

# CRYSTALLINITY OF THE HOMOGENIZED ENERGY DENSITY OF PERIODIC LATTICE SYSTEMS

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**ABSTRACT.** We study the homogenized energy densities of periodic ferromagnetic Ising systems. We prove that, for finite range interactions, the homogenized energy density, identifying the effective limit, is crystalline, i.e. its Wulff crystal is a polytope, for which we can (exponentially) bound the number of vertices. This is achieved by deriving a dual representation of the energy density through a finite cell formula. This formula also permits easy numerical computations: we show a few experiments where we compute periodic patterns which minimize the anisotropy of the surface tension.

## 1. INTRODUCTION

The study of discrete interfacial energies has attracted widespread attention in the mathematical community over last decades, with applications in various contexts such as computer vision [8], crystallization problems [9], fracture mechanics [7, 19, 36], or statistical physics [41, 42]. To give examples, in computer vision the understanding of these energies allows to investigate functional correctness of segmentation algorithms [23]. Whereas for crystallization problems it gives fluctuation estimates on the macroscopic shape of the crystal cluster of ground state configurations [28, 34, 35, 40].

In this work, we consider energies defined on discrete periodic sets  $\mathcal{L} \subset \mathbb{R}^d$  and corresponding Ising systems. We refer to [1, 13, 21, 22, 29, 31, 37, 38, 39] for an abundant literature on the derivation of continuum limits of such systems and their effective behavior. More precisely, we consider  $\mathcal{L}$  satisfying the following two conditions (see Figure 1)

- (i) (Discreteness) There exists  $c > 0$  such that  $\text{dist}(x, \mathcal{L} \setminus \{x\}) \geq c$  for all  $x \in \mathcal{L}$ ;
- (ii) (Periodicity) There exists  $T \in \mathbb{N}$  such that for all  $z \in \mathbb{Z}^d$ , it holds that  $\mathcal{L} + Tz = \mathcal{L}$ ;

To each function  $u : \mathcal{L} \rightarrow \{0, 1\}$  and each  $A \subset \mathbb{R}^d$  we associate an energy

$$E(u, A) = \sum_{i \in \mathcal{L} \cap A} \sum_{j \in \mathcal{L}} c_{i,j} (u(i) - u(j))^+, \quad (1)$$

where  $(z)^+$  denotes the positive part of  $z \in \mathbb{R}$ ,  $c_{i,j} : \mathcal{L} \times \mathcal{L} \rightarrow [0, +\infty)$  are  $T$ -periodic, that is  $c_{i+Tz, j+Tz} = c_{i,j}$  for all  $i, j \in \mathcal{L}$  and  $z \in \mathbb{Z}^d$  and satisfy the following decay assumption

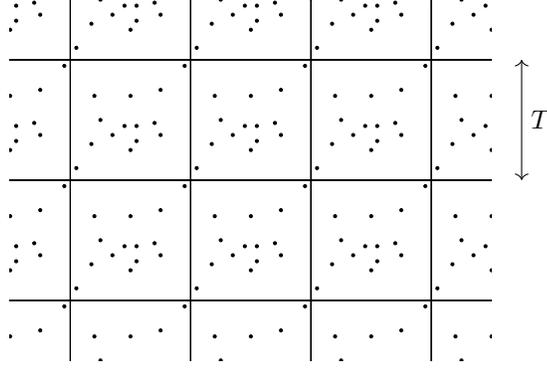
- (iii) (Decay of interactions) For all  $i \in \mathcal{L}$  there holds

$$\sum_{j \in \mathcal{L}} c_{i,j} |i - j| < +\infty.$$

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FIGURE 1. An example of the set  $\mathcal{L}$ 

Assuming conditions (i)-(iii) (and some additional coercivity assumption) ensures that the asymptotic behavior of (1) is well described (in a variational sense) by a continuum perimeter energy. More precisely, let us introduce a scaling parameter  $\varepsilon > 0$ . We consider the scaled energies

$$E_\varepsilon(u) = \sum_{i,j \in \varepsilon\mathcal{L}} \varepsilon^{d-1} c_{i,j}^\varepsilon (u(i) - u(j))^+,$$

where  $c_{i,j}^\varepsilon = c_{i/\varepsilon, j/\varepsilon}$  and  $u: \varepsilon\mathcal{L} \rightarrow \{0, 1\}$ . By identifying  $u$  with its piecewise constant interpolation taking the value  $u(i)$  on the Voronoi cell centered at  $i \in \varepsilon\mathcal{L}$  we may regard the energies as defined on  $L_{\text{loc}}^1(\mathbb{R}^d, \{0, 1\})$ . Integral representation results [2, 4, 18] then guarantee that the energies  $E_\varepsilon$   $\Gamma$ -converge (see [12, 30] for an introduction to that subject) with respect to the  $L_{\text{loc}}^1(\mathbb{R}^d)$ -topology to a continuum energy of the form

$$E_0(u) = \int_{\partial^* \{u=1\}} \varphi(\nu_u(x)) \, d\mathcal{H}^{d-1} \quad u \in BV_{\text{loc}}(\mathbb{R}^d; \{0, 1\}).$$

Here,  $BV_{\text{loc}}(\mathbb{R}^d; \{0, 1\})$  denotes the space of functions with (locally) bounded variation and values in  $\{0, 1\}$ ,  $\partial^* \{u = 1\}$  denotes the reduced boundary of the level set  $\{u = 1\}$ ,  $\nu_u(x)$  its measure theoretic normal at the point  $x \in \partial^* \{u = 1\}$ , and  $\mathcal{H}^{d-1}$  denotes the  $(d-1)$ -dimensional Hausdorff measure, see [5] for the precise definitions of these notions. The energy density  $\varphi: \mathbb{R}^d \rightarrow [0, +\infty)$  can be recovered via the asymptotic cell formula

$$\varphi(\nu) := \lim_{\delta \rightarrow 0} \lim_{S \rightarrow +\infty} \frac{1}{S^{d-1}} \inf \left\{ E(u, Q_S^\nu) : u: \mathcal{L} \rightarrow \{0, 1\}, u(i) = u_\nu(i) \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S}^\nu \right\}, \quad (2)$$

where

$$u_\nu(x) = \begin{cases} 1 & \text{if } \langle x, \nu \rangle \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Here,  $Q_S^\nu$  is a suitable rotation of the coordinate cube with side-length  $S$  such that two faces are parallel to  $\{\nu = 0\}$ . In the case  $\mathcal{L} = \mathbb{Z}^2$ ,  $c_{i,j} = 1$  if  $|i - j| = 1$  and  $c_{i,j} = 0$  otherwise, we have that  $\varphi(\nu) = 2\|\nu\|_1$ , see Figure 2.

The goal of this article is to investigate the energy density  $\varphi$ . In particular we show, that for finite interaction range  $c_{i,j}$ , that is there exists  $R > 0$  such that  $c_{i,j} = 0$  if  $|i - j| > R$ , then  $\varphi$  is

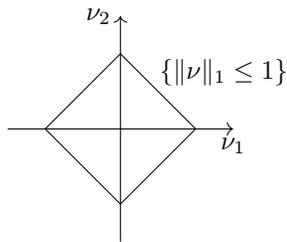


FIGURE 2. The energy density in the case of nearest neighbor interactions on  $\mathbb{Z}^2$

*crystalline*, see [32, Definition 3.2]. This means that the solution to

$$\min \left\{ \int_{\partial^* A} \varphi(\nu_A(x)) \, d\mathcal{H}^{d-1} : |A| = 1 \right\}$$

is a convex polytope. The finite range of interaction is crucial. Indeed, example 2.8 shows that for infinite range interactions this is in general not true. In [16, 17] it is shown that, as the periodicity  $T$  of the interactions tends to  $+\infty$ , it is possible to approximate any norm as surface energy density satisfying suitable growth conditions. We refer to [2] for a random setting where it is shown that an isotropic energy density (and thus non-crystalline) can be obtained in the limit.

The proof of the crystallinity in the case of finite range interactions relies on the following alternative representation formula of the density, proven in Proposition 2.6. Namely, we prove that

$$\varphi(\nu) = \frac{1}{T^d} \inf \{ E(u, Q_T) : u : \mathcal{L} \rightarrow \mathbb{R}, u(\cdot) - \langle \nu, \cdot \rangle \text{ is } T\text{-periodic} \}. \quad (3)$$

This representation formula is reminiscent of the representation formula of the energy density of integral functionals obtained via homogenization of  $T$ -periodic integral functionals in  $W^{1,p}$  [15]. To motivate this, consider the positively 1-homogeneous extensions of  $E_\varepsilon$  defined by

$$F_\varepsilon(u) = \sum_{i,j \in \varepsilon\mathcal{L}} \varepsilon^{d-1} c_{i,j}^\varepsilon (u(i) - u(j))^+,$$

for  $u : \varepsilon\mathcal{L} \rightarrow \mathbb{R}$ . The  $\Gamma$ -limit  $F_0$  of the above sequence is clearly positively 1-homogeneous and convex as the sequence of functionals satisfies these properties. Thus,  $F_0$  admits an integral representation of the form

$$F_0(u) = \int f_0(\nabla u) \, dx + \int f_0 \left( \frac{dD_s u}{|dD_s u|} \right) \, d|D_s u|,$$

where  $f_0 : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is convex and positively 1-homogeneous, see [14]. (We like to stress however, that this integral representation for the spin energies considered above is not proven in the literature.) Here, the important point is that the density of the singular part and the density of absolutely continuous parts agree. In the continuous setting, in [24, 27] it has been shown that for continuous and convex densities, that satisfy a coarea formula, the  $\Gamma$ -convergence of sets of finite perimeter or in the space of  $BV$ -functions is equivalent. Thus also in their setting, the densities agree. The density of the absolutely continuous part can be calculated via (3). This property eventually allows us to express  $\varphi$  via (3) since the density of the absolutely continuous part can be calculated via (3) and the density of the singular part agrees with the energy density in (2), see Proposition 2.6. Using convex duality (see [43]) and using (3) we show in Theorem 2.7 that  $\varphi$  is crystalline, and estimate an upper bound on the number of extreme points of the

corresponding Wulff shape. We would like to stress that (3) is not only a useful tool in our proof but it can be used also for computational purposes as it is a finite and not an asymptotic cell formula.

The paper is organized as follows. In Section 2 we describe the mathematical setting and state the main theorems of our paper. In Section 3 we prove Proposition 2.6, the alternate representation formula for  $\varphi$ . In Section 4 we show that, in the case of finite range interactions, the density  $\varphi$  is always crystalline. In Section 5 we discuss some differentiability properties of  $\varphi$ . We present some numerical simulations of our findings in the last chapter.

## 2. SETTING OF THE PROBLEM AND STATEMENT OF THE MAIN RESULT

**2.1. Notation.** We denote by  $\mathcal{B}(\mathbb{R}^d)$  the collection of all Borel-Sets in  $\mathbb{R}^d$ . For every  $A \subset \mathbb{R}^d$  we denote by  $|A|$  its  $d$ -dimensional Lebesgue measure. Given  $r > 0$ , we denote by  $(A)_r := \{x \in \mathbb{R}^d : \text{dist}(x, A) < r\}$  the  $r$ -neighbourhood of  $A$ . Given  $\tau \in \mathbb{R}^d$ , we set  $A + \tau := \{x + \tau : x \in A\}$ . The set  $\mathbb{S}^{d-1} := \{\nu \in \mathbb{R}^d : |\nu| = 1\}$  is the set of all  $d$ -dimensional unit vectors. For  $v, w \in \mathbb{R}^d$  we denote by  $\langle v, w \rangle$  their scalar product. We denote by  $\{e_1, \dots, e_d\} \subset \mathbb{R}^d$  the standard orthonormal basis of  $\mathbb{R}^d$ . Given  $C \subset \mathbb{R}^d$  convex, we denote by  $\text{extreme}(C)$  its extreme points. Given  $\rho > 0$ , we denote by  $Q_\rho := [-\rho/2, \rho/2]^d$  the half open cube centred in 0 with side-length  $\rho$ . For  $\nu \in \mathbb{S}^{d-1}$ , we set  $Q_\rho^\nu := R^\nu Q_\rho$ , where  $R^\nu$  is a rotation such that  $R^\nu e_d = \nu$ . Furthermore, given  $x \in \mathbb{R}^d$  we set  $Q_\rho^\nu(x) := x + Q_\rho^\nu$  (resp.  $Q_\rho(x) = x + Q_\rho$ ). Given  $x \in \mathbb{R}^d$  and  $r > 0$  we denote by  $B_r(x)$  the open ball with radius  $r > 0$  and center  $x$ . For  $A \subset \mathbb{R}^d$  we denote by  $\chi: \mathbb{R}^d \rightarrow \{0, 1\}$  the characteristic function of the set  $A$  given by

$$\chi_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

We denote by  $\omega_d$  the volume of the unit ball in  $\mathbb{R}^d$ . Given  $\nu \in \mathbb{S}^{d-1}$  we define

$$u_\nu(x) := \begin{cases} 1 & \text{if } \langle \nu, x \rangle \geq 0, \\ 0 & \text{otherwise.} \end{cases} \quad (5)$$

For  $z \in \mathbb{R}$  we denote by  $(z)^+ := \max\{z, 0\}$  the positive part of  $z$ .

**2.2. Discrete energies and homogenized surface energy density.** In this paragraph we define the discrete energies we want to consider and the homogenized surface energy density.

Let  $\mathcal{L} \subset \mathbb{R}^d$  satisfy the following two conditions:

(L1) (Discreteness) There exists  $c > 0$  such that for all  $x \in \mathcal{L}$  there holds

$$\text{dist}(x, \mathcal{L} \setminus \{x\}) \geq c.$$

(L2) (Periodicity) There exists  $T \in \mathbb{N}$  such that for all  $z \in \mathbb{Z}^d$  there holds

$$\mathcal{L} + Tz = \mathcal{L}.$$

Note that the two assumptions (L1) and (L2) include multi-lattices, such as the hexagonal closed packing lattice in three dimensions, and Bravais lattices, such as  $\mathbb{Z}^d$ , or the face-centered cubic lattice in three dimensions.

We consider interaction coefficients  $c_{i,j} : \mathcal{L} \times \mathcal{L} \rightarrow [0, +\infty)$  and the corresponding (localized) ferromagnetic spin energies of the form

$$E(u, A) := \sum_{i \in \mathcal{L} \cap A} \sum_{j \in \mathcal{L}} c_{i,j} (u(i) - u(j))^+, \quad (6)$$

where  $u : \mathcal{L} \rightarrow \mathbb{R}$  and  $A \in \mathcal{B}(\mathbb{R}^d)$ . If  $A = \mathbb{R}^d$  we omit the dependence on the set and write  $E(u) := E(u, \mathbb{R}^d)$ . We want to remark that we are considering interactions on the directed graph instead of the undirected graph.

We introduce the following three hypotheses on the interaction coefficients  $c_{i,j} : \mathcal{L} \times \mathcal{L} \rightarrow [0, +\infty)$ :

(H1) (Periodicity) There holds

$$c_{i+Tz, j+Tz} = c_{i,j}$$

for all  $i, j \in \mathcal{L}$ ,  $z \in \mathbb{Z}^d$ .

(H2) (Decay of Interactions) For all  $i \in \mathcal{L}$  there holds

$$\sum_{j \in \mathcal{L}} c_{i,j} |i - j| < +\infty.$$

(H3) (Finite Range Interactions) There exists  $R > 1$  such that

$$c_{i,j} = 0$$

for all  $i, j \in \mathcal{L}$  such that  $|i - j| \geq R$ .

It is obvious, that hypothesis (H3) implies hypothesis (H2). Note that, if (H1) and (H2) are satisfied then

$$\max_{i \in \mathcal{L}} \sum_{j \in \mathcal{L}} c_{i,j} |i - j| = \max_{i \in \mathcal{L} \cap Q_T} \sum_{j \in \mathcal{L}} c_{i,j} |i - j| < +\infty$$

and for all  $R > 0$ , there exists  $C_R > 0$  such that  $C_R \rightarrow 0$  as  $R \rightarrow +\infty$  and

$$\max_{i \in \mathcal{L}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \geq R}} c_{i,j} |i - j| \leq C_R.$$

**Definition 2.1.** Let  $c_{i,j}$  satisfy (H1) and (H2). We then define the *homogenized surface energy density*  $\varphi : \mathbb{R}^d \rightarrow [0, +\infty)$  as the convex positively homogeneous function of degree one such that for all  $\nu \in \mathbb{S}^{d-1}$  we have

$$\varphi(\nu) := \lim_{\delta \rightarrow 0} \lim_{S \rightarrow +\infty} \frac{1}{S^{d-1}} \inf \left\{ E(u, Q_S^\nu) : u : \mathcal{L} \rightarrow \{0, 1\}, u(i) = u_\nu(i) \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S}^\nu \right\}, \quad (7)$$

with  $u_\nu$  defined in (5).

**Remark 2.2.** The definition above can be interpreted as a passage from discrete to continuum description as follows. Given  $\varepsilon > 0$ , we consider the scaled energies

$$E_\varepsilon(u) := \sum_{i \in \mathcal{L}} \sum_{j \in \mathcal{L}} \varepsilon^{d-1} c_{i,j} (u(\varepsilon i) - u(\varepsilon j))^+,$$

where  $u : \varepsilon\mathcal{L} \rightarrow \{0, 1\}$ . Upon identifying  $u$  with its piecewise-constant interpolation, we can regard these energies to be defined on  $L^1_{\text{loc}}(\mathbb{R}^d)$ . We know that their  $\Gamma$ -limit is infinite outside the space  $BV_{\text{loc}}(\mathbb{R}^d; \{0, 1\})$ , where it has the form

$$E_0(u) := \int_{\partial^*\{u=1\}} \varphi(\nu) \, d\mathcal{H}^{d-1}$$

with  $\varphi$  given by (7), see for example [2].<sup>1</sup> Here,  $\partial^*\{u = 1\}$  denotes the reduced boundary of the set  $\{u = 1\}$  and  $\mathcal{H}^{d-1}$  denotes the  $(d - 1)$ -dimensional Hausdorff measure in  $\mathbb{R}^d$  (cf. [5], Chapters 2.8 and 3.5).

**Remark 2.3.** Testing with  $u_\nu$  in (7), using (L1) and (H2), it is easy to see that  $\varphi(\nu) \leq C$  for all  $\nu \in \mathbb{S}^{d-1}$ . Therefore, due to the convexity and the fact that it is a positively one homogeneous function of degree one,  $\varphi$  is Lipschitz continuous.

**2.3. Statement of the main result.** In this section we state the main result.

**Definition 2.4.** Given  $\varphi : \mathbb{R}^d \rightarrow [0, +\infty)$  convex, positively homogeneous of degree one, we define the Wulff set of  $\varphi$  by

$$W_\varphi := \{\zeta \in \mathbb{R}^d : \langle \zeta, \nu \rangle \leq \varphi(\nu) \text{ for all } \nu \in \mathbb{S}^{d-1}\}.$$

We say that  $\varphi$  is crystalline, if  $W_\varphi$  is a polytope.

**Remark 2.5.** From the definition of the Wulff set, it is clear that

$$\varphi(\nu) = \sup_{\zeta \in W_\varphi} \langle \nu, \zeta \rangle.$$

Furthermore, one can check, that if  $\varphi$  is crystalline, then the set  $\{\varphi \leq 1\}$  is a polytope.

The next proposition shows that, we obtain a finite cell formula in order to calculate  $\varphi$  instead of the asymptotic one, given in (7). We think that this result in itself is interesting, since it allows for calculations on finite size systems in order to compute  $\varphi$  for general Ising systems. This result is in spirit very close to [10, 14], where convex and positively 1-homogenous continuum energies are considered. In this case, the surface energy density and the energy with respect to the absolutely continuous part coincide. For  $k \in \mathbb{N}$  let  $\Lambda := Tk$  and denote by

$$\mathcal{A}_{\text{per}}(Q_\Lambda; \mathbb{R}) := \{u : \mathcal{L} \rightarrow \mathbb{R} : u(x + \Lambda z) = u(x) \text{ for all } z \in \mathbb{Z}^d\} \quad (8)$$

be the space of  $\Lambda$ -periodic functions.

**Proposition 2.6.** Let  $c_{i,j} : \mathcal{L} \times \mathcal{L} \rightarrow [0, +\infty)$  be interaction coefficients such that (H1) and (H2) hold true. Then

$$\varphi(\nu) = \frac{1}{T^d} \inf \{E(u, Q_T) : u : \mathcal{L} \rightarrow \mathbb{R}, u(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R})\}. \quad (9)$$

**Theorem 2.7.** Let  $c_{i,j} : \mathcal{L} \times \mathcal{L} \rightarrow [0, +\infty)$  be interaction coefficients such that (H1) and (H3) hold true. Then, the homogenized surface energy density  $\varphi$  is crystalline. Denote by

$$N := \#\{(i, j) \in \mathcal{L} \cap Q_T \times \mathcal{L} : c_{i,j} \neq 0\}.$$

Then,

$$\#\text{extreme}(W_\varphi) \leq 3^N.$$

The next example shows that without assumption (H3) Theorem 2.7 fails to hold true.

<sup>1</sup>Actually, the integral representation for the  $\Gamma$ -limit has only been shown for undirected graphs. However, a slight modification of the proof shows that it is still true for directed graphs.

**Example 2.8.** To construct the example we first observe that if  $f: \mathbb{R}^d \rightarrow [0, +\infty)$  is crystalline, then  $D^2f$  is a Radon-measure with support contained in finitely many hyper-planes. To see this, note that if  $f: \mathbb{R}^d \rightarrow [0, +\infty)$  is crystalline, then there exist  $\{\xi_k\}_{k=1}^N \subset \mathbb{R}^d$  such that

$$f(\nu) = \max_{1 \leq k \leq N} \langle \xi_k, \nu \rangle.$$

Here, we assume that  $\{\xi_k\}_{k=1}^N$  is chosen minimal, i.e. if we set  $V_k = \{\nu \in \mathbb{R}^d: f(\nu) = \langle \xi_k, \nu \rangle\}$ , then  $|V_k| > 0$  for all  $k = 1, \dots, N$ . This assumption ensures that all the vectors  $\xi_k$  play an active role in the definition of  $f$ . Now,  $Df \in BV_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d)$  is given by

$$Df(\nu) = \sum_{k=1}^N \chi_{V_k}(\nu) \xi_k,$$

with  $V_k$  defined above. Then

$$D^2f(\nu) = \sum_{1 \leq k < j \leq N} (\xi_k - \xi_j) \otimes \nu_{kj} \mathcal{H}^{d-1} \llcorner_{\partial V_k \cap \partial V_j},$$

where  $\partial V_k \cap \partial V_j = \{\nu \in \mathbb{R}^d: f(\nu) = \langle \xi_k, \nu \rangle = \langle \xi_j, \nu \rangle\}$  and  $\nu_{kj} \in \mathbb{S}^{d-1}$  denotes the normal pointing towards the set  $V_k$ .

Let now  $\mathcal{L} = \mathbb{Z}^d$  and  $c_{i,j} = c_{j-i} = c_{i-j}$  (in the following denoted by  $\{c_\xi\}_{\xi \in \mathbb{Z}^d}$ ) be such that  $c_\xi > 0$  for all  $\xi \in \mathbb{Z}^d$  and

$$\sum_{\xi \in \mathbb{Z}^d} c_\xi |\xi| < +\infty.$$

It is then obvious that  $c_{i,j}$  is 1-periodic, (H1) and (H2) hold true, but (H3) is violated. Therefore, due to Proposition 2.6, we have

$$\varphi(\nu) = \sum_{\xi \in \mathbb{Z}^d} c_\xi |\langle \xi, \nu \rangle|.$$

This is true, since the only admissible functions in the minimum problem given by Proposition 2.6 are  $u_\nu(i) = \langle \nu, i \rangle + c$  for some  $c \in \mathbb{R}$ . We claim that

$$D\varphi(\nu) = \sum_{\xi \in \mathbb{Z}^d} \text{sign}(\langle \xi, \nu \rangle) c_\xi \xi,$$

where  $\text{sign}: \mathbb{R} \rightarrow \mathbb{R}$  is defined by

$$\text{sign}(t) = \begin{cases} 1 & t \geq 0; \\ -1 & t < 0. \end{cases}$$

Therefore

$$D^2\varphi = 2 \sum_{\xi \in \mathbb{Z}^d} c_\xi \xi \otimes \frac{\xi}{|\xi|} \mathcal{H}^{d-1} \llcorner_{\{\nu: \langle \xi, \nu \rangle = 0\}}. \quad (10)$$

This can be seen by approximation. Consider  $\varphi_R: \mathbb{R}^d \rightarrow \mathbb{R}$  defined by

$$\varphi_R(\nu) = \sum_{\substack{\xi \in \mathbb{Z}^d \\ |\xi| \leq R}} c_\xi |\langle \xi, \nu \rangle|, \quad D\varphi_R(\nu) = \sum_{\substack{\xi \in \mathbb{Z}^d \\ |\xi| \leq R}} \text{sign}(\langle \xi, \nu \rangle) c_\xi \xi,$$

Then

$$D^2\varphi_R = 2 \sum_{\substack{\xi \in \mathbb{Z}^d \\ |\xi| \leq R}} c_\xi \xi \otimes \frac{\xi}{|\xi|} \mathcal{H}^{d-1} \llcorner_{\{\nu: \langle \xi, \nu \rangle = 0\}}.$$

Now

$$|D^2\varphi_R|(B_r) \leq Cr^{d-1} \sum_{\xi \in \mathbb{Z}^d} c_\xi |\xi|,$$

so the total variation of  $D^2\varphi_R$  is (locally) uniformly bounded with limiting measure  $D^2\varphi$  and  $D\varphi_R \rightarrow D\varphi$  in  $L^1_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d)$ , actually weakly in  $BV$ . Hence, (10) is shown. Now, since  $c_\xi > 0$  for all  $\xi \in \mathbb{Z}^d$  it is obvious that  $D^2\varphi$  is not supported on finitely many hyper-planes. Thus  $\varphi$  cannot be crystalline.

Note that  $\varphi$  is differentiable in totally irrational directions.<sup>2</sup> A similar property is known to hold, in the continuous setting [6, 25], for homgenized surface tensions. We can state a result showing that this is still the case in the discrete setting, under assumptions (H1) and (H2).

**Proposition 2.9.** *Under the assumptions of Proposition 2.6,  $\varphi$  is differentiable in any totally irrational direction.*

It is expected that it should be, “in general”, not differentiable in the other directions, at least whenever the minimizers  $u$  in (9) are constant on an infinite set, however the proofs in [6, 25] rely on ellipticity properties of the problem and are less easy to transfer to the discrete case. The proof of Proposition 2.9, which mimicks the proof in [25], is postponed to Section 5, and relies on the dual representation (95) introduced later on.

### 3. PROOF OF PROPOSITION 2.6

This section is devoted to the proof of Proposition 2.6. We assume throughout this section that assumptions (L1), (L2) and (H1), (H2) are satisfied. The proof consists in showing that  $\varphi$  can be characterized by several (equivalent) cell-formulas and therefore passing from (7) to (9).

First, we will state and prove some elementary properties of  $E$  that will be used throughout this section.

**Lemma 3.1.** *Let  $A \in \mathcal{B}(\mathbb{R}^d)$  and let  $c > 0$  be as in (L1).*

- (i) *There exists a universal constant  $C > 0$  (depending only on  $c_{i,j}$  and  $c$  in (L1)) such that for all  $\nu \in \mathbb{R}^d$  we have*

$$E(\langle \nu, \cdot \rangle, A) \leq C|\nu|(A)_c.$$

- (ii) *Let  $u: \mathcal{L} \rightarrow \mathbb{R}$ . For all  $t \in \mathbb{R}, \lambda > 0$  there holds*

$$E(\lambda u + t, A) = \lambda E(u, A)$$

*and  $u \mapsto E(u, A)$  is convex. In particular,*

$$E(u + v, A) \leq E(u, A) + E(v, A)$$

*for all  $u, v: \mathcal{L} \rightarrow \mathbb{R}$ .*

- (iii) *Let  $u: \mathcal{L} \rightarrow \mathbb{R}$  and  $B \in \mathcal{B}(\mathbb{R}^d)$  be such that  $A \subset B$ . Then*

$$E(u, A) \leq E(u, B).$$

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<sup>2</sup> $p$  is totally irrational if there is no  $q \in \mathbb{Z}^d \setminus \{0\}$  such that  $\langle q, p \rangle = 0$ .

(iv) Let  $u: \mathcal{L} \rightarrow \mathbb{R}$  and  $B \in \mathcal{B}(\mathbb{R}^d)$  be such that  $A \cap B = \emptyset$ . Then

$$E(u, A \cup B) = E(u, A) + E(u, B).$$

(v) We have

$$\#\{i \in \mathcal{L} \cap A\} \leq \frac{1}{c^d \omega_d} |(A)_c|.$$

(vi) Let  $u: \mathcal{L} \rightarrow \mathbb{R}$ . Then, the function  $\tau \mapsto E(u(\cdot - \tau), A + \tau)$  is  $T$ -periodic.

*Proof.* We start by proving (ii)-(iv) in Step 1, then (v) and (vi) in Step 2 and Step 3 respectively, and finally (i) in Step 4.

**Step 1.**(Proof of (ii) - (iv)) All the three statements are a direct consequence of (6) and the fact that  $c_{i,j} \geq 0$ .

**Step 2.** (Proof of (v)) Note that

$$\bigcup_{i \in \mathcal{L} \cap A} B_c(i) \subset (A)_c$$

and therefore, due to (L1),

$$c^d \omega_d \#\{i \in \mathcal{L} \cap A\} = \left| \bigcup_{i \in \mathcal{L} \cap A} B_c(i) \right| \leq |(A)_c|.$$

This is the claim.

**Step 3.**(Proof of (vi)) Let  $u: \mathcal{L} \rightarrow \mathbb{R}$  and  $z \in \mathbb{Z}^d$ . Then, using (H1) and (L2),

$$\begin{aligned} E(u(\cdot - Tz), A + Tz) &= \sum_{i \in \mathcal{L} \cap (A+Tz)} \sum_{j \in \mathcal{L}} c_{i,j} (u(i - Tz) - u(j - Tz))^+ \\ &= \sum_{i \in \mathcal{L} \cap A} \sum_{j \in (\mathcal{L} + Tz)} c_{i+Tz, j+Tz} (u(i) - u(j))^+ \\ &= \sum_{i \in \mathcal{L} \cap A} \sum_{j \in \mathcal{L}} c_{i,j} (u(i) - u(j))^+ = E(u, A). \end{aligned}$$

**Step 4.**(Proof of (i)) Let  $\nu \in \mathbb{R}^d$ , then, due to (v), (L1), (L2), (H1), and (H2), we have

$$E(\langle \nu, \cdot \rangle, A) = \sum_{i \in \mathcal{L} \cap A} \sum_{j \in \mathcal{L}} c_{i,j} |\langle \nu, i - j \rangle| \leq |\nu| \#\{i \in \mathcal{L} \cap A\} \max_{i \in \mathcal{L}} \sum_{j \in \mathcal{L}} c_{i,j} |i - j| \leq C |\nu| |(A)_c|.$$

□

The next Lemma shows that our energy satisfies a generalized coarea formula [24, 45].

**Lemma 3.2.** Let  $u: \mathcal{L} \rightarrow \mathbb{R}$  and  $A \subset \mathbb{R}^d$ . Then

$$E(u, A) = \int_{-\infty}^{+\infty} E(\chi_{\{u>t\}}(i), A) dt. \quad (11)$$

*Proof.* For  $a, b \in \mathbb{R}$  we have

$$(a - b)^+ = \int_{-\infty}^{+\infty} (\chi_{\{a>t\}} - \chi_{\{b>t\}})^+ dt.$$

Therefore,

$$E(u, A) = \sum_{i \in \mathcal{L} \cap A} \sum_{j \in \mathcal{L}} c_{i,j} (u(i) - u(j))^+ = \sum_{i \in \mathcal{L} \cap A} \sum_{j \in \mathcal{L}} c_{i,j} \int_{-\infty}^{+\infty} (\chi_{\{u(i) > t\}} - \chi_{\{u(j) > t\}})^+ dt.$$

Now, by Fubini's Theorem (note that  $c_{i,j} \geq 0$ ), we obtain

$$\begin{aligned} \sum_{i \in \mathcal{L} \cap A} \sum_{j \in \mathcal{L}} c_{i,j} \int_{-\infty}^{+\infty} (\chi_{\{u(i) > t\}} - \chi_{\{u(j) > t\}})^+ dt &= \int_{-\infty}^{+\infty} \sum_{i \in \mathcal{L} \cap A} \sum_{j \in \mathcal{L}} c_{i,j} (\chi_{\{u(i) > t\}} - \chi_{\{u(j) > t\}})^+ dt \\ &= \int_{-\infty}^{+\infty} E(\chi_{\{u > t\}}, A) dt \end{aligned}$$

and thus the claim.  $\square$

**Lemma 3.3.** *Let  $S > 0$ ,  $\delta > 0$  and  $\nu \in \mathbb{S}^{d-1}$ . Then*

$$\begin{aligned} \inf \left\{ E(u, Q_S^\nu) : u : \mathcal{L} \rightarrow \mathbb{R}, u(i) = u_\nu(i) \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S}^\nu \right\} \\ = \inf \left\{ E(u, Q_S^\nu) : u : \mathcal{L} \rightarrow \{0, 1\}, u(i) = u_\nu(i) \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S}^\nu \right\}. \end{aligned}$$

*Proof.* The infimum on the left hand side is taken over a larger class of admissible function, since here the image of the competitor  $u$  is a subset of  $\mathbb{R}$  and not just of  $\{0, 1\}$ . Hence, one inequality is trivial. The other inequality follows from a coarea formula satisfied by our energies.

**Step 1.** (Proof of ' $\leq$ ') This inequality is clear, since the infimum on the left hand side is taken over a larger class of functions.

**Step 2.** (Proof of ' $\geq$ ') Let us take  $u : \mathcal{L} \rightarrow \mathbb{R}$  such that  $u = u_\nu$  on  $\mathcal{L} \setminus Q_{(1-\delta)S}^\nu$  and denote by  $u^s = \chi_{u > s}$ . Then, using Lemma 3.2, there exists  $t \in (0, 1)$  such that

$$E(u^t, Q_S^\nu) \leq \int_0^1 E(u^s, Q_S^\nu) ds \leq \int_{-\infty}^{+\infty} E(u^s, Q_S^\nu) ds = E(u, Q_S^\nu).$$

Noting that  $u^t(i) \in \{0, 1\}$  for all  $i \in \mathcal{L}$  and  $u^t = u_\nu$  on  $\mathcal{L} \setminus Q_{(1-\delta)S}^\nu$ , this concludes Step 2.  $\square$

Let  $\phi : \mathbb{R}^d \rightarrow [0, +\infty]$  be defined by

$$\phi(\nu) = \lim_{\delta \rightarrow 0} \lim_{S \rightarrow +\infty} \frac{1}{S^d} \inf \left\{ E(u, Q_S) : u : \mathcal{L} \rightarrow \mathbb{R}, u(i) = \langle \nu, i \rangle \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S} \right\}. \quad (12)$$

$\phi_{\text{per}} : \mathbb{R}^d \rightarrow [0, +\infty]$  is defined by

$$\phi_{\text{per}}(\nu) = \liminf_{k \rightarrow +\infty} \frac{1}{(kT)^d} \inf \left\{ E(u, Q_{kT}) : u : \mathcal{L} \rightarrow \mathbb{R}, u(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_{kT}; \mathbb{R}) \right\}. \quad (13)$$

The next lemma shows that  $\phi_{\text{per}}$  can be calculated via a finite cell formula. Additionally, it shows that the liminf in the definition of (13) is actually a limit.

**Lemma 3.4.** *Let  $\nu \in \mathbb{S}^{d-1}$ . For all  $k \in \mathbb{N}$  there holds*

$$\begin{aligned} \frac{1}{(kT)^d} \inf \left\{ E(u, Q_{kT}) : u : \mathcal{L} \rightarrow \mathbb{R}, u(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_{kT}; \mathbb{R}) \right\} \\ = \frac{1}{T^d} \inf \left\{ E(u, Q_T) : u : \mathcal{L} \rightarrow \mathbb{R}, u(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R}) \right\}. \end{aligned} \quad (14)$$

In particular,

$$\phi_{\text{per}}(\nu) = \frac{1}{T^d} \inf \left\{ E(u, Q_T) : u : \mathcal{L} \rightarrow \mathbb{R}, u(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R}) \right\} \quad (15)$$

and the lim inf in (13) is actually a limit since the sequence, in fact, does not depend on  $k$ .

The following example shows that, without any further assumption on  $c_{i,j}$ , the minimum in

$$\inf \{E(u, Q_T) : u : \mathcal{L} \rightarrow \mathbb{R}, u(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R})\} \quad (16)$$

is not achieved by any  $u : \mathcal{L} \rightarrow \mathbb{R}$ .

**Example 3.5.** Let  $\mathcal{L} = \mathbb{Z}$  and let  $c_{i,j}$  be 2-periodic. We set

$$c_{i,j} = \begin{cases} 2^{-k} & \text{if } j - i = 2k + 1, k \in \mathbb{N}, i \text{ even,} \\ 0 & \text{otherwise.} \end{cases}$$

Let  $s \in \mathbb{R}$ ,  $u(0) = 0$  and  $u(1) = s$  be 2-periodic. Then, for  $\nu = -1$  we have

$$E(u, Q_2) = \sum_{k=0}^{\infty} c_{0,2k+1} (u(0) - u(2k+1) + 2k+1)^+ = \sum_{k=0}^{\infty} 2^{-k} (2k+1-s)^+.$$

We have that

$$\inf_{u \in \mathcal{A}_{\text{per}}(Q_2; \mathbb{R})} E(u, Q_2) = 0.$$

However, clearly, for all  $u \in \mathcal{A}_{\text{per}}(Q_2; \mathbb{R})$ ,  $E(u, Q_2) > 0$ . In order to ensure the existence in (16) and the other minimum problems a coercivity condition might be: For any  $i, j \in \mathcal{L}$  there exists a path  $\gamma = (i_0, \dots, i_N)$  such that  $i_0 = i$ ,  $i_N = j$  and such that  $c_{i_k, i_{k+1}} > 0$ . By considering the directed graph  $G = (\mathcal{L}, \mathcal{E})$ , where  $\mathcal{E} := \{(i, j) \in \mathcal{L} \times \mathcal{L} : \text{there exists } z \in \mathbb{Z}^d \text{ such that } c_{i, j+Tz} > 0\}$  this condition ensures that for two vertices  $i, j \in G$  there always exists a path of edges (in the infinite graph) with positive weights connecting them.

*Proof of Lemma 3.4.* We split the proof into two steps by first observing the (obvious) inequality that the right hand side in (14) is less than or equal to the left hand side. Then, we prove the converse inequality by using a superposition argument.

**Step 1.** (Proof of ' $\leq$ ') given  $u$  with  $u(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R})$ , then obviously  $u(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_{kT}; \mathbb{R})$  and

$$E(u, Q_{kT}) = \frac{1}{k^d} E(u, Q_T)$$

so this inequality is obvious.

**Step 2.** (Proof of ' $\geq$ ') This is a standard convexity argument: given  $u$  now with  $u(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_{kT}; \mathbb{R})$ , then for  $i \in Q_T$  we let:

$$u_T(i) = \langle \nu, i \rangle + \frac{1}{k^d} \sum_{z \in \{0, \dots, k-1\}^d} (u(i + Tz) - \langle \nu, i + Tz \rangle).$$

Then clearly by construction,  $u_T \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R})$  and by convexity,

$$E(u_T, Q_T) = \frac{1}{k^d} E(u_T, Q_{kT}) \leq \frac{1}{k^d} E(u, Q_{kT}),$$

which shows the lemma. □

The following lemma uses a standard cutoff-argument. However, due to the infinite range of interactions, the arguments for the case of finite range interactions need to be adapted.

**Lemma 3.6.** *Let  $\nu \in \mathbb{R}^d$ . Then:  $\phi_{\text{per}}(\nu) = \phi(\nu)$ .*

*Proof.* We first show  $\phi_{\text{per}}(\nu) \leq \phi(\nu)$  in Step 1, and then the reverse inequality. In order to do so, we modify competitors of the respective cell formulas in order to obtain a competitor for the other formula. Due to the one homogeneity of both functions, we may assume that  $\nu \in \mathbb{S}^{d-1}$ .

**Step 1.**(Proof of ' $\leq$ ') Due to Lemma 3.4, the limit in the definition of  $\phi_{\text{per}}(\nu)$  (resp.  $\phi(\nu)$ ) exists. Thus, we can assume without loss of generality that  $S = kT$  for some  $k \in \mathbb{N}$  with  $k$  large. Let  $\delta > 0$ ,  $\varepsilon > 0$ , and let  $u_k^\delta: \mathcal{L} \rightarrow \mathbb{R}$  be such that  $u_k^\delta(i) = \langle \nu, i \rangle$  on  $\mathcal{L} \setminus Q_{(1-\delta)kT}$  and

$$E(u_k^\delta, Q_{kT}) \leq \inf \{E(u, Q_{kT}): u: \mathcal{L} \rightarrow \mathbb{R}, u(i) = \langle \nu, i \rangle \text{ on } \mathcal{L} \setminus Q_{(1-\delta)kT}\} + \varepsilon. \quad (17)$$

We assume that

$$\|u_k^\delta\|_{L^\infty(Q_{(1+\delta)kT})} \leq 2kT. \quad (18)$$

If that were not true we perform the following construction with  $\tilde{u}_k^\delta(i) = (u_k^\delta(i) \vee (-2kT)) \wedge (2kT)$ . Note that still  $\tilde{u}_k^\delta(i) = \langle \nu, i \rangle$  on  $Q_{(1+\delta)kT}$  for  $\delta$  small enough. We define  $v_k^\delta: \mathcal{L} \rightarrow \mathbb{R}$  by setting

$$v_k^\delta(i) - \langle \nu, i \rangle = u_k^\delta(i_0) - \langle \nu, i_0 \rangle \text{ if } i = i_0 + kTz, i_0 \in Q_{kT}, z \in \mathbb{Z}^d \quad (19)$$

so that  $v_k^\delta(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_{kT}; \mathbb{R})$ . Then clearly, writing  $i = i_0 + kTz$  and  $j = j_0 + k'Tz$  as above:

$$|v_k^\delta(i) - v_k^\delta(j)| \leq |u_k^\delta(i_0) - u_k^\delta(j_0)| + |i_0 - j_0| + |i - j| \leq CkT + |i - j|, \quad (20)$$

and

$$v_k^\delta(i) = u_k^\delta(i) \text{ for } i \in Q_{(1+\delta)kT}, \quad (21)$$

since  $u_k^\delta(i) - \langle \nu, i \rangle = 0$  for  $i \notin Q_{(1-\delta)kT}$ . Additionally,

$$\inf \{E(u, Q_{kT}): u: \mathcal{L} \rightarrow \mathbb{R}, u(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_{kT}; \mathbb{R})\} \leq E(v_k^\delta, Q_{kT}). \quad (22)$$

We are finished with Step 1 if we prove

$$E(v_k^\delta, Q_{kT}) \leq E(u_k^\delta, Q_{kT}) + \frac{C_k^\delta}{\delta} (kT)^d, \quad (23)$$

where  $C_k^\delta \rightarrow 0$  as  $k \rightarrow +\infty$ . In fact, using (17), (22), (23), dividing by  $(kT)^d$ , letting  $k \rightarrow +\infty$ , and then  $\delta \rightarrow 0$ , we obtain the claim by noting that  $\varepsilon > 0$  is chosen arbitrarily. Let us prove (23). We have, using (21),

$$\begin{aligned} E(v_k^\delta, Q_{kT}) &= \sum_{i \in \mathcal{L} \cap Q_{kT}} \sum_{j \in \mathcal{L}} c_{i,j} (v_k^\delta(i) - v_k^\delta(j))^+ \\ &= \sum_{i \in \mathcal{L} \cap Q_{kT}} \sum_{j \in \mathcal{L} \cap Q_{(1+\delta)kT}} c_{i,j} (v_k^\delta(i) - v_k^\delta(j))^+ + \sum_{i \in \mathcal{L} \cap Q_{kT}} \sum_{j \in \mathcal{L} \setminus Q_{(1+\delta)kT}} c_{i,j} (v_k^\delta(i) - v_k^\delta(j))^+ \\ &\leq E(u_k^\delta, Q_{kT}) + \sum_{i \in \mathcal{L} \cap Q_{kT}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \geq \delta kT/2}} c_{i,j} |v_k^\delta(i) - v_k^\delta(j)|. \end{aligned}$$

Hence, in order to show (23), it remains to prove

$$\sum_{i \in \mathcal{L} \cap Q_{kT}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \geq \delta kT/2}} c_{i,j} |v_k^\delta(i) - v_k^\delta(j)| \leq \frac{C_k^\delta}{\delta} (kT)^d, \quad (24)$$

where  $C_k^\delta \rightarrow 0$  as  $k \rightarrow +\infty$ . Using (20), (H2), and Lemma 3.1(v), we have

$$\begin{aligned} \sum_{i \in \mathcal{L} \cap Q_{kT}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \geq \delta kT/2}} c_{i,j} |v_k^\delta(i) - v_k^\delta(j)| &\leq \sum_{i \in \mathcal{L} \cap Q_{kT}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \geq \delta kT/2}} c_{i,j} (CkT + |i-j|) \\ &\leq \left( \frac{C}{\delta} + 1 \right) \sum_{i \in \mathcal{L} \cap Q_{kT}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \geq \delta kT/2}} c_{i,j} |i-j| \\ &\leq \frac{C}{\delta} \#(\mathcal{L} \cap Q_{kT}) \max_{i \in \mathcal{L}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \geq \delta kT/2}} c_{i,j} |i-j| \leq \frac{C_k^\delta}{\delta} (kT)^d, \end{aligned}$$

where  $C_k^\delta \rightarrow 0$  as  $k \rightarrow +\infty$ . This yields (24) and therefore the claim of Step 1.

**Step 2.** (Proof of ' $\geq$ ') Let  $\varepsilon > 0$  and  $u: \mathcal{L} \rightarrow \mathbb{R}$  be such that  $u(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R})$  and

$$E(u, Q_T) \leq \inf \{ E(u, Q_T) : u: \mathcal{L} \rightarrow \mathbb{R}, u(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R}) \} + \varepsilon.$$

Fix  $\delta > 0$  and  $S \in \mathbb{N}$  such that  $S = kT$  for some  $k \in \mathbb{N}$ ,  $k \gg 1$  and  $\delta S \gg 1$ . Since  $u(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R})$ , we have

$$E(u, Q_T(x_0)) = E(u, Q_T) \text{ for all } x_0 \in T\mathbb{Z}^d$$

and therefore

$$E(u, Q_S) = \frac{S^d}{T^d} E(u, Q_T). \quad (25)$$

There exists a constant  $C > 0$  (we omit the dependence on  $T$ ) such that, due to the fact that  $u(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R})$ , there holds

$$\max_{i \in \mathcal{L}} |u(i) - \langle \nu, i \rangle| = \max_{i \in \mathcal{L} \cap Q_T} |u(i) - \langle \nu, i \rangle| \leq C_\varepsilon. \quad (26)$$

Let  $\zeta_S \in C_c^\infty(\mathbb{R}^d; [0, 1])$  be a cut-off function such that

$$\zeta_S(x) = 1 \text{ for } x \in Q_{(1-3\delta)S}, \quad \text{supp } \zeta_S(x) \subset Q_{(1-2\delta)S}, \quad \text{and } \|\nabla \zeta_S\|_\infty \leq \frac{C}{\delta S}.$$

Define  $u_S: \mathcal{L} \rightarrow \mathbb{R}$  by

$$u_S(i) = \zeta_S(i)u(i) + (1 - \zeta_S(i))\langle \nu, i \rangle.$$

Then,  $u_S(i) = \langle \nu, i \rangle$  for  $i \in \mathcal{L} \setminus Q_{(1-\delta)S}$  and therefore

$$\inf \{ E(u, Q_S) : u: \mathcal{L} \rightarrow \mathbb{R}, u(i) = \langle \nu, i \rangle \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S} \} \leq E(u_S, Q_S). \quad (27)$$

For all  $i, j \in \mathcal{L}$  there holds

$$u_S(i) - u_S(j) = \zeta_S(i)(u(i) - u(j)) + (1 - \zeta_S(i))\langle \nu, i - j \rangle + (\zeta_S(i) - \zeta_S(j))(u(j) - \langle \nu, j \rangle),$$

which, together with (26), implies for all  $i, j \in \mathcal{L}$

$$\begin{aligned} (u_S(i) - u_S(j))^+ &\leq (u(i) - u(j))^+ + |i-j| + \frac{C}{\delta S} |u(j) - \langle \nu, j \rangle| |i-j| \\ &\leq (u(i) - u(j))^+ + C|i-j|, \end{aligned} \quad (28)$$

where we assume that  $S\delta \geq C_\varepsilon$  (we will first send  $k$  to  $+\infty$ , then  $\delta$  to 0, and finally  $\varepsilon$  to 0). For all  $i, j \in Q_{(1-3\delta)S}$  we have

$$(u_S(i) - u_S(j))^+ = (u(i) - u(j))^+. \quad (29)$$

Using (25), (28), and (29), we obtain

$$\begin{aligned}
E(u_S, Q_S) &\leq \sum_{i \in \mathcal{L} \cap Q_S} \sum_{j \in \mathcal{L}} c_{i,j} (u(i) - u(j))^+ + C \sum_{i \in \mathcal{L} \cap Q_{(1-6\delta)S}} \sum_{j \in \mathcal{L} \setminus Q_{(1-3\delta)S}} c_{i,j} |i - j| \\
&\quad + C \sum_{i \in \mathcal{L} \cap Q_S \setminus Q_{(1-6\delta)S}} \sum_{j \in \mathcal{L}} c_{i,j} |i - j| \\
&= \frac{S^d}{(kT)^d} E(u_k, Q_{kT}) + C \sum_{i \in \mathcal{L} \cap Q_{(1-6\delta)S}} \sum_{j \in \mathcal{L} \setminus Q_{(1-3\delta)S}} c_{i,j} |i - j| \\
&\quad + C \sum_{i \in \mathcal{L} \cap Q_S \setminus Q_{(1-6\delta)S}} \sum_{j \in \mathcal{L}} c_{i,j} |i - j|.
\end{aligned} \tag{30}$$

We show that

$$\sum_{i \in \mathcal{L} \cap Q_{(1-6\delta)S}} \sum_{j \in \mathcal{L} \setminus Q_{(1-3\delta)S}} c_{i,j} |i - j| \leq C_S^\delta S^d, \tag{31}$$

and

$$\sum_{i \in \mathcal{L} \cap Q_S \setminus Q_{(1-6\delta)S}} \sum_{j \in \mathcal{L}} c_{i,j} |i - j| \leq C\delta S^d, \tag{32}$$

where  $C_S^\delta \rightarrow 0$  as  $S \rightarrow +\infty$  and  $C > 0$  is a universal constant. Note that, due to Lemma 3.4, since  $\varepsilon > 0$  is chosen arbitrary, from (31) and (32) we obtain the claim of Step 2 by using (27), (30), dividing by  $S^d$ ,  $k \rightarrow +\infty$  and then  $\delta \rightarrow 0$ .

We first prove (31). Note that, for  $S$  big enough, due to (H2) and Lemma 3.1(v), we have

$$\begin{aligned}
\sum_{i \in \mathcal{L} \cap Q_{(1-6\delta)S}} \sum_{j \in \mathcal{L} \setminus Q_{(1-3\delta)S}} c_{i,j} |i - j| &\leq \sum_{i \in \mathcal{L} \cap Q_{(1-6\delta)S}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \geq \delta S}} c_{i,j} |i - j| \\
&\leq \#(\mathcal{L} \cap Q_S) \max_{i \in \mathcal{L}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \geq \delta S}} c_{i,j} |i - j| \\
&\leq C_S^\delta S^d,
\end{aligned}$$

where  $C_S^\delta \rightarrow 0$  as  $S \rightarrow +\infty$ . Next, we show (32). Using (H2), and Lemma 3.1(v), we obtain

$$\begin{aligned}
\sum_{i \in \mathcal{L} \cap Q_S \setminus Q_{(1-6\delta)S}} \sum_{j \in \mathcal{L}} c_{i,j} |i - j| &\leq \sum_{i \in \mathcal{L} \cap Q_S \setminus Q_{(1-6\delta)S}} \sum_{j \in \mathcal{L}} c_{i,j} |i - j| \\
&\leq \#(\mathcal{L} \cap Q_S \setminus Q_{(1-6\delta)S}) \max_{i \in \mathcal{L}} \sum_{j \in \mathcal{L}} c_{i,j} |i - j| \\
&\leq C\delta S^d.
\end{aligned}$$

This is (32) and hence the claim of Step 2.  $\square$

Let  $\psi: \mathbb{R}^d \rightarrow [0, +\infty]$  be defined as the positively homogeneous function of degree one that for  $\nu \in \mathbb{S}^{d-1}$  is defined by

$$\psi(\nu) = \lim_{\delta \rightarrow 0} \lim_{S \rightarrow +\infty} \frac{1}{S^d} \inf \left\{ E(u, Q_S^\nu) : u: \mathcal{L} \rightarrow \mathbb{R}, u(i) = \langle \nu, i \rangle \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S}^\nu \right\}. \tag{33}$$

The function  $\psi$  differs from the function  $\phi$  in the domain where one calculates the energy. For the function  $\phi$  we take the coordinate cube  $Q_T$  whereas for  $\nu$  we take the cube  $Q_T^\nu$ .

**Remark 3.7.** The existence of the limits in (12) and (33) can be deduced from standard subadditivity arguments, see e.g. [3, Proposition 4.2].

**Lemma 3.8.**  $\psi: \mathbb{R}^d \rightarrow [0, +\infty]$  satisfies the following properties:

- (i) There exists  $C > 0$  such that  $\psi(\nu) \leq C|\nu|$  for all  $\nu \in \mathbb{R}^d$ ,
- (ii)  $\psi$  is a continuous function.

*Proof.* We divide the proof into two steps. We first prove (i) and then (ii). Throughout the proofs let  $1 \ll S$ .

**Step 1.**(Proof of (i)) Let  $\nu \in \mathbb{S}^{d-1}$ ; it suffices to prove

$$\psi(\nu) \leq C.$$

The general case then follows by one-homogeneity. In order to prove (i) we insert  $u(i) = \langle \nu, i \rangle$  for all  $i \in \mathcal{L}$  as a competitor in the cell formula. Using Lemma 3.1(i), we then have

$$E(u, Q_S^\nu) = E(\langle \nu, \cdot \rangle, Q_S^\nu) \leq C|\nu| |(Q_S)_c| \leq CS^d.$$

Dividing by  $S^d$  and letting  $S \rightarrow +\infty$  yields the claim.

**Step 2.**(Proof of (ii)) Due to the one-homogeneity, it suffices to consider the case where  $\nu_1, \nu_2 \in \mathbb{S}^{d-1}$ . Let  $\eta > 0$  and  $\nu_1, \nu_2 \in \mathbb{S}^{d-1}$  be such that  $|\nu_1 - \nu_2| \leq \eta$ . Our goal is to prove that there exists  $C > 0$  independent of  $\nu_1$  and  $\nu_2$  such that

$$|\psi(\nu_1) - \psi(\nu_2)| \leq C\eta. \quad (34)$$

We only prove

$$\psi(\nu_1) - \psi(\nu_2) \leq C\eta, \quad (35)$$

since then (34) follows by exchanging  $\nu_1$  and  $\nu_2$  in (35). To this end let  $\delta > 0$  small enough,  $S > 0$  big enough,  $u_1: \mathcal{L} \rightarrow \mathbb{R}$  be such that  $u_1(i) = \langle \nu_1, i \rangle$  on  $\mathcal{L} \setminus Q_{(1-\delta)S}^{\nu_1}$  and

$$\frac{1}{S^d} E(u_1, Q_S^{\nu_1}) \leq \phi(\nu_1) + \eta. \quad (36)$$

We assume that

$$\|u_1\|_{L^\infty(Q_S^{\nu_1})} \leq S. \quad (37)$$

If this were not the case, we consider

$$\tilde{u}_1(i) = \begin{cases} (u_1(i) \wedge S) \vee (-S) & i \in Q_{2S}^{\nu_1}, \\ u_1(i) & \text{otherwise.} \end{cases}$$

Note that for  $i, j \in Q_{2S}^{\nu_1}$ , due to truncation,  $(\tilde{u}_1(i) - \tilde{u}_1(j))^+ \leq (u_1(i) - u_1(j))^+$ , whereas in general there holds  $|\tilde{u}_1(i) - \tilde{u}_1(j)| \leq CS + |i - j|$ . From this, using Lemma 3.1(v) and (H2), we deduce

$$\begin{aligned} E(\tilde{u}_1, Q_S^{\nu_1}) &= \sum_{i \in \mathcal{L} \cap Q_S^{\nu_1}} \sum_{j \in \mathcal{L} \cap Q_{2S}^{\nu_1}} c_{i,j} (\tilde{u}_1(i) - \tilde{u}_1(j))^+ + \sum_{i \in \mathcal{L} \cap Q_S^{\nu_1}} \sum_{j \in \mathcal{L} \setminus Q_{2S}^{\nu_1}} c_{i,j} (\tilde{u}_1(i) - \tilde{u}_1(j))^+ \\ &\leq \sum_{i \in \mathcal{L} \cap Q_S^{\nu_1}} \sum_{j \in \mathcal{L} \cap Q_{2S}^{\nu_1}} c_{i,j} (u_1(i) - u_1(j))^+ + C \sum_{i \in \mathcal{L} \cap Q_S^{\nu_1}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \geq S/2}} c_{i,j} |i - j| \\ &\leq E(u_1, Q_S^{\nu_1}) + C \#(\mathcal{L} \cap Q_S^{\nu_1}) \max_{i \in \mathcal{L}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \geq S/2}} c_{i,j} |i - j| \leq E(u_1, Q_S^{\nu_1}) + CS^d, \end{aligned}$$

where  $C_S \rightarrow 0$  as  $S \rightarrow \infty$ . In particular  $C_S \leq \eta$  for  $S$  big enough. Hence, we can assume (37). There exists  $C > 0$  such that for  $\tilde{S} = (1 + C\eta)S$  there holds  $Q_{(1-\delta)\tilde{S}}^{\nu_2} \supset Q_{(1+\delta)S}^{\nu_1}$ . We now define  $u_2: \mathcal{L} \rightarrow \mathbb{R}$  by

$$u_2(i) = \langle \nu_2 - \nu_1, i \rangle + u_1(i).$$

First, note that  $u_2(i) = \langle \nu_2, i \rangle$  for all  $i \in \mathcal{L} \setminus Q_{(1-\delta)\tilde{S}}^{\nu_2}$  and therefore

$$\inf \left\{ E(u, Q_{\tilde{S}}^{\nu_2}) : u: \mathcal{L} \rightarrow \mathbb{R}, u(i) = \langle \nu_2, i \rangle \text{ on } \mathcal{L} \setminus Q_{(1-\delta)\tilde{S}}^{\nu_2} \right\} \leq E(u_2, Q_{\tilde{S}}^{\nu_2}). \quad (38)$$

We claim that

$$E(u_2, Q_{\tilde{S}}^{\nu_2}) \leq E(u_1, Q_S^{\nu_1}) + \frac{C_S^\delta}{\delta} S^d + C\eta S^d + C\delta S^d, \quad (39)$$

where  $C_S^\delta \rightarrow 0$  as  $S \rightarrow +\infty$ . We postpone the proof of (39) and show first how it implies (35). Dividing (39) by  $\tilde{S}^d$ , letting  $\tilde{S}$  (therefore also  $S$ ) tend to  $+\infty$ ,  $\delta \rightarrow 0$ , and using (38) as well as (36), we get

$$\phi(\nu_2) \leq \phi(\nu_1) + C\eta \leq \phi(\nu_1) + C\eta.$$

This is (35). We now prove (39). Due to Lemma 3.1(ii), there holds

$$E(u_2, Q_{\tilde{S}}^{\nu_2}) \leq E(u_1, Q_{\tilde{S}}^{\nu_2}) + E(\langle \nu_2 - \nu_1, \cdot \rangle, Q_{\tilde{S}}^{\nu_1}). \quad (40)$$

Now, due to Lemma 3.1(i) and the fact that  $\tilde{S} \leq 2S$ , there holds

$$E(\langle \nu_2 - \nu_1, \cdot \rangle, Q_{\tilde{S}}^{\nu_1}) \leq C|\nu_2 - \nu_1|S^d \leq C\eta S^d. \quad (41)$$

Next, we prove

$$E(u_1, Q_{\tilde{S}}^{\nu_2}) \leq E(u_1, Q_S^{\nu_1}) + C\delta S^d + \frac{C_S^\delta}{\delta} S^d, \quad (42)$$

where  $C_S^\delta \rightarrow 0$  as  $S \rightarrow +\infty$ . We use Lemma 3.1(iv), to obtain

$$E(u_1, Q_{\tilde{S}}^{\nu_2}) = E(u_1, Q_S^{\nu_1}) + E(u_1, Q_{\tilde{S}}^{\nu_2} \setminus Q_S^{\nu_1}).$$

In order to prove (42) it suffices to prove

$$E(u_1, Q_{\tilde{S}}^{\nu_2} \setminus Q_S^{\nu_1}) \leq C\eta S^d + \frac{C_S^\delta}{\delta} S^d, \quad (43)$$

where  $C_S^\delta \rightarrow 0$  as  $S \rightarrow +\infty$ . To see this we write

$$\begin{aligned} E(u_1, Q_{\tilde{S}}^{\nu_2} \setminus Q_S^{\nu_1}) &= \sum_{i \in \mathcal{L} \cap Q_{\tilde{S}}^{\nu_2} \setminus Q_S^{\nu_1}} \sum_{j \in \mathcal{L} \cap Q_{(1-\delta)S}^{\nu_1}} c_{i,j} (u_1(i) - u_1(j))^+ \\ &+ \sum_{i \in \mathcal{L} \cap Q_{\tilde{S}}^{\nu_2} \setminus Q_S^{\nu_1}} \sum_{j \in \mathcal{L} \setminus Q_{(1-\delta)S}^{\nu_1}} c_{i,j} (u_1(i) - u_1(j))^+. \end{aligned} \quad (44)$$

To estimate the first term, note that due to (37), we have  $|u_1(i) - u_1(j)| \leq CS + |i - j|$ , and therefore, up to changing  $C$ , using (H2), and Lemma 3.1(iv), we get

$$\begin{aligned} \sum_{i \in \mathcal{L} \cap Q_{\tilde{S}}^{\nu_2} \setminus Q_S^{\nu_1}} \sum_{j \in \mathcal{L} \cap Q_{(1-\delta)S}^{\nu_1}} c_{i,j} (u_1(i) - u_1(j))^+ &\leq \frac{C}{\delta} \sum_{i \in \mathcal{L} \cap Q_{\tilde{S}}^{\nu_2} \setminus Q_S^{\nu_1}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \geq \delta S/2}} c_{i,j} |i - j| \\ &\leq \frac{C}{\delta} \#(\mathcal{L} \cap Q_{\tilde{S}}^{\nu_2}) \max_{i \in \mathcal{L}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \geq \delta S/2}} c_{i,j} |i - j| \leq \frac{C_S^\delta}{\delta} S^d, \end{aligned} \quad (45)$$

where  $C_S^\delta \rightarrow 0$  as  $S \rightarrow +\infty$ . To estimate the first term, we use the fact that  $u_1(i) = \langle \nu_1, i \rangle$  on  $\mathcal{L} \setminus Q_{(1-\delta)S}^{\nu_1}$ , and Lemma 3.1 (i), to obtain

$$\sum_{i \in \mathcal{L} \cap Q_S^{\nu_2} \setminus Q_S^{\nu_1}} \sum_{j \in \mathcal{L} \setminus Q_{(1-\delta)S}^{\nu_1}} c_{i,j} (u_1(i) - u_1(j))^+ \leq E(\langle \nu_1, \cdot \rangle, Q_S^{\nu_2} \setminus Q_S^{\nu_1}) \leq C|\nu_1| |(Q_S^{\nu_2} \setminus Q_S^{\nu_1})_c| \leq C\eta S^d.$$

This together with (44) and (45) implies (43) which in turn, together with (40) and (41) implies (39) and therefore the conclusion of Step 2.  $\square$

**Lemma 3.9.**  $\phi: \mathbb{R}^d \rightarrow [0, +\infty]$  satisfies the following properties:

- (i) There exists  $C > 0$  such that  $\phi(\nu) \leq C|\nu|$  for all  $\nu \in \mathbb{R}^d$ ,
- (ii)  $\phi$  is a positively homogeneous function of degree one,
- (iii)  $\phi$  is a convex function. In particular,  $\phi$  is Lipschitz continuous.

*Proof.* We divide the proof into two steps. Throughout the proofs let  $1 \ll S$ .

**Step 1.**(Proof of (i) and (ii)) In order to prove (i) we insert  $u(i) = \langle \nu, i \rangle$  for all  $i \in \mathcal{L}$  as a competitor in the cell formula. Using Lemma 3.1(i), we then have

$$E(u, Q_S^\nu) = E(\langle \nu, \cdot \rangle, Q_S^\nu) \leq C|\nu| |(Q_S)_c| \leq CS^d.$$

Dividing by  $S^d$  and letting  $S \rightarrow +\infty$  yields the claim. (ii) follows by using Lemma (3.1)(ii) to obtain  $E(\lambda u, Q_S) = \lambda E(u, Q_S)$  for all  $\lambda > 0$  and by noting that, given  $\nu \in \mathbb{R}^d$ , if  $u: \mathcal{L} \rightarrow \mathbb{R}$  satisfies  $u(i) = \langle \nu, i \rangle$  on  $\mathcal{L} \setminus Q_{(1-\delta)S}$ , then  $\lambda u(i) = \langle \lambda \nu, i \rangle$  on  $\mathcal{L} \setminus Q_{(1-\delta)S}$ . Employing this in (12) it is easy to see that  $\phi$  is a positively homogeneous function of degree one.

**Step 2.**(Proof of (iii)) We show that for every  $S > 0$  and  $\delta > 0$ ,  $S\delta \gg 1$  the function  $\phi_S^\delta: \mathbb{R}^d \rightarrow [0, +\infty]$  given by

$$\phi_S^\delta(\nu) := \frac{1}{S^d} \inf \{ E(u, Q_S) : u: \mathcal{L} \rightarrow \mathbb{R}, u(i) = \langle \nu, i \rangle \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S} \} \quad (46)$$

is a convex function. Note that

$$\phi(\nu) = \lim_{\delta \rightarrow 0} \lim_{S \rightarrow +\infty} \phi_S^\delta(\nu) \text{ for all } \nu \in \mathbb{R}^d.$$

Thus, the convexity for  $\phi_S^\delta$  also implies the convexity of  $\phi$ . This together with (i) and (ii) implies that  $\phi$  is also Lipschitz continuous. Now we prove that  $\phi_S^\delta$  is a convex function. Given  $\lambda \in [0, 1]$ ,  $\nu_1, \nu_2 \in \mathbb{R}^d$ , let  $\varepsilon > 0$   $u_1: \mathcal{L} \rightarrow \mathbb{R}$ ,  $u_2: \mathcal{L} \rightarrow \mathbb{R}$  be such that  $u_k(i) = \langle \nu_k, i \rangle$  on  $\mathcal{L} \setminus Q_{(1-\delta)S}$  and

$$E(u_k, Q_S) \leq \phi_S^\delta(\nu_k) + \frac{1}{2}\varepsilon \text{ for } k = 1, 2.$$

We have that  $u(i) := \lambda u_1(i) + (1 - \lambda)u_2$  is admissible for  $\phi_S^\delta(\lambda \nu_1 + (1 - \lambda)\nu_2)$  and by Lemma 3.1(ii) we obtain

$$\begin{aligned} \phi_S^\delta(\lambda \nu_1 + (1 - \lambda)\nu_2) &\leq E(u, Q_S) = E(\lambda u_1 + (1 - \lambda)u_2, Q_S) \leq \lambda E(u_1, Q_S) + (1 - \lambda)E(u_2, Q_S) \\ &\leq \lambda \phi_S^\delta(\nu_1) + (1 - \lambda)\phi_S^\delta(\nu_2) + \varepsilon. \end{aligned}$$

Since  $\varepsilon > 0$  is arbitrary. This yields the claim.  $\square$

The next Lemma shows that the asymptotic cell-formula describing the surface energy density is equal to the asymptotic cell-formula with affine boundary conditions. In Lemma 3.11 and Lemma 3.12 we use the following remark.

**Remark 3.10.** We point out that  $\mathbb{S}^{d-1} \cap \mathbb{Q}^d$  is dense in  $\mathbb{S}^{d-1}$ . This follows from the fact that  $\mathbb{Q}^{d-1}$  is dense in  $\mathbb{R}^{d-1}$  and that the inverse of the stereographic projection  $P_d: \mathbb{R}^{d-1} \rightarrow \mathbb{S}^{d-1} \setminus \{e_d\}$  is a rational and continuous function.

**Lemma 3.11.** *Let  $\nu \in \mathbb{R}^d$ . Then:  $\psi(\nu) = \varphi(\nu)$ .*

*Proof.* Due the fact that both  $\psi$  and  $\varphi$  are positively homogeneous functions of degree one, it suffices to consider the case where  $\nu \in \mathbb{S}^{d-1}$ . Furthermore, since both functions are continuous, see Lemma 3.8(ii) and Remark 2.3, it suffices to prove the claim for  $\nu \in \mathbb{S}^{d-1} \cap \mathbb{Q}^d$ . For each such vector we can find  $\nu_1, \dots, \nu_{d-1} \in \mathbb{S}^{d-1} \cap \mathbb{Q}^d$  such that the set  $\{\nu_1, \dots, \nu_{d-1}, \nu\}$  forms an orthonormal basis of  $\mathbb{R}^d$ . Then, there exists  $\lambda \in \mathbb{N}$  such that

$$\lambda \nu_n = z_n \text{ for some } z_n \in \mathbb{Z}^d \text{ for all } n \in \{1, \dots, d\}. \quad (47)$$

For  $t \in (-1/2, 1/2)$  we define an auxiliary function  $\varphi_t: \mathbb{R}^d \rightarrow [0, +\infty)$  and for  $\nu \in \mathbb{S}^{d-1}$  given by

$$\varphi_t(\nu) := \lim_{\delta \rightarrow 0} \lim_{S \rightarrow +\infty} \frac{1}{S^{d-1}} \inf \{E(u, Q_S^\nu) : u: \mathcal{L} \rightarrow \{0, 1\}, u(i) = \chi_{\{\langle \nu, i \rangle > tS\}} \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S}^\nu\}. \quad (48)$$

Note that  $\varphi_0(\nu) = \varphi(\nu)$ .

**Step 1.** ( $\varphi_t \geq \psi$  for all  $t$ ) We show that for all  $t \in (-1/2, 1/2)$  we have

$$\varphi_t(\nu) \geq \psi(\nu). \quad (49)$$

To this end let  $\{\nu_1, \dots, \nu_{d-1}, \nu_d = \nu\} \subset \mathbb{S}^{d-1} \cap \mathbb{Q}^d$  be an orthonormal basis as previously described and let  $1 \ll S_1 \ll S_2$ . We assume that  $S_1 = \lambda T$ , where  $\lambda$  satisfies (47) and  $T$  is given by (H1). Note that if  $\lambda$  satisfies (47), also  $k\lambda$  satisfies (47) and therefore we can find a sequence  $S_k = k\lambda T$  such that  $S_k \rightarrow +\infty$  of the desired form. The existence of the limit in definition (7) of  $\varphi$  permits us to assume that  $S$  is of the specific form. Let  $t \in (-1/2, 1/2)$ ,  $\delta > 0$  and  $u_1: \mathcal{L} \rightarrow \{0, 1\}$  be such that  $u_1(i) = \chi_{\{\langle \nu, i \rangle > tS\}}$  on  $\mathcal{L} \setminus Q_{(1-\delta)S_1}^\nu$  and

$$E(u_1, Q_{S_1}^\nu) \leq \inf \{E(u, Q_{S_1}^\nu) : u: \mathcal{L} \rightarrow \{0, 1\}, u(i) = \chi_{\{\langle \nu, i \rangle > tS\}} \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S_1}^\nu\} + 1. \quad (50)$$

Due to the assumption on  $S_1$  and Lemma 3.1(vi), we have

$$E(u_1(\cdot - z), Q_{S_1}^\nu(z)) = E(u_1, Q_{S_1}^\nu) \text{ for all } z = \lambda T \sum_{n=1}^d k_n \nu_n, k \in \mathcal{L}. \quad (51)$$

Set (omitting the dependence on  $S_1$  and  $S_2$ )

$$\mathcal{Z} := \left\{ z = S_1 \sum_{n=1}^d k_n \nu_n : k \in \mathbb{Z}^d, Q_{S_1}^\nu(z) \subset Q_{(1-\delta)S_2}^\nu \right\}.$$

We define  $u_2: \mathcal{L} \rightarrow \mathbb{R}$  by

$$u_2(i) = \begin{cases} S_1 \left( u_1(i - z) - \frac{1}{2} \right) + \langle \nu, z \rangle & \text{if } z \in \mathcal{Z}, i \in Q_{S_1}^\nu(z), \\ \langle \nu, i \rangle & \text{otherwise.} \end{cases} \quad (52)$$

We claim that

$$|u_2(i) - u_2(j)| \leq C(S_1 + |i - j|). \quad (53)$$

We postpone the proof of (53) to the end of Step 1. By the definition of  $u_2$ , it is clear that

$$\inf \left\{ E(u, Q_{S_2}^\nu) : u: \mathcal{L} \rightarrow \mathbb{R}, u(i) = \langle \nu, i \rangle \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S_2}^\nu \right\} \leq E(u_2, Q_{S_2}^\nu). \quad (54)$$

It remains to show that

$$E(u_2, Q_{S_2}^\nu) \leq \frac{S_2^d}{S_1^{d-1}} E(u_1, Q_{S_1}^\nu) + C\delta S_2^d + CS_1^2 S_2^{d-1} + \frac{S_2^d}{\delta} C_{S_1}^\delta, \quad (55)$$

where  $C_{S_1}^\delta \rightarrow 0$  as  $S_1 \rightarrow +\infty$ . In fact, once we have shown (55), Step 1 follows from (54) and (50) by dividing by  $S_2^d$  and letting first  $S_2 \rightarrow +\infty$ , then  $S_1 \rightarrow +\infty$ , and finally  $\delta \rightarrow 0$ . We are left to prove (55). In order to prove (55) we introduce

$$\begin{aligned} r_1 &= r_1(S_1, S_2, \delta) = (1 - \delta)S_2 - 3\sqrt{d}S_1, \\ r_2 &= r_2(S_1, S_2, \delta) = (1 - \delta)S_2 + 3\sqrt{d}S_1. \end{aligned} \quad (56)$$

We use Lemma 3.1(iv) to obtain

$$E(u_2, Q_{S_2}^\nu) = E(u_2, Q_{r_1}^\nu) + E(u_2, Q_{r_2}^\nu \setminus Q_{r_1}^\nu) + E(u_2, Q_{S_2}^\nu \setminus Q_{r_2}^\nu) \quad (57)$$

and we estimate the three terms on the right hand side separately. We claim that

$$E(u_2, Q_{r_1}^\nu) \leq \frac{S_2^d}{S_1^{d-1}} E(u_1, Q_{S_1}^\nu) + \frac{S_2^d}{\delta} C_{S_1}^\delta, \quad (58)$$

where  $C_{S_1}^\delta \rightarrow 0$  as  $S_1 \rightarrow +\infty$ . Indeed, if  $i \in Q_{S_1}^\nu(z)$  such that  $z = S_1 \sum_{n=1}^d k_n \nu_n \in \mathcal{Z}$  and  $Q_{S_1}^\nu(z) \cap Q_{r_1}^\nu \neq \emptyset$  then

$$u_2(i) = S_1 \left( u_1(i - z) - \frac{1}{2} \right) + \langle \nu, z \rangle \text{ for all } i \in Q_{(1+\delta)S_1}^\nu(z). \quad (59)$$

Due to (52), this is clearly true for  $i \in Q_{S_1}^\nu(z)$ , while if  $i \in Q_{(1+\delta)S_1}^\nu(z) \setminus Q_{S_1}^\nu(z)$  there exists  $z' = S_1 \sum_{n=1}^d k_n \nu_n \in \mathcal{Z}$ ,  $k \in \mathbb{Z}^d$  such that  $\|k - k'\|_\infty = 1$ ,  $Q_{S_1}^\nu(z') \subset Q_{r_1}^\nu$ , and  $i \in Q_{S_1}^\nu(z') \setminus Q_{(1-\delta)S_1}^\nu(z')$ . Then, due to the boundary conditions of  $u_1$ , we have

$$\begin{aligned} u_2(i) &= S_1 \left( u_1(i - z') - \frac{1}{2} \right) + \langle \nu, z' \rangle = S_1 \left( \chi_{\{\langle \nu, i - z' \rangle > tS\}} - \frac{1}{2} \right) + \langle \nu, z \rangle + \langle \nu, z' - z \rangle \\ &= S_1 \left( \chi_{\{\langle \nu, i - z \rangle > tS\}} - \frac{1}{2} \right) + \langle \nu, z \rangle = S_1 \left( u_1(i - z) - \frac{1}{2} \right) + \langle \nu, z \rangle. \end{aligned}$$

Here, the third equality follows, from the fact that  $\|k - k'\|_\infty = 1$  and therefore  $\langle \nu, z' - z \rangle \in \{-S_1, 0, S_1\}$ . To obtain the previous equality, we distinguish the following two cases:

$$\begin{aligned} \langle \nu, z' - z \rangle &= \pm S_1 \implies \chi_{\{\langle \nu, i - z' \rangle > tS\}} - \chi_{\{\langle \nu, i - z \rangle > tS\}} = \mp 1 \text{ and} \\ \langle \nu, z' - z \rangle &= 0 \implies \chi_{\{\langle \nu, i - z' \rangle > tS\}} = \chi_{\{\langle \nu, i - z \rangle > tS\}}. \end{aligned}$$

Now (59) together with (51) implies for  $z \in \mathcal{Z}$  such that  $Q_{S_1}^\nu(z) \cap Q_{r_1}^\nu \neq \emptyset$

$$\begin{aligned} E(u_2, Q_{S_1}^\nu(z)) &= \sum_{i \in \mathcal{L} \cap Q_{S_1}^\nu(z)} \sum_{j \in \mathcal{L} \cap Q_{(1+\delta)S_1}^\nu(z)} c_{i,j} (u_2(i) - u_2(j))^+ \\ &\quad + \sum_{i \in \mathcal{L} \cap Q_{S_1}^\nu(z)} \sum_{j \in \mathcal{L} \setminus Q_{(1+\delta)S_1}^\nu(z)} c_{i,j} (u_2(i) - u_2(j))^+ \\ &\leq S_1 E(u_1, Q_{S_1}^\nu(z)) + \sum_{i \in \mathcal{L} \cap Q_{S_1}^\nu(z)} \sum_{j \in \mathcal{L} \setminus Q_{(1+\delta)S_1}^\nu(z)} c_{i,j} |u_2(i) - u_2(j)| \\ &= S_1 E(u_1, Q_{S_1}^\nu(z)) + \sum_{i \in \mathcal{L} \cap Q_{S_1}^\nu(z)} \sum_{j \in \mathcal{L} \setminus Q_{(1+\delta)S_1}^\nu(z)} c_{i,j} |u_2(i) - u_2(j)|. \end{aligned} \quad (60)$$

We estimate the second term on the right hand side of (60) to obtain (58). Note for  $i \in Q_{S_1}^\nu(z)$ ,  $j \in \mathcal{L} \setminus Q_{(1+\delta)S_1}^\nu$  we have  $|i - j| \geq \delta S_1/2$  and thus, due to (53), we obtain

$$|u_2(i) - u_2(j)| \leq \frac{C}{\delta} |i - j| \text{ for all } i \in Q_{S_1}^\nu(z), j \in \mathcal{L} \setminus Q_{(1+\delta)S_1}^\nu(z) \quad (61)$$

for some  $C > 0$  independent of  $S_1$ ,  $S_2$  and  $\delta$ . Now, we get

$$\begin{aligned} \sum_{i \in Q_{S_1}^\nu(z)} \sum_{j \in \mathcal{L} \setminus Q_{(1+\delta)S_1}^\nu(z)} c_{i,j} |u_2(i) - u_2(j)| &\leq \frac{C}{\delta} \sum_{i \in \mathcal{L} \cap Q_{S_1}^\nu(z)} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \geq \delta S_1/2}} c_{i,j} |i-j| \\ &\leq \#(\mathcal{L} \cap Q_{S_1}^\nu(z)) \max_{i \in \mathcal{L}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \geq \delta S_1/2}} c_{i,j} |i-j| \leq \frac{C_{S_1}^\delta}{\delta} S_1^d, \end{aligned}$$

where  $C_{S_1}^\delta \rightarrow 0$  as  $S_1 \rightarrow +\infty$ . Hence, noting that for  $z, z' \in \mathcal{Z}$  such that  $z \neq z'$ , we have  $Q_{S_1}^\nu(z) \cap Q_{S_1}^\nu(z') = \emptyset$  and therefore  $\#\mathcal{Z} \leq S_2^d/S_1^d$ , we get

$$\begin{aligned} E(u_2, Q_{r_1}^\nu) &\leq \sum_{\substack{z \in \mathcal{Z} \\ Q_{S_1}^\nu(z) \cap Q_{r_1}^\nu \neq \emptyset}} E(u_2, Q_{S_1}^\nu(z)) \leq \#\mathcal{Z} (S_1 E(u_1, Q_{S_1}^\nu) + \frac{C_{S_1}^\delta}{\delta} S_1^d) \\ &\leq \frac{S_2^d}{S_1^{d-1}} E(u_1, Q_{S_1}^\nu) + \frac{C_{S_1}^\delta}{\delta} S_2^d, \end{aligned}$$

where  $C_{S_1}^\delta \rightarrow 0$  as  $S_1 \rightarrow +\infty$ . This is (58). Next, we prove

$$E(u_2, Q_{r_2}^\nu \setminus Q_{r_1}^\nu) \leq C S_1^2 S_2^{d-1}. \quad (62)$$

We use (53) to obtain

$$E(u_2, Q_{r_2}^\nu \setminus Q_{r_1}^\nu) = \sum_{i \in Q_{r_2}^\nu \setminus Q_{r_1}^\nu} \sum_{j \in \mathcal{L}} c_{i,j} (u_2(i) - u_2(j))^+ \leq C S_1 \#(\mathcal{L} \cap Q_{r_2}^\nu \setminus Q_{r_1}^\nu) \max_{i \in \mathcal{L}} \sum_{j \in \mathcal{L}} c_{i,j} |i-j|,$$

where we used that, owing to (L1), we have  $|i-j| \geq c$  if  $i \neq j$ . Using Lemma 3.1(v), (H2), and (56) we have that  $\#(\mathcal{L} \cap Q_{r_2}^\nu \setminus Q_{r_1}^\nu) \leq C S_1 S_2^{d-1}$  we obtain (62). As for the third term on the right hand side we prove

$$E(u_2, Q_{S_2}^\nu \setminus Q_{r_2}^\nu) \leq C \delta S_2^d + \frac{C_{S_1}^\delta}{\delta} S_2^d, \quad (63)$$

where  $C_{S_1}^\delta \rightarrow 0$  as  $S_1 \rightarrow +\infty$ . To this end we split the summation over  $j$  to obtain

$$\begin{aligned} E(u_2, Q_{S_2}^\nu \setminus Q_{r_2}^\nu) &= \sum_{i \in \mathcal{L} \cap Q_{S_2}^\nu \setminus Q_{r_2}^\nu} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \leq S_1 \delta/2}} c_{i,j} (u_2(i) - u_2(j))^+ \\ &\quad + \sum_{i \in \mathcal{L} \cap Q_{S_2}^\nu \setminus Q_{r_2}^\nu} \sum_{\substack{j \in \mathcal{L} \\ |i-j| > S_1 \delta/2}} c_{i,j} (u_2(i) - u_2(j))^+. \end{aligned} \quad (64)$$

Let us note first that  $Q_{S_2}^\nu \setminus Q_{r_2}^\nu \subset Q_{S_2}^\nu \setminus Q_{(1-2\delta)S_2}^\nu$  and therefore, due to 3.1(v), we have

$$\#(\mathcal{L} \cap Q_{S_2}^\nu \setminus Q_{r_2}^\nu) \leq C \delta S_2^d. \quad (65)$$

Now, for the first term on the right hand side of (64), employing (52), we note that  $u_2(i) = \langle \nu, i \rangle$  and  $u_2(j) = \langle \nu, j \rangle$ . Hence,

$$\sum_{i \in \mathcal{L} \cap Q_{S_2}^\nu \setminus Q_{r_2}^\nu} \sum_{\substack{j \in \mathcal{L} \\ |i-j| \leq S_1 \delta/2}} c_{i,j} |u_2(i) - u_2(j)| \leq \#(\mathcal{L} \cap Q_{S_2}^\nu \setminus Q_{r_2}^\nu) \max_{i \in \mathcal{L}} \sum_{j \in \mathcal{L}} c_{i,j} |i-j| \leq C \delta S_2^d, \quad (66)$$

where we used (H2) and (65). For the second term on the right hand side of (64), we use (53) and (65) to obtain

$$\begin{aligned} \sum_{i \in \mathcal{L} \cap Q_{S_2}^\nu \setminus Q_{r_2}^\nu} \sum_{\substack{j \in \mathcal{L} \\ |i-j| > S_1 \delta / 2}} c_{i,j} |u_2(i) - u_2(j)| &\leq \frac{C}{\delta} \sum_{i \in \mathcal{L} \cap Q_{S_2}^\nu \setminus Q_{r_2}^\nu} \sum_{\substack{j \in \mathcal{L} \\ |i-j| > S_1 \delta / 2}} c_{i,j} |i - j| \\ &\leq \frac{C}{\delta} \#(\mathcal{L} \cap Q_{S_2}^\nu \setminus Q_{r_2}^\nu) \max_{i \in \mathcal{L}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| > S_1 \delta / 2}} c_{i,j} |i - j| \quad (67) \\ &\leq C S_2^d C_{S_1}^\delta, \end{aligned}$$

where  $C_{S_1}^\delta \rightarrow 0$  as  $S_1 \rightarrow +\infty$ . Inequality (62) follows from (64), (66), and (67). Now (57), (58), (62), (63) give (55). To conclude Step 1, it remains to prove (53). There are four cases to consider:

- (a)  $i = i_0 + z, j = j_0 + z, i_0 \in Q_{S_1}^\nu, j_0 \in Q_{S_1}^\nu, z \in \mathcal{Z}$ ;
- (b)  $i = i_0 + z, j = j_0 + z', i_0 \in Q_{S_1}^\nu(z), j_0 \in Q_{S_1}^\nu(z'), z, z' \in \mathcal{Z}$ ;
- (c)  $i = i_0 + z, i_0 \in Q_{S_1}^\nu(z), z \in \mathcal{Z}, j_0 \notin Q_{S_1}^\nu(z')$  for any  $z' \in \mathcal{Z}$ ;
- (d)  $i \notin Q_{S_1}^\nu(z)$  for any  $z \in \mathcal{Z}$  and  $j \notin Q_{S_1}^\nu(z')$  for any  $z' \in \mathcal{Z}$ .

*Case (a):* This case follows since  $\|u_1\|_{L^\infty(Q_{S_1}^\nu)} \leq 1$ .

*Case (b):* Note that in the case where  $i = i_0 + z, j = j_0 + z'$  for some  $i_0, j_0 \in Q_{S_1}^\nu$  and for some  $z, z' \in \mathcal{Z}$ , we have

$$|u_2(i) - u_2(j)| \leq |\langle \nu, z - z' \rangle| + C S_1 \leq |\langle \nu, z + i_0 - z' - j_0 \rangle| + |i_0 - j_0| + C S_1 \leq |i - j| + C S_1$$

and therefore (53) holds true.

*Case (c):* Note that in the case where  $i = i_0 + z, i_0 \in Q_{S_1}^\nu, z \in \mathcal{Z}$  and  $j \notin Q_{S_1}^\nu(z)$  for any  $z \in \mathcal{Z}$ , we have

$$|u_2(i) - u_2(j)| \leq C S_1 + |\langle \nu, z - j \rangle| \leq C S_1 + |\langle \nu, i - j \rangle| + |i_0| \leq C S_1 + |i - j|.$$

Also here (53) holds true.

*Case (d):* In this case  $u_2(i) = \langle \nu, i \rangle$  and  $u_2(j) = \langle \nu, j \rangle$  and therefore (53) holds true. This shows (53) in general.

**Step 2.** ( $\varphi_t = \psi$  for almost all  $t$ ) Given  $\delta > 0, t \in (-1/2, 1/2)$  and  $S \gg 1$  and set

$$\varphi(t, S, \delta) := \frac{1}{S^{d-1}} \inf \{ E(u, Q_S^\nu) : u : \mathcal{L} \rightarrow \{0, 1\}, u(i) = \chi_{\{\langle \nu, i \rangle > tS\}} \text{ on } \mathcal{L} \setminus Q_{(1-\delta)S}^\nu \}. \quad (68)$$

Then, for  $\varepsilon > 0$  we find  $u_1 : \mathcal{L} \rightarrow \mathbb{R}$  be such that  $u_1(i) = \langle \nu, i \rangle$  for  $i \in \mathcal{L} \setminus Q_{(1-\delta)S}^\nu$  and

$$E(u_1, Q_S^\nu) \leq S^d (\psi(\nu) + \varepsilon). \quad (69)$$

Due to Lemma 3.2, there holds

$$\begin{aligned} E(u_1, Q_S^\nu) &= \int_{-\infty}^{+\infty} E(\chi_{\{u_1 > t\}}, Q_S^\nu) dt \geq \int_{-S(1-\delta)/2}^{S(1-\delta)/2} E(\chi_{\{u_1 > t\}}, Q_S^\nu) dt \\ &= S \int_{-(1-\delta)/2}^{(1-\delta)/2} E(\chi_{\{u_1 > tS\}}, Q_S^\nu) dt. \end{aligned} \quad (70)$$

Note that for all  $t \in (-(1-\delta)/2, (1-\delta)/2)$ , due to  $u_1(i) = \langle \nu, i \rangle$  we have that  $\chi_{\{u_1 > tS\}}(i) = \chi_{\{\langle \nu, i \rangle > tS\}}$  for  $i \in \mathcal{L} \setminus Q_S^\nu$  and thus

$$E(\chi_{\{u_1 > tS\}}, Q_S^\nu) \geq S^{d-1} \varphi(t, S, \delta). \quad (71)$$

Therefore,

$$\varepsilon + \psi(\nu) \geq \int_{-(1-\delta)/2}^{(1-\delta)/2} \varphi(t, S, \delta) dt$$

and then thanks to Fatou's lemma we deduce for  $\delta_0 > 0$

$$\varepsilon + \psi(\nu) \geq \int_{-(1-\delta_0)/2}^{(1-\delta_0)/2} \varphi_t(\nu) dt.$$

After letting  $\delta_0 \rightarrow 0$  and  $\varepsilon \rightarrow 0$ , using Step 1, we obtain

$$\int_{-1/2}^{1/2} \varphi_t(\nu) dt \leq \psi(\nu) \leq \varphi_t(\nu) \text{ for all } t \in (-1/2, 1/2).$$

Hence,  $\varphi_t(\nu) = \psi(\nu)$  for almost all  $t \in (-1/2, 1/2)$ . This concludes Step 2.

**Step 3.** ( $t \mapsto \varphi_t(\nu)$  is constant) To this end, let  $t_1, t_2 \in (-1/2, 1/2)$ ,  $t_2 < t_1$ , let  $\varepsilon > 0, \delta > 0$ ,  $S \gg 1$ , and let  $u_1: \mathcal{L} \rightarrow \{0, 1\}$  be such that  $u_1(i) = \chi_{\{\langle \nu, i \rangle > t_1 S\}}$  on  $\mathcal{L} \setminus Q_S^\nu$  and

$$E(u_1, Q_S^\nu) \leq S^{d-1} (\varphi_{t_1}(\nu) + \varepsilon). \quad (72)$$

We set  $u_\nu^s(i) = \chi_{\{\langle \nu, i \rangle > s\}}$  and define  $u_2: \mathcal{L} \rightarrow \{0, 1\}$  by

$$u_2(i) = u_1(i) + (u_\nu^{t_2 S}(i) - u_\nu^{t_1 S}(i)) \chi_{(Q_{(1-\delta)S}^\nu)^c}(i). \quad (73)$$

It is obvious that  $u_2(i) = u_\nu^{t_2 S}(i)$  on  $\mathcal{L} \setminus Q_{(1-\delta)S}^\nu$  and therefore

$$E(u_2, Q_S^\nu) \geq S^{d-1} \varphi(t_2, S, \delta). \quad (74)$$

Next, we show that

$$E(u_2, Q_S^\nu) \leq E(u_1, Q_S^\nu) + C(\delta + |t_1 - t_2|) S^{d-1} + \frac{C_S^\delta}{\delta} S^{d-1}, \quad (75)$$

where  $C_S^\delta \rightarrow 0$  as  $S \rightarrow +\infty$ . Now the claim follows by (72), (74), and (75) by dividing with  $S^{d-1}$  and letting first  $S \rightarrow +\infty$ , then  $\delta \rightarrow 0$ , and eventually  $\varepsilon \rightarrow 0$ . It remains to prove (75). Here, we exploit Lemma 3.1(ii) and (73) to deduce

$$E(u_2, Q_S^\nu) \leq E(u_1, Q_S^\nu) + E((u_\nu^{t_2 S} - u_\nu^{t_1 S}) \chi_{(Q_{(1-\delta)S}^\nu)^c}, Q_S^\nu). \quad (76)$$

We call  $v = (u_\nu^{t_2 S} - u_\nu^{t_1 S}) \chi_{(Q_{(1-\delta)S}^\nu)^c}$  and note that it suffices to prove

$$E(v, Q_S^\nu) \leq C(\delta + |t_1 - t_2|) S^{d-1} + \frac{C_S^\delta}{\delta} S^{d-1}. \quad (77)$$

We observe that

$$\{v = -1\} = \mathcal{L} \cap B_{t_1, t_2}^S, \text{ where } B_{t_1, t_2}^S := \{x \in \mathbb{R}^d \setminus Q_{(1-\delta)S}^\nu : t_1 S \leq \langle \nu, x \rangle < t_2 S\} \quad (78)$$

and  $\{v = 0\} = \mathcal{L} \setminus B_{t_1, t_2}^S$ . Therefore

$$E(v, Q_S^\nu) = \sum_{i \in \{v=0\} \cap Q_S^\nu} \sum_{j \in \{v=-1\}} c_{i,j} \leq \sum_{i \in \mathcal{L} \cap Q_S^\nu} \sum_{\substack{j \in \mathcal{L} \\ |i-j| > \delta S}} c_{i,j} + \sum_{i \in \{v=0\} \cap Q_S^\nu} \sum_{\substack{j \in \{v=-1\} \\ |i-j| \leq \delta S}} c_{i,j}. \quad (79)$$

As for the first term on the right hand side of (79), we point out that, due to Lemma 3.1(v), we have

$$\sum_{i \in \mathcal{L} \cap Q_S^\nu} \sum_{\substack{j \in \mathcal{L} \\ |i-j| > \delta S}} c_{i,j} \leq \frac{1}{\delta S} \#(\mathcal{L} \cap Q_S^\nu) \max_{i \in \mathcal{L}} \sum_{\substack{j \in \mathcal{L} \\ |i-j| > \delta S}} c_{i,j} |i-j| \leq \frac{C_S^\delta}{\delta} S^{d-1}, \quad (80)$$

where  $C_S^\delta \rightarrow 0$  as  $S \rightarrow +\infty$ . Now, let us consider  $j \in \mathcal{L}$  such that  $|i - j| < \delta S$ . For  $i_0, j_0 \in Q_T$  and  $\xi \in \mathbb{Z}^d$  set

$$A_{i_0, j_0}^\xi := \{(z, z') \in T\mathbb{Z}^d \times T\mathbb{Z}^d : z' - z = \xi, i = i_0 + z \in \{v = 0\} \cap Q_S^\nu, \\ j = j_0 + z' \in \{v = -1\}\}. \quad (81)$$

We observe that  $A_{i_0, j_0}^\xi \subset \{(z, z + \xi) : z \in T\mathbb{Z}^d, \text{dist}(z, \partial B_{t_1, t_2}^S \cap Q_S^\nu) \leq \sqrt{d}T + |\xi|\}$  and thus for  $|\xi| \leq C\delta S$

$$\#A_{i_0, j_0}^\xi \leq C(\delta + |t_1 - t_2|)(|\xi| + T)S^{d-1}.$$

Therefore, there holds

$$\begin{aligned} \sum_{i \in \{v=0\} \cap Q_S^\nu} \sum_{\substack{j \in \{v=-1\} \\ |i-j| \leq \delta S}} c_{i,j} &\leq \sum_{i_0, j_0 \in Q_T} \sum_{\substack{\xi \in T\mathbb{Z}^d \\ |\xi| \leq \delta S}} \sum_{(z, z') \in A_{i_0, j_0}^\xi} c_{i_0+z, j_0+z'} \\ &\leq C(\delta + |t_1 - t_2|)S^{d-1} \sum_{i_0, j_0 \in Q_T} \sum_{\substack{\xi \in T\mathbb{Z}^d \\ |\xi| \leq \delta S_1}} c_{i_0, j_0 + \xi} (|\xi| + T) \\ &\leq C(\delta + |t_1 - t_2|)S^{d-1} \max_{i \in \mathcal{L} \cap Q_T} \sum_{j \in \mathcal{L}} c_{i,j} |i - j| \leq C(\delta + |t_1 - t_2|)S^{d-1}. \end{aligned} \quad (82)$$

Here we used that  $|\xi| \leq |i - j| + |i_0 - j_0| \leq |i - j| + \sqrt{d}T$  and the fact that  $|i - j| \geq c = T \cdot (c/T)$  for all  $i \neq j$ , where  $c/T > 0$  is a fixed constant. This together with (79) and (80) implies (77) and therefore  $\varphi_{t_2}(\nu) \leq \varphi_{t_1}(\nu) + C|t_1 - t_2|$  for  $t_2 < t_1$ . Due to Step 2, for any  $t \in (-1/2, 1/2)$  we can find  $t_n \rightarrow t$  such that  $\varphi_{t_n}(\nu) = \psi(\nu)$  and  $\varphi_t(\nu) \leq \varphi_{t_n}(\nu) + |t - t_n| = \psi(\nu) + |t - t_n|$ . Letting  $t_n \rightarrow t$  we obtain  $\varphi_t(\nu) \leq \psi(\nu)$ . This together with Step 1 shows  $\varphi_t(\nu) = \psi(\nu)$ .

Due to Step 3 we have that  $\varphi(\nu) = \varphi_0(\nu) = \psi(\nu)$ . This concludes the proof.  $\square$

In the next Lemma we show that, assuming affine boundary conditions, the calculation of the asymptotic cell formula with respect to the coordinate cube and the calculation of the asymptotic cell formula with respect to the rotated cube are equivalent.

**Lemma 3.12.** *Let  $\nu \in \mathbb{R}^d$ . Then:  $\psi(\nu) = \phi(\nu)$ .*

*Proof.* After reducing to rational directions, we define a sequence of cell problems defined on cubes  $Q_{\lambda T}^\nu$  and exploit Lemma 3.6 to show that it suffices to compare the two sequences of cell problems with periodic boundary conditions.

Since both  $\psi$  and  $\phi$  are positively homogeneous functions of degree one (cf. (33) and Lemma 3.9(ii)) it suffices to consider the case where  $\nu \in \mathbb{S}^{d-1}$ . Thanks to Lemma 3.8(ii) and Lemma 3.9(iii) both functions are continuous. Thus it suffices to prove the claim for  $\nu \in \mathbb{S}^{d-1} \cap \mathbb{Q}^d$ . For each such vector we can find  $\{\nu_1, \dots, \nu_{d-1}\} \in \mathbb{S}^{d-1} \cap \mathbb{Q}^d$  such that the set  $\{\nu_1, \dots, \nu_{d-1}, \nu_d = \nu\}$  forms an orthonormal basis of  $\mathbb{R}^d$ . For such an orthonormal basis, it is clear that there exists  $\lambda \in \mathbb{N}$  such that

$$\lambda \nu_n = z_n \text{ for some } z_n \in \mathbb{Z}^d \text{ for all } n \in \{1, \dots, d\}. \quad (83)$$

Let  $\{\nu_1, \dots, \nu_{d-1}, \nu_d = \nu\} \subset \mathbb{S}^{d-1} \cap \mathbb{Q}^d$  be the orthonormal basis described previously. For  $\nu \in \mathbb{S}^{d-1} \cap \mathbb{Q}^d$  fixed, we set

$$\psi_{\text{per}}(\nu) := \lim_{k \rightarrow \infty} \frac{1}{(\lambda k T)^d} \inf \{E(u, Q_{\lambda k T}^\nu) : u : \mathcal{L} \rightarrow \mathbb{R},$$

$$u(\cdot + \lambda k T \nu_n) - \langle \nu, \cdot + \lambda k T \nu_n \rangle = u(\cdot) - \langle \nu, \cdot \rangle \text{ for all } n = 1, \dots, d\}$$
(84)

**Step 1.** ( $\psi_{\text{per}} = \psi$ ) We claim that

$$\psi_{\text{per}}(\nu) = \psi(\nu). \quad (85)$$

This follows exactly as the proof of Lemma 3.6 by replacing the coordinate cubes with the cubes  $Q_{\lambda k T}^\nu$ .

**Step 2.** ( $\psi_{\text{per}} = \phi_{\text{per}}$ ) Due to Step 1 and Lemma 3.6 it suffices to show that

$$\psi_{\text{per}}(\nu) = \phi_{\text{per}}(\nu). \quad (86)$$

**Step 2.1.** (Proof of ' $\geq$ ') To this end, let  $\varepsilon > 0$ ,  $k \in \mathbb{N}$  be big enough, and  $u_k : \mathcal{L} \rightarrow \mathbb{R}$  be such that  $u_k(\cdot + \lambda k T \nu_n) - \langle \nu, \cdot + \lambda k T \nu_n \rangle = u_k(\cdot) - \langle \nu, \cdot \rangle$  for all  $n = 1, \dots, d$  and

$$E(u_k, Q_{\lambda k T}^\nu) \leq (\lambda k T)^d (\psi_{\text{per}}(\nu) + \varepsilon). \quad (87)$$

Thanks to (83) and the fact that  $\{\nu_1, \dots, \nu_d\}$  is a basis, we have that  $u_k \in \mathcal{A}_{\text{per}}(Q_{k m T}; \mathbb{R})$  for some  $m \in \mathbb{N}$  depending only on  $\{\nu_1, \dots, \nu_d\}$  and  $\lambda$ . Fix now  $M \in \mathbb{N}$  such that  $M \gg m$  and let

$$\mathcal{Z} := \left\{ z = \lambda k T \sum_{n=1}^d \mu_n \nu_n : \mu \in \mathbb{Z}^d, Q_{\lambda k T}^\nu(z) \cap Q_{M k T} \neq \emptyset \right\}.$$

Since,  $u_k \in \mathcal{A}_{\text{per}}(Q_{k m T}; \mathbb{R})$  we have that  $u_k \in \mathcal{A}_{\text{per}}(Q_{k M T}; \mathbb{R})$  and thus

$$\inf \{E(u, Q_{k M T}) : u : \mathcal{L} \rightarrow \mathbb{R}, u(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_{k M T}; \mathbb{R})\} \leq E(u_k, Q_{k M T}). \quad (88)$$

Due to the periodicity of  $u_k$ , the assumption on  $\lambda$ , and Lemma 3.1(vi), we have that

$$E(u_k, Q_{\lambda k T}^\nu(z)) = E(u_k(\cdot - z), Q_{\lambda k T}^\nu(z)) = E(u_k, Q_{\lambda k T}^\nu) \text{ for all } z = \lambda k T \sum_{n=1}^d \mu_n \nu_n, \mu \in \mathbb{Z}^d. \quad (89)$$

Note that for  $z \in \mathcal{Z}$  we have  $Q_{\lambda k T}^\nu(z) \subset Q_{(M + \sqrt{d}\lambda)kT}$  and thus  $\#\mathcal{Z} \leq (M^d + CM^{d-1}\lambda k T)/\lambda^d$ . Therefore,

$$E(u_k, Q_{k M T}) \leq \sum_{z \in \mathcal{Z}} E(u_k, Q_{\lambda k T}^\nu(z)) \leq \#\mathcal{Z} E(u_k, Q_{\lambda k T}^\nu) \leq \frac{M^d + CM^{d-1}\lambda k T}{\lambda^d} E(u_k, Q_{\lambda k T}^\nu). \quad (90)$$

Dividing by  $(k M T)^d$ , letting first  $M$  tend to  $+\infty$ , then  $k$  to  $+\infty$ , and lastly  $\varepsilon \rightarrow 0$ , and noting (87) as well as (88), we obtain the conclusion of Step 2.1.

**Step 2.2.** (Proof of ' $\leq$ ') The reverse inequality follows as in Step 2.1 by noting that if  $u \in \mathcal{A}_{\text{per}}(Q_{k T}; \mathbb{R})$  for some  $k \in \mathbb{N}$ , then, due to (83), we have that  $u(\cdot - \lambda m T \nu_n) - \langle \nu, \cdot - \lambda m T \nu_n \rangle = u(\cdot) - \langle \nu, \cdot \rangle$  for some  $m \in \mathbb{N}$  and all  $n = 1, \dots, d$ . This allows us to perform the same construction as in Step 2.1. This concludes the proof.  $\square$

*Proof of Proposition 2.6.* Our goal is to prove

$$\varphi(\nu) = \frac{1}{T^d} \inf \{E(u, Q_T) : u : \mathcal{L} \rightarrow \mathbb{R}, u(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R})\} \quad (91)$$

for all  $\nu \in \mathbb{R}^d$ . Due to Lemma 3.12, Lemma 3.11, and Lemma 3.6, we have

$$\varphi(\nu) = \psi(\nu) = \phi(\nu) = \phi_{\text{per}}(\nu). \quad (92)$$

Additionally, Lemma 3.4 ensures that

$$\phi_{\text{per}}(\nu) = \frac{1}{T^d} \inf \{E(u, Q_T) : u : \mathcal{L} \rightarrow \mathbb{R}, u(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R})\}.$$

This shows (91) and concludes the proof.  $\square$

#### 4. CRYSTALLINITY OF THE HOMOGENIZED SURFACE ENERGY DENSITY

This section is devoted to the proof of Theorem 2.7. We assume throughout this section that assumptions (L1), (L2) and (H1), (H3) are satisfied.

We define the set of edges  $\mathcal{E}$  by

$$\mathcal{E} = \{(i, j) \in (\mathcal{L} \cap Q_T) \times \mathcal{L} : c_{i,j} \neq 0\} \text{ and } N = \#\mathcal{E}. \quad (93)$$

*Proof of Theorem 2.7.* We divide the proof into three steps. First, we derive a dual representation of  $\varphi$ . Then, using this representation, we show that  $\varphi$  is crystalline.

**Step 1.**(Dual representation) We define

$$\mathcal{C} = \left\{ \alpha_{i,j} \in [0, c_{i,j}] : \alpha_{i+Tz, j+Tz} = \alpha_{i,j} \text{ for all } z \in \mathcal{L}, \right. \\ \left. \sum_{j \in \mathcal{L}} (\alpha_{j,i} - \alpha_{i,j}) = 0 \text{ for all } i \in Q_T \cap \mathcal{L} \right\}. \quad (94)$$

Our goal is to prove

$$\varphi(\nu) = \frac{1}{T^d} \sup_{(\alpha_{i,j})_{i,j} \in \mathcal{C}} \left\langle \nu, \sum_{i \in \mathcal{L} \cap Q_T} \sum_{j \in \mathcal{L}} \alpha_{i,j} (i - j) \right\rangle. \quad (95)$$

Let  $\nu \in \mathbb{R}^d$ . Due to Proposition, 2.6 there holds

$$\begin{aligned} \varphi(\nu) &= \phi_{\text{per}}(\nu) = \frac{1}{T^d} \inf \{E(u, Q_T) : u(\cdot) - \langle \nu, \cdot \rangle \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R})\} \\ &= \frac{1}{T^d} \inf \{E(u + \langle \nu, \cdot \rangle, Q_T) : u \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R})\} \\ &= \frac{1}{T^d} \inf_{u \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R})} \sum_{i \in Q_T} \sum_{j \in \mathcal{L}} c_{i,j} (u(i) - u(j) + \langle \nu, i - j \rangle)^+. \end{aligned}$$

Note that, we can write

$$\phi_{\text{per}}(\nu) = \frac{1}{T^d} \inf_{u \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R})} \sup_{\substack{0 \leq \alpha_{i,j} \leq c_{i,j} \\ \alpha_{i+Tz, j+Tz} = \alpha_{i,j}}} \sum_{i \in \mathcal{L} \cap Q_T} \sum_{j \in \mathcal{L}} \alpha_{i,j} (u(i) - u(j) + \langle \nu, i - j \rangle). \quad (96)$$

To see this, we observe that for all  $u \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R})$  and all  $0 \leq \alpha_{i,j} \leq c_{i,j}$  such that  $\alpha_{i+Tz, j+Tz} = \alpha_{i,j}$  for all  $z \in \mathbb{Z}^d$  we have

$$\alpha_{i,j} (u(i) - u(j) + \langle \nu, i - j \rangle) \leq c_{i,j} (u(i) - u(j) + \langle \nu, i - j \rangle)^+$$

with equality for

$$\alpha_{i,j} = \begin{cases} c_{i,j} & \text{if } (u(i) - u(j) + \langle \nu, i - j \rangle) \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Given  $0 \leq \alpha_{i,j} \leq c_{i,j}$  such that  $\alpha_{i+Tz,j+Tz} = \alpha_{i,j}$  for all  $z \in \mathbb{Z}^d$ , and  $u: \mathcal{L} \rightarrow \mathbb{R}$   $T$ -periodic, we have

$$\begin{aligned}
\sum_{i \in \mathcal{L} \cap Q_T} \sum_{j \in \mathcal{L}} \alpha_{i,j} (u(i) - u(j)) &= \sum_{i \in \mathcal{L} \cap Q_T} \sum_{j \in \mathcal{L}} \alpha_{i,j} u(i) - \sum_{i \in \mathcal{L} \cap Q_T} \sum_{j \in \mathcal{L}} \alpha_{i,j} u(j) \\
&= \sum_{i \in \mathcal{L} \cap Q_T} \sum_{j \in \mathcal{L}} \alpha_{i,j} u(i) - \sum_{j \in \mathcal{L} \cap Q_T} \sum_{z \in \mathbb{Z}^d} \sum_{i \in \mathcal{L} \cap Q_T} \alpha_{i,j+Tz} u(j+Tz) \\
&= \sum_{i \in \mathcal{L} \cap Q_T} \sum_{j \in \mathcal{L}} \alpha_{i,j} u(i) - \sum_{j \in \mathcal{L} \cap Q_T} \sum_{z \in \mathbb{Z}^d} \sum_{i \in \mathcal{L} \cap Q_T} \alpha_{i-Tz,j} u(j) \\
&= \sum_{i \in \mathcal{L} \cap Q_T} \sum_{j \in \mathcal{L}} \alpha_{i,j} u(i) - \sum_{j \in \mathcal{L} \cap Q_T} \sum_{i \in \mathcal{L}} \alpha_{i,j} u(j) \\
&= \sum_{i \in \mathcal{L} \cap Q_T} \sum_{j \in \mathcal{L}} (\alpha_{i,j} - \alpha_{j,i}) u(i).
\end{aligned}$$

Note that, since in all steps the sum over  $i$  and, due to (H3), the sum over  $j$  runs over a finite index set, the order of summation can be changed without changing the value of the various sums. This implies that, given  $0 \leq \alpha_{i,j} \leq c_{i,j}$  such that  $\alpha_{i+Tz,j+Tz} = \alpha_{i,j}$  for all  $z \in \mathcal{L}$ , we have

$$\inf_{u \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R})} \sum_{i \in \mathcal{L} \cap Q_T} \sum_{j \in \mathcal{L}} \alpha_{i,j} (u(i) - u(j)) = \begin{cases} 0 & \text{if } \sum_{j \in \mathcal{L}} (\alpha_{i,j} - \alpha_{j,i}) = 0 \text{ for all } i \in Q_T \cap \mathcal{L}, \\ -\infty & \text{otherwise.} \end{cases} \quad (97)$$

Hence, using (92), (94), (96), and (97), we obtain

$$\varphi(\nu) \geq \frac{1}{T^d} \sup_{(\alpha_{i,j})_{i,j \in \mathcal{C}}} \left\langle \nu, \sum_{i \in \mathcal{L} \cap Q_T} \sum_{j \in \mathcal{L}} \alpha_{i,j} (i - j) \right\rangle. \quad (98)$$

As for the other inequality in finite dimension, that is when  $c_{i,j}^R$  is such that  $c_{i,j}^R = c_{i,j}$  if  $|i - j| < R$  and  $c_{i,j}^R = 0$  if  $|i - j| \geq R$ , the equality is true due to [43, Corollary 31.2.1]. More precisely, we obtain:

$$\begin{aligned}
\varphi_R(\nu) &= \frac{1}{T^d} \inf_{u \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R})} \sum_{i \in Q_T} \sum_{j \in \mathcal{L}} c_{i,j}^R (u(i) - u(j) + \langle \nu, i - j \rangle)^+ \\
&= \frac{1}{T^d} \sup_{(\alpha_{i,j})_{i,j \in \mathcal{C}_R}} \left\langle \nu, \sum_{i \in \mathcal{L} \cap Q_T} \sum_{j \in \mathcal{L}} \alpha_{i,j} (i - j) \right\rangle, \quad (99)
\end{aligned}$$

where  $\mathcal{C}_R = \mathcal{C} \cap \prod_{(i,j)} [0, c_{i,j}^R]$ . Note that  $\mathcal{C}_R \subset \mathcal{C}$  and thus by (98) and (99), we have

$$\varphi_R(\nu) \leq \frac{1}{T^d} \sup_{(\alpha_{i,j})_{i,j \in \mathcal{C}}} \left\langle \nu, \sum_{i \in \mathcal{L} \cap Q_T} \sum_{j \in \mathcal{L}} \alpha_{i,j} (i - j) \right\rangle \leq \varphi(\nu).$$

In addition, for all  $u \in \mathcal{A}_{\text{per}}(Q_T; \mathbb{R})$  we have

$$\lim_{R \rightarrow +\infty} \sum_{i \in Q_T} \sum_{j \in \mathcal{L}} c_{i,j}^R (u(i) - u(j) + \langle \nu, i - j \rangle)^+ = \sum_{i \in Q_T} \sum_{j \in \mathcal{L}} c_{i,j} (u(i) - u(j) + \langle \nu, i - j \rangle)^+,$$

monotonically in  $R$ . Hence, due to  $\Gamma$ -convergence of monotone sequences we have

$$\varphi(\nu) = \lim_{R \rightarrow +\infty} \varphi_R(\nu) \leq \frac{1}{T^d} \sup_{(\alpha_{i,j})_{i,j \in \mathcal{C}}} \left\langle \nu, \sum_{i \in \mathcal{L} \cap Q_T} \sum_{j \in \mathcal{L}} \alpha_{i,j} (i - j) \right\rangle.$$

This shows (95).

**Step 2.**(Crystallinity) By Remark 2.5, we have

$$\varphi(\nu) = \sup_{\zeta \in W_\varphi} \langle \nu, \zeta \rangle.$$

So that, by (95)

$$W_\varphi = \left\{ \frac{1}{T^d} \sum_{i \in \mathcal{L} \cap Q_T} \sum_{j \in \mathcal{L}} \alpha_{i,j}(i-j) : (\alpha_{i,j})_{i,j} \in \mathcal{C} \right\}, \quad (100)$$

with  $\mathcal{C}$  given in (94). Recall  $N$  and  $\mathcal{E}$  defined in (93). Define  $L: \mathbb{R}^N \rightarrow \mathbb{R}^d$  by

$$L(\alpha_{i,j})_{(i,j) \in \mathcal{E}} = \frac{1}{T^d} \sum_{(i,j) \in \mathcal{E}} \alpha_{i,j}(i-j). \quad (101)$$

Hence, we observe that

$$W := \left\{ \frac{1}{T^d} \sum_{i \in \mathcal{L} \cap Q_T} \sum_{j \in \mathcal{L}} \alpha_{i,j}(i-j) : (\alpha_{i,j})_{i,j} \in \mathcal{C} \right\} = L(\mathcal{C}),$$

where  $\mathcal{C}$  is a convex closed set as the intersection of two convex and closed sets. Thus, as the image of the convex and closed set  $\mathcal{C}$  through the linear map  $L$ ,  $W$  is a closed and convex set. Then (100) follows by fenchel duality, since

$$\varphi(\nu) = \mathbf{I}_W^*(\nu) = \mathbf{I}_{W_\varphi}^*(\nu),$$

where  $\mathbf{I}_A: \mathbb{R}^d \rightarrow [0, +\infty]$  denotes the support function of the set  $A$  given by

$$\mathbf{I}_A(\zeta) = \begin{cases} 0 & \text{if } \zeta \in A, \\ +\infty & \text{otherwise.} \end{cases}$$

Now, since both  $W_\varphi$  and  $W$  are closed and convex, we have

$$\mathbf{I}_{W_\varphi}(\zeta) = \mathbf{I}_{W_\varphi}^{**}(\zeta) = \varphi^*(\zeta) = \mathbf{I}_W^{**} = \mathbf{I}_W.$$

This shows (100). Furthermore, we find

$$W_\varphi = L \left( V \cap \prod_{(i,j) \in \mathcal{E}} [0, c_{i,j}] \right), \quad (102)$$

where  $V \subset \mathbb{R}^N$  is a linear subspace of co-dimension  $k := \#(Q_T \cap \mathcal{L}) - 1$  given by

$$V = \left\{ \alpha_{i,j} \in \mathbb{R}^N : \sum_{j \in \mathcal{L}} (\alpha_{i,j} - \alpha_{j,i}) = 0 \text{ for all } i \in Q_T \cap \mathcal{L} \right\}. \quad (103)$$

Hence, due to (102),  $W_\varphi$  is the image of the linear map  $L$ , given in (101), of a  $N$ -dimensional polytope  $\prod_{(i,j) \in \mathcal{E}} [0, c_{i,j}]$  intersected with the linear subspace  $V$ , given in (103). The intersection of a cube with a linear subspace is a polytope, and thus also its image through a linear map. This proves that  $\varphi$  is crystalline.

**Step 3.**(Estimate on the number of vertices) Our goal is to prove that

$$\#\text{extreme}(W_\varphi) \leq 3^N, \quad (104)$$

where we recall  $N$  defined in (93). Let us note that, due to the Krein-Milman Theorem (cf. [20], Theorem 1.13) and (102), it is easy to see that there holds

$$\#\text{extreme}(W_\varphi) = \#\text{extreme} \left( L \left( V \cap \prod_{(i,j) \in \mathcal{E}} [0, c_{i,j}] \right) \right) \leq \#\text{extreme} \left( V \cap \prod_{(i,j) \in \mathcal{E}} [0, c_{i,j}] \right).$$

In order to show (104), it remains to show

$$\#\text{extreme} \left( V \cap \prod_{(i,j) \in \mathcal{E}} [0, c_{i,j}] \right) \leq 3^N. \quad (105)$$

In order to obtain this estimate we note that the extreme points of  $V \cap \prod_{(i,j) \in \mathcal{E}} [0, c_{i,j}]$  lie on the  $k$ -dimensional (here  $k$  is the co-dimension of the linear subspace  $V$ ) facets of  $\prod_{(i,j) \in \mathcal{E}} [0, c_{i,j}]$ . Furthermore, we can find an injective relation between extreme points and these facets. In fact, if the matrix determining the intersection of the facet with  $V$  is full rank, then the point of intersection is unique. If that is not the case, then the solution set is itself a subspace and one can add one additional condition to obtain the extreme point. This implies that here the extreme point is shared by more  $k$ -dimensional facets on a lower dimensional facet. Note that there are at most  $\binom{N}{k} 2^{N-k}$  such facets and by the binomial formula we have that

$$\binom{N}{k} 2^{N-k} \leq \sum_{j=0}^N \binom{N}{j} 2^{N-j} = 3^N.$$

This concludes Step 3.  $\square$

## 5. DIFFERENTIABILITY OF THE EFFECTIVE SURFACE TENSION

In this Section, we prove Proposition 2.9 which states that  $\varphi$  is differentiable in totally irrational directions. It is a corollary of the two lemmas which we state and prove below.

**Lemma 5.1.** *Let  $\nu \in \mathbb{S}^{d-1}$ , let  $u$  be a minimizer in (9) and assume that for any  $s \in \mathbb{R}$ , the set  $\{u = s\}$  is finite. Then  $\varphi$  is differentiable in  $\nu$ .*

*Proof.* The expression (95) shows that  $\varphi$  is a convex, one-homogeneous function with subgradient at  $\nu$  given by

$$\partial\varphi(\nu) = \left\{ \frac{1}{T^d} \sum_{i \in \mathcal{L} \cap Q_T} \sum_{j \in \mathcal{L}} \alpha_{i,j} (i - j) : \alpha = (\alpha_{i,j})_{i,j \in \mathcal{L}} \text{ maximizer in (95)} \right\}$$

It is differentiable at  $\nu$  if and only if the above set has exactly one element.

Let  $\alpha, \alpha' \in \mathcal{C}$  be two maximizers in (95). Classical optimality conditions guarantee that for any  $i, j$ , if  $u(i) \neq u(j)$ , then:

$$\alpha_{i,j} = \alpha'_{i,j} = \begin{cases} c_{i,j} & \text{if } u(i) - u(j) > 0 \\ 0 & \text{if } u(i) - u(j) < 0. \end{cases} \quad (106)$$

Let us denote by  $p, p' \in \partial\varphi(\nu)$  the subgradients given by the dual variables, respectively,  $\alpha$  and  $\alpha'$ , we claim that  $p = p'$ . One has:

$$p - p' = \frac{1}{T^d} \sum_{i \in \mathcal{L} \cap Q_T} \sum_{j: u(j) = u(i)} (\alpha_{i,j} - \alpha'_{i,j})(i - j). \quad (107)$$

Let  $s \in \mathbb{R}$ ,  $i_0 \in \mathcal{L} \cap Q_T$  with  $u(i_0) = s$  and such that the finite set  $J_s := \{j : u(j) = s\}$  has more than one element. For any  $i, j$ , let  $\beta_{i,j} := \alpha_{i,j} - \alpha'_{i,j}$ . Then

$$\begin{aligned} \sum_{i \in J_s} \sum_{j \in J_s} \beta_{i,j}(i-j) &= \sum_{z \in \mathbb{Z}^d} \sum_{i \in J_s \cap (Tz + Q_T)} \sum_{j \in J_s} \beta_{i,j}(i-j) \\ &= \sum_{z \in \mathbb{Z}^d} \sum_{i \in (J_s - Tz) \cap Q_T} \sum_{j \in J_s - Tz} \beta_{i,j}(i-j) \end{aligned}$$

where for the last line we have substituted  $(i, j)$  with  $(i - Tz, j - Tz)$  and used that  $\beta$  is  $Q_T$ -periodic. In addition, we have that  $u(i) = u(j)$  if and only if  $u(i - Tz) = u(j - Tz)$  so that this can be rewritten:

$$\sum_{i \in J_s} \sum_{j \in J_s} \beta_{i,j}(i-j) = \sum_{z \in \mathbb{Z}^d} \sum_{i \in (J_s - Tz) \cap Q_T} \sum_{j: u(j)=u(i)} \beta_{i,j}(i-j)$$

By assumption, the sets  $(J_s - Tz) \cap Q_T$ ,  $z \in \mathbb{Z}^d$  are all disjoint. Otherwise, there would be  $i, z$  with  $s = u(i - Tz) = u(i) + T\langle \nu, z \rangle = s$ , yielding in particular that  $\langle \nu, z \rangle = 0$ , and one would deduce that  $i - kTz \in J_s$  for all  $k \in \mathbb{Z}$ , a contradiction since we assumed  $J_s$  was finite. As a consequence, showing that (107) vanishes is equivalent to showing that

$$\sum_{i \in J_s} \sum_{j \in J_s} \beta_{i,j}(i-j) = 0 \quad (108)$$

for any  $s \in \mathbb{R}$  (such that  $J_s$  is not empty and contains more than one point). Obviously, the expression in (108) is also

$$\sum_{i \in J_s} \sum_{j \in J_s} (\beta_{i,j} - \beta_{j,i})i$$

Thanks to the definition (94) of  $\mathcal{C}$ , one has for any  $i$  that  $\sum_j \beta_{i,j} - \beta_{j,i} = 0$ , so that:

$$\sum_{i \in J_s} \sum_{j \in J_s} (\beta_{i,j} - \beta_{j,i})i = \sum_{i \in J_s} \sum_{j \notin J_s} (\beta_{j,i} - \beta_{i,j})i = 0$$

thanks to (106). Hence, (108) holds and we deduce  $p = p'$ , which shows the lemma.  $\square$

**Lemma 5.2.** *Let  $\nu \in \mathbb{S}^{d-1}$  be totally irrational and let  $u$  be a minimizer in (9). Then for any  $s \in \mathbb{R}$ , the set  $\{u = s\}$  is finite.*

*Proof.* Recalling the notation in the previous proof, let  $s \in \mathbb{R}$  and consider the set  $J_s := \{u = s\}$ . For  $z \in \mathbb{Z}^d$ , let  $J_s^z = J_s \cap (Q_T + Tz) - Tz \subset Q_T$ . For  $i \in J_s^z$ ,  $u(i) = s + T\langle z, \nu \rangle$ . Since  $\nu$  is totally irrational, we deduce that  $J_s^z \cap J_s^{z'} = \emptyset$  for any  $z \neq z'$ , showing that all sets  $J_s^z$  but a finite number must be empty. Hence  $J_s$  is finite.  $\square$

## 6. NUMERICAL ILLUSTRATION

**6.1. A simplified framework.** In this section, we address, as an illustrative experiment, the following issue. We consider a basic 2D cartesian graph  $\{(i, j) : 0 \leq i \leq M-1, 0 \leq j \leq N-1\}$ , representing for instance the pixels of an image, and we want to approximate on this discrete grid the two-dimensional total variation  $\int_{\Omega} |Du|$ ,  $u \in BV(\Omega)$ . Here it is assumed that  $\Omega \subset \mathbb{R}^2$  is a rectangle and that  $\{0, \dots, M-1\} \times \{0, \dots, N-1\}$  is a discretization of  $\Omega$  at a length scale  $\sim 1/N \sim 1/M$ .

There are of course many ways to do this, but we propose here to consider a family of discrete “graph” total variations, defined for a family  $(u_{i,j})_{i,j} \in \mathbb{R}^{M \times N}$  by:

$$J(u) = \sum_{i,j} c_{i+\frac{1}{2},j}^+ (u_{i+1,j} - u_{i,j})^+ + c_{i+\frac{1}{2},j}^- (u_{i,j} - u_{i+1,j})^+ \\ + c_{i,j+\frac{1}{2}}^+ (u_{i,j+1} - u_{i,j})^+ + c_{i,j+\frac{1}{2}}^- (u_{i,j} - u_{i,j+1})^+ \quad (109)$$

and which involves only nearest-neighbour interactions in horizontal and vertical directions.

We assume in addition that the weight  $c^\pm$  are  $T$ -periodic for some  $T \in \mathbb{N}$ ,  $T > 0$ , that is,  $C_{a+kT,b+lT}^\pm = c_{a,b}^\pm$  for any  $(k,l) \in \mathbb{Z}^2$ ,  $(a,b) = (i + \frac{1}{2}, j)$  or  $(i, j + \frac{1}{2})$ , as long as the points fall inside the grid.

For  $T = 1$ ,  $c_{a,b}^\pm \equiv 1$ , it is standard that (109) approximates, in the continuum limit, the anisotropic total variation  $\int_\Omega |\partial_1 u| + |\partial_2 u|$ , which, if used for instance as a regularizer for image denoising or reconstruction, may produce undesired artefacts (although hardly visible on standard applications, see Figure 8).

A standard way to mitigate this issue (besides, of course, resorting to numerical analysis based on finite differences or elements in order to define more refined discretizations), is to add to (109) diagonal interactions, with appropriate weights, in order to improve the isotropy of the limit (see for instance [11]), with the drawback of complexifying the graph and the optimization. We show here that a similar effect can be attained by homogenization. To illustrate this, let us first consider the simplest situation, for  $T = 2$ .

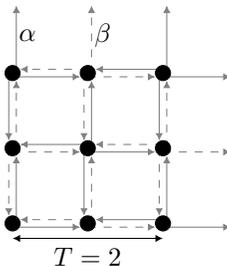


FIGURE 3. The alternating 2-periodic coefficients yielding the smallest anisotropy

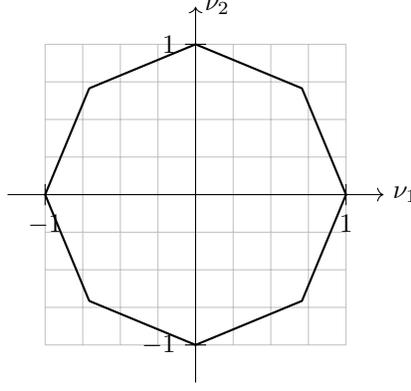
In that case, one can explicitly build coefficients  $c_{a,b}^\pm$ , taking two values  $\alpha, \beta$  (see Figure 3), which will yield the homogenized surface tension

$$\varphi(\nu) = (\sqrt{2} - 1) \left( |\nu_1| + |\nu_2| + \frac{|\nu_1 + \nu_2|}{\sqrt{2}} + \frac{|\nu_1 - \nu_2|}{\sqrt{2}} \right) \quad (110)$$

whose 1-level set (or *Frank diagram*) is shown in Figure 4. Observe that this is the same anisotropy which would be obtained by using constant coefficients and adding interactions along the edges  $((i,j), (i+1, j+1))$  and  $((i,j), (i+1, j-1))$ .

In order to obtain (110), one needs to tune  $\alpha, \beta$  so that a vertical edge and a diagonal edge, in the most favorable position, have the same length (with a  $\sqrt{2}$  factor for the diagonal, whose intersection with the periodicity cell is of course longer). This is ensured if  $\alpha + \beta = 4\alpha/\sqrt{2}$ , that is,  $\beta = (2\sqrt{2} - 1)\alpha$ . We find that choosing

$$\begin{cases} \alpha = \frac{1}{4\sqrt{2}} \approx 0.1768 \\ \beta = (2\sqrt{2} - 1)\alpha \approx 0.3232 \end{cases} \quad (111)$$

FIGURE 4. The Frank diagram  $\{\nu : \varphi(\nu) \leq 1\}$  given by (110)

yields (110), as an effective homogenized anisotropy.

For larger periodicity cells, it seems difficult to do a similar analysis, first of all, because one should not expect the optimal minimizers, in most directions (if not all), to be given by straight lines, but rather by periodic perturbations of straight lines. We propose an optimization process in order to compute the optimal weights  $c_{a,b}^\pm$ .

**6.2. The optimization method.** The effective surface tension is obtained by solving the cell problem:

$$\begin{aligned} \phi(\nu) = \min_u \left\{ \sum_{(i,j) \in Y} c_{i+\frac{1}{2},j}^+ (u_{i+1,j} - u_{i,j})^+ + c_{i+\frac{1}{2},j}^- (u_{i,j} - u_{i+1,j})^+ \right. \\ \left. + c_{i,j+\frac{1}{2}}^+ (u_{i,j+1} - u_{i,j})^+ + c_{i,j+\frac{1}{2}}^- (u_{i,j} - u_{i,j+1})^+ : \right. \\ \left. u_{i,j} - \nu \cdot \begin{pmatrix} i \\ j \end{pmatrix} Y\text{-periodic} \right\} \end{aligned} \quad (112)$$

where  $Y = \mathbb{Z}^2 \cap ([0, T) \times [0, T))$  is the periodicity cell. This is easily solved, for instance by a saddle-point algorithm [26] which aims at finding a solution to:

$$\begin{aligned} \phi(\nu) = \min_{\nu \text{ } Y\text{-periodic}} \max_{0 \leq w_{i,j}^\pm \leq 1} \sum_{(i,j) \in Y} (w_{i+\frac{1}{2},j}^+ c_{i+\frac{1}{2},j}^+ - w_{i+\frac{1}{2},j}^- c_{i+\frac{1}{2},j}^-) (v_{i+1,j} - v_{i,j} + \nu_1) \\ + (w_{i,j+\frac{1}{2}}^- c_{i,j+\frac{1}{2}}^- - w_{i,j+\frac{1}{2}}^+ c_{i,j+\frac{1}{2}}^+) (v_{i,j+1} - v_{i,j} + \nu_2), \end{aligned}$$

where we have replaced the variable  $u$  with the periodic vector  $v_{i,j} = u_{i,j} - \nu \cdot (i, j)^T$ . For technical reasons, we need to “regularize” slightly this problem in order to make it differentiable with respect to the coefficients  $\mathbf{c} = (c_{i,j}^\pm)$ . This is done by introducing  $\varepsilon > 0$  a (very) small parameter and adding to the previous objective the penalization

$$-\frac{\varepsilon}{2} \sum_{(i,j) \in Y} (w_{i+\frac{1}{2},j}^+)^2 + (w_{i+\frac{1}{2},j}^-)^2 + (w_{i,j+\frac{1}{2}}^-)^2 + (w_{i,j+\frac{1}{2}}^+)^2 + \frac{\varepsilon}{2} \sum_{(i,j) \in Y} v_{i,j}^2$$

which makes the problem strongly convex/concave and the solutions  $w, v$  unique. We call  $\phi_\varepsilon(\nu)[\mathbf{c}]$  the corresponding value. The advantage of this regularization is that one can easily

show that  $\mathbf{c} \mapsto \phi_\varepsilon(\nu)[\mathbf{c}]$  is locally  $C^{1,1}$ , with a gradient given by:

$$\lim_{t \rightarrow 0} \frac{\phi_\varepsilon(\nu)[\mathbf{c} + t\mathbf{d}] - \phi_\varepsilon(\nu)[\mathbf{c}]}{t} = \sum_{(i,j) \in Y} (w_{i+\frac{1}{2},j}^+ d_{i+\frac{1}{2},j}^+ - w_{i+\frac{1}{2},j}^- d_{i+\frac{1}{2},j}^-)(v_{i+1,j} - v_{i,j} + \nu_1) \\ + (w_{i,j+\frac{1}{2}}^- d_{i,j+\frac{1}{2}}^- - w_{i,j+\frac{1}{2}}^+ d_{i,j+\frac{1}{2}}^+)(v_{i,j+1} - v_{i,j} + \nu_2)$$

where  $(w, v)$  solves the saddle-point problem which defines  $\phi_\varepsilon(\nu)[\mathbf{c}]$ .

Then, to find coefficients which ensure that  $\phi$  is as “isotropic” as possible, one fixes a finite set of directions  $(\nu_1, \dots, \nu_k)$  (typically,  $(\cos(2\ell\pi/k), \sin(2\ell\pi/k))$  for  $\ell = 1, \dots, k$ ), and uses a first order gradient descent algorithm to optimize:

$$\mathcal{L}(\mathbf{c}) = \sum_{\ell=1}^k (\phi_\varepsilon(\nu_\ell)[\mathbf{c}] - 1)^2$$

The problem is easily solved for  $Y = \{0, 1\} \times \{0, 1\}$ ,  $k = 8$  and  $(\nu_\ell)_{\ell=1}^8$  given as above. For larger periodicity cells and more directions, it easily gets trapped in local minima and we use a random initialization in order to be able to find satisfactory solutions. We then test the result by computing the un-regularized surface tension  $\phi$  with the resulting coefficients  $\mathbf{c}$ . We show some results in the next section. Of course, taking a large value of  $\varepsilon$  will make the problem easier to solve, but the learned coefficients will not allow to reconstruct a satisfactory surface tension: we need to choose  $\varepsilon$  small, an order of magnitude below the error which we expect on the anisotropy of  $\phi$ .

**6.3. Numerical results.** We show the outcome of the optimization, in the periodicity cell  $Y = \{0, \dots, T-1\} \times \{0, \dots, T-1\}$  for  $T = 2, 4, 6, 8$ . We plot first the set  $\{\varphi \leq 1\}$  or Frank diagram for the effective surface tensions. Figure 5 shows the diagram obtained, for  $T = 2, 4, 8$ .

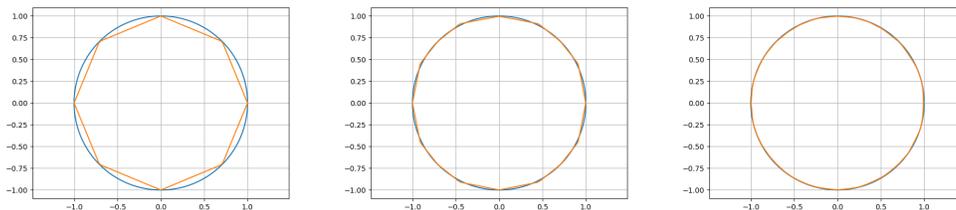


FIGURE 5. Frank diagrams of the effective anisotropies for  $T = 2, 4, 8$ .

For  $T = 2$ , the optimization yields the same anisotropy as our construction in (111) which gives the anisotropy (110) and which we conjecture to be optimal for  $\mathcal{L}(\mathbf{c})$  taking  $k = 8$  (compare with Fig. 4). However, except when initialized with the values in (111), the algorithm usually outputs different values with the same effective anisotropy, see Fig. 6 (the values in (111) are in some sense better, as for instance a vertical edge will always have the same effective energy with these values, while with the computed values displayed in Fig. 6, it will need to pass through the edges in the second column of the cell in order to get the minimal energy).

For  $T = 4$ , one sees that the behavior is almost isotropic, while for  $T = 8$ , the relative error with the perfect unit disk is about 1%. Here, we estimated this error as  $(\max_\ell \phi(\nu_\ell) - \min_\ell \phi(\nu_\ell)) / \min_\ell \phi(\nu_\ell)$ , where  $\ell \in \{1, \dots, k\}$  and  $k = 180$ . We illustrate this on an “inpainting” example, which consists in finding the minimal line in a given direction. We consider as an

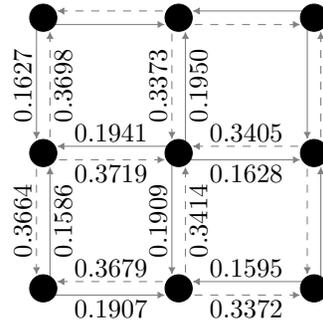


FIGURE 6. An example of optimized 2-periodic coefficients yielding the same anisotropy as the choice (111)



FIGURE 7. A minimal half-plane in the orientation  $(\cos 3\pi/8, \sin 3\pi/8)$ . Top left, boundary datum, the region where the perimeter is minimized is in gray. Top, middle:  $\varphi(\nu) = |\nu_1| + |\nu_2|$ . Top, right: optimal effective  $\varphi$  for  $T = 2$ . Bottom: for  $T = 4, 6, 8$ .

example the direction  $(\cos 3\pi/8, \sin 3\pi/8)$ , which is irrational, so that there cannot be a fully periodic solution. The figure 7 displays several minimal half-planes in this orientation. Observe that for this orientation, the results for  $T = 4$  or  $6$  look nicer than the result obtained for  $T = 8$ .

We also show a denoising example based on the “ROF” method (which consists simply in minimizing the total variation (defined by the surface tension  $\varphi$ ) of an image with a quadratic penalization of the distance to a noisy data, in order to produce a denoised version, see [44]) with the anisotropic tension  $\varphi(\nu) = |\nu_1| + |\nu_2|$  (“ $T = 1$ ”) and the optimized homogenized surface tension for  $T = 4$ . The original image is degraded with a Gaussian noise with 10% standard deviation (with respect to the range of the values). Here, the difference between the two regularizers is hardly perceptible (since the data term strongly influences the position of the

discontinuities), yet a close-up (bottom row) allows to see a slight difference, for instance on the cheek where the  $T = 1$  anisotropy produces block structures.



FIGURE 8. “ROF” denoising example. Left: noisy image. Middle, denoised with  $\varphi(\nu) = |\nu_1| + |\nu_2|$ . Right: with the effective tension computed for  $T = 4$ .

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