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A NOTE ON EXISTENCE OF SOLUTIONS TO CONTROL PROBLEMS OF SEMILINEAR PARTIAL DIFFERENTIAL EQUATIONS*

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Abstract. In this paper, we study optimal control problems of semilinear elliptic and parabolic equations. A tracking cost functional, quadratic in the control and state variables, is considered. No control constraints are imposed. We prove that the corresponding state equations are well posed for controls in L^2. However, it is well known that in the L^2 framework the mappings involved in the control problem are not Frechet differentiable in general, which makes any analysis of the optimality conditions challenging. Nevertheless, we prove that every L^2 optimal control belongs to L^∞, and consequently standard optimality conditions are available.

Key words. optimal control, existence of solutions, semilinear partial differential equations

MSC codes. 35J61, 35K58, 49J20, 49K20

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1. Introduction.

In this paper, we study the optimal control problem

$$\text{(P)} \qquad \inf_{u\in L^2(Q)} J(u) := \frac{1}{2}\int_Q [(y_u - y_d)^2 + \alpha u^2]\,\mathrm{d}x\,\mathrm{d}t,$$

where $\alpha > 0$ and y_u is the solution of the semilinear parabolic equation

$$\text{(1.1)} \qquad \begin{cases} \dfrac{\partial y}{\partial t} + Ay + f(y) = u & \text{in } Q = \Omega \times (0,T), \\ y = 0 \ \text{ on } \Sigma = \Gamma \times (0,T), & y(x,0) = y_0(x) \ \text{ in } \Omega. \end{cases}$$

Here, A denotes an elliptic operator in the bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, whose boundary is denoted by Γ, $T \in (0,\infty)$ is fixed, $y_0 \in L^\infty(\Omega)$, and $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a given function. Additionally, we assume that $y_d \in L^p(0,T;L^q(\Omega))$ with $p,q \in [2,\infty]$ and $\frac{1}{p} + \frac{n}{2q} < 1$ is a given function. Assumptions on the nonlinear term f in the state equation will be established later. Let us emphasize that we do not impose an upper bound on n nor a growth condition on f.

In many papers, the authors assume box control constraints in the formulation of the problem (P); see, for instance, [8, 11, 14, 18], [23, Chapter 5]. That is because bounded controls u lead to solutions y_u of (1.1) that are functions of $L^\infty(Q)$. This boundedness of the state is crucial to derive first and second order optimality conditions for local or global minimizers. Indeed, the C^1 or C^2 differentiability of the superposition operator $y \to f(y)$ for highly nonlinear functions f requires the

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boundedness of y. Moreover, as far as we know, the well posedness of the state equation (1.1) has not been studied for controls $u \in L^2(Q)$. In some recent papers (see [6, 9, 12]), the existence of global minimizers to (P) in $L^\infty(Q)$ has been proven in the absence of control constraints or for unbounded control sets with the restriction $n \leq 3$ on the dimension. The novelties of our paper with respect to these previous results are the following: first we prove that the state equation (1.1) is well posed for $L^2(Q)$ controls, and the associated control problem (P) has at least one global minimizer $\bar{u}$ in $L^2(Q)$; second we prove that any local minimizer of (P) in the $L^2(Q)$ sense is an element of $L^\infty(Q)$. Usually, this regularity follows from the optimality conditions satisfied by $\bar{u}$, but we cannot get such conditions due to the lack of differentiability of the mapping $y \to f(y)$, since the boundedness of the state $\bar{y}$ corresponding to $\bar{u}$ cannot be deduced for $L^2(Q)$ controls. Therefore, our approach is necessarily different from the one used in the previous papers.

In the second part of the paper, we will prove similar results for a Neumann boundary control problem of a semilinear elliptic equation. The approach used for a Neumann boundary control can be applied to the case of a distributed control problem; see Remark 3.5. Classical results on existence of optimal controls subject to box constraints can be found in [23, section 4.4]. The reader is referred to [10] for the proof of existence of an optimal control in $L^\infty(\Omega)$ for distributed control problems of arbitrary space dimension without box constraints. However, in [10] the analysis of the state equation for the controls in $L^2(\Omega)$ is not performed and, consequently, the existence of minimizers in $L^2(\Omega)$ is not proven, which are ultimately functions of $L^\infty(\Omega)$.

The plan of this paper is as follows. In section 2 we investigate (P). First, we analyze the well posedness of the state equation (1.1); see section 2.1, Theorem 2.1, where for every control $u \in L^2(Q)$ the existence and uniqueness of a solution in $W(0,T)$ is established. We also provide an example showing that the state associated with a control of $L^2(Q)$ does not need to be a bounded function if $n > 1$; see section 2.2. In section 2.3, we prove that (P) has at least one global minimizer $\bar{u}$ in $L^2(Q)$. Then, we demonstrate that any local or global minimizer of (P) is an element of $L^\infty(Q)$ in section 2.4. In the last section of the paper, the same study is applied to a Neumann boundary control problem for a semilinear elliptic equation in dimension $n > 2$.

2. Optimal distributed control of a semilinear parabolic equation.

2.1. Analysis of the state equation. We make the following assumptions on (1.1), which are assumed to hold throughout the section.

(A1) We assume that Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with boundary denoted by Γ, and A denotes a second order elliptic operator in Ω of the form

$$Ay = -\sum_{i,j=1}^{n} \partial_{x_j}(a_{ij}(x)\partial_{x_i} y) + a_0(x)$$

with coefficients $a_{ij}, a_0 \in L^\infty(\Omega)$ satisfying for some $\Lambda_A > 0$

$$\Lambda_A |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \quad \forall \xi \in \mathbb{R}^n \text{ and } a_0(x) \geq 0 \text{ for a.e. } x \in \Omega.$$

(A2) $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a function of class C^1 satisfying that

$$f(0) = 0 \text{ and } \exists \Lambda_f \geq 0 \text{ such that } f'(s) \geq -\Lambda_f \ \forall s \in \mathbb{R}. \tag{2.1}$$

(A3) $\alpha > 0$, $y_0 \in L^\infty(\Omega)$, $y_d \in L^p(0,T;L^q(\Omega))$ with $p,q \in [2,\infty]$ and $\frac{1}{p} + \frac{n}{2q} < 1$.

For convenience, we work with the norm $\|y\|_{H_0^1(\Omega)} := \|\nabla y\|_{L^2(\Omega)}$. As usual, we denote $W(0,T) = L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$. Then, we have the following existence and uniqueness result for a solution to (1.1).

THEOREM 2.1. *For every $u \in L^2(Q)$, (1.1) has a unique solution $y_u \in W(0,T)$. Moreover, $f(y) \in L^2(Q)$, and there exists a constant C depending on $\|y_0\|_{L^\infty(\Omega)}$ but independent of u such that*

$$\|y_u\|_{W(0,T)} + \|f(y_u)\|_{L^2(Q)} \le C(\|u\|_{L^2(Q)} + \|y_0\|_{L^\infty(\Omega)}). \tag{2.2}$$

In addition, if $u_k \rightharpoonup u$ in $L^2(Q)$, then $y_{u_k} \rightharpoonup y_u$ in $W(0,T)$ and $f(y_{u_k}) \rightharpoonup f(y)$ in $L^2(Q)$ hold.

Proof. For every integer $k \ge \|y_0\|_{L^\infty(\Omega)}$ we set $f_k(s) = f(P_k(s))$ with $P_k(s) = \min\{\max\{-k,s\},+k\}$. By a standard application of Schauder's fixed point theorem we infer the existence of a function $y_k \in W(0,T)$ satisfying

$$\begin{cases} \dfrac{\partial y_k}{\partial t} + Ay_k + f_k(y_k) = u \ \text{ in } Q, \\ y_k = 0 \ \text{ on } \Sigma, \quad y_k(0,x) = y_0(x) \ \text{ in } \Omega; \end{cases} \tag{2.3}$$

see, for instance, [7] or [23, Theorem 5.5]. Moreover, testing (2.3) with $\mathrm{e}^{-2\Lambda_f s} y_k(s)$ and integrating with respect to s, we infer for every $t \in (0,T]$

$$\begin{aligned}
\frac{1}{2}\mathrm{e}^{-2\Lambda_f t}\|y_k(t)\|^2_{L^2(\Omega)} + \Lambda_f \int_0^t \mathrm{e}^{-2\Lambda_f s}\|y_k(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s + \mathrm{e}^{-2\Lambda_f T}\Lambda_A\|y_k\|^2_{L^2(0,t;H_0^1(\Omega))}& \\
+ \int_0^t\int_\Omega \mathrm{e}^{-2\Lambda_f s} f_k(y_k) y_k\,\mathrm{d}x\,\mathrm{d}s& \\
\le \int_0^t\int_\Omega \mathrm{e}^{-2\Lambda_f s} u y_k\,\mathrm{d}x\,\mathrm{d}s + \frac{1}{2}\|y_0\|^2_{L^2(\Omega)}& \\
\le C_\Omega \|u\|_{L^2(0,t;L^2(\Omega))}\|y_k\|_{L^2(0,t;H_0^1(\Omega))} + \frac{1}{2}\|y_0\|^2_{L^2(\Omega)}& \\
\le \frac{C_\Omega^2}{2\Lambda_A}\mathrm{e}^{2\Lambda_f T}\|u\|^2_{L^2(0,t;L^2(\Omega))} + \frac{\Lambda_A}{2}\mathrm{e}^{-2\Lambda_f T}\|y_k\|^2_{L^2(0,t;H_0^1(\Omega))} + \frac{1}{2}\|y_0\|^2_{L^2(\Omega)}&.
\end{aligned}$$

With (2.1) and the mean value theorem, we get that $f_k(y_k)y_k \ge -\Lambda_f y_k^2$. Inserting this lower bound into the inequality above, we obtain that the sum of the second and fourth integrals of the left-hand side is nonnegative, i.e.,

$$\Lambda_f \int_0^t \mathrm{e}^{-2\Lambda_f s}\|y_k(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s + \int_0^t\int_\Omega \mathrm{e}^{-2\Lambda_f s} f_k(y_k) y_k\,\mathrm{d}x\,\mathrm{d}s \ge 0.$$

This leads to

$$\|y_k\|_{L^\infty(0,T;L^2(\Omega))} + \|y_k\|_{L^2(0,T;H_0^1(\Omega))} \le C_1\Big(\|u\|_{L^2(Q)} + \|y_0\|_{L^2(\Omega)}\Big), \tag{2.4}$$

where C_1 is independent of u and y_0. Now, we prove that $\{f_k(y_k)\}_{k=1}^\infty$ is bounded in $L^2(Q)$. To this end, we test (2.3) with $f_k(y_k)$ and integrate in Q

$$\begin{aligned}
\int_0^T \left\langle \frac{\partial y_k}{\partial t}, f_k(y_k)\right\rangle \mathrm{d}t - \Lambda_f C\|y_k\|^2_{L^2(0,T;H_0^1(\Omega))} + \|f_k(y_k)\|^2_{L^2(Q)}& \\
\le \frac{1}{2}\|u\|^2_{L^2(Q)} + \frac{1}{2}\|f_k(y_k)\|^2_{L^2(Q)}&,
\end{aligned} \tag{2.5}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. We define the function $F_k(\rho)=\int_0^\rho f_k(s)\,\mathrm{d}s$ for $\rho\in\mathbb{R}$. Then, we have

$$\int_0^T\left\langle\frac{\partial y_k}{\partial t},f_k(y_k)\right\rangle\mathrm{d}t=\int_0^T\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega F_k(y_k)\,\mathrm{d}x\,\mathrm{d}t=\int_\Omega F_k(y_k(T))\,\mathrm{d}x-\int_\Omega F_k(y_0)\,\mathrm{d}x.$$

By the mean value theorem we get a function $\theta:\mathbb{R}\longrightarrow[0,1]$ such that for $\rho>0$

$$F_k(\rho)=\int_0^\rho f_k(s)\,\mathrm{d}s=\int_0^\rho f'\big(\theta(s)P_k(s)\big)P_k(s)\,\mathrm{d}s\geq-\Lambda_f\int_0^\rho s\,\mathrm{d}s=-\Lambda_f\frac{\rho^2}{2}.$$

We establish the same inequality for $\rho<0$:

$$F_k(\rho)=\int_0^\rho f_k(s)\,\mathrm{d}s=-\int_\rho^0 f'\big(\theta(s)P_k(s)\big)P_k(s)\,\mathrm{d}s\geq\Lambda_f\int_\rho^0 s\,\mathrm{d}s=-\Lambda_f\frac{\rho^2}{2}.$$

Moreover, since $k\geq\|y_0\|_{L^\infty(Q)}$ we have

$$\begin{aligned}|F_k(y_0(x))|&\leq\left|\int_0^{y_0(x)}f(s)\,\mathrm{d}s\right|\leq\|y_0\|_{L^\infty(Q)}\max\{|f(s)|:|s|\leq\|y_0\|_{L^\infty(Q)}\}\\&=C_{f,y_0}\|y_0\|_{L^\infty(Q)}.\end{aligned}$$

From the last two estimates we infer

$$\int_0^T\left\langle\frac{\partial y_k}{\partial t},f_k(y_k)\right\rangle\mathrm{d}t\geq-\frac{\Lambda_f}{2}\|y_k(T)\|^2_{L^2(\Omega)}-C_{f,y_0}\|y_0\|_{L^\infty(\Omega)}.$$

Using this fact in (2.5) we obtain with (2.4)

$$\|f_k(y_k)\|_{L^2(Q)}\leq C_2\Big(\|u\|_{L^2(Q)}+\|y_0\|_{L^\infty(\Omega)}\Big).$$

Hence, from (2.3), (2.4), and this estimate we deduce that $\{y_k\}_{k=1}^\infty$ is bounded in $W(0,T)$. Therefore, we can take a subsequence, denoted in the same way, such that $y_k\rightharpoonup y$ in $W(0,T)$, $y_k(x,t)\to y(x,t)$ for almost all $(x,t)\in Q$, and $f_k(y_k)\rightharpoonup f(y)$ in $L^2(Q)$. Then, we can pass to the limit in (2.3) and deduce that $y=y_u$ is a solution of (1.1). Moreover, (2.2) follows from the estimates established for y_k. The uniqueness is obtained in the standard way. Indeed, if y_1 and y_2 are two solutions of (1.1) such that $f(y_i)\in L^2(Q)$ for $i=1,2$, then we test (1.1) with $\mathrm{e}^{-2\Lambda_f t}(y_2-y_1)$ and, arguing as above, we deduce that $y_2-y_1=0$. Finally, the convergence property stated in the theorem follows easily from the estimate (2.2). ∎

Let us remark that the crucial part of the proof was to establish the uniform boundedness of $\{f_k(y_k)\}_{k=1}^\infty$ in $L^2(Q)$, which was used to establish the boundedness of $\{y_k\}_{k=1}^\infty$ in $W(0,T)$. Here, the assumptions **(A2)** on f were essential.

Now, we prove some extra $L^p(Q)$ regularity of the solution y_u. First we state the following lemma.

LEMMA 2.2. *The following properties are satisfied:*

I. *The space* $L^2(0,T;H_0^1(\Omega))\cap L^\infty(0,T;L^2(\Omega))$ *is continuously embedded in* $L^p(Q)$ *with* $p=\frac{2(n+2)}{n}$.

II. *If* $u\in L^2(Q)$ *and* $y_0\in H_0^1(\Omega)\cap L^\infty(\Omega)$, *then* $y_u\in H^1(Q)$ *holds.*

III. *Let* $\frac{1}{\sigma}+\frac{n}{2\gamma}<1$ *with* $\sigma,\gamma\in[2,\infty]$, *and* $y_0\in L^\infty(\Omega)$ *be given. Then there exists a constant* C *independent of* y_0 *such that for all* $u\in L^\sigma(0,T;L^\gamma(\Omega))$ *it holds that*

$$\|y_u\|_{L^\infty(Q)}\leq C\big(\|u\|_{L^\sigma(0,T;L^\gamma(\Omega))}+\|y_0\|_{L^\infty(\Omega)}\big).\tag{2.6}$$

Proof. I. It is enough to apply the Gagliardo–Nirenberg inequality (see, e.g., [19, p. 125]) with $p=\frac{2(n+2)}{n}$, $a=\frac{2}{p}$, $r=q=2$, and $m=0$ to get

$$\|y\|_{L^p(\Omega)} \le C_1\|\nabla y\|_{L^2(\Omega)}^{\frac{2}{p}}\|y\|_{L^2(\Omega)}^{1-\frac{2}{p}}.$$

Integrating this inequality on $(0,T)$ implies the claim.

II. Since $f(y_u)\in L^2(Q)$ by Theorem 2.1, the $H^1(Q)$ regularity follows from the classical results for linear parabolic equations; see, for instance, [21, section III.2].

III. By the change of variables $\phi=\mathrm{e}^{-\Lambda_f t}y_u$, (1.1) is transformed in

$$\begin{cases}\dfrac{\partial\phi}{\partial t}+A\phi+\hat f(t,\phi)=\mathrm{e}^{-\Lambda_f t}u \ \text{ in } Q=\Omega\times(0,T),\\ \phi=0 \ \text{ on } \Sigma=\Gamma\times(0,T), \quad \phi(x,0)=y_0(x) \ \text{ in } \Omega,\end{cases}$$

where $\hat f:[0,T]\times\mathbb{R}\longrightarrow\mathbb{R}$ is given by $\hat f(t,s)=\Lambda_f s+\mathrm{e}^{-\Lambda_f t}f(\mathrm{e}^{\Lambda_f t}s)$. We note that (2.1) implies $\frac{\partial\hat f}{\partial s}(t,s)=\Lambda_f+f'(\mathrm{e}^{\Lambda_f t}s)\ge 0$ and $\hat f(t,0)=0$.

We set $\beta=\|u\|_{L^\sigma(0,T;L^\gamma(\Omega))}+\|y_0\|_{L^\infty(\Omega)}$. We assume that $\beta>0$; otherwise $\phi=0$ and (2.6) holds. We also set $\phi_\beta=\frac{1}{\beta}\phi$, $u_\beta=\frac{1}{\beta}u$, and $y_{0\beta}=\frac{1}{\beta}y_0$. Then, ϕ_β satisfies the equation

$$\begin{cases}\dfrac{\partial\phi_\beta}{\partial t}+A\phi_\beta+\dfrac{1}{\beta}\hat f(t,\phi)=\mathrm{e}^{-\Lambda_f t}u_\beta \ \text{ in } Q=\Omega\times(0,T),\\ \phi_\beta=0 \ \text{ on } \Sigma=\Gamma\times(0,T), \quad \phi_\beta(x,0)=y_{0\beta}(x) \ \text{ in } \Omega.\end{cases}$$

Let $k\ge 1$ be given. Define $\phi_{\beta,k}=\phi_\beta-P_k(\phi_\beta)$. Testing the above equation with $\phi_{\beta,k}$, integrating in $(0,t)$ with $t\in(0,T)$, and using that $\frac{\partial\phi_\beta}{\partial t}\phi_{\beta,k}=\frac{\partial\phi_{\beta,k}}{\partial t}\phi_{\beta,k}$, $\nabla\phi_\beta\cdot\nabla\phi_{\beta,k}=|\nabla\phi_{\beta,k}|^2$, and $\hat f(t,\phi(x,t))\phi_{\beta,k}(x,t)\ge 0$, we infer

$$\frac{1}{2}\|\phi_{\beta,k}(t)\|_{L^2(\Omega)}^2+\Lambda_A\|\phi_{\beta,k}\|_{L^2(0,t;H_0^1(\Omega))}^2\le\int_0^t\int_\Omega \mathrm{e}^{-\Lambda_f s}u_\beta\phi_{\beta,k}\,\mathrm{d}x\,\mathrm{d}s.$$

The proof now follows the lines of the one of [15, Theorem III.7.1] to deduce the existence of a constant $C>0$ independent of (u,y_0) such that $\|\phi_\beta\|_{L^\infty(Q)}\le C$. Therefore, we have

$$\|\phi\|_{L^\infty(Q)}=\beta\|\phi_\beta\|_{L^\infty(Q)}\le C(\|u\|_{L^\sigma(0,T;L^\gamma(\Omega))}+\|y_0\|_{L^\infty(\Omega)}),$$

which implies (2.6). ∎

THEOREM 2.3. *Let $u\in L^r(Q)$ and $y_0\in L^\infty(\Omega)$ be given such with $r\in[2,1+\frac{n}{2}]$. Then the solution y_u of (1.1) belongs to $L^q(Q)$, where q has to be chosen as follows:*

1. *if $r<1+\frac{n}{2}$, then*

$$q=r\frac{n+2}{n+2-2r}\ge r\frac{n+2}{n}, \tag{2.7}$$

2. *if $r=1+\frac{n}{2}$, then $q<+\infty$ is arbitrary.*

In particular, there exists $C=C(q,r)>0$ independent of u and y_0 such that

$$\|y_u\|_{L^q(Q)}\le C(\|u\|_{L^r(Q)}+\|y_0\|_{L^\infty(\Omega)}). \tag{2.8}$$

Proof. For $r < 1 + \frac{n}{2}$ we set $p = \frac{rn}{n+2-2r}$. Due to the assumptions on r, it follows that $n \geq 3$ and $p \geq r\frac{n}{n-2} \geq r \geq 2$. In the critical case $r = 1 + \frac{n}{2}$, we can choose $p \geq 2$ arbitrarily. Then in both cases, we have $p \geq 2$ and p satisfies

$$\frac{1}{r} + \frac{p-1}{p}\frac{n}{n+2} \leq 1. \tag{2.9}$$

In addition, (2.7) yields $q = p\frac{n+2}{n}$.

Throughout the proof we abbreviate $y := y_u$.

1. *Estimates for regular* y. Let us assume for the moment that $y_t \in L^2(Q)$ and $y \in L^\infty(Q)$. Then, we have that $|y|^{p-2}y \in H^1(Q) \cap L^\infty(Q)$. Note that

$$\begin{aligned}\int_\Omega \sum_{i,j=1}^n a_{ij}\partial_{x_i} y \partial_{x_j}(|y|^{p-2}y)\,\mathrm{d}x &= \int_\Omega (p-1)\sum_{i,j=1}^n a_{ij}\partial_{x_i} y\partial_{x_j} y \cdot |y|^{p-2}\,\mathrm{d}x \\ &= \int_\Omega \frac{4(p-1)}{p^2}\sum_{i,j=1}^n a_{ij}\partial_{x_i}(|y|^{p/2})\partial_{x_j}(|y|^{p/2})\,\mathrm{d}x \\ &\geq \frac{4(p-1)}{p^2}\Lambda_A \int_\Omega |\nabla(|y|^{\frac{p}{2}})|^2\,\mathrm{d}x.\end{aligned}$$

Taking $|y|^{p-2}y$ as a test function in the weak formulation of (1.1), integrating on $(0,t)\times\Omega$, and using the above inequality results in

$$\begin{aligned}&\frac{1}{p}(\|y(t)\|^p_{L^p(\Omega)} - \|y_0\|^p_{L^p(\Omega)}) + \frac{4(p-1)}{p^2}\Lambda_A\int_0^t\int_\Omega |\nabla(|y|^{\frac{p}{2}})|^2\,\mathrm{d}x\,\mathrm{d}s \\ &\qquad + \int_0^t\int_\Omega f(y)y|y|^{p-2}\,\mathrm{d}x\,\mathrm{d}s \leq \int_0^t\int_\Omega uy|y|^{p-2}\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

Since $f(y)y \geq -\Lambda_f y^2$, we obtain

$$\begin{aligned}&\frac{1}{p}\|y(t)\|^p_{L^p(\Omega)} + \frac{4(p-1)}{p^2}\Lambda_A\int_0^t\int_\Omega |\nabla(|y|^{\frac{p}{2}})|^2\,\mathrm{d}x\,\mathrm{d}s \\ &\qquad \leq \int_Q |u|\cdot|y|^{p-1}\,\mathrm{d}x\,\mathrm{d}s + \frac{1}{p}\|y_0\|^p_{L^p(\Omega)} + \Lambda_f\int_0^t \|y(t)\|^p_{L^p(\Omega)}\,\mathrm{d}s.\end{aligned}$$

By the Gronwall inequality, we obtain

$$\|y\|^p_{L^\infty(0,T;L^p(\Omega))} + \|\nabla(|y|^{\frac{p}{2}})\|^2_{L^2(Q)} \leq C_1\left(\int_Q |u|\cdot|y|^{p-1}\,\mathrm{d}x\,\mathrm{d}s + \|y_0\|^p_{L^p(\Omega)}\right),$$

which is an estimate of $|y|^{\frac{p}{2}}$ in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$. Using Lemma 2.2, property I, this space embeds continuously into $L^{\frac{2(n+2)}{n}}(Q)$, which implies $y \in L^{p\frac{n+2}{n}}(Q) = L^q(Q)$ together with the corresponding estimate

$$\|y\|^p_{L^q(Q)} \leq C_2\left(\int_Q |u|\cdot|y|^{p-1}\,\mathrm{d}x\,\mathrm{d}s + \|y_0\|^p_{L^p(\Omega)}\right).$$

Due to the property (2.9), we can apply Hölder and Young inequalities, and we get

$$\|y\|^p_{L^q(Q)} \leq C_3(\|u\|^p_{L^r(Q)} + \|y_0\|^p_{L^p(\Omega)}),$$

which is the claim. In the critical case $r = 1 + \frac{n}{2}$, we can chose p and thus q arbitrarily large. In any case, the constant C in the inequality (2.8) depends on p and q.

2. *General case.* Given $u \in L^r(Q)$ we set $u_k = P_k(u)$. For $y_0 \in L^\infty(\Omega)$, we take a sequence $\{\hat{y}_{0k}\}_{k=1}^\infty \subset H_0^1(\Omega)$ such that $\hat{y}_{0k}(x) \to y_0(x)$ for almost every $x \in \Omega$. Now, we define $y_{0k} = P_{M_0}(\hat{y}_k)$ with $M_0 = \|y_0\|_{L^\infty(\Omega)}$. We still have that $\{y_{0k}\}_{k=1}^\infty \subset H_0^1(\Omega)$ and $\|y_{0k}\|_{L^\infty(\Omega)} \le \|y_0\|_{L^\infty(\Omega)}$. Then, the solution y_k of (1.1) associated with (u_k, y_{0k}) is an element of $H^1(Q) \cap L^\infty(Q)$; see Lemma 2.2. From Theorem 2.1 we infer that $y_k \rightharpoonup y$ in $W(0,T)$. Moreover, every function y_k satisfies the inequality (2.8) with y_k in the left-hand side and u_k and y_{0k} on the right. Now, it is easy to pass to the limit in this inequality and to deduce that y satisfies (2.8) as well. ◻

The reader is referred to [22] for other L^p estimates in the case of linear equations, which were proven using semigroup theory.

Remark 2.4. The regularity $y_0 \in L^\infty(\Omega)$ was used in the proof to be able to perform the approximation procedure in the second part, as the existence of $L^\infty(Q)$ solutions for the nonlinear equation requires this regularity of y_0. The estimates themselves only used L^p-norms of y_0, $p < \infty$.

2.2. Example. Let us show by means of a small counterexample that the solution of (1.1) is not necessarily an element of $L^\infty(Q)$ if the control u is just an element of $L^2(Q)$. Actually, we prove something more general: for $n \ge 2$ and smooth domain Ω the space $L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$ is not contained in $L^\infty(Q)$.

For $r,s>0$, let $Q_{r,s} := B_r(0) \times [1-s, 1+s] \subset \mathbb{R}^{n+1}$, where $B_r(0)$ is the open ball of radius r. Let us choose $\phi \in C_c^\infty(\mathbb{R}^{n+1})$ such that $0 \le \phi(x,t) \le 1$, $\phi = 1$ on $Q_{1,1}$, and $\phi = 0$ on $\mathbb{R}^{n+1} \setminus Q_{2,2}$. We set $Q = \Omega \times (0,T) = B_2(0) \times (0,2)$ and define the function y in Q by

$$y(x,t) := \sum_{k=1}^\infty k^{-1} \phi(2^k x, 2^{2k}(t-1)).$$

Note that for $(x,t) \ne (0,1)$ only finitely many summands are nonzero. The derivatives of $(x,t) \mapsto \phi(2^k x, 2^{2k}(t-1))$ are supported on $Q_{2^{1-k},2^{1-2k}} \setminus Q_{2^{-k},2^{-2k}}$, hence the supports of the derivatives of the terms in the sum are disjoint. Due to this fact, and using the coordinate transform $(\hat{x},\hat{t}) = (2^k x, 2^{2k}(t-1))$, we deduce

$$\|\partial_t y\|_{L^2(Q)}^2 = \|\partial_t y\|_{L^2(\mathbb{R}^{n+1})}^2 = \sum_{k=1}^\infty k^{-2} 2^{k(2-n)} \|\partial_t \phi\|_{L^2(\mathbb{R}^{n+1})}^2 < +\infty$$

and similarly

$$\|\partial_{x_i}\partial_{x_j} y\|_{L^2(Q)}^2 = \|\partial_{x_i}\partial_{x_j} y\|_{L^2(\mathbb{R}^{n+1})}^2 = \sum_{k=1}^\infty k^{-2} 2^{k(2-n)} \|\partial_{x_i}\partial_{x_j} \phi\|_{L^2(\mathbb{R}^{n+1})}^2 < +\infty.$$

Since y vanishes in $Q \setminus Q_{1,1}$, it follows $y \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$ and $y(x,0) = 0$. For $m \in \mathbb{N}$, let $(x,t) \in Q_{2^{-m},2^{-2m}}$. Then

$$y(x,t) \ge \sum_{k=1}^m k^{-1} \phi(2^k x, 2^{2k}(t-1)) = \sum_{k=1}^m k^{-1}.$$

Clearly, $Q_{2^{-m},2^{-2m}}$ has positive measure, and $y \notin L^\infty(Q)$. Now, setting $u = \frac{\partial y}{\partial t} - \Delta y$, we infer that y is the unique solution of

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y = u & \text{in } Q, \\ y = 0 \ \text{ on } \Sigma, \quad y(x,0) = 0 & \text{in } \Omega. \end{cases}$$

Moreover, since $L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))\cap H^1(0,T;L^2(\Omega))\subset C([0,T];H_0^1(\Omega))$ and $H_0^1(\Omega)\subset L^6(\Omega)$ if $n\le 3$, for $f(y)=y^3$ we have that

$$\|f(y)\|_{L^2(Q)}\le C\|y\|^2_{C([0,T];H_0^1(\Omega))}\|y\|_{L^2(0,T;H_0^1(\Omega))}<\infty.$$

Therefore, $u=\frac{\partial y}{\partial t}-\Delta y+f(y)\in L^2(Q)$ for $n=2$ or 3, and $y\not\in L^\infty(Q)$ solves (1.1).

2.3. Existence of solutions in $L^2(Q)$. In this section, we prove the existence of at least one solution to (P). Below we will prove that any local solution of (P) belongs to $L^\infty(Q)$. Here, local solutions are intended in the sense of $L^2(Q)$. Let us start proving the existence of optimal controls in $L^2(Q)$. The proof is standard, and we only give a brief sketch.

THEOREM 2.5. *Problem* (P) *admits a global solution.*

Proof. Due to the structure of the cost functional J, a minimizing sequence $\{u_k\}_{k=1}^\infty$ is bounded in $L^2(Q)$, and hence we can assume (after passing to a subsequence if necessary) that $u_k\rightharpoonup\bar u$ in $L^2(Q)$. Due to Theorem 2.1, we can pass to the limit in the state equation. Using the weak sequentially lower semicontinuity of the cost functional J, we can prove that $\bar u$ is a global solution of (P). ∎

2.4. Local solutions are in $L^\infty(Q)$. In order to prove that local solutions of (P) are in $L^\infty(Q)$, we employ the following auxiliary problems, which are localized and contain box constraints parametrized by M. Let a local minimizer $\bar u$ of (P) be given. Let $\rho>0$ be such that $J(\bar u)\le J(u)$ for all u with $\|u-\bar u\|_{L^2(Q)}\le\rho$. We define the following problem:

$$(\mathrm{P}_M)\qquad \min J(u)+\frac{1}{2}\|u-\bar u\|^2_{L^2(Q)}$$

subject to $\|u-\bar u\|_{L^2(Q)}\le\rho$, $|u(x,t)|\le M$ f.a.a. $(x,t)\in Q$.

Similar to Theorem 2.5, we obtain solvability of (P_M).

LEMMA 2.6. *Let $\{u_M\}_{M>0}$ be a family of solutions of* (P_M). *Then $u_M\to\bar u$ in $L^2(Q)$ for $M\to\infty$.*

Proof. Let $M_k\to\infty$ and set $u_k:=u_{M_k}$. We can assume (after passing to a subsequence if necessary) that $u_k\rightharpoonup u^*$ in $L^2(Q)$. Let us define the truncation $\bar u_k=P_{M_k}(\bar u)$. Then $\bar u_k\to\bar u$ in $L^2(Q)$. Hence, $\bar u_k$ is a feasible control for problem (P_{M_k}) for k large enough and, consequently, $J(u_k)+\frac12\|u_k-\bar u\|^2_{L^2(Q)}\le J(\bar u_k)+\frac12\|\bar u_k-\bar u\|^2_{L^2(Q)}$. Due to the weak lower semicontinuity of J on $L^2(Q)$, we can pass to the limit in this inequality to obtain $J(u^*)+\frac12\|u^*-\bar u\|^2_{L^2(Q)}\le J(\bar u)$. Since $\|u^*-\bar u\|_{L^2(Q)}\le\rho$, it follows $\bar u=u^*$ by the optimality of $\bar u$ in the ball $B_\rho(\bar u)$. By the properties of limit inferior and superior, we have

$$\begin{aligned}
J(\bar u)&=\lim_{k\to\infty}\left(J(\bar u_k)+\frac12\|\bar u_k-\bar u\|^2_{L^2(Q)}\right)\\
&\ge\limsup_{k\to\infty}\left(J(u_k)+\frac12\|u_k-\bar u\|^2_{L^2(Q)}\right)\\
&\ge\liminf_{k\to\infty}J(u_k)+\limsup_{k\to\infty}\frac12\|u_k-\bar u\|^2_{L^2(Q)}\\
&\ge J(\bar u)+\limsup_{k\to\infty}\frac12\|u_k-\bar u\|^2_{L^2(Q)}\\
&\ge J(\bar u)+\liminf_{k\to\infty}\frac12\|u_k-\bar u\|^2_{L^2(Q)}\ge J(\bar u).
\end{aligned}$$

Hence $\|u_k - \bar{u}\|^2_{L^2(Q)} \to 0$. Since the limit is independent of the chosen subsequence, the claim follows. ∎

From this lemma we infer the existence of M_0 such that $\|u_M - \bar{u}\|_{L^2(Q)} < \rho$ for all $M > M_0$. Hence, u_M is a local minimizer of $J(u) + \frac{1}{2}\|u - \bar{u}\|^2_{L^2(Q)}$ on the set of controls of $u \in L^2(Q)$ such that $|u| \leq M$. Since the set of feasible controls for (P_M) is bounded in $L^\infty(Q)$, then a classical proof [23, Chapter 5] establishes the following optimality conditions for the local minimizers.

THEOREM 2.7. *Let u_M be a local minimizer of* (P_M) *for $M > M_0$. Then, there exists $\varphi_M \in H^1(Q) \cap L^\infty(Q)$ satisfying*

$$\begin{cases} -\dfrac{\partial \varphi_M}{\partial t} + A^*\varphi_M + f'(y_M)\varphi_M = y_M - y_d & \textit{in } Q, \\ \varphi_M = 0 \quad \textit{on } \Sigma, \quad \varphi_M(x,T) = 0 & \textit{in } \Omega, \end{cases} \tag{2.10}$$

$$\int_Q (\varphi_M + \alpha u_M + u_M - \bar{u})(v - u_M)\,\mathrm{d}x\,\mathrm{d}t \geq 0 \quad \forall v \in L^2(Q): \ |v| \leq M, \tag{2.11}$$

where y_M is the state associated with u_M and

$$A^*\varphi = -\sum_{i,j=1}^n \partial_{x_j}(a_{ji}(x)\partial_{x_i}\varphi) + a_0(x)\varphi.$$

From (2.10) and due to $y_d \in L^p(0,T;L^q(\Omega))$ with $p,q \in [2,\infty]$ and $\frac{1}{p} + \frac{n}{2q} < 1$, the boundedness of φ_M follows from [15, Theorem III.7.1]. The $H^1(Q)$ regularity is classical; see [21, section III.2].

THEOREM 2.8. *Let $\bar{u}$ be a local minimizer of* (P). *Then $\bar{u} \in L^\infty(Q)$ holds.*

Proof. From Lemma 2.6 we know that there exists a number $M_0 > 0$ and a family $\{u_M\}_{M>M_0}$ of local minimizers of problems (P_M) such that (2.10)–(2.11) hold and $u_M \to \bar{u}$ in $L^2(Q)$ as $M \to \infty$. Denote by y_M the state associated with u_M. From (2.10) we deduce that $\{\varphi_M\}_{M>M_0}$ is bounded in $W(0,T)$. Hence, there exists a sequence $\{M_k\}_{k=1}^\infty$ converging to infinity and a function $\varphi \in W(0,T)$ such that $\varphi_k = \varphi_{M_k} \rightharpoonup \varphi$ in $W(0,T)$. Due to the compactness of the embedding $W(0,T) \subset L^2(Q)$ [16, Theorem 5.1], we have that $\varphi_k \to \varphi$ in $L^2(Q)$. Let us denote $u_k = u_{M_k}$ and $y_k = y_{M_k}$. Taking a new subsequence, we can also assume that $\varphi_k(x,t) \to \varphi(x,t)$ and $u_k(x,t) \to \bar{u}(x,t)$ for almost all $(x,t) \in Q$.

Now, from (2.11) we infer

$$u_k = P_{M_k}\left(-\frac{1}{\alpha}[\varphi_k + u_k - \bar{u}]\right). \tag{2.12}$$

Passing pointwise to the limit in the above identity we deduce that $\bar{u} = -\frac{1}{\alpha}\varphi$. We are going to prove that $\varphi \in L^\infty(Q)$. First, (2.10) is split into two equations:

$$\begin{cases} -\dfrac{\partial \phi_k}{\partial t} + A^*\phi_k + f'(y_k)\phi_k = y_k & \text{in } Q, \\ \phi_k = 0 \quad \text{on } \Sigma, \quad \phi_k(x,T) = 0 & \text{in } \Omega \end{cases} \tag{2.13}$$

and

$$\begin{cases} -\dfrac{\partial \psi_k}{\partial t} + A^*\psi_k + f'(y_k)\psi_k = y_d & \text{in } Q, \\ \psi_k = 0 \quad \text{on } \Sigma, \quad \psi_k(x,T) = 0 & \text{in } \Omega. \end{cases} \tag{2.14}$$

Then, we have $\varphi_k = \phi_k - \psi_k$, $\phi_k \rightharpoonup \phi$ and $\psi_k \rightharpoonup \psi$ in $W(0,T)$, and $\varphi = \phi - \psi$. Due to our assumptions on y_d, we know that $\{\psi_k\}_{k=1}^\infty$ is uniformly bounded in $L^\infty(Q)$; see Lemma 2.2, property III. As a consequence, we get that $\psi \in L^\infty(Q)$. We are going to prove that $\{\phi_k\}_{k=1}^\infty$ is also bounded in $L^\infty(Q)$. Since $\{u_k\}_{k=1}^\infty$ is bounded in $L^2(Q)$, we infer from Theorem 2.3

$$\|y_k\|_{L^{2\frac{n+2}{n-2}}(Q)} \le C\big(\|u_k\|_{L^2(Q)} + \|y_0\|_{L^\infty(Q)}\big) \le C_1 \quad \forall k \ge 1.$$

If $n \le 5$, then the inequality $2\frac{n+2}{n-2} > 1 + \frac{n}{2}$ holds. Therefore, applying again Lemma 2.2, property III, to (2.13), we deduce the existence of a constant $C_2 > 0$ such that $\|\phi_k\|_{L^\infty(Q)} \le C_2$ for every $k \ge 1$. This yields $\varphi \in L^\infty(Q)$ and $\bar u \in L^\infty(Q)$ as well.

For $n > 5$ we can repeat the arguments of Theorem 2.3 for (2.13) and deduce

$$\|\phi_k\|_{L^{2\frac{(n+2)^2}{(n-2)^2}}(Q)} \le C\|y_k\|_{L^{2\frac{n+2}{n-2}}(Q)} \le CC_1 \quad \forall k \ge 1.$$

This implies that $\phi \in L^{2\frac{(n+2)^2}{(n-2)^2}}(Q)$ and consequently $\bar u = -\frac{1}{\alpha}\varphi \in L^{2\frac{(n+2)^2}{(n-2)^2}}$ holds. Using (2.11) we get

$$u_k = P_{M_k}\left(\frac{-1}{1+\alpha}[\varphi_k - \bar u]\right).$$

This implies

$$\|u_k\|_{L^{2\frac{(n+2)^2}{(n-2)^2}}(Q)} \le \frac{1}{1+\alpha}\left(\|\varphi_k\|_{L^{2\frac{(n+2)^2}{(n-2)^2}}(Q)} + \|\bar u\|_{L^{2\frac{(n+2)^2}{(n-2)^2}}(Q)}\right) \le C_3.$$

A second application of Theorem 2.3 yields

$$\|y_k\|_{L^{2\frac{(n+2)^3}{(n-2)^3}}(Q)} \le C\left(\|u_k\|_{L^{2\frac{(n+2)^2}{(n-2)^2}}} + \|y_0\|_{L^\infty(Q)}\right) \le C_4 = C\big(C_3 + \|y_0\|_{L^\infty(Q)}\big) \quad \forall k \ge 1.$$

If $2\frac{(n+2)^3}{(n-2)^3} > 1 + \frac{n}{2}$, then we argue as before to deduce that $\bar u = -\frac{1}{\alpha}\varphi \in L^\infty(Q)$. If not then we can repeat the arguments and increase the $L^p(Q)$ regularity of y_k until we obtain the desired regularity for φ after finitely many steps. $\square$

We proved that any local solution of (P) is a function belonging to $L^\infty(Q)$. Hence, the problem (P) is equivalent to the minimization of J on $L^\infty(Q)$. It is well known that the mapping $u \mapsto y_u$ from $L^\infty(Q)$ to $W(0,T) \cap L^\infty(Q)$ is of class C^1. Then, we can write the necessary optimality conditions satisfied by any local minimizer $\bar u$ of (P) as follows (see [23, Chapter 5]):

$$\begin{cases} \dfrac{\partial \bar y}{\partial t} + A\bar y + f(\bar y) = \bar u & \text{in } Q, \\ \bar y = 0 \quad \text{on } \Sigma, \quad \bar y(x,0) = y_0(x) & \text{in } \Omega, \end{cases} \tag{2.15}$$

$$\begin{cases} -\dfrac{\partial \bar\varphi}{\partial t} + A^*\bar\varphi + f'(\bar y)\bar\varphi = \bar y - y_d & \text{in } Q, \\ \bar\varphi = 0 \quad \text{on } \Sigma, \quad \bar\varphi(x,T) = 0 & \text{in } \Omega, \end{cases} \tag{2.16}$$

$$\bar\varphi + \alpha\bar u = 0, \tag{2.17}$$

where $\bar y \in W(0,T) \cap L^\infty(Q)$ and $\bar\varphi \in H^1(Q) \cap C^{\mu,\frac{\mu}{2}}(\bar Q)$ for some $\mu \in (0,1)$. The reader is referred to [15, Theorem III.10.1] for the Hölder regularity of $\bar\varphi$. Then, as a consequence of (2.17), we deduce that any local solution of (P) also belongs to $H^1(Q) \cap C^{\mu,\frac{\mu}{2}}(\bar Q)$.

Remark 2.9. Given a measurable subset $\omega \subset \Omega$ with positive Lebesgue measure, all the results of this paper are valid if we replace u in (1.1) by $u\chi_\omega$ with $u \in L^2(\omega \times (0,T))$ and χ_ω being the characteristic function of ω. The changes in the proofs are obvious.

We also observe that in real-world applications the case $u(x,t) = \sum_{j=1}^m u_j(t)g_j(x)$ with $\{u_j\}_{j=1}^m \subset L^2(0,T)$ and $\text{supp}(g_i)\cap\text{supp}(g_j) = \emptyset$ for $i \neq j$ is very interesting. In this case, if $\{g_j\}_{j=1}^m \subset L^q(\Omega)$ for $q > n$, we deduce from (2.6) that the solution of (1.1) belongs to $L^\infty(Q)$. Consequently, the mapping $(u_1,\dots,u_m) \to y$ is differentiable from $L^2(0,T)^m$ to $L^\infty(Q) \cap W(0,T)$. Hence, it is obvious to prove the existence of an optimal control and to deduce the optimality system for every local solution $\{\bar u_j\}_{j=1}^m$. Moreover, since the states belong to $L^\infty(Q)$, the adjoint states belong to $L^\infty(Q)$ as well. In this context, the optimality condition (2.17) is replaced by

$$\int_\Omega g_j(x)\bar\varphi(x,t)\,\mathrm{d}x + \alpha \bar u_j(t) = 0 \quad \text{for } 1 \le j \le m \text{ and almost all } t \in (0,T).$$

This implies that $\{\bar u_j\}_{j=1}^m \subset L^\infty(0,T)$.

3. Optimal Neumann boundary control of a semilinear elliptic equation. In this section we study the control problem

$$(\mathrm{P_{ell}}) \qquad \inf_{u \in L^2(\Gamma)} J(u) := \frac{1}{2}\int_\Omega (y_u - y_d)^2\,\mathrm{d}x + \frac{\alpha}{2}\int_\Gamma u^2\,\mathrm{d}x,$$

where y_u is the solution of the semilinear elliptic equation

$$(3.1) \qquad \begin{cases} Ay + f(\cdot,y) = g & \text{in } \Omega, \\ \partial_{\nu_A} y = u & \text{on } \Gamma. \end{cases}$$

Here, $\Omega \subset \mathbb{R}^n$ with $n > 2$ is a bounded domain with Lipschitz boundary Γ. A denotes the same operator as in section 2 and $\partial_{\nu_A} y = \sum_{i,j=1}^n a_{ij}(x)\partial_{x_i} y \nu_j(x)$, where $\nu(x)$ is the unit outward normal vector to Γ at the point x. We make the following assumptions on $(\mathrm{P_{ell}})$:

(B1) The coefficients of the operator A satisfy the conditions in (A1) with the additional requirement that $a_0 \not\equiv 0$.

(B2) $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function that is of class C^1 with respect to the second parameter satisfying

$$(3.2) \qquad f(x,0) = 0 \;\text{ and }\; \frac{\partial f}{\partial y}(x,y) \ge 0 \;\text{ for a.a. } x \in \Omega\, \forall y \in \mathbb{R}.$$

In addition, for every $M > 0$ there is $C_{f,M} > 0$ such that $|f(x,y)| + |\frac{\partial f}{\partial y}(x,y)| \le C_{f,M}$ for almost all $x \in \Omega$ and all $|y| \le M$.

(B3) $\alpha > 0$, $g, y_d \in L^p(\Omega)$ with $p > \frac{n}{2}$.

The condition $f(\cdot,0) = 0$ was imposed to shorten the presentation. It can be replaced by the condition $f(\cdot,0) \in L^p(\Omega)$ with $p > \frac{n}{2}$. In the analysis, we can then replace f and g by $f(\cdot,y) - f(\cdot,0)$ and $g - f(\cdot,0)$.

Analogously to the control problem analyzed in section 2, here we will prove that $(\mathrm{P_{ell}})$ is well posed and has at least one global minimizer in $L^2(\Gamma)$. Then, we establish that any local minimizer of $(\mathrm{P_{ell}})$ in $L^2(\Gamma)$ is actually a function of $L^\infty(\Gamma)$. This regularity implies the $C(\bar\Omega)$ regularity of the locally optimal states, which allows us to derive first and second order optimality conditions for $(\mathrm{P_{ell}})$. We recall that, under the above conditions, for $n = 2$ and for every $u \in L^2(\Gamma)$ there exists a unique solution $y_u \in H^1(\Omega) \cap C(\bar\Omega)$. Therefore, we can differentiate the relation $u \to f(y_u)$ and derive first order optimality conditions for $(\mathrm{P_{ell}})$. From these conditions we infer as usual the $C(\bar\Omega)$ regularity of the adjoint state and, consequently, the $C(\Gamma)$ regularity of the locally optimal controls. This is why we have selected $n > 2$ in this section.

3.1. Analysis of the state equation. Associated with A, we define the bilinear form $B : H^1(\Omega) \times H^1(\Omega) \longrightarrow \mathbb{R}$ by

$$B(y,z) = \int_\Omega \left(\sum_{i,j=1}^{n} a_{ij} \partial_{x_i} y \partial_{x_j} z + a_0 y z \right) \mathrm{d}x.$$

From assumption **(B1)** we get

$$\exists \Lambda_B > 0 \ \text{ such that } \ \Lambda_B \|y\|^2_{H^1(\Omega)} \le B(y,y) \quad \forall y \in H^1(\Omega). \tag{3.3}$$

In the following, $\langle \cdot,\cdot \rangle_\Omega$ and $\langle \cdot,\cdot \rangle_\Gamma$ denote the duality pairing between $H^1(\Omega)^*$ and $H^1(\Omega)$ and $H^{-\frac{1}{2}}(\Gamma)$ and $H^{\frac{1}{2}}(\Gamma)$, respectively. Let us first state the existence result for weak solutions of the state equation. We will give its proof below.

THEOREM 3.1. *Given $u \in H^{-\frac{1}{2}}(\Gamma)$ and $g \in H^1(\Omega)^*$, there exists a unique function $y_u \in H^1(\Omega)$ such that $f(\cdot, y_u) \in L^1(\Omega) \cap H^1(\Omega)^*$ and*

$$B(y_u, z) + \langle f(\cdot, y_u), z \rangle_\Omega = \langle g, z \rangle_\Omega + \langle u, z \rangle_\Gamma \quad \forall z \in H^1(\Omega). \tag{3.4}$$

Furthermore, if $u_k \to u$ in $H^{-\frac{1}{2}}(\Gamma)$, then $y_{u_k} \to y_u$ in $H^1(\Omega)$ and $f(\cdot, y_{u_k}) \to f(\cdot, y_u)$ in $L^1(\Omega) \cap H^1(\Omega)^*$ hold.

According to this result, we call $y_u \in H^1(\Omega)$ a weak solution of (3.1) if $f(\cdot, y_u) \in L^1(\Omega) \cap H^1(\Omega)^*$ and (3.4) is satisfied.

If $h \in H^1(\Omega)^*$ and there exists $\phi \in L^1(\Omega)$ such that

$$\langle h, z \rangle = \int_\Omega \phi(x) z(x) \, \mathrm{d}x \quad \forall z \in H^1(\Omega) \cap L^\infty(\Omega),$$

we say that $h \in L^1(\Omega) \cap H^1(\Omega)^*$. In this case, we identify h with ϕ.

If $z \in H^1(\Omega)$ satisfies $hz \in L^1(\Omega)$ we also have that $\langle h, z \rangle_\Omega = \int_\Omega h(x) z(x) \, \mathrm{d}x$. Indeed, define $z_k = P_k(z)$ for every integer $k \ge 1$. Then, $z_k \in H^1(\Omega) \cap L^\infty(\Omega)$ and $z_k \to z$ in $H^1(\Omega)$ holds. Moreover, since $hz \in L^1(\Omega)$, $h(x) z_k(x) \to h(x) z(x)$ for almost all $x \in \Omega$, and $|h(x) z_k(x)| \le |h(x) z(x)|$, Lebesgue's dominated convergence theorem implies that $h z_k \to hz$ in $L^1(\Omega)$. These arguments yield

$$\int_\Omega h(x) z(x) \, \mathrm{d}x = \lim_{k\to\infty} \int_\Omega h(x) z_k(x) \, \mathrm{d}x = \lim_{k\to\infty} \langle h, z_k \rangle_\Omega = \langle h, z \rangle_\Omega.$$

LEMMA 3.2. *The following properties are satisfied:*

1. *If $f(\cdot, y) \in H^1(\Omega)^* \cap L^1(\Omega)$, then $f(\cdot, y) y \in L^1(\Omega)$ holds.*
2. *If $y, z \in H^1(\Omega)$ and $f(\cdot, y), f(\cdot, z) \in H^1(\Omega)^* \cap L^1(\Omega)$, then the inequality $\langle f(\cdot, y) - f(\cdot, z), y - z \rangle_\Omega \ge 0$ is fulfilled.*

Proof. To prove the first statement, we define $y_k = P_k(y)$ for every integer $k \ge 1$. Then, we have that $y_k \to y$ in $H^1(\Omega)$, $y_k(x) \to y(x)$ for almost all $x \in \Omega$, and $\{y_k\}_{k=1}^\infty \subset L^\infty(\Omega)$. Hence, we also have $f(\cdot, y_k(x)) \to f(\cdot, y(x))$ for almost all $x \in \Omega$. Moreover, (3.2) implies that $f(\cdot, s) s \ge 0$ for every $s \in \mathbb{R}$. Therefore, using Fatou's lemma we get

$$\begin{aligned} \int_\Omega f(\cdot, y) y \, \mathrm{d}x &\le \liminf_{k\to\infty} \int_\Omega f(\cdot, y_k) y_k \, \mathrm{d}x \le \liminf_{k\to\infty} \int_\Omega f(\cdot, y) y_k \, \mathrm{d}x \\ &= \lim_{k\to\infty} \langle f(\cdot, y), y_k \rangle_\Omega = \langle f(\cdot, y), y \rangle_\Omega < \infty. \end{aligned}$$

Thus, we have that $f(\cdot,y)y \in L^1(\Omega)$. For the second part of the lemma we consider the projections $y_k = P_k(y)$ and $z_k = P_k(z)$ and use the monotonicity of f as follows:

$$\begin{aligned}\langle f(\cdot,y) - f(\cdot,z), y - z\rangle_\Omega &= \lim_{k\to\infty} \langle f(\cdot,y) - f(\cdot,z), y_k - z_k\rangle_\Omega \\ &= \lim_{k\to\infty} \int_\Omega (f(\cdot,y) - f(\cdot,z))(y_k - z_k)\,\mathrm{d}x \ge 0.\end{aligned}$$ ∎

Now, we have everything at hand to prove Theorem 3.1.

Proof of Theorem 3.1. For every integer $k \ge 1$ we define the truncation $f_k(x,s) = f(x, P_k(s))$. Applying the monotone operator theory or Schauder's fixed point theorem we infer the existence of a function $y_k \in H^1(\Omega)$ such that

$$\begin{cases} Ay_k + f_k(\cdot,y_k) = g & \text{in } \Omega, \\ \partial_{\nu_A} y_k = u & \text{on } \Gamma \text{ in } \Omega; \end{cases} \tag{3.5}$$

see [5, Theorem 3.1, Lemma 3.2] or [13]. Testing this equation with y_k and using $f_k(\cdot,s)s \ge 0$, we infer with (3.3)

$$\|y_k\|_{H^1(\Omega)} \le C_1\big(\|g\|_{H^1(\Omega)^*} + \|u\|_{H^{-\frac{1}{2}}(\Gamma)}\big).$$

Therefore, we take a subsequence, denoted in the same way, such that $y_k \rightharpoonup y$ in $H^1(\Omega)$, $y_k \to y$ in $L^2(\Omega)$, and $y_k(x) \to y(x)$ for almost all $x \in \Omega$. This implies that $f_k(\cdot, y_k(x)) \to f(\cdot, y(x))$ for almost all $x \in \Omega$. By **(B2)**, there exists $C_{f,1} > 0$ such that $|f(x,s)| \le C_{f,1}$ for almost all $x \in \Omega$ and all $|s| \le 1$. Using the weak formulation, we can derive the bound

$$\begin{aligned}\int_\Omega |f(\cdot,y_k)|\,\mathrm{d}x &\le |\Omega|C_{f,1} + \int_\Omega f_k(\cdot,y_k)y_k\,\mathrm{d}x \\ &= |\Omega|C_{f,1} + \langle g, y_k\rangle_\Omega + \langle u, y_k\rangle_\Gamma - B(y_k,y_k) \le C_2 < \infty \quad \forall k \ge 1,\end{aligned}$$

and $\{f_k(\cdot,y_k)y_k\}_{k=1}^\infty$ is bounded in $L^1(\Omega)$. Then, from Fatou's lemma we deduce

$$\int_\Omega |f(\cdot,y)|\,\mathrm{d}x \le \liminf_{k\to\infty} \int_\Omega |f(\cdot,y_k)|\,\mathrm{d}x \le C_2.$$

Thus, $f(\cdot,y) \in L^1(\Omega)$ holds. Let us prove that $\{f_k(\cdot,y_k)\}_{k=1}^\infty$ is equi-integrable. Given $\varepsilon > 0$ we select $M > 0$ such that $\frac{C_2}{M} < \frac{\varepsilon}{2}$. Let $C_{f,M}$ be given by (B2) and take $\delta > 0$ such that $\delta C_{f,M} < \frac{\varepsilon}{2}$. Then, for every measurable set $E \subset \Omega$ with $|E| < \delta$ and every $k \ge 1$ we have

$$\int_E |f_k(\cdot,y_k)|\,\mathrm{d}x \le \frac{1}{M}\int_\Omega f_k(\cdot,y_k)y_k\,\mathrm{d}x + C_{f,M}|E| \le \frac{C_2}{M} + C_{f,M}\delta < \varepsilon.$$

Therefore, from Vitali's theorem we deduce that $f_k(\cdot,y_k) \to f(\cdot,y)$ in $L^1(\Omega)$. Moreover, we have

$$\langle f_k(\cdot,y_k), z\rangle_\Omega = \int_\Omega f_k(\cdot,y_k)z\,\mathrm{d}x = \langle g, z\rangle_\Omega + \langle u, z\rangle_\Gamma - B(y_k,z) \quad \forall z \in H^1(\Omega),$$

which implies the boundedness of $\{f_k(\cdot,y_k)\}_{k=1}^\infty$ in $H^1(\Omega)^*$. All together this yields $f(\cdot,y) \in H^1(\Omega)^*$ and $f_k(\cdot,y_k) \rightharpoonup f(\cdot,y)$ in $H^1(\Omega)^*$. Further, passing to the limit in the above identity we obtain that y satisfies (3.4).

Let us prove the uniqueness. If y_1 and y_2 are solutions of (3.1), subtracting the identities (3.4) for y_2 and y_1 and taking $z = y_2 - y_1$ we infer with (3.3) and Lemma 3.2, property 2,

$$\Lambda_B \|y_2 - y_1\|^2_{H^1(\Omega)} \le B(y_2 - y_1, y_2 - y_1) + \langle f(\cdot, y_2) - f(\cdot, z_1), y_2 - y_1\rangle_\Omega = 0.$$

Finally, we prove the continuous dependence of y_u with respect to u. Let be $\{u_k\}_{k=1}^\infty$ be a sequence converging strongly to u in $H^{-\frac{1}{2}}(\Gamma)$. Taking $u = u_k$ and $z = y_{u_k}$ in (3.4) we infer with (3.3) and Lemma 3.2, property 2,

$$\begin{aligned}\Lambda_B \|y_{u_k}\|^2_{H^1(\Omega)} &\le B(y_{u_k}, y_{u_k}) + \langle f(\cdot, y_{u_k}), y_{u_k}\rangle_\Omega \\ &\le C_3\big(\|g\|_{H^1(\Omega)^*} + \|u_k\|_{H^{-\frac{1}{2}}(\Gamma)}\big)\|y_{u_k}\|_{H^1(\Omega)}.\end{aligned}$$

This implies the boundedness of $\{y_{u_k}\}_{k=1}^\infty$ in $H^1(\Omega)$ and, consequently, the convergence $y_{u_k} \rightharpoonup y$ in $H^1(\Omega)$ for a subsequence, denoted in the same way. Moreover, using Lemma 3.2, property 1, the above inequality also leads to the uniform boundedness of the integral $\int_\Omega f(\cdot, y_k) y_k \,\mathrm{d}x$. Hence, we can argue as above and deduce the equi-integrability of $\{f(\cdot, y_k)\}_{k=1}^\infty$ and the convergence $f(\cdot, y_k) \to f(\cdot, y)$ in $L^1(\Omega)$ for a subsequence, again denoted in the same way. We also have that $f(\cdot, y_k) \rightharpoonup f(\cdot, y)$ in $H^1(\Omega)^*$. Now, it is easy to pass to the limit in the equations satisfied by y_{u_k} and to deduce that $y = y_u$. From the uniqueness of the solution of (3.4) we get that the whole sequence $\{y_{u_k}\}_{k=1}^\infty$ converges weakly to y_u in $H^1(\Omega)$. Finally, the strong convergence follows with (3.3) and Lemma 3.2, property 2,

$$\begin{aligned}\Lambda_B \|y_{u_k} - y_u\|^2_{H^1(\Omega)} &\le B(y_{u_k} - y_u, y_{u_k} - y_u) + \langle f(\cdot, y_{u_k}) - f(\cdot, y_u), y_{u_k} - y_u\rangle_\Omega \\ &= \langle u_k - u, y_{u_k} - y_u\rangle_\Gamma \to 0 \quad \text{as } k \to \infty.\end{aligned}$$

Now, we prove the convergence of $f(\cdot, y_{u_k}) \to f(\cdot, y_u)$ in $H^1(\Omega)^*$ as follows:

$$\begin{aligned}\|f(\cdot, y_{u_k}) - f(\cdot, y_u)\|_{H^1(\Omega)^*} &= \sup_{\|z\|_{H^1(\Omega)} \le 1} |\langle f(\cdot, y_{u_k}) - f(\cdot, y_u), z\rangle_\Omega| \\ &= \sup_{\|z\|_{H^1(\Omega)} \le 1} |\langle u_k - u, z\rangle_\Gamma - B(y_{u_k} - y_u, z)| \\ &\le C\big(\|u_k - u\|_{H^{-\frac{1}{2}}(\Gamma)} + \|y_{u_k} - y_u\|_{H^1(\Omega)}\big) \to 0 \text{ as } k \to \infty.\end{aligned}$$

The proof of the convergence $f(\cdot, y_{u_k}) \to f(\cdot, y_u)$ in $L^1(\Omega)$ follows from Vitali's theorem as above taking into account again that $\int_\Omega f(\cdot, y_{u_k}) y_{u_k} \,\mathrm{d}x = \langle f(\cdot, y_{u_k}), y_{u_k}\rangle_\Omega \le C'$ for every k. ∎

The reader is referred to [2] for the study of the Dirichlet problem corresponding to (3.1). See also [3].

In the next theorem we establish some L^q estimates for the solution of (3.1).

THEOREM 3.3. *Let $u \in L^r(\Gamma)$ and $g \in L^s(\Omega)$ with*

$$r \in \left[2\frac{n-1}{n}, n-1\right), \quad s \in \left[\frac{2n}{n+2}, \frac{n}{2}\right)$$

satisfying

$$(n-1)\left(\frac{1}{r} - \frac{1}{n-1}\right) = n\left(\frac{1}{s} - \frac{2}{n}\right) \tag{3.6}$$

be given. Let q and $\tilde{q}$ be defined by

$$\frac{1}{q} = \frac{1}{r} - \frac{1}{n-1}, \quad \frac{1}{\tilde{q}} = \frac{1}{s} - \frac{2}{n}. \tag{3.7}$$

Then $y_u \in L^{\tilde{q}}(\Omega)$ and its trace $y_{u|_\Gamma} \in L^q(\Gamma)$ hold. Moreover, there exists a constant $C = C(r,s)$ independent of g and u such that

$$\|y_u\|_{L^{\tilde{q}}(\Omega)} + \|y_u\|_{L^q(\Gamma)} \le C\big(\|g\|_{L^s(\Omega)} + \|u\|_{L^r(\Gamma)}\big).$$

If $u \in L^{n-1}(\Gamma)$ and $g \in L^{\frac{n}{2}}(\Omega)$, then the above estimates are valid for every q and $\tilde{q}$ smaller than ∞.

The conditions in (3.7) show the improvements in the integrability, while (3.6) enforces some compatibility between all these exponents.

Proof. The proof is similar to the one of Theorem 2.3. We test the weak formulation with $|y|^{p-2}y$ for suitable p to obtain H^1-estimates of $|y|^{p/2}$. Then the exponents q and $\tilde{q}$ are derived by applying embedding and trace theorems for $|y|^{p/2} \in H^1(\Omega)$, respectively. Let us set

$$\frac{1}{p} = \frac{n-1}{n-2}\left(\frac{1}{r} - \frac{1}{n-1}\right) = \frac{n}{n-2}\left(\frac{1}{s} - \frac{2}{n}\right), \tag{3.8}$$

which is well-defined due to (3.6), and $p \ge 2$ holds due to $r \ge 2\frac{n-1}{n}$.

We define $y_k = P_k(y_u)$ for $k \ge 1$. Then, $|y_k|^{p-2}y_k \in H^1(\Omega) \cap L^\infty(\Omega)$ can be used as a test function in the weak formulation, leading to

$$B(y, |y_k|^{p-2}y_k) + \int_\Omega f(\cdot,y)|y_k|^{p-2}y_k \,\mathrm{d}x = \int_\Omega g|y_k|^{p-2}y_k \,\mathrm{d}x + \int_\Gamma u|y_k|^{p-2}y_k \,\mathrm{d}x.$$

Using (3.3) we get

$$\begin{aligned}
B(y, |y_k|^{p-2}y_k) &= \int_\Omega \sum_{i,j=1}^n a_{ij}\partial_{x_i} y \partial_{x_j}(|y_k|^{p-2}y_k) + a_0 y|y_k|^{p-2}y_k \,\mathrm{d}x \\
&\ge \int_\Omega (p-1)\sum_{i,j=1}^n a_{ij}\partial_{x_i} y_k \partial_{x_j} y_k \cdot |y_k|^{p-2} + a_0|y_k|^p \,\mathrm{d}x \\
&= \int_\Omega \frac{4(p-1)}{p^2}\sum_{i,j=1}^n a_{ij}\partial_{x_i}(|y_k|^{p/2})\partial_{x_j}(|y_k|^{p/2}) + a_0(|y_k|^{p/2})^2 \,\mathrm{d}x \\
&\ge \frac{4(p-1)}{p^2} B(|y_k|^{p/2}, |y_k|^{p/2}) \ge \Lambda_B \frac{4(p-1)}{p^2}\||y_k|^{p/2}\|^2_{H^1(\Omega)},
\end{aligned}$$

where we used $\frac{4(p-1)}{p^2} \le 1$ for $p \ge 2$. In addition, we have $f(\cdot,y)|y_k|^{p-2}y_k \ge 0$. Hence, we arrive at the inequality

$$\Lambda_B \frac{4(p-1)}{p^2}\||y_k|^{p/2}\|^2_{H^1(\Omega)} \le \int_\Omega |g| \cdot |y_k|^{p-1}\,\mathrm{d}x + \int_\Gamma |u| \cdot |y_k|^{p-1}\,\mathrm{d}x.$$

Using the continuity of the embedding $H^1(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$ and of the trace $H^1(\Omega) \hookrightarrow L^{\frac{2n-2}{n-2}}(\Gamma)$, we infer

$$\begin{aligned}
&\|y_k\|^p_{L^{\frac{pn}{n-2}}(\Omega)} + \|y_k\|^p_{L^{\frac{p(n-1)}{n-2}}(\Gamma)} \\
&\qquad \le \frac{C_1}{\Lambda_B}\frac{p^2}{4(p-1)}\left(\int_\Omega |g| \cdot |y_k|^{p-1}\,\mathrm{d}x + \int_\Gamma |u| \cdot |y_k|^{p-1}\,\mathrm{d}x\right),
\end{aligned} \tag{3.9}$$

where $C_1 = C_1(n,\Omega)$. By definition of p in (3.8), we find[1]

$$\frac{1}{r} + \frac{p-1}{p}\frac{n-2}{n-1} = 1, \quad \frac{1}{s} + \frac{p-1}{p}\frac{n-2}{n} = 1,$$

and we can apply the Hölder and Young inequalities to obtain

$$\|y_k\|^p_{L^{\frac{pn}{n-2}}(\Omega)} + \|y_k\|^p_{L^{\frac{p(n-1)}{n-2}}(\Gamma)} \le C_2\big(\|g\|^p_{L^s(\Omega)} + \|u\|^p_{L^r(\Gamma)}\big). \tag{3.10}$$

The exponents in the above inequality satisfy $\frac{p(n-1)}{n-2} = q$ and $\frac{pn}{n-2} = \tilde{q}$ by construction. The claim now follows by taking the limit $k \to \infty$ and Lebesgue's dominated convergence theorem. The last statement of the theorem is an immediate consequence of the first part. □

This theorem is similar to [4, Theorem 18], where $L^{\tilde{q}}(\Omega)$-$L^s(\Omega)$ estimates are proven for a problem with a homogeneous Dirichlet boundary condition. Note that the above proof cannot be used to derive L^∞-estimates of y; see (3.9).

3.2. Analysis of the control problem. By the usual approach of taking a minimizing sequence, it is immediate to establish the existence of a global minimizer of problem $(\mathrm{P_{ell}})$ with the help of Theorem 3.1. Observe that the weak convergence $u_k \rightharpoonup u$ in $L^2(\Gamma)$ implies the strong convergence $u_k \to u$ in $H^{-\frac{1}{2}}(\Gamma)$. The goal of this section is to prove that any local (global) minimizer of $(\mathrm{P_{ell}})$ in the $L^2(\Gamma)$ sense is a function of $L^\infty(\Gamma)$. For this purpose we follow the steps of section 2.4. Given a local minimizer $\bar{u}$, we take $\rho > 0$ such that $J(\bar{u}) \le J(u)$ for all u with $\|u - \bar{u}\|_{L^2(\Gamma)} \le \rho$. Now, we define the control problems:

$$\min J(u) + \frac{1}{2}\|u - \bar{u}\|^2_{L^2(\Gamma)} \tag{$\mathrm{P_{ell,\mathit{M}}}$}$$

subject to $\|u - \bar{u}\|_{L^2(\Gamma)} \le \rho$ and $|u(x)| \le M$ f.a.a. $x \in \Gamma$. $(\mathrm{P}_{\mathrm{ell},M})$ has at least one solution u_M. Moreover, arguing as in Lemma 2.6, we get that $u_M \to \bar{u}$ in $L^2(\Gamma)$ as $M \to \infty$. Then, we select $M_0 > 0$ such that $\|u_M - \bar{u}\|_{L^2(\Gamma)} < \rho$ for every $M > M_0$. For $M > M_0$, the optimality conditions satisfied by u_M are written as follows:

$$\begin{cases} A^*\varphi_M + \dfrac{\partial f}{\partial y}(\cdot, y_M)\varphi_M = y_M - y_d & \text{in } \Omega, \\ \partial_{\nu_{A^*}}\varphi_M = 0 & \text{on } \Gamma, \end{cases} \tag{3.11}$$

$$\int_\Gamma (\varphi_M + \alpha u_M + u_M - \bar{u})(v - u_M)\,\mathrm{d}x\,\mathrm{d}t \ge 0 \quad \forall v \in L^2(\Gamma):\ |v| \le M, \tag{3.12}$$

where y_M is the state associated with u_M and $\varphi_M \in H^1(\Omega) \cap L^\infty(\Omega)$; see [23, Chapter 4]. Observe that $y_M \in H^1(\Omega) \cap L^\infty(\Omega)$ holds due to the assumption **(B3)** on g and the fact that $u_M \in L^\infty(\Gamma)$. As a consequence, we also get with **(B3)** that $\varphi_M \in L^\infty(\Omega)$.

Analogously to Theorem 2.8 we have the following result.

THEOREM 3.4. *Let $\bar{u}$ be a local minimizer of* $(\mathrm{P_{ell}})$. *Then, $\bar{u} \in L^\infty(\Gamma)$ holds.*

The proof of this theorem follows the same arguments used to prove Theorem 2.8 with the obvious changes. The only difference is that we use the estimates established in Theorem 3.3 instead of the ones provided in Theorem 2.3. First we get $L^p(\Omega)$ estimates for the states y_M and with them we derive $L^q(\Gamma)$ estimates for the adjoint state φ_M.

[1] The condition here can be written equivalently as $1 - \frac{1}{p} = (1 - \frac{1}{r})\frac{n-1}{n-2} = (1 - \frac{1}{s})\frac{n}{n-2}$.

Once the $L^\infty(\Gamma)$ regularity is proved for any local minimizer of $(\mathrm{P}_{\mathrm{ell}})$, using the differentiability of the mapping $G: L^\infty(\Gamma) \longrightarrow H^1(\Omega) \cap L^\infty(\Omega)$, we can get the first order optimality conditions satisfied by any local minimizer $\bar u$:

$$\begin{cases} A\bar y + f(\cdot, \bar y) = g & \text{in } \Omega, \\ \partial_{\nu_A} \bar y = \bar u & \text{on } \Gamma, \end{cases} \tag{3.13}$$

$$\begin{cases} A^*\bar\varphi + \dfrac{\partial f}{\partial y}(\cdot, \bar y)\bar\varphi = \bar y - y_d & \text{in } \Omega, \\ \partial_{\nu_{A^*}} \bar\varphi = 0 & \text{on } \Gamma, \end{cases} \tag{3.14}$$

$$\bar\varphi_{|\Gamma} + \alpha \bar u = 0. \tag{3.15}$$

The reader is referred to [23, Chapter 4]. We have the regularity $\bar y \in H^1(\Omega) \cap C^\mu(\bar\Omega)$ and $\bar\varphi \in H^1(\Omega) \cap C^\mu(\bar\Omega)$ for some $\mu \in (0,1)$; see [1, 17, 20] for the Hölder regularity. Moreover, from (3.15) the $H^{\frac12}(\Gamma) \cap C^\mu(\Gamma)$ regularity of $\bar u$ follows.

Remark 3.5. The arguments used in this section can be applied to the study of the distributed control problem

$$\inf_{u \in L^2(\Omega)} J(u) := \frac12 \int_\Omega [(y_u - y_d)^2 + \alpha u^2]\,\mathrm{d}x,$$

where y_u is the solution of the state equation

$$\begin{cases} Ay + f(\cdot, y) = u & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma. \end{cases}$$

The problem is again well posed in $L^2(\Omega)$ and any local minimizer is a function of $H^1(\Omega) \cap C^\mu(\bar\Omega)$. To establish the $L^\infty(\Omega)$ boundedness of the control, the arguments relies on the $L^p(\Omega)$ estimates for the states and adjoint states proved in [4]. The reader is referred to [10] for the analysis of this problem with $L^\infty(\Omega)$ controls.

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