}\right)$ such that $B \in \mathcal{L}(U; X_{-\gamma})$, where $X_{-\gamma}$ is the completion of X with respect to the norm $\|z\|_{-\gamma} := \|(\rho_0 I - A)^{-\gamma} z\|_X$, $z \in X$ (where $\rho_0 \in \rho(A) \cap \mathbb{R}$ is arbitrarily fixed).

Then the following conclusions are true:

(i) The assumptions $(\widetilde{H_1})$-$(\widetilde{H_3})$ in Section 3.1 hold for $[A, B]$.

(ii) The operator $(\rho_0 I - A)^\gamma$ has a unique extension $(\widetilde{\rho_0 I - A})^\gamma \in \mathcal{L}(X; X_{-\gamma})$. Moreover, this extension is invertible and $((\widetilde{\rho_0 I - A})^\gamma)^{-1} B \in \mathcal{L}(U; X)$.

Proof. We organize the proof by two steps.

Step 1. We show the conclusion (i).

First, $(\widetilde{H_1})$ is clearly true.

Second, one can directly check that $X_{-\gamma}$ is continuously embedded into X_{-1}. Then by the condition (b), we get $B \in \mathcal{L}(U; X_{-1})$ which leads to $(\widetilde{H_2})$.

We are now going to show $(\widetilde{H_3})$. First of all, one can directly check the following two facts:

(f_1) The operator $(\rho_0 I - A)^\gamma$ belongs to $\mathcal{L}(D((\rho_0 I - A)^\gamma); X)$ (Here, the norm of $D((\rho_0 I - A)^\gamma)$ is as: $\|z\|_{D((\rho_0 I - A)^\gamma)} = \|(\rho_0 I - A)^\gamma z\|_X$, $z \in D((\rho_0 I - A)^\gamma)$.) and has a unique extension $(\widetilde{\rho_0 I - A})^\gamma \in \mathcal{L}(X; X_{-\gamma})$ which is invertible (see [41, Chapter 2, Proposition 2.10.3]);

(f_2) $B \in \mathcal{L}(U; X_{-\gamma})$ if and only if $\left((\widetilde{\rho_0 I - A})^\gamma\right)^{-1} B \in \mathcal{L}(U; X)$.

The above facts (f_1) and (f_2), together with $D(A^*) \subset D((\rho_0 I - A^*)^\gamma)$ and the analyticity of $S(t)$ $(t \geq 0)$ (which means that $S(t)^* x \in D(A^*)$ for any $x \in X$ when $t > 0$), yield that for each $t > 0$,

$$
\begin{aligned}
B^* S(t)^* x &= B^* \left((\widetilde{\rho_0 I - A})^\gamma{}^*\right)^{-1} (\widetilde{\rho_0 I - A})^\gamma{}^* S(t)^* x \\
&= B^* \left((\widetilde{\rho_0 I - A})^\gamma{}^*\right)^{-1} (\rho_0 I - A^*)^\gamma S(t)^* x \quad \text{for any } x \in X. \tag{4.11}
\end{aligned}
$$

Here, we notice that $B^* \in \mathcal{L}(D(A^*); U)$ since $B \in \mathcal{L}(U; X_{-1})$ is proved and X_{-1} is the dual space of $D(A^*)$ with respect to the pivot space X (see (d_3) in Remark 3.1).

Meanwhile, we have the following observations: First, the analytic semigroup $S(t)^* e^{-\rho_0 t}$ $(t \geq 0))$ is generated by $-\rho_0 I + A^*$; Second, since $\rho_0 \in \rho(A) \cap \mathbb{R}$, we have $0 \in \rho(\rho_0 I - A^*)$ (see [34, Chapter 1, Lemma 10.2]). From these observations, we can use [34, Chapter 2, Theorem 6.13] to find $C(\gamma) > 0$ such that

$$
\|(\rho_0 I - A^*)^\gamma S(t)^*\|_{\mathcal{L}(X)} \leq C(\gamma) e^{\rho_0 t} t^{-\gamma} \quad \text{for all } t > 0. \tag{4.12}
$$

Now, it follows by (4.12) and (4.11) that, when $T > 0$ and $x \in X$,

$$
\int_0^T \|B^* S(t)^* x\|_U^2 dt = \int_0^T \left\| B^* \left((\widetilde{\rho_0 I - A})^\gamma{}^*\right)^{-1} (\rho_0 I - A^*)^\gamma S(t)^* x \right\|_U^2 dt
$$

$$\leq \; \left\|B^*\left((\widetilde{\rho_0 I - A})^\gamma\right)^{*}{}^{-1}\right\|_{\mathcal{L}(X;U)}^2 C(\gamma)^2 e^{2\rho_0 T}\int_0^T \frac{1}{t^{2\gamma}}dt\|x\|_X^2 \leq C(T,\gamma)\|x\|_X^2,$$

where $C(T,\gamma) := \left\|B^*\left((\widetilde{\rho_0 I - A})^\gamma\right)^{*}{}^{-1}\right\|_{\mathcal{L}(X;U)}^2 C(\gamma)^2 e^{2\rho_0 T}\frac{T^{1-2\gamma}}{1-2\gamma}$. This leads to $(\widetilde{H_3})$ obviously.

Step 2. The conclusion (ii) follows from the above (f_1) and (f_2) at once. $\qquad\square$

Theorem 4.4. *Assume that the conditions (a) and (b) in Lemma 4.3 hold. Suppose that there is a family of orthogonal projections $\{P_k\}_{k\in\mathbb{N}}$ on X satisfying the commutative condition:*

$$P_k S(\cdot) = S(\cdot)P_k; \tag{4.13}$$

the dissipative condition (a) in Theorem 4.1; and the following observability condition:

(b_2') *there exists $T_0 > 0$ such that for each $k \in \mathbb{N}$, there is $C(k,T_0) > 0$ such that*

$$\|P_k S(T_0)^*\varphi\|_X^2 \leq C(k,T_0)\int_0^{T_0}\|B^* P_k S(t)^*\varphi\|_U^2 dt \;\; \text{for any } \varphi \in D(A^*). \tag{4.14}$$

Then the statement (iii) in Theorem 3.4 is true.

Proof. Let T_0 and $C(k,T_0)$ be given in (b_2'). Let M_k and α_k be given in (a) of Theorem 4.1. Let M and δ_0 be given by (4.4) which clearly holds in the current case. Arbitrarily fix $\alpha > 0$ and $T \geq 2T_0$. Then, there is a natural number N with $N \geq 2$ such that $NT_0 \leq T < (N+1)T_0$.

By Lemma 4.3, (4.13) and (4.14), and by the same way as that used in the proof of (4.8) (in the proof of Theorem 4.1), we can easily get

$$
\|S(T)^*\varphi\|_X^2 \;\leq\; 2M^2 e^{2\delta_0 T_0}C(k,T_0)\left(\int_{(N-1)T_0}^{NT_0}\|B^*S(t)^*\varphi\|_U^2\,dt + \int_{(N-1)T_0}^{NT_0}\|B^*S(t)^*(I-P_k)\varphi\|_U^2\,dt\right)
$$
$$
+ M^2 e^{2\delta_0 T_0}M_k^2 e^{-2\alpha_k NT_0}\|\varphi\|_X^2 \;\;\text{for any } \varphi \in D(A^*). \tag{4.15}
$$

(Notice that it follows by (4.13) and the property of C_0-semigroup that $P_k A^* = A^* P_k$ in $D(A^*)$ for each $k \in \mathbb{N}$, which implies that if $\varphi \in D(A^*)$ then $P_k\varphi \in D(A^*)$.) At the same time, we clearly have (4.11) and (4.12). These, along with (4.1) and the conclusion (ii) in Lemma 4.3, yield

$$
\int_{(N-1)T_0}^{NT_0}\|B^*S(t)^*(I-P_k)\varphi\|_U^2\,dt
$$
$$
= \int_{(N-1)T_0}^{NT_0}\|(((\widetilde{\rho_0 I - A})^\gamma)^{-1}B)^*(\rho_0 I - A^*)^\gamma S(T_0)^*S(t-T_0)^*(I-P_k)\varphi\|_U^2 dt
$$
$$
\leq \|((\widetilde{\rho_0 I - A})^\gamma)^{-1}B\|_{\mathcal{L}(U;X)}^2 C(\gamma)^2 e^{2\rho_0 T_0}T_0^{-2\gamma}\int_{(N-2)T_0}^{(N-1)T_0}\|(I-P_k)S(t)^*\varphi\|_X^2\,dt
$$
$$
\leq \|((\widetilde{\rho_0 I - A})^\gamma)^{-1}B\|_{\mathcal{L}(U;X)}^2 C(\gamma)^2 e^{2\rho_0 T_0}T_0^{-2\gamma}M_k^2\int_{(N-2)T_0}^{(N-1)T_0}e^{-2\alpha_k t}dt\|\varphi\|_X^2
$$
$$
\leq \|((\widetilde{\rho_0 I - A})^\gamma)^{-1}B\|_{\mathcal{L}(U;X)}^2 C(\gamma)^2 e^{2\rho_0 T_0}T_0^{1-2\gamma}M_k^2 e^{-2\alpha_k(N-2)T_0}\|\varphi\|_X^2 \;\;\text{for any } \varphi \in D(A^*). \tag{4.16}
$$

Now, by (4.15) and (4.16), using the facts $\alpha_k > \alpha$ and $NT_0 \leq T < (N+1)T_0$, we obtain

$$\|S(T)^*\varphi\|_X^2 \leq D(\alpha)^2\int_0^T\|B^*S(t)^*\varphi\|_U^2\,dt + C(\alpha)^2 e^{-2\alpha T}\|\varphi\|_X^2 \;\;\text{for any } \varphi \in D(A^*),$$

where $D(\alpha) := Me^{\delta_0 T_0}\sqrt{2C(k,T_0)}$ and

$$C(\alpha) := MM_k e^{(\delta_0+\alpha)T_0}\sqrt{2C(k,T_0)\|((\widetilde{\rho_0 I - A})^\gamma)^{-1}B\|_{\mathcal{L}(U;X)}^2 C(\gamma)^2 e^{2(\rho_0+2\alpha)T_0}T_0^{1-2\gamma} + 1}.$$

This leads to the statement (iii) in Theorem 3.4. $\qquad\square$

4.2 Examples

This subsection presents several concrete controlled equations which are not null controllable but can be checked to be completely stabilizable, via the weak observability inequalities. The first two examples are under the framework in Section 1, the third example is under the framework in Subsection 3.1, while the last example is under the framework in Subsection 3.2.

Example 1. (Fractional heat equations in $\mathbb{R}^n$.) Let $c \geq 0$ and $s \in (0,1)$. Let E be a thick set in $\mathbb{R}^n$ such that $E \subset B^c(\bar{x}, R)$ for some $\bar{x} \in \mathbb{R}^n$ and $R > 0$ (where $B^c(\bar{x}, R)$ denotes the complementary set of the closed ball centered at $\bar{x} \in \mathbb{R}^n$ and of radius R). Here, by E being a thick set in $\mathbb{R}^n$, we mean that there is $\gamma > 0$ and $L > 0$ such that

$$|E \cap Q_L(x)| \geq \gamma L^n \text{ for each } x \in \mathbb{R}^n,$$

where $Q_L(x)$ is the closed cube in $\mathbb{R}^n$ (centered at x and of side-length L) and $|E \cap Q_L(x)|$ denotes the Lebesgue measure of $E \cap Q_L(x)$. We consider the controlled equation:

$$\begin{cases} \partial_t y(t,x) + (-\triangle)^{\frac{s}{2}} y(t,x) - cy(t,x) = \chi_E(x) u(t,x), & \text{in } \mathbb{R}^+ \times \mathbb{R}^n, \\ y(0,\cdot) \in L^2(\mathbb{R}^n). \end{cases} \tag{4.17}$$

Here, χ_E is the characteristic function of E, $u \in L^2(\mathbb{R}^+; L^2(\mathbb{R}^n))$ and the fractional Laplacian $(-\triangle)^{\frac{s}{2}}$ is defined by

$$(-\triangle)^{\frac{s}{2}} \varphi := \mathcal{F}^{-1}(|\xi|^s \mathcal{F}(\varphi)), \quad \varphi \in C_0^\infty(\mathbb{R}^n),$$

where and throughout this example, we use $\mathcal{F}$ and $\mathcal{F}^{-1}$ to denote the Fourier transform and its inverse respectively.

For the system (4.17), we have the following conclusions:

- The equation (4.17) can be put into our framework (in Section 1) in the following manner: Let $X = U := L^2(\mathbb{R}^n)$ and $A := -(-\triangle)^{\frac{s}{2}} + c$, with $D(A) = H^s(\mathbb{R}^n)$. Let $B : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be defined by $Bv := \chi_E v$ for each $v \in L^2(\mathbb{R}^n)$. It is well known that A generates a C_0-semigroup $\{S(t)\}_{t \geq 0}$ on X. One can directly check that $B \in \mathcal{L}(U; X)$.

- The equation (4.17) with the null control is unstable (see [18, (1.6), as well as the note (d_1)]).

- Since $E \subset B^c(\bar{x}, R)$, it follows by [20, Theorem 1.3 and its generalization in Section 4.3] that the equation (4.17) is not null controllable.

- The equation (4.17) is completely stabilizable. (See Theorem 4.5 below).

The next Theorem 4.5 may have independent interest.

Theorem 4.5. *Let $\hat{E}$ be a measurable subset of $\mathbb{R}^n$. Then the equation (4.17) (where E is replaced by $\hat{E}$) is completely stabilizable if and only if $\hat{E}$ is a thick set.*

Remark 4.6. *Theorem 4.5 extends [18, Theorem 1.1], which shows that the equation (4.17) (where E is replaced by $\hat{E}$) is stabilizable if and only if $\hat{E}$ is a thick set.*

The proof of Theorem 4.5. We first show the *only if* part. Suppose that the equation (4.17) (where E is replaced by $\hat{E}$) is completely stabilizable, then it is stabilizable. Thus it follows by [18, Theorem 1.1] that $\hat{E}$ is a thick set.

We next show the *if* part. Assume that $\hat{E}$ is a thick set. We will use Theorem 4.1 to show the statement *(iii)* of Theorem 1.1 in the following manner: Define, for each $k \in \mathbb{N}$, the linear operator on $L^2(\mathbb{R}^n)$ by

$$P_k \varphi := \mathcal{F}^{-1} \left(\chi_{\{\xi \in \mathbb{R}^n : |\xi|^s - c \leq k\}} \mathcal{F}(\varphi) \right), \quad \varphi \in L^2(\mathbb{R}^n).$$

By the Plancherel theorem, one can easily check that $\{P_k\}_{k \in \mathbb{N}}$ are bounded and self-adjoint.

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Firstly, we show that $\{P_k\}_{k\in\mathbb{N}}$ are orthogonal projections. To this aim, we first prove $\{P_k\}_{k\in\mathbb{N}}$ are projections. It is suffices to prove that $P_k(I - P_k) = 0$ for each $k \in \mathbb{N}$. Indeed, for any $\varphi, \psi \in L^2(\mathbb{R}^n)$, by the Plancherel theorem and the fact $\{P_k\}_{k\in\mathbb{N}}$ are self-adjoint, we have that, for each $k \in \mathbb{N}$,

$$
\begin{aligned}
\langle P_k(I - P_k)\varphi, \psi \rangle_{L^2(\mathbb{R}^n)} &= \langle (I - P_k)\varphi, P_k\psi \rangle_{L^2(\mathbb{R}^n)} = \langle \mathcal{F}(\varphi) - \mathcal{F}(P_k\varphi), \mathcal{F}(P_k\psi) \rangle_{L^2(\mathbb{R}^n)} \\
&= \langle \chi_{\{\xi\in\mathbb{R}^n:|\xi|^s-c>k\}}\mathcal{F}(\varphi), \chi_{\{\xi\in\mathbb{R}^n:|\xi|^s-c\le k\}}\mathcal{F}(\psi) \rangle_{L^2(\mathbb{R}^n)} = 0.
\end{aligned}
$$

This, together with the arbitrariness of φ, ψ, means that $P_k(I - P_k) = 0$ for each $k \in \mathbb{N}$. Thus, $\{P_k\}_{k\in\mathbb{N}}$ are projections. We next show $\{P_k\}_{k\in\mathbb{N}}$ are orthogonal. Indeed, since $\{P_k\}_{k\in\mathbb{N}}$ are projections and self-adjoint, we have that, for each $k \in \mathbb{N}$, $\text{Range}(P_k) \perp \text{Range}(I - P_k)$. Because of P_k being projection, it is clear that $\text{Range}(I - P_k) = \text{Ker}(P_k)$ for each $k \in \mathbb{N}$. (Indeed, if $f \in \text{Ker}(P_k)$, then $f = P_k f + (I - P_k)f = (I - P_k)f$. Thus, $f \in \text{Range}(I - P_k)$ and then $\text{Ker}(P_k) \subset \text{Range}(I - P_k)$. Conversely, if $f \in \text{Range}(I - P_k)$, i.e., there exists a $g \in L^2(\mathbb{R}^n)$ such that $f = (I - P_k)g$, then, by the fact $P_k(I - P_k) = 0$, we have $P_k f = P_k(I - P_k)g = 0$, i.e., $f \in \text{Ker}(P_k)$. Thus, $\text{Range}(I - P_k) \subset \text{Ker}(P_k)$. In summary, we can conclude that $\text{Range}(I - P_k) = \text{Ker}(P_k)$.) Therefore, $\text{Range}(P_k) \perp \text{Ker}(P_k)$ for each $k \in \mathbb{N}$, i.e., $\{P_k\}_{k\in\mathbb{N}}$ are orthogonal.

Secondly, by [18, Lemma 3.1 and the inequality (4.1)], there exists a $C > 0$ such that, for each $k \in \mathbb{N}$,

$$
\|P_k\varphi\|_{L^2(\mathbb{R}^n)} \le e^{Ck^{\frac{1}{s}}}\|B^*P_k\varphi\|_{L^2(\mathbb{R}^n)} \text{ for any } \varphi \in L^2(\mathbb{R}^n);
$$

$$
\|(I - P_k)S(t)^*\varphi\|_{L^2(\mathbb{R}^n)} \le e^{-kt}\|\varphi\|_{L^2(\mathbb{R}^n)} \text{ for all } t \in \mathbb{R}^+ \text{ and } \varphi \in L^2(\mathbb{R}^n).
$$

The above two inequalities clearly imply the conditions (b_1) and (a) in Theorem 4.1 respectively. Then, by Theorem 4.1, we have (iii) of Theorem 1.1.

Finally, by Theorem 1.1, we see that the equation (4.17) (where E is replaced by $\hat{E}$) is completely stabilizable. $\qquad\square$

Example 2. (Heat equation with Hermite operator in $\mathbb{R}^n$.) Let $c \ge n$ and let E be a subset of positive measure in a half-space of $\mathbb{R}^n$. We consider the equation:

$$
\begin{cases}
\partial_t y(t, x) - \triangle y(t, x) + |x|^2 y(t, x) - cy(t, x) = \chi_E(x)u(t, x), & \text{in } \mathbb{R}^+ \times \mathbb{R}^n, \\
y(0, \cdot) = y_0(\cdot) \in L^2(\mathbb{R}^n),
\end{cases} \tag{4.18}
$$

where $u \in L^2(\mathbb{R}^+; L^2(\mathbb{R}^n))$.

For the equation (4.18), we have the following conclusions:

- The equation (4.18) can be put into the framework in Section 1 by the following manner: Let $X = U := L^2(\mathbb{R}^n)$ and $A := \triangle - |x|^2 + c$, with $D(A) = \{f \in L^2(\mathbb{R}^n) : -\triangle f + |x|^2 f \in L^2(\mathbb{R}^n)\}$. Let $B : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be defined in the same way as that used in Example 1. Then A generates a C_0-semigroup $\{S(t)\}_{t\ge0}$ on X (see [38]) and $B \in \mathcal{L}(U; X)$.

- The equation (4.18) with the null control is unstable (see [18, (1.7), as well as the note (e_1)]).

- It follows by [33, Theorem 1.10] that the equation (4.18) is not null controllable.

- The equation (4.18) is completely stabilizable. (See Theorem 4.7 below.)

The next Theorem 4.7 may have independent interest.

Theorem 4.7. *Let $\hat{E}$ be a measurable subset in a half-space of $\mathbb{R}^n$. Then the equation (4.18) (where E is replaced by $\hat{E}$) is completely stabilizable if and only if $\hat{E}$ has a positive measure.*

Remark 4.8. *Theorem 4.7 extends [18, Theorem 1.2], which shows that the equation (4.18) (where E is replaced by $\hat{E}$) is stabilizable if and only if $\hat{E}$ has a positive measure.*

The proof of Theorem 4.7. We first show the *only if* part. Suppose that the equation (4.18) (where E is replaced by $\hat{E}$) is completely stabilizable, then it is stabilizable. Thus, it follows by [18, Theorem 1.2] that $\hat{E}$ has a positive measure.

We next show the *if* part. Assume that $\hat{E}$ has a positive measure. We will use Theorem 4.1 to show the statement (*iii*) of Theorem 1.1. Indeed, according to [38], the operator $A_0 := -\triangle + |x|^2$ has discrete spectrum set $\sigma(A_0) = \{2k + n, k \in \mathbb{N}\}$. For each $k \in \mathbb{N}$, we let π_k be the orthogonal projection onto the linear space spanned by the eigenfunctions of A_0 associated with the eigenvalue $2k + n$. We then define the orthogonal projection P_k on $L^2(\mathbb{R}^n)$ by

$$P_k\varphi := \sum_{0 \leq j \leq (k+c-n)/2} \pi_j\varphi, \quad \varphi \in \mathbb{R}^n.$$

By [18, Lemma 3.2 and inequality (4.17)], there exists a $C > 0$ such that, for each $k \in \mathbb{N}$,

$$\|P_k\varphi\|_{L^2(\mathbb{R}^n)} \leq e^{\frac{n}{2}k \ln k + Ck}\|B^*P_k\varphi\|_{L^2(\mathbb{R}^n)} \text{ for any } \varphi \in L^2(\mathbb{R}^n);$$

$$\|(I - P_k)S(t)^*\varphi\|_{L^2(\mathbb{R}^n)} \leq e^{-kt}\|\varphi\|_{L^2(\mathbb{R}^n)} \text{ for all } t \in \mathbb{R}^+ \text{ and } \varphi \in L^2(\mathbb{R}^n).$$

The above two inequalities imply the conditions (b_1) and (a) (with $\alpha_k = k$) in Theorem 4.1 respectively. Then, by Theorem 4.1, we have (*iii*) of Theorem 1.1. Finally, by Theorem 1.1, we find that the equation (4.18) (where E is replaced by $\hat{E}$) is completely stabilizable. $\qquad\square$

Example 3. (Point-wise control of 1-D heat equation.) Consider the point-wise controlled 1-D heat equation:

$$\begin{cases} y_t(t,x) - y_{xx}(t,x) - cy(t,x) = \delta(x - x_0)u(t), & \text{in } \mathbb{R}^+ \times (0,1), \\ y(t,0) = y(t,1) = 0, & \text{in } \mathbb{R}^+, \\ y(0,x) = y_0(x) \in L^2(0,1), & \text{in } (0,1). \end{cases} \quad (4.19)$$

Here, $c > \pi^2$, $\delta(\cdot - x_0)$ is the usual Dirac function, while $x_0 \in (0,1)$ is given in the following manner:

First of all, given $x \in \mathbb{R}$, we let $\|x\| := \inf_{n \in \mathbb{Z}} |x-n|$ and write $[x]$ for the integer so that $x-1 < [x] \leq x$. We next construct a continued fraction $[a_1, a_2, \cdots]$ by setting

$$a_1 = 2, q_0 = 0, q_1 = 1,$$

and defining successively $q_n, a_n, n \geq 2$ as

$$q_{n+1} = a_nq_n + q_{n-1}, \quad a_{n+1} = [e^{q_{n+1}^3}] + 1, \quad n \geq 1. \quad (4.20)$$

(About the definition of continued fractions, we refer readers to [9, Chapter 1].) According to [9, Chapter 1, Theorem II and Theorem III], there is an one-to-one correspondence between continued fractions and real numbers in $(0,1)$. We now let x_0 be the real number corresponding to the above $[a_1, a_2, \cdots]$ (for the construction of x_0, one can refer to the proof of [9, Chapter 1, Theorem III]). It is clear that x_0 is an irrational number since the sequence $\{a_n\}_{n \in \mathbb{N}}$ is infinite.

For the equation (4.19) when x_0 is chosen as above, we have the following conclusions:

- The equation (4.19) can be put into the framework in Section 3.1. (See the proof of Theorem 4.9 below.)

- The equation (4.19) with the null control is unstable, since $c > \pi^2$.

- The equation (4.19) is not null controllable in any time interval. To see this, we first notice that the eigenvalues of operator $-(\partial_x^2 + c)$ with domain $H_0^1(0,1) \cap H^2(0,1)$ are $\lambda_n := (n\pi)^2 - c$ ($n \in \mathbb{N}$); the corresponding normalized eigenfunctions are $\phi_n(x) := \sqrt{2}\sin(n\pi x)$, $x \in (0,1)$ ($n \in \mathbb{N}$). We next have from [14] and [9, (15) in Chapter 1] that

$$|\phi_n(x_0)| = |\sqrt{2}\sin(n\pi x_0)| \leq \sqrt{2}\pi\|nx_0\|; \quad \|q_nx_0\| < \frac{1}{q_{n+1}} \text{ for all } n \in \mathbb{N}. \quad (4.21)$$

We then arbitrarily fix $T > 0$ and consider the series $\sum_{n=1}^{\infty} \frac{\exp(-\lambda_{q_n} T)}{|\phi_{q_n}(x_0)|}$. By (4.20), we see that $\lambda_{q_n} = \pi q_n^2$ and $q_n \to +\infty$, as $n \to +\infty$. These, along with the definition of a_n, yield that there is $N := N(T) > 0$ so that when $n > N$, $\exp(-\lambda_{q_n} T) > 1/a_n$, which, together with (4.21) and the first equation in (4.20), implies

$$\frac{\exp(-\lambda_{q_n} T)}{|\phi_{q_n}(x_0)|} \geq \frac{1}{a_n} \cdot \frac{q_{n+1}}{\sqrt{2\pi}} \geq \frac{q_n}{\sqrt{2\pi}} \quad \text{for all} \ \ n > N. \tag{4.22}$$

Since $q_n \to +\infty$, as $n \to +\infty$, it follows from (4.22) that the series $\sum_{n=1}^{\infty} \frac{exp(-\lambda_{q_n} T)}{|\phi_{q_n}(x_0)|}$ is divergent. Thus, we can use [14, Theorem 1] to see that the equation (4.19) is not null controllable over $[0, \widehat{T}]$ for any $\widehat{T} < T$. Since $T > 0$ was arbitrarily taken, the equation (4.19) is not null controllable in any time interval.

- The equation (4.19) is completely stabilizable. (See Theorem 4.9 below.)

The next Theorem 4.9 may have independent interest.

Theorem 4.9. *Let $\hat{x}_0 \in (0, 1)$. Then the equation (4.19) (where x_0 is replaced by $\hat{x}_0$) is completely stabilizable if and only if $\hat{x}_0$ is irrational.*

Proof. We organize the proof in several steps.

Step 1. We put the equation (4.19) (where x_0 is replaced by $\hat{x}_0$) into the framework in Section 3.1.
    Let $X := L^2(0, 1)$ and $U := \mathbb{R}$. Let

$$Ay := (\partial_x^2 + c)y, \ \ y \in D(A) = H_0^1(0, 1) \cap H^2(0, 1); \ \ Bu = \delta(\cdot - \hat{x}_0)u, \ \ u \in U.$$

First, we will prove that the above $[A, B]$ satisfies the assumptions (a) and (b) in Lemma 4.3. For this purpose, we arbitrarily fix $\rho_0 > c$. The assumption (a) can be checked easily. Indeed, one can directly check that $\rho_0 \in \rho(A) \cap \mathbb{R}$ and the operator A, with its domain $D(A)$, generates an analytic semigroup $S(t)$ $(t \geq 0)$ on X. From these, it follows that (a) in Lemma 4.3 is true. To show (b), we notice the following facts:

- **Fact One.** $H_0^{2\gamma}(0, 1) \subset C[0, 1]$ continuously for each $\gamma > 1/4$. (See [29, Chapter1, Theorem 9.8].)

- **Fact Two.** $D((\rho_0 I - A)^\gamma) = H_0^{2\gamma}(0, 1)$ for each $1/4 < \gamma < 3/4$, where the norm of $D((\rho_0 I - A)^\gamma)$ is as: $\|z\|_{D((\rho_0 I - A)^\gamma)} = \|(\rho_0 I - A)^\gamma z\|_X$, $z \in D((\rho_0 I - A)^\gamma)$. (See [29, Chapter 1, Definition 2.1 and Theorem 11.6].)

- **Fact Three.** $B \in \mathcal{L}(U; [C[0, 1]]')$. (This can be directly checked.)

- **Fact Four.** $[D((\rho_0 I - A)^\gamma)]' = X_{-\gamma}$ for each $\gamma > 0$, where $[D((\rho_0 I - A)^\gamma)]'$ is the dual space with the pivot space $X(= L^2(0, 1))$ and $X_{-\gamma}$ is defined in (b) of Lemma 4.3. (This follows from [41, Chapter 2, Section 2.9] or [29, Chapter 1, Theorem 6.2 and Theorem 12.2].)

From these facts, we obtain that $B \in \mathcal{L}(U; X_{-\gamma})$ for each $1/4 < \gamma < 3/4$, which leads to (b) in Lemma 4.3. In summary, the assumptions (a) and (b) in Lemma 4.3 hold for the above $[A, B]$.
    Next, we can use Lemma 4.3 to see that the assumptions $(\widetilde{H_1})$-$(\widetilde{H_3})$ in Section 3.1 are satisfied by $[A, B]$ in the current case. Consequently, the equation (4.19) has been put into the framework in Section 3.1 by the above way.

Step 2. We prove the sufficiency.

Assume that $\hat{x}_0$ is irrational. Then we have

$$\phi_n(\hat{x}_0) \neq 0 \ \ \text{for all} \ \ n \in \mathbb{N}. \tag{4.23}$$

Here ϕ_n is the normalized eigenfunctions defined above (4.21). We will verify that all the assumptions in Theorem 4.4 are true for the current case. *When this is done, we can apply Theorem 4.4 to get (iii) in Theorem 3.4, and then use Theorem 3.4 to see that the equation (4.19) is completely stabilizable.*

To this end, we first recall that the assumptions (a) and (b) in Lemma 4.3 have been checked in Step 1. We then define, for each $k \in \mathbb{N}$,

$$P_k \varphi := \sum_{i=1}^{k} \langle \varphi, \phi_i \rangle_X \phi_i, \ \ \varphi \in X.$$

It is clear that, for each $k \in \mathbb{N}$, P_k is the orthogonal projections of X onto the linear span $\mathcal{S}_k := \mathrm{span}\{\phi_i : i = 1, 2, \ldots, k\}$, and $P_k S(\cdot) = S(\cdot) P_k$. From these, we see that the assumption (4.13) in Theorem 4.4 holds. Next, since $S(t)^* \phi_n = e^{-\lambda_n t} \phi_n$ for all $n \in \mathbb{N}$ and $t \in \mathbb{R}^+$, one can directly check that for each $k \in \mathbb{N}^+$,

$$\|(I - P_k)S(t)^*\phi\|_X \le e^{-\lambda_k t}\|\phi\|_X \ \text{ for all } \ \phi \in X, \ t \in \mathbb{R}^+.$$

This, together with the definition of λ_k, implies that the dissipative condition (a) in Theorem 4.1. We finally show the assumption (b_2') in Theorem 4.4. For this purpose, we arbitrarily fix $T_0 > 0$. Define, for each $k \in \mathbb{N}$, the function $p_k(t) := e^{-\lambda_k t}$, $t \in (0, T_0)$. Let $E(n, T_0)$, with $n \in \mathbb{N}$, be the subspace (in $L^2(0, T_0)$), spanned by the functions $\{p_k\}_{k \in \mathbb{N} \setminus \{n\}}$. Let d_n be the distance of p_n to $E(n, T_0)$ in $L^2(0, T_0)$. Then there are $K > 0$ and $\varepsilon > 0$ which are independent of n such that (see [16, Theorem 1.1])

$$d_n \ge K \exp(-\varepsilon \lambda_n) \ \text{ for all } \ n \in \mathbb{N}.$$

Thus, it follows by (4.23) that when $k \in \mathbb{N}$ and $\varphi = \sum_{n=1}^{+\infty} a_n \phi_n \in D(A^*)$,

$$
\begin{aligned}
\int_0^{T_0} \|B^* P_k S(s)^* \varphi\|_U^2 ds &= \int_0^{T_0} \left\| e^{-\lambda_j s} a_j \phi_j(\hat{x}_0) + \sum_{1 \le n \le k, n \ne j} e^{-\lambda_n s} a_n \phi_n(\hat{x}_0) \right\|_U^2 ds \\
&= |a_j|^2 |\phi_j(\hat{x}_0)|^2 \int_0^{T_0} \left\| e^{-\lambda_j s} - \sum_{1 \le n \le k, n \ne j} \frac{-a_n \phi_n(\hat{x}_0)}{a_j \phi_j(\hat{x}_0)} e^{-\lambda_n s} \right\|_U^2 ds \\
&\ge |a_j|^2 |\phi_j(\hat{x}_0)|^2 d_j^2 \ \text{ for all } \ j \in \{1, 2, \ldots, k\}.
\end{aligned}
$$

This, together with (4.23), gives

$$|a_j|^2 \le \frac{1}{d_j^2 |\phi_j(\hat{x}_0)|^2} \int_0^{T_0} \|B^* P_k S^*(s)\varphi\|_U^2 ds \ \text{ for all } \ j \in \{1, 2, \ldots, k\}.$$

From the above, we see that when $k \in \mathbb{N}$,

$$\|P_k S(T_0)^* \varphi\|_X^2 = \sum_{j=1}^{k} |a_j|^2 e^{-2\lambda_j T_0} \le \sum_{j=1}^{k} \left(\frac{e^{-2\lambda_j T_0}}{d_j^2 |\phi_j(\hat{x}_0)|^2} \right) \int_0^{T_0} \|B^* P_k S(s)^* \varphi\|_U^2 ds \ \text{ for all } \ \varphi \in D(A^*).$$

This leads to the condition (b_2') (in Theorem 4.4) with $C(k, T_0) := \sum_{j=1}^{k} \left(\frac{e^{-2\lambda_j T_0}}{d_j^2 |\phi_j(\hat{x}_0)|^2} \right)$.

Hence, all assumptions in Theorem 4.4 are satisfied for the current case.

Step 2. We prove the necessity.

By contradiction, we suppose that (4.19) (where x_0 is replaced by $\hat{x}_0$) is completely stabilizable, but $\hat{x}_0$ is a rational number. Then there is $n_0 \in \mathbb{N}$ such that $\phi_{n_0}(\hat{x}_0) = 0$. Write $\varphi(x) := \phi_{n_0}(x)$, $x \in (0, 1)$. Arbitrarily fix $T > 0$. Then we have that $\|\varphi\|_X = 1$;

$$\|S(T)^* \varphi\|_X = \|e^{-\lambda_{n_0} T} \phi_{n_0}\|_X = e^{-\lambda_{n_0} T};$$

$$\|B^* S(T-t)^* \varphi\|_{L^2(0,T;U)} = \left(\int_0^T |e^{-\lambda_{n_0} t} \phi_{n_0}(\hat{x}_0)|^2 dt \right)^{1/2} = 0. \tag{4.24}$$

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Since the equation (4.19) (where x_0 is replaced by $\hat{x}_0$) is completely stabilizable, we obtain from (iii) of Theorem 3.4 (see (1.3)) and (4.24) that for each $\alpha > 0$, there exists $C(\alpha) > 0$, which is independent on T, such that

$$e^{-\lambda_{n_0}T} \le C(\alpha)e^{-\alpha T} \text{ for all } T > 0,$$

which is equivalent to

$$C(\alpha) \ge e^{(\alpha - \lambda_{n_0})T} \text{ for all } T > 0. \tag{4.25}$$

However, if we take $\alpha = \lambda_{n_0} + 1$, then there is no $C(\alpha) > 0$ so that (4.25) is true, because the right hand side tends to $+\infty$ as T tends to infinity. This leads to a contradiction. Thus, $\hat{x}_0$ must be irrational. $\square$

Example 4. (A periodic controlled system.) Let $X = U := l^2$ and $\mathbb{T} := 1$. Define a sequence $\{\tau_n\}_{n \in \mathbb{N}}$ in the manner: $\tau_n := \frac{1}{\alpha} \sum_{k=n+1}^{\infty} a_k$, where $a_k := e^{-k^2}$ and $\alpha := \sum_{k=1}^{\infty} a_k$. (It is clear that $\alpha \in (0, 1)$ and $\tau_n \in (0, 1)$ for each $n \in \mathbb{N}$). Let

$$A := -\text{diag}\{1, 2, \cdots, n, \cdots\}; \quad B(t) := \text{diag}\left\{\chi_{(\tau_1, \tau_0)}(\{t\}), \cdots, \chi_{(\tau_n, \tau_{n-1})}(\{t\}), \cdots\right\}, \ t \in (0, +\infty),$$

where $\{t\}$ denotes the fractional part of t, i.e., $\{t\} = t - [t]$, where $[t]$ is the integer so that $t - 1 < [t] \le t$. Consider the following 1-periodic system:

$$\frac{d}{dt}y(t) := \frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \\ \vdots \end{pmatrix}(t) = A\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \\ \vdots \end{pmatrix}(t) + B(t)\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \\ \vdots \end{pmatrix}(t), \tag{4.26}$$

where $u = (u_1, u_2, \cdots)^\top$ is taken from $L^2(\mathbb{R}^+; l^2)$. For the equation (4.26), we have the conclusions:

- One can directly check that the equation (4.26) can be put into the framework in Subsection 3.2.

- The equation (4.26) is not null controllable (see Theorem 4.10 given later).

- The equation (4.26) is periodically completely stabilizable (see Theorem 4.10 given later).

Theorem 4.10. *The system* (4.26) *is not null controllable but completely stabilizable.*

Proof. We organize the proof in two steps.

Step 1. We show that (4.26) *is not null controllable.*

We only need to prove the following *Statement A*: For each $m \in \mathbb{N}$, the system (4.26) is not null controllable over $[0, m]$. To this end, we arbitrarily fix $m \in \mathbb{N}$. Write $\varphi_m(\cdot; \psi)$ for the solution to the dual system:

$$\begin{cases} \frac{d}{dt}\varphi_m(t) \triangleq \frac{d}{dt}\begin{pmatrix} \varphi_{m,1} \\ \varphi_{m,2} \\ \vdots \\ \varphi_{m,n} \\ \vdots \end{pmatrix}(t) = \begin{pmatrix} 1 & & & & \\ & 2 & & & \\ & & \ddots & & \\ & & & n & \\ & & & & \ddots \end{pmatrix}\begin{pmatrix} \varphi_{m,1} \\ \varphi_{m,2} \\ \vdots \\ \varphi_{m,n} \\ \vdots \end{pmatrix}(t), \ t \in [0, m], \\ \varphi_m(m) = \psi = (\psi_1, \psi_2, \cdots)^\top. \end{cases}$$

Because of the equivalence between the controllability and the observability, (4.26) is not null controllable over $[0, m]$ if and only if for any $C > 1$, there is $\psi \in l^2$ so that

$$\|\varphi_m(0; \psi)\|_X > C\|B^*(\cdot)\varphi_m(\cdot; \psi)\|_{L^2(0,m;U)}. \tag{4.27}$$

To show (4.27), we arbitrarily fix $C > 1$. Take $n = n(C) \in \mathbb{N}$ such that

$$n \geq m + \sqrt{m^2 + 2\ln C + \ln\frac{2}{\alpha}}. \tag{4.28}$$

Then we take $\psi = (\psi_1, \psi_2, \cdots)^\top \in l^2$ with $\psi_n = 1$ and $\psi_k = 0$, when $k \neq n$. By a direct calculation, we find that $\|\varphi_m(0; \psi)\| = e^{-nm}$ and that

$$\|B^*(\cdot)\varphi_m(\cdot; \psi)\|^2_{L^2(0,m;U)} = \int_{\tau_n}^{\tau_{n-1}} e^{-2n\tau}\mathrm{d}\tau \sum_{k=0}^{m-1} e^{-2nk} \leq \frac{e^{-n^2}}{\alpha(1 - e^{-2n})} < \frac{2}{\alpha}e^{-n^2}.$$

These, together with (4.28), leads to (4.27). Therefore, the system (4.26) is not null controllable.

Step 2. We show that (4.26) *is completely stabilizable.*

It is sufficient to prove (ii) of Theorem 3.11. To this end, we arbitrarily fix $k \in \mathbb{N}$. Let $n_k = 1$ and $C(k) = \sqrt{\alpha}e^{k^2/2}$. Then we have that for each $\psi = (\psi_1, \psi_2, \cdots)^\top \in l^2$,

$$\|\varphi_1(0; \psi)\|^2_X = \sum_{n=1}^{k} e^{-2n}\psi_n^2 + \sum_{n=k+1}^{\infty} e^{-2n}\psi_n^2 \leq \sum_{n=1}^{k} e^{-2n}\psi_n^2 + e^{-2k}\sum_{n=k+1}^{\infty} \psi_n^2 \leq \sum_{n=1}^{k} e^{-2n}\psi_n^2 + e^{-2k}\|\psi\|^2_X;$$

$$\|B^*(\cdot)\varphi_1(\cdot; \psi)\|^2_{L^2(0,1;U)} = \sum_{n=1}^{\infty} \int_{\tau_n}^{\tau_{n-1}} e^{-2n\tau}\mathrm{d}\tau \psi_n^2 \geq \sum_{n=1}^{k} \frac{a_n}{\alpha}e^{-2n}\psi_n^2.$$

These, along with the fact that $C(k)^2 a_n = e^{k^2} a_n \geq 1$ when $1 \leq n \leq k$, yield

$$\|\varphi_1(0; \psi)\|_X \leq C(k)\|B^*(\cdot)\varphi_1(\cdot; \psi)\|_{L^2(0,1;U)} + e^{-k}\|\psi\|_X \text{ for any } \psi \in X,$$

which, along with Remark 3.13, leads to the statement (ii) of Theorem 3.11. Then it follows from Theorem 3.11 that the system (4.26) is periodically completely stabilizable. □

References

[1] K. Ammari, and M. Tucsnak, *Stabilization of second order evolution equations by a class of unbounded feedbacks*, ESAIM Control Optim. Calc. Var., 6 (2001), pp. 361-386.

[2] B. Azmi, and K. Kunisch, *A hybrid finite-dimensional RHC for stabilization of time-varying parabolic equations*. SIAM J. Control Optim., 57(5) (2019), pp. 3496-3526.

[3] M. Badra, D. Mitra, M. Ramaswamy, and J. P. Raymond, *Stabilizability of time-periodic evolution equations by finite dimensional controls*, SIAM J. Control Optim., 58 (2020), pp. 1735-1768.

[4] M. Badra, and T. Takahashi, *On the Fattorini criterion for approximate controllability and stabilizability of parabolic systems*, ESAIM Control Optim. Calc. Var., 20 (2014), pp. 924-956.

[5] M. Badra, D. Mitra, M. Ramaswamy, and J. P. Raymond, *Local feed- back stabilization of time-periodic evolution equations by finite dimensional controls*, ESAIM Control Optim. Calc. Var., (2020), 26: Paper No. 101, 2020.

[6] V. Barbu, *Stabilization of Navier-Stokes flows*, Communications and Control Engineering Series, Springer, London, 2011.

[7] V. Barbu, S. S. Rodrigues, and A. Shirikyan, *Internal exponential stabilization to a nonstationary solution for 3D Navier-Stokes equations*, SIAM J. Control Optim., 49 (2011), pp. 1454-1478.

[8] G. Bastin, and J. M. Coron, *Stability and boundary stabilization of 1-D hyperbolic systems*, Progress in Nonlinear Differential Equations and their Applications, 88, Subseries in Control, Birkhäuser/Springer, 2016.

[9] J. W. S. Cassels, *An introduction to diophantine approximation*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 45, Cambridge University Press, New York, 1957.

[10] J. M. Coron, *Control and nonlinearity*, Mathematical surveys and monographs, Vol. 136, American Mathematical Society, 2007.

[11] J. M. Coron, and Q. Lü, *Local rapid stabilization for a Korteweg-de Vries equation with a Neumann boundary control on the right*, J. Math. Pures Appl., 102(9) (2014), pp. 1080-1120.

[12] R. F. Curtain, and G. Weiss, *Exponential stabilization of well-posed systems by colocated feedback*, SIAM J. Control Optim., 45 (2006), pp. 273-297.

[13] R. F. Curtain, and H. Zwart, *An introduction to infinite-dimensional linear systems theory*, Texts in Applied Mathematics, 21, Springer-Verlag, New York, 1995.

[14] S. Dolecki, *Observability for the one-dimensional heat equation*, Studia Math., 48 (1973), pp. 291-305.

[15] F. Flandoli, I. Lasiecka, and R. Triggiani, *Algebraic Riccati equations with nonsmoothing observation arising in hyperbolic and Euler-Bernoulli boundary control problems*, Ann. Mat. Pura Appl., 153(4) (1988), pp. 307-382.

[16] H. O. Fattorini, and D. L. Russell, *Uniform Bounds on Biorthogonal Functions for Real Exponentials with an Application to the Control Theory of Parabolic Equations*, Quart. Appl. Math., 32 (1974), pp. 45-69.

[17] B. Guo, and J. Wang, *Control of wave and beam PDEs. The Riesz basis approach*, Communications and Control Engineering Series, Springer, Cham, 2019.

[18] S. Huang, G. Wang, and M. Wang, *Characterizations of stabilizable sets for some parabolic equations in $\mathbb{R}^n$*, J. Differential Equations, 272 (2021), pp. 255-288.

[19] S. Kerschbaum, and J. Deutscher, *Backstepping control of coupled linear parabolic PDEs with space and time dependent coefficients*, IEEE Trans. Automat. Control, 65 (2020), pp. 3060-3067.

[20] A. Koenig, *Lack of null-controllability for the fractional heat equation and related equations*, SIAM J. Control Optim. 58 (2020), pp. 3130-3160.

[21] V. Komornik, *Exact controllability and stabilization. The multiplier methods*, Research in Applied Mathematics, Masson, Paris, John Wiley & Sons Ltd., Chichester, 1994.

[22] V. Komornik, *Rapid boundary stabilization of linear distributed systems*, SIAM J. Control Optim., 35 (1997), pp. 1591-1613.

[23] M. KRSTIC, AND A. SMYSHLYAEV, *Boundary control of PDEs. A course on backstepping designs*, Advances in Design and Control, 16. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2008.

[24] K. KUNISCH, AND S. S. RODRIGUES, *Explicit exponential stabilization of nonautonomous linear parabolic-like systems by a finite number of internal actuators*, ESAIM Control Optim. Calc. Var., 25 (2019), Paper No. 67, pp. 38.

[25] J. LAGNESE, *Control of wave processes with distributed controls supported on a subregion*, SIAM J. Control Optim., 21 (1983), pp. 68-85.

[26] I. LASIECKA, AND R. TRIGGIANI, *Differential and algebraic Riccati Equations with application to Boundary/Point Control Problems: Continuous Theory and Approximation Theory*, Lecture Notes in Control and Information Sciences, 164, Springer-Verlag, Berlin, 1992.

[27] X. LI, AND J. YONG, *Optimal Control Theory for Infinite-Dimensional Systems*, Systems & Control: Foundations & Applications, Birkhäuser Boston, Inc., Boston, MA, 1995.

[28] J. L. LIONS, *Exact controllability, stabilization and perturbations for distributed systems*, SIAM Rev., 30 (1988), pp. 1-68.

[29] J. L. LIONS, AND E. MAGENES, *Non-homogeneous Boundary Value Problems and Applications*, Vol. I, Die Grundlehren der mathematischen Wissenschaften, Band 181, Springer-Verlag, New York, 1972.

[30] K. LIU, *Locally distributed control and damping for the conservative systems.* SIAM J. Control Optim., 35 (1997), pp. 1574-1590.

[31] A. LUNARDI, *Stabilizability of time-periodic parabolic equations*, SIAM J. Control Optim., 29 (1991), pp. 810-828.

[32] L. MILLER, *A direct Lebeau-Robbiano strategy for the observability of heat-like semigroups.* Discrete Contin. Dynam. Systems, 14(4) (2010), pp. 1465-1485.

[33] L. MILLER, *Unique continuation estimates for sums of semiclassical eigenfunctions and null-controllability from cones*, hal-00411840v2, 2008.

[34] A. PAZY, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[35] A. J. PRITCHARD, AND J. ZABCZYK, *Stability and stabilizability of infinite-dimensional systems*, SIAM Rev., 23 (1981), pp. 25-52.

[36] K. RAMDANI, M. TUCSNAK, AND G. WEISS, *Recovering the initial state of an infinite-dimensional system using observers*, Automatica, 46 (2010), pp. 1616-1625.

[37] D. L. RUSSELL, *Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions*, SIAM Rev., 20 (1978), pp. 639-739.

[38] E. C. TITCHMARSH, *Eigenfunction expansions associated with second-order differential equations. Part I*, Second edition, Clarendon Press, Oxford 1962.

[39] E. TRÉLAT, G. WANG, AND Y. XU, *Characterization by observability inequalities of controllability and stabilization properties*, Pure Appl. Anal., 2 (2020), pp. 93-122.

[40] R. TRIGGIANI, *On the stabilizability problem in Banach space*, J. Math. Anal. Appl., 52 (1975), pp. 383-403.

[41] M. Tucsnak, and G. Weiss, *Observation and Control for Operator Semigroups*, Birkhäuser Verlag, Basel, 2009.

[42] J. M. Urquiza, *Rapid exponential feedback stabilization with unbounded control operators*, SIAM J. Control Optim., 43 (2005), pp. 2233-2244.

[43] A. Vest, *Rapid stabilization in a semigroup framework*, SIAM J. Control Optim., 51 (2013), pp. 4169-4188.

[44] G. Wang, and Y. Xu, *Equivalent conditions on periodic feedback stabilization for linear periodic evolution equations*, J. Funct. Anal., 266 (2014), pp. 5126-1573.

[45] G. Wang, and Y. Xu, *Periodic Feedback Stabilization for Linear Periodic Evolution Equations*, Springer, Cham, BCAM Basque Center for Applied Mathematics, Bilbao, 2016.

[46] G. Weiss, and R. Rebarber, *Optimizability and Estimatability for Infinite-Dimensional Linear Systems*, SIAM J. Control Optim., 39 (2000), pp. 1204-1232.

[47] Y. Xu, *Characterization by detectability inequality for periodic stabilization of linear time-periodic evolution systems*, Systems & Control Letters, 149 (2021), 104871.