

# Surface subgroups of graph products of groups

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**ABSTRACT.** A graph product kernel means the kernel of the natural surjection from a graph product to the corresponding direct product. We prove that a graph product kernel of countable groups is special, and a graph product of finite or cyclic groups is virtually cocompact special in the sense of Haglund and Wise. The proof of this yields conditions for a graph over which the graph product of arbitrary nontrivial groups (or some cyclic groups, or some finite groups) contains a hyperbolic surface group. In particular, the graph product of arbitrary nontrivial groups over a cycle of length at least five, or over its opposite graph, contains a hyperbolic surface group. For the case when the defining graphs have at most seven vertices, we completely characterize right-angled Coxeter groups with hyperbolic surface subgroups.

## 1. INTRODUCTION

By a *graph*, we mean a simplicial 1–complex. Throughout this paper, we will let  $\Gamma$  be a finite graph. The vertex set and the edge set of  $\Gamma$  are denoted as  $V(\Gamma)$  and  $E(\Gamma)$ , respectively. Suppose  $\mathcal{G} = \{G_v : v \in V(\Gamma)\}$  is a collection of groups indexed by  $V(\Gamma)$ . We define  $GP(\Gamma, \mathcal{G})$  to be the free product of the groups in  $\mathcal{G}$  quotient by the normal closure of the set  $\{[g, h] : g \in G_u, h \in G_v \text{ for some } \{u, v\} \in E(\Gamma)\}$ . We call  $GP(\Gamma, \mathcal{G})$  as the *graph product of the groups in  $\mathcal{G}$  over  $\Gamma$* , and each  $G_v$  as a *vertex group* of  $GP(\Gamma, \mathcal{G})$ . The kernel of the natural surjection  $GP(\Gamma, \mathcal{G}) \rightarrow \prod_{v \in V(\Gamma)} G_v$  is called as the *graph product kernel of  $\mathcal{G}$  over  $\Gamma$*  and denoted as  $KP_0(\Gamma, \mathcal{G})$ .

By a *hyperbolic surface group*, we mean the fundamental group of a closed hyperbolic surface. For abbreviation, we let  $\mathcal{S}$  be the class of groups that contain hyperbolic surface groups. Our main question is the following.

**Question 1.** *For which graph  $\Gamma$  and which collection of groups  $\mathcal{G}$ , is  $GP(\Gamma, \mathcal{G})$  in  $\mathcal{S}$ ?*

Let us briefly explain some motivation for Question 1. Gromov asked the following intriguing question [21, p.277].

**Question 2.** *Is every one-ended word-hyperbolic group in  $\mathcal{S}$ ?*

Question 2 has been answered for only a few cases, all affirmatively. These include graphs of free groups with cyclic edge groups with nontrivial second rational homology [5], doubles of rank-two free groups symmetrically amalgamated along cyclic edge groups [19, 34, 33], and most remarkably, the fundamental groups of closed hyperbolic 3–manifolds [29]. We note that these groups are all *virtually cocompact special* in the sense that each one is virtually the fundamental group of a compact *special* cube complex [24, 25]<sup>a</sup>; see Definition 10. So very broadly, we may ask under which conditions a one-ended, virtually cocompact special group belongs to  $\mathcal{S}$ . On the other hand,

**Theorem 3.** (1) *Graph product kernels of countable groups are special.*  
(2) *Graph products of finite or cyclic groups are virtually cocompact special.*

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<sup>a</sup>The fact that closed hyperbolic 3–manifold groups are virtually cocompact special is recently announced by Agol.

The proof of Theorem 3 will reveal inclusion relations between certain subgroups of graph products, and so, provide an important tool for this paper. In some sense, a graph product kernel will “remember” only the order of each vertex group, while “forgetting” the group structure of it.

For  $2 \leq m \leq \infty$ , we let  $GP_m(\Gamma)$  denote the graph product of cyclic groups of order  $m$  over  $\Gamma$ . We write  $A(\Gamma) = GP_\infty(\Gamma)$  and  $C(\Gamma) = GP_2(\Gamma)$ . We will call  $A(\Gamma)$  and  $C(\Gamma)$  as a *right-angled Artin group* and a *right-angled Coxeter group* on  $\Gamma$ , respectively [7]. Question 1 has a close relation to the question of whether  $A(\Gamma) \in \mathcal{S}$  or  $C(\Gamma) \in \mathcal{S}$  as described below.

**Theorem 4.** (1) *We have  $C(\Gamma) \in \mathcal{S}$  if and only if the graph product of arbitrary nontrivial groups over  $\Gamma$  is in  $\mathcal{S}$ .*  
 (2) *We have  $A(\Gamma) \in \mathcal{S}$  if and only if the graph product of some cyclic groups over  $\Gamma$  is in  $\mathcal{S}$ .*  
 (3) *We have  $[A(\Gamma), A(\Gamma)] \in \mathcal{S}$  if and only if the graph product of some finite groups over  $\Gamma$  is in  $\mathcal{S}$ , if and only if  $GP_m(\Gamma) \in \mathcal{S}$  for some  $2 \leq m < \infty$ .*

We denote by  $C_m$  the cycle of length  $m$ . The *opposite graph*  $\Gamma^{\text{opp}}$  of  $\Gamma$  is defined by  $V(\Gamma^{\text{opp}}) = V(\Gamma)$  and  $E(\Gamma^{\text{opp}}) = \{\{u, v\} : u \text{ and } v \text{ are non-adjacent vertices of } \Gamma\}$ . If there is a finite sequence of edge-contractions [12, p.20] from  $\Gamma_1^{\text{opp}}$  to  $\Gamma_2^{\text{opp}}$ , we say  $\Gamma_1$  *co-contracts* onto  $\Gamma_2$ . In [31], it was shown that a co-contraction  $\Gamma_1 \rightarrow \Gamma_2$  induces an embedding  $A(\Gamma_2) \hookrightarrow A(\Gamma_1)$ .

**Theorem 5.** *Suppose  $\Gamma_1$  and  $\Gamma_2$  are finite graphs such that  $\Gamma_1$  co-contracts onto  $\Gamma_2$ . If  $2 \leq m \leq \infty$ , then  $GP_m(\Gamma_2)$  embeds into  $GP_m(\Gamma_1)$ .*

It is well-known that  $C(C_m)$  and  $A(C_m)$  are in  $\mathcal{S}$  for  $m \geq 5$ ; see [37]. Also, it was shown that  $A(C_m^{\text{opp}}) \in \mathcal{S}$  for  $m \geq 5$  in [31, 9]; see [3] for an alternative proof. Using Theorem 5, we generalize these results.

**Corollary 6** ([30, 27], cf. [15]). *For  $m \geq 5$ , the graph product of arbitrary nontrivial groups over  $C_m$  is in  $\mathcal{S}$ .*

**Corollary 7.** *For  $m \geq 5$ , the graph product of arbitrary nontrivial groups over  $C_m^{\text{opp}}$  is in  $\mathcal{S}$ .*

Suppose  $X \subseteq V(\Gamma)$ . The *induced subgraph* of  $\Gamma$  on  $X$  is the maximal subgraph of  $\Gamma$  whose vertex set is  $X$ . If  $\Lambda$  is isomorphic to an induced subgraph of  $\Gamma$ , we simply write  $\Lambda \leq \Gamma$  and say that  $\Gamma$  has an *induced*  $\Lambda$ . We also use the notation  $H \leq G$  for two groups  $G$  and  $H$ , if there exists an embedding from  $H$  into  $G$ . We say  $\Gamma$  is *weakly chordal* if  $\Gamma$  does not contain an induced  $C_m$  or  $C_m^{\text{opp}}$  for  $m \geq 5$ . For each finite graph  $\Gamma_1$ , there exists a (algorithmically constructible) graph  $\Gamma_2 \geq \Gamma_1$  such that  $[C(\Gamma_2) : A(\Gamma_1)] < \infty$  [11]. In particular,  $A(\Gamma_1) \in \mathcal{S}$  if and only if  $C(\Gamma_2) \in \mathcal{S}$ . Hence, the classification of  $\Gamma$  satisfying  $C(\Gamma) \in \mathcal{S}$  is presumably “harder” than that of  $\Gamma$  satisfying  $A(\Gamma) \in \mathcal{S}$ . Complete classification of the graphs  $\Gamma$  with  $|V(\Gamma)| \leq 8$  and  $A(\Gamma) \in \mathcal{S}$  is given in [9]. We will classify all the graphs  $\Gamma$  with  $|V(\Gamma)| \leq 7$  and  $C(\Gamma) \in \mathcal{S}$ .

**Theorem 8.** *Suppose  $\Gamma$  has at most seven vertices. Then  $C(\Gamma) \in \mathcal{S}$  if and only if  $\Gamma$  is not weakly chordal.*

In particular, the proof of Theorem 8 will exhibit graphs  $\Gamma$  such that  $A(\Gamma) \in \mathcal{S}$  and  $C(\Gamma) \notin \mathcal{S}$ . When  $\Gamma$  has more than seven vertices,  $C(\Gamma) \in \mathcal{S}$  does not necessarily imply that  $\Gamma \geq C_m$  or  $\Gamma \geq C_m^{\text{opp}}$  for some  $m \geq 5$ ; see Remark 5. Lastly, we will make an observation that the class of finitely generated groups that “conform” to an affirmative answer to Question 2 is closed under graph products.

Here is the organization of this paper. In Section 2, we summarize basic facts on cube complexes and label-reading maps. We describe two special cube complexes whose fundamental groups are specific subgroups of graph products and use these complexes to prove Theorems 3, 4 and Corollary 6 in Section 3. Section 4 introduces a general, combinatorial group theoretic lemma, which yields nontrivial embeddings between graph products. Theorem 5 and Corollary 7 will follow. In Section 5, we investigate seven-vertex graphs and prove Theorem 8. We discuss a role of graph products in relation to Question 2 in Section 6.

*Note on the literature.* (1) Haglund has shown that a graph product of *finite* groups is virtually cocompact special [23] by considering a certain cube complex for its graph product kernel; see also [8]. The cube complex discussed in Section 3 is not a generalized version of his complex.

- (2) While it is unknown whether Coxeter groups are virtually *cocompact* special, they are virtually special [25, Problem 9.2, Theorem 1.2]. This already implies that graph products of finite or cyclic groups are virtually special, since these graph products embed into Coxeter groups. Note that for Coxeter groups, Question 2 has an affirmative answer as well [18].
- (3) Holt and Rees constructed a complex  $Z$  for a graph product kernel of *cyclic* groups [27, Theorem 3.1]. Their complex  $Z$  is different from ours in that  $Z$  is not cubical and not necessarily aspherical. Theorem 4 (1) can also be proved using the construction of Holt and Rees, combined with Droms' description of a complex for  $[C(\Gamma), C(\Gamma)]$ ; see [14].
- (4) Corollary 6 was first proved in the author's Ph.D thesis [30, Theorem 3.6], but never published by the author. After that, the same result was proved again by Futer and Thomas (for  $m \geq 6$ ) [15, Corollary 1.3] and by Holt and Rees [27, Theorem 3.1].
- (5) Theorem 5 and Corollary 7 were proved in the author's Ph.D thesis [30, Corollaries 4.11 and 4.12], but never published anywhere. We will give new accounts of these results.

The methods presented in this article do not depend on the above mentioned works.

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## 2. PRELIMINARY ON CUBE COMPLEXES AND LABEL-READING MAPS

**2.1. Local isometries and special cube complexes.** By a *cube complex*, we mean a CW-complex obtained from unit Euclidean cubes of various dimensions by isometrically gluing some of the faces. A *flag complex* is a simplicial complex  $L$  such that each complete subgraph in  $L^{(1)}$  is the 1-skeleton of some simplex in  $L$ . We say a cube complex  $X$  is *nonpositively curved*, or simply *NPC*, if the link of each vertex is a flag complex; this is equivalent to saying that the piecewise Euclidean length metric induced on the universal cover of  $X$  is CAT(0) [21].

We denote by  $X_\Gamma$  the *Salvetti complex* of  $A(\Gamma)$  [7]. This means that  $X_\Gamma^{(2)}$  is the presentation 2-complex of  $A(\Gamma)$ , and for each maximal complete subgraph  $K$  of  $\Gamma$  with  $k$  vertices, a  $k$ -torus  $T$  is glued to  $X_\Gamma^{(2)}$  so that the 1-skeleton of  $T$  is the bouquet of the circles corresponding to the vertices of  $K$ . Note that  $X_\Gamma$  is an NPC cube complex such that  $\pi_1(X_\Gamma) \cong A(\Gamma)$ ; see [7].

If  $X$  is a cube complex and  $v$  is a vertex of  $X$ , we denote the link of  $X$  at  $v$  by  $\text{Lk}(X; v)$ . Let us consider a combinatorial map  $f : X \rightarrow Y$  between cube complexes  $X$  and  $Y$ . The map  $f$  induces a simplicial map  $\text{Lk}(f; v)$  between  $\text{Lk}(X; v)$  and  $\text{Lk}(Y; f(v))$  for each vertex  $v$  of  $X$ . Following [6], we call  $f$  as a *local isometry* if

- (i)  $\text{Lk}(f; v)$  is injective, and
- (ii) the image of  $\text{Lk}(f; v)$  is a full subcomplex of  $\text{Lk}(Y; f(v))$ .

**Lemma 9** ([6, 10, 4]). *Suppose  $X$  and  $Y$  are cube complexes and  $f : X \rightarrow Y$  is a combinatorial map. If  $Y$  is NPC and  $f$  is a local isometry, then  $X$  is also NPC and  $f$  is  $\pi_1$ -injective.*

**Definition 10** ([24, 25]). (1) A cube complex  $X$  is called *special* if  $X$  combinatorially maps to a Salvetti complex by a local isometry.

- (2) A group  $G$  is *special* if  $G \cong \pi_1(X)$  for some special cube complex  $X$ . Furthermore, if  $X$  can be chosen to be compact, then we say  $G$  is *cocompact special*.

We remark that Definition 10 (1) is different from, but equivalent to, the original definition in [24]; see [25, Proposition 3.2]. For a group theoretic property  $P$ , we say a group  $G$  is *virtually  $P$*  if a finite-index subgroup of  $G$  is  $P$ . Virtually special groups are of particular interest in 3-manifold theory [1].

**2.2. Label-reading maps.** By a *curve* on a surface, we will mean a simple closed curve or a properly embedded arc. Let  $S$  be a compact surface possibly with boundary. Suppose  $\mathcal{V}$  is a finite set of transversely intersecting curves on  $S$  and  $\lambda: \mathcal{V} \rightarrow V(\Gamma)$  is a map such that two curves  $\alpha$  and  $\beta$  in  $\mathcal{V}$  are intersecting only if  $\{\lambda(\alpha), \lambda(\beta)\} \in E(\Gamma)$ . Following [10], we say that  $(\mathcal{V}, \lambda)$  is a *label-reading pair* on  $S$  with the *underlying graph*  $\Gamma$ ; and for each  $\alpha \in \mathcal{V}$ , we call  $\lambda(\alpha)$  as the *label* of  $\alpha$ . If an arc  $\alpha$  is labeled by  $a \in V(\Gamma)$ , we say  $\alpha$  is an  $a$ -arc. For each oriented path  $\gamma$  transverse to  $\mathcal{V}$ , we follow  $\gamma$  and read off the labels of the curves in  $\mathcal{V}$  that intersect  $\gamma$ . The word  $w(\gamma)$  thus obtained will be called the *label-reading* of  $\gamma$  with respect to  $(\mathcal{V}, \lambda)$ . The word  $w(\gamma)$  represents an element of  $C(\Gamma)$ . If there exists a group homomorphism  $\phi: \pi_1(S) \rightarrow C(\Gamma)$  satisfying that  $\phi([\gamma]) = w(\gamma)$  for each  $[\gamma] \in \pi_1(S)$ , we call  $\phi$  as a *label-reading map* with respect to  $(\mathcal{V}, \lambda)$ .

Recall that a word  $w$  representing an element in  $C(\Gamma)$  is *reduced* if no shorter word represents the same element. It is *cyclically reduced* if  $w$  and each of its cyclic conjugations are reduced. If a curve  $\gamma$  on a compact surface  $S$  is homotopic to a subset of  $\partial S$  by a homotopy fixing  $\partial\gamma$ , then we say  $\gamma$  is *homotopic into*  $\partial S$ .

Crisp and Wiest proved that the fundamental group of a closed hyperbolic surface  $S$  embeds into some right-angled Artin group if and only if  $\chi(S) \neq -1$  [10]. A critical tool for the proof was the realization of an arbitrary group homomorphism  $\phi: \pi_1(S) \rightarrow A(\Gamma)$  as a label-reading map (using  $A(\Gamma)$  instead of  $C(\Gamma)$ ). The following is a simple variation of the results in [10] combined with [32].

**Theorem 11** ([10, 32]). *Let  $S$  be a compact surface.*

- (1) *Suppose  $(\mathcal{V}, \lambda)$  is a label-reading pair on  $S$  with the underlying graph  $\Gamma$ . Then for each choice of the base point of  $S$ , there uniquely exists a label-reading map  $\phi: \pi_1(S) \rightarrow C(\Gamma)$  with respect to  $(\mathcal{V}, \lambda)$ .*
- (2) *Conversely, every group homomorphism  $\phi: \pi_1(S) \rightarrow C(\Gamma)$  can be realized as a label-reading map with respect to some label-reading pair  $(\mathcal{V}, \lambda)$  that has the underlying graph  $\Gamma$ .*
- (3) *Possibly after composing  $\phi$  with an inner automorphism of  $C(\Gamma)$ , we can choose  $(\mathcal{V}, \lambda)$  in (2) further satisfying the following:*
  - (i) *curves in  $\mathcal{V}$  are minimally intersecting;*
  - (ii) *curves in  $\mathcal{V}$  are neither null-homotopic nor homotopic into  $\partial S$ ;*
  - (iii) *for each component  $\partial_i S$  of  $\partial S$ , the label-reading  $w(\partial_i S)$  is cyclically reduced.*

*Proof.* (1) and (2) are proved in [10] for right-angled Artin groups. The proofs for right-angled Coxeter groups are very similar, except that we now allow  $\mathcal{V}$  to contain orientation-reversing closed curves and also that curves in  $\mathcal{V}$  are *not* assigned with transverse orientations. (3) is obtained by lexicographically minimizing the complexity  $(|(\cup \mathcal{V}) \cap \partial S|, |\mathcal{V}|, \sum_{\alpha \neq \beta \in \mathcal{V}} |\alpha \cap \beta|)$ , possibly after changing  $\phi$  by  $\text{Inn}(C(\Gamma))$ ; see [10] and [32] for discussion on the same technique.  $\square$

### 3. SPECIAL CUBE COMPLEXES FOR CERTAIN SUBGROUPS OF GRAPH PRODUCTS

In this section, we write  $V(\Gamma) = \{1, 2, \dots, n\}$  and assume  $\mathcal{G} = \{G_1, G_2, \dots, G_n\}$  is a collection of groups indexed by  $V(\Gamma)$ . Choose  $k$  such that  $|G_i|$  is finite if and only if  $1 \leq i \leq k$ . Set  $p_i: GP(\Gamma, \mathcal{G}) \rightarrow G_i$  to be the natural projection map for  $i = 1, 2, \dots, n$ . Recall that we have defined the *graph product kernel* as  $KP_0(\Gamma, \mathcal{G}) = \cap_{1 \leq i \leq n} \ker p_i$ . We also define  $KP_f(\Gamma, \mathcal{G}) = \cap_{1 \leq i \leq k} \ker p_i$ . Note that  $KP_0(\Gamma, \mathcal{G}) \leq KP_f(\Gamma, \mathcal{G}) \leq GP(\Gamma, \mathcal{G})$  and that  $GP(\Gamma, \mathcal{G})/KP_f(\Gamma, \mathcal{G}) \cong \prod_{1 \leq i \leq k} G_i$  is finite. If all the groups in  $\mathcal{G}$  are abelian,  $KP_0(\Gamma, \mathcal{G})$  is the commutator subgroup of  $GP(\Gamma, \mathcal{G})$ .

Let us regard  $\mathbb{R}^n$  as a cube complex whose vertices are the lattice points and whose 1-skeleton consists of the grid lines. We set  $e_i$  to be the  $i$ -th standard basis vector. Following [37],  $Y_\Gamma \leq \mathbb{R}^n$  is defined to be the lift of  $X_\Gamma \subseteq (S^1)^n$  with respect to the covering  $\mathbb{R}^n \rightarrow (S^1)^n$ . Concretely,  $(Y_\Gamma)^{(1)} = (\mathbb{R}^n)^{(1)}$  and for each complete subgraph of  $\Gamma$  having the vertex set  $\{i_1, \dots, i_k\}$ , the

following collection of the unit  $k$ -cubes is contained in  $Y_\Gamma$ :

$$\left\{ \sum_{j=1}^k t_j e_{i_j} : t_j \in [0, 1] \right\} + \mathbb{Z}^n.$$

We define  $Z_0(\Gamma, \mathcal{G}) = Y_\Gamma \cap \left( \prod_{i=1}^k [0, |G_i| - 1] \times \mathbb{R}^{n-k} \right) \subseteq \mathbb{R}^n$ . We let  $Z_\Gamma$  denote the preimage of  $X_\Gamma \subseteq (S^1)^n$  with respect to the covering  $\mathbb{R}^k \times (S^1)^{n-k} \rightarrow (S^1)^n$  and put  $Z_f(\Gamma, \mathcal{G}) = Z_\Gamma \cap \left( \prod_{i=1}^k [0, |G_i| - 1] \times (S^1)^{n-k} \right)$ . See Figure 1.

$$\begin{array}{ccccc} Z_0(\Gamma, \mathcal{G}) = Y_\Gamma \cap \left( \prod_{i=1}^k [0, |G_i| - 1] \times \mathbb{R}^{n-k} \right) & \hookrightarrow & Y_\Gamma & \hookrightarrow & \mathbb{R}^n \\ \downarrow & & \downarrow p & & \downarrow \\ Z_f(\Gamma, \mathcal{G}) = Z_\Gamma \cap \left( \prod_{i=1}^k [0, |G_i| - 1] \times (S^1)^{n-k} \right) & \hookrightarrow & Z_\Gamma & \hookrightarrow & \mathbb{R}^k \times (S^1)^{n-k} \\ & & \downarrow & & \downarrow \\ & & X_\Gamma & \hookrightarrow & (S^1)^n \end{array}$$

FIGURE 1. Cube complexes in Theorem 12. The horizontal maps are inclusions and the vertical maps are coverings.

It is well-known that  $\pi_1(Y_\Gamma) = [A(\Gamma), A(\Gamma)]$  and that  $\pi_1(Y_\Gamma \cap [0, 1]^n) \cong [C(\Gamma), C(\Gamma)]$ ; see [37, 14]. We generalize these observations as follows.

**Theorem 12.** (1) If  $\mathcal{G}$  consists of countable groups, then  $\pi_1(Z_0(\Gamma, \mathcal{G})) \cong KP_0(\Gamma, \mathcal{G})$ .  
(2) If  $\mathcal{G}$  consists of finite or cyclic groups, then  $\pi_1(Z_f(\Gamma, \mathcal{G})) \cong KP_f(\Gamma, \mathcal{G})$ .

*Proof.* (1) We use the notation described so far in this section. Choose the origin  $O \in \mathbb{R}^n$  as the base point. Let  $q_j: \mathbb{R}^n \rightarrow \mathbb{R}$  denotes the natural projection onto the  $j$ -th component. Enumerate  $G_j = \{g_0^{(j)} = 1, g_1^{(j)}, g_2^{(j)}, \dots, g_{|G_j|-1}^{(j)}\}$  for each  $j \leq k$ , and  $G_j = \{\dots, g_{-1}^{(j)}, g_0^{(j)} = 1, g_1^{(j)}, \dots\}$  for each  $j > k$ . We let  $\mathcal{H}$  be the homotopy classes of the edge-paths in  $Z_0(\Gamma, \mathcal{G})$  starting from  $O$ .

We will first define a map  $\Phi: \mathcal{H} \rightarrow GP(\Gamma, \mathcal{G})$ . Let us consider an edge-path  $\gamma: [0, l] \rightarrow Z_0(\Gamma, \mathcal{G})$  such that  $\gamma(0) = O$  and  $\gamma^{-1}(\mathbb{Z}^n \cap Z_0(\Gamma, \mathcal{G})) = \mathbb{Z} \cap [0, l]$ . For each  $i = 1, 2, \dots, l$ , there uniquely exists  $j$  such that  $\gamma[i, i+1]$  is parallel to  $\mathbb{R}e_j$ ; then we put  $k = q_j \circ \gamma(i)$ ,  $k' = q_j \circ \gamma(i+1)$  and  $x_i = (g_k^{(j)})^{-1} g_{k'}^{(j)} \in G_j$ . Note that  $|k - k'| = 1$ . If  $\gamma[i, i+1]$  and  $\gamma[i+1, i+2]$  span a 2-cell in  $Z_0(\Gamma, \mathcal{G})$ , then the groups containing  $x_i$  and  $x_{i+1}$  commute; that is,  $x_i x_{i+1} = x_{i+1} x_i$ . So we can define a map  $\Phi: \mathcal{H} \rightarrow GP(\Gamma, \mathcal{G})$  by setting  $\Phi([\gamma]) = x_1 x_2 \cdots x_l$ .

Conversely, suppose  $1 \neq g \in GP(\Gamma, \mathcal{G})$  is given. The normal form theorem for graph products [20, 28] implies that we can write  $g = g_{k_1}^{(j_1)} g_{k_2}^{(j_2)} \cdots g_{k_l}^{(j_l)}$  such that:

- (i)  $k_i \neq 0$  for each  $i = 1, 2, \dots, l$ ;
- (ii)  $j_i \neq j_{i+1}$  for each  $i = 1, 2, \dots, l-1$ ;
- (iii) if  $j_i = j_{i'}$  for some  $i < i'$ , then there exists  $i < i'' < i'$  such that  $\{j_i, j_{i''}\} = \{j_{i'}, j_{i''}\} \notin E(\Gamma)$ .

Let us fix  $i \in \{1, 2, \dots, l\}$  and put  $j = j_i$ . There exist  $k$  and  $k'$  such that

$$g_k^{(j)} = \prod_{1 \leq t < i \text{ and } j_t = j} g_{k_t}^{(j)} \quad \text{and} \quad g_{k'}^{(j)} = \prod_{1 \leq t \leq i \text{ and } j_t = j} g_{k_t}^{(j)} = g_k^{(j)} g_{k_i}^{(j)}.$$

We inductively define  $\gamma_i$  to be the edge-path in  $\mathbb{R}^n$  starting from the end point of  $\gamma_{i-1}$  and changing only its  $j$ -th coordinate from  $k$  to  $k'$ . We set  $O$  as the initial point of  $\gamma_1$ . By defining  $\Psi(g) =$

$[\gamma_1\gamma_2\cdots\gamma_l]$ , we have a map  $\Psi: GP(\Gamma, \mathcal{G}) \rightarrow \mathcal{H}$ . Note that  $\Psi$  is well-defined since two normal forms differ only by a finite sequence of swapping certain consecutive terms, which can also be realized as a homotopy in  $Z_0(\Gamma, \mathcal{G})$ . It is clear that  $\Psi$  is the (set-theoretic) inverse of  $\Phi$ .

Now if  $1 \neq g \in KP_0(\Gamma, \mathcal{G})$ , then we further have:

(iv)  $\prod_{1 \leq t \leq l \text{ and } j_t=j} g_{k_t}^{(j)}$  is trivial in  $G_j$  for each  $j = 1, 2, \dots, n$ .

It is clear that  $\Psi$  restricts to a group isomorphism from  $KP_0(\Gamma, \mathcal{G})$  onto  $\pi_1(Z_0(\Gamma, \mathcal{G}))$ .

(2) In the case when  $1 \neq g \in KP_f(\Gamma, \mathcal{G})$ , we have the following condition instead of (iv) above:

(iv)'  $\prod_{1 \leq t \leq l \text{ and } j_t=j} g_{k_t}^{(j)}$  is trivial in  $G_j$  for each  $j = 1, 2, \dots, k$ .

Hence, the covering  $Y_\Gamma \rightarrow Z_\Gamma$  projects  $\Psi(g)$  to a homotopy class of a loop in  $Z_f(\Gamma, \mathcal{G})$ . We can check that the restriction of  $\psi$  onto  $KP_f(\Gamma, \mathcal{G})$  is a group isomorphism onto  $\pi_1(Z_f(\Gamma, \mathcal{G}))$ .  $\square$

*Remark.* The  $\mathcal{H}$  in the above proof depends only on the orders of the groups in  $\mathcal{G}$ . Hence,  $\Phi$  determines a bijection between  $GP(\Gamma, \{\mathbb{Z}_{|G_i|} : i = 1, 2, \dots, n\})$  and  $GP(\Gamma, \mathcal{G})$ . In particular, if all the groups in  $\mathcal{G}$  are infinite, then  $\Phi$  induces a bijection between  $A(\Gamma)$  and  $GP(\Gamma, \mathcal{G})$ .

*Example 13.* Let us consider  $G = (\mathbb{Z}_3 * \mathbb{Z}_4) \times \mathbb{Z}$ , which is regarded as the graph product in Figure 2 (a). Then  $KP_0(\Gamma, \mathcal{G}) = [G, G]$  since the vertex groups are abelian. Also,  $KP_f(\Gamma, \mathcal{G}) = [\mathbb{Z}_3 * \mathbb{Z}_4, \mathbb{Z}_3 * \mathbb{Z}_4] \times \mathbb{Z}$  is a subgroup of  $G$  with index  $|\mathbb{Z}_3| \cdot |\mathbb{Z}_4| = 12$ . If  $\Lambda$  is the graph shown in Figure 2 (b), then we have  $Z_0(\Gamma, \mathcal{G}) = \Lambda \times \mathbb{R}$  and  $Z_f(\Gamma, \mathcal{G}) = \Lambda \times S^1$ .



FIGURE 2. Example 13.

*Proof of Theorem 3.* Let us use the notations in the proof of Theorem 12. Note that the compositions  $Z_0(\Gamma, \mathcal{G}) \hookrightarrow Y_\Gamma \rightarrow X_\Gamma$  and  $Z_f(\Gamma, \mathcal{G}) \hookrightarrow Z_\Gamma \rightarrow X_\Gamma$  are local isometries, and that  $Z_f(\Gamma, \mathcal{G})$  is compact. Moreover,  $[GP(\Gamma, \mathcal{G}) : KP_f(\Gamma, \mathcal{G})] = \prod_{i=1}^k |G_i|$  is finite.  $\square$

If  $C \subseteq D$  are (possibly infinite) rectangular boxes in  $\mathbb{R}^n$  whose vertices are lattice points, then the inclusion  $Y_\Gamma \cap C \hookrightarrow Y_\Gamma \cap D$  is a local isometry. We note two immediate corollaries of Theorem 12.

**Corollary 14.** (1) *The graph product kernel of countable groups embeds into  $[A(\Gamma), A(\Gamma)]$ .*

(2) *The graph product of finite or cyclic groups virtually embeds into  $A(\Gamma)$ .*

**Corollary 15.** *Let  $\mathcal{G} = \{G_v : v \in V(\Gamma)\}$  and  $\mathcal{G}' = \{G'_v : v \in V(\Gamma)\}$  be collection of countable groups.*

(1) *If  $|G_v| \leq |G'_v| \leq \infty$  for each vertex  $v$ , then  $KP_0(\Gamma, \mathcal{G})$  embeds into  $KP_0(\Gamma, \mathcal{G}')$ . If we further assume that  $\mathcal{G}$  and  $\mathcal{G}'$  consist of finite or cyclic groups, then  $KP_f(\Gamma, \mathcal{G})$  embeds into  $KP_f(\Gamma, \mathcal{G}')$ .*

(2) *If  $|G_v| = |G'_v| \leq \infty$  for each vertex  $v$ , then  $KP_0(\Gamma, \mathcal{G}) \cong KP_0(\Gamma, \mathcal{G}')$ .*

**Lemma 16.** *Each finitely generated subgroup of  $[A(\Gamma), A(\Gamma)]$  embeds into  $[GP_m(\Gamma), GP_m(\Gamma)]$  for some  $2 \leq m < \infty$ .*

*Proof.* Let  $H$  be a finitely generated subgroup of  $[A(\Gamma), A(\Gamma)]$  and  $B$  be the union of edge-paths in  $Y_\Gamma$  that correspond to the generators of  $H$ . For  $n = |V(\Gamma)|$ , there exists  $2 \leq m < \infty$  such that  $B$  is contained in  $[0, m-1]^n$ . Then  $H \rightarrow [A(\Gamma), A(\Gamma)]$  factors as  $H \rightarrow \pi_1(Y_\Gamma \cap [0, m-1]^n) = [GP_m(\Gamma), GP_m(\Gamma)] \rightarrow \pi_1(Y_\Gamma) = [A(\Gamma), A(\Gamma)]$ .  $\square$

*Proof of Theorem 4.* For each of (1) and (2), only one direction of the assertion is not obvious. We will follow the notations in the proof of Theorem 12.

(1) Suppose  $C(\Gamma) \in \mathcal{S}$ . If  $\mathcal{G}$  consists of nontrivial groups, Corollary 15 implies that  $[C(\Gamma), C(\Gamma)]$  embeds into  $KP_0(\Gamma, \mathcal{G})$ . Note that  $[C(\Gamma), C(\Gamma)]$  is in  $\mathcal{S}$  since  $[C(\Gamma): [C(\Gamma), C(\Gamma)]] < \infty$ .

(2) Suppose  $\mathcal{G}$  consists of cyclic groups and  $GP(\Gamma, \mathcal{G}) \in \mathcal{S}$ . Then  $KP_f(\Gamma, \mathcal{G})$  is in  $\mathcal{S}$  since it is a finite-index subgroup of  $GP(\Gamma, \mathcal{G})$ . The conclusion follows since  $KP_f(\Gamma, \mathcal{G}) \leq \pi_1(Z_\Gamma) \leq A(\Gamma)$ .

(3) By Lemma 16,  $[A(\Gamma), A(\Gamma)] \in \mathcal{S}$  if and only if  $GP_m(\Gamma) \in \mathcal{S}$  for sufficiently large  $m$ . Also note that if  $\mathcal{G}$  consists of finite groups, then  $KP_0(\Gamma, \mathcal{G})$  embeds into  $GP_m(\Gamma)$  for sufficiently large  $m$ .  $\square$

*Proof of Corollary 6 (1).* Since  $C(C_m)$  is a cocompact Fuchsian group, it contains a hyperbolic surface group. Apply Theorem 4 (1).  $\square$

#### 4. DOUBLES AND CO-CONTRACTIONS

Suppose  $A$  and  $B$  are groups. For an isomorphism  $\psi: C \rightarrow D$  where  $C \leq A$  and  $D \leq B$ , we let  $A *_\psi B$  denote the free product of  $A$  and  $B$  amalgamated along  $\psi$ . If  $\psi: C \rightarrow C'$  is an isomorphism for some  $C, C' \leq A$ , then the HNN extension of  $A$  along  $\psi$  is denoted as  $A *_\psi$ .

**Lemma 17.** *Suppose  $G$  is a group,  $\psi: H \rightarrow H$  is an isomorphism for some  $H \leq G$  and  $2 \leq k < \infty$ . Let  $G_k = G *_\psi \cdots *_\psi G$  where there are  $k$  copies of  $G$ . We denote the stable generator of  $G *_\psi$  by  $t$ .*

(1) *The group  $G_k$  embeds into  $G *_\psi / \langle\langle t^k \rangle\rangle$  as a subgroup of index  $k$ .*

(2) *The group  $G *_\psi / \langle\langle t^k \rangle\rangle$  virtually embeds into  $G *_\psi$ .*

*Proof.* Let  $L_k = G_k *_\psi$ , whose stable generator is denoted by  $s$ . The groups in the first line of Figure 3 are illustrated as graphs of groups, where each vertex corresponds to  $G$  and each directed edge corresponds to  $\psi: H \rightarrow H$ ; (b) shows the case when  $k = 5$  as an example. The number  $1 \times k$  in (e) means that  $G *_\psi / \langle\langle t^k \rangle\rangle$  is obtained from  $G *_\psi$  by gluing (to a classifying space of  $G *_\psi$ )  $D^2$  such that  $[\partial D^2]$  and  $t^k$  are identified. Similarly,  $G_k \cong L_k / \langle\langle s \rangle\rangle$  in (d) is obtained from  $L_k$  by attaching  $k$  copies of  $D^2$ , whose boundary curves are all identified with  $s$ ; this is described as  $k \times 1$ . The figure shows a commutative diagram, where  $p_1$  and  $p_2$  are induced by covering maps. Now for (1), note that  $p_2$  is injective. (2) follows from that  $G_k$  embeds into  $L_k$  and that  $p_1$  is injective.  $\square$

The following is a special case of Lemma 17 (2).

**Corollary 18.** *Let  $G$  and  $K$  be graph products of groups such that  $K$  is obtained from  $G$  by replacing a finite cyclic vertex group of  $G$  by  $\mathbb{Z}$ . Then  $G$  virtually embeds into  $K$ .*

*Example 19.* (1) Consider the graph product  $G$  shown in Figure 2 (a). Suppose  $K$  is obtained from  $G$  by substituting  $\mathbb{Z}$  for  $\mathbb{Z}_4$ . Then  $G$  virtually embeds into  $K$ .

(2) We can apply Lemma 17 (2) repeatedly to see that  $C(\Gamma)$  virtually embeds into  $A(\Gamma)$ . This also follows from the well-known fact that  $[C(\Gamma), C(\Gamma)] \leq [A(\Gamma), A(\Gamma)]$ ; see [14].

We let  $G *_H G$  denote the free product of two copies of  $G$  amalgamated along  $id_H$ . The HNN extension of  $G$  along  $id_H$  is denoted as  $G *_H$ .

**Lemma 20.** *Let  $G$  be a group,  $H \leq G$  and  $k \in \mathbb{Z} \setminus \{1, -1\}$ . Then  $G *_H G$  embeds into  $G *_H / \langle\langle t^k \rangle\rangle$ .*

*Proof.* The case when  $k \neq 0$  follows from Lemma 17 (1). For  $k = 0$ , consider the first line of Figure 3. We remark that  $k = 0$  case is already well-known; see [35, p.187].  $\square$

Let us consider a graph isomorphism  $\mu: \Gamma \rightarrow \Gamma'$ . Fix a vertex  $t \in V(\Gamma)$  and put  $\Lambda$  the subgraph of  $\Gamma$  induced by  $\text{Lk}(t)$ . Define  $\Gamma''$  as the graph obtained from the union of  $\Gamma \setminus \text{St}(t)$  and  $\Gamma' \setminus \text{St}(\mu(t))$  after identifying  $\Lambda$  and  $\mu(\Lambda)$  by  $\mu$ . Here,  $\text{St}(v)$  denotes the open star of a vertex  $v$ . We call  $\Gamma''$  as the *double of  $\Gamma \setminus \text{St}(t)$  along the link of  $t$* . There is a natural projection map  $\rho: \Gamma'' \rightarrow \Gamma \setminus \text{St}(t)$ .

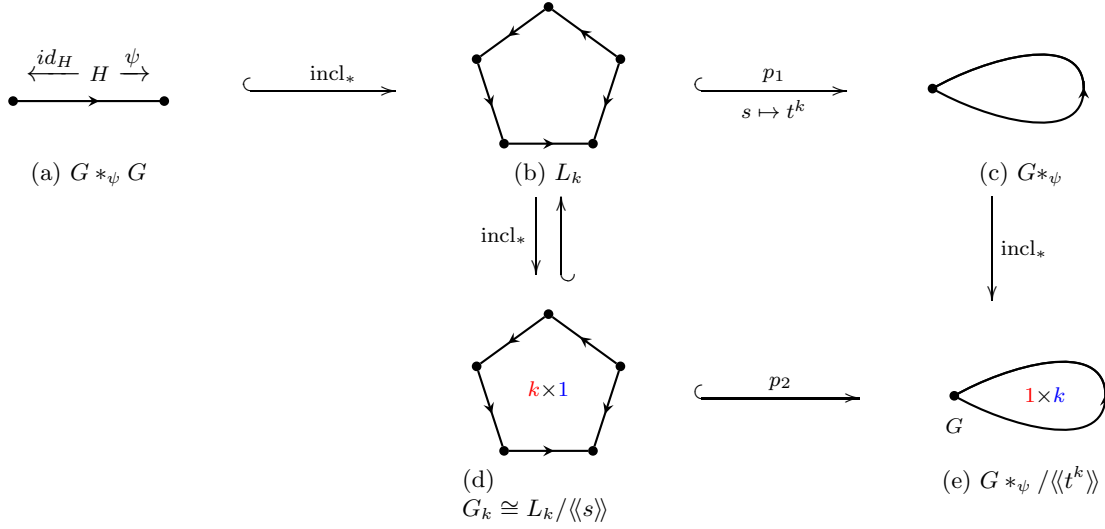


FIGURE 3. Lemma 17 and 21.

**Lemma 21.** *If  $\mathcal{G} = \{G_v : v \in V(\Gamma)\}$  is a collection of nontrivial groups and  $\Gamma''$  is the double of  $\Gamma \setminus \text{St}(t)$  along the link of  $t$ , then there is an embedding from  $GP(\Gamma'', \{G_{\rho(u)} : u \in V(\Gamma'')\})$  into  $GP(\Gamma, \mathcal{G})$ .*

*Proof.* Since  $G_t$  contains a nontrivial cyclic subgroup, we may assume  $G_t = \mathbb{Z}/k\mathbb{Z}$  for some  $2 \leq k < \infty$  or  $k = 0$ . Put  $G = GP(\Gamma \setminus \text{St}(t), \mathcal{G} \setminus \{G_t\})$  and  $H = \langle\{G_v : v \in \text{Lk}(t)\}\rangle \leq G$ . Then we have  $GP(\Gamma, \mathcal{G}) = G *_H / \langle\langle t^k \rangle\rangle$  and Lemma 21 applies.  $\square$

For  $e \in E(\Gamma)$ , we let  $\Gamma/e$  denote the contraction of  $e$  [12, p.20].

**Corollary 22.** *Suppose  $e = \{x, t\}$  is an edge of  $\Gamma^{\text{opp}}$ . We put  $\Gamma_0^{\text{opp}} = \Gamma^{\text{opp}}/e$  and denote by  $y \in V(\Gamma_0) = V(\Gamma_0^{\text{opp}})$  the contracted vertex of  $e$ . If  $\mathcal{G} = \{G_v : v \in V(\Gamma)\}$  is a collection of nontrivial groups and  $G_y = G_x$ , then  $GP(\Gamma_0, (\mathcal{G} \setminus \{G_x, G_t\}) \cup \{G_y\})$  embeds into  $GP(\Gamma, \mathcal{G})$ .*

*Proof.* As in Figure 4 (a), we partition  $V(\Gamma) \setminus \{x, t\}$  into  $P, Q, R, S$  where  $P = \text{Lk}(x) \cap \text{Lk}(t)$ ,  $Q = \text{Lk}(t) \setminus \text{Lk}(x)$ ,  $R = \text{Lk}(x) \setminus \text{Lk}(t)$  and  $S = V(\Gamma) \setminus (P \cup Q \cup R \cup \{x, t\})$ . Let  $\Gamma_1$  be the double of  $\Gamma \setminus \text{St}(t)$  along the link of  $t$  as shown in Figure 4 (c); we write  $R', S'$  and  $x'$  for the copies of  $R, S$  and  $x$ . Figure 4 (b) shows that  $\Gamma_0$  is isomorphic to the subgraph of  $\Gamma_1$  induced by  $P \cup Q \cup R \cup S \cup \{x'\}$ .  $\square$

Theorem 5 now follows from Corollary 22. Since there is a finite sequence of edge-contractions from  $C_m$  onto  $C_5$  for each  $m \geq 5$ , we have an embedding from  $C(C_5) \cong C(C_5^{\text{opp}})$  into  $C(C_m^{\text{opp}})$ . Hence, Corollary 7 also follows.

*Remark.* A special case of Corollary 22 is when all the vertex groups are infinite cyclic, and was proved in [31]. Bell gave a shorter proof of this case using the same decomposition of a graph as shown in Figure 4 [3], independently from this writing.

## 5. RIGHT-ANGLED COXETER GROUPS ON SEVEN VERTEX GRAPHS

For a group  $G$  and  $H \leq G$ , we say that  $g \in G$  is *conjugate into*  $H$  if  $g$  is conjugate to an element of  $H$ .

**Definition 23.** Let a group  $G$  and its subgroup  $H$  be given. We say  $(G, H)$  is *big* if there exists a compact hyperbolic surface  $S$  and a monomorphism  $\phi: \pi_1(S) \rightarrow G$  such that  $\phi([\gamma])$  is conjugate into  $H$  whenever  $\gamma$  is homotopic into  $\partial S$ . The pair  $(G, H)$  will be called *small* if it is not big.



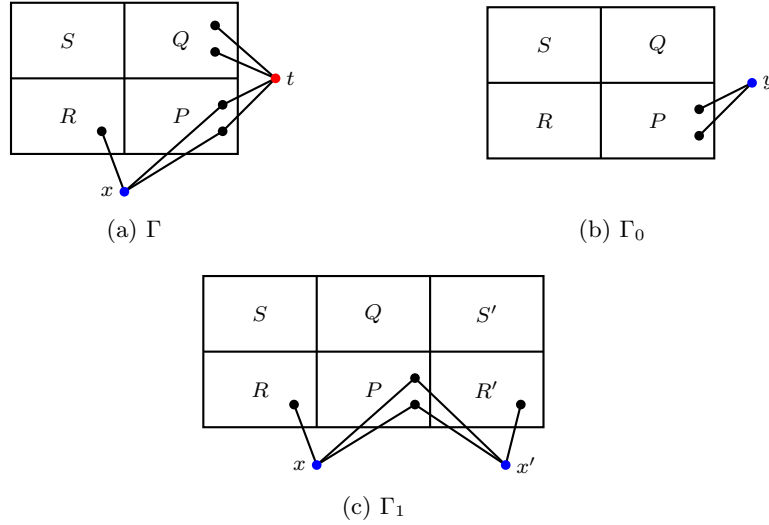


FIGURE 4. Co-contraction is contained in the double along a link.

- Remark.*
- (1) If  $(G, H)$  is big and  $H \leq K \leq G$ , then  $(G, K)$  is also big.
  - (2) If  $G \in \mathcal{S}$ , then  $(G, H)$  is big for each  $H \leq G$ .
  - (3) If a group  $G$  does not contain  $F_2$ , then  $(G, H)$  is small for each  $H \leq G$ .

The following lemma is obvious from typical transversality arguments.

- Lemma 24** ([32]).
- (1) Let  $A$  and  $B$  be groups and  $\psi: C \rightarrow D$  be an isomorphism where  $C \leq A$  and  $D \leq B$ . If  $A *_\psi B \in \mathcal{S}$ , then either  $(A, C)$  or  $(B, D)$  is big.
  - (2) Let  $A$  be a group and  $\psi: C \rightarrow C'$  be an isomorphism where  $C, C' \leq A$ . If  $A *_\psi \in \mathcal{S}$ , then  $(A, \langle C, C' \rangle)$  is big.

**Lemma 25.** Suppose  $a, b \in V(\Gamma)$  and  $\text{Lk}(b) \subseteq \text{Lk}(a)$ . If  $C(\Gamma) \notin \mathcal{S}$ , then  $(C(\Gamma), \langle a, b \rangle)$  is small.

*Proof.* Suppose  $S$  is a compact hyperbolic surface and  $\phi: \pi_1(S) \rightarrow C(\Gamma)$  is a monomorphism such that  $\phi([\gamma])$  is conjugate into  $\langle a, b \rangle$  whenever  $\gamma$  is homotopic into  $\partial S$ . Since  $C(\Gamma) \notin \mathcal{S}$ , we have  $\partial S \neq \emptyset$ . Let  $\partial_1 S, \partial_2 S, \dots$  be the boundary components of  $S$ . Since  $\langle a, b \rangle$  contains  $\mathbb{Z}$ , we see that  $a$  and  $b$  are distinct and non-adjacent in  $\Gamma$ . We realize  $\phi$  as a label-reading map with respect to a label-reading pair  $(\mathcal{V}, \lambda)$  satisfying the three conditions in Theorem 11 (3). Then each  $\partial_i S$  intersects with both  $a$ -arcs and  $b$ -arcs, and no arcs with labels other than  $a$  or  $b$  intersect  $\partial S$ . Choose a  $b$ -arc  $\beta$  joining say,  $\partial_i S$  and  $\partial_j S$ . These two boundary components may coincide; however,  $\beta$  is never homotopic into  $\partial S$ . With suitable choices of the base point and the orientations,  $\phi([\beta \cdot \partial_j S \cdot \beta^{-1}]) = w(ab)^l w^{-1}$  for some  $l \neq 0$  where  $w$  is the label-reading of  $\beta$ . Since  $w \in \langle \text{Lk}(b) \rangle = \langle \text{Lk}(a) \cap \text{Lk}(b) \rangle$ , we have  $\phi([\beta \cdot \partial_j S \cdot \beta^{-1}]) = (ab)^l$ . Also,  $\phi([\partial_i S]) = (ab)^m$  for some  $m \neq 0$ . Since  $[\beta \cdot \partial_j S \cdot \beta^{-1}]$  and  $[\partial_i S]$  do not commute, we have a contradiction.  $\square$

**Lemma 26.** Let  $H, G_1, G_2$  be groups such that  $H$  is torsion-free word-hyperbolic. Denote by  $p_i: G_1 \times G_2 \rightarrow G_i$  the natural projection for  $i = 1, 2$ . If  $\phi: H \rightarrow G_1 \times G_2$  is injective, then  $p_1 \circ \phi$  or  $p_2 \circ \phi$  is also injective.

*Proof.* Suppose  $1 \neq x_1 \in \ker(p_2 \circ \phi)$  and  $1 \neq x_2 \in \ker(p_1 \circ \phi)$  so that  $\phi(x_i) \in G_i$  for  $i = 1, 2$ . Since  $\phi([x_1, x_2]) = [\phi(x_1), \phi(x_2)] = 1$ , we have  $x_1^M = x_2^N$  for some  $M, N \neq 0$  [4, Corollary 3.10]. So,  $\phi(x_1^M) = \phi(x_2^N) \in G_1 \cap G_2 = 1$ .  $\square$

A repeated application of Lemma 26 easily implies the following.

**Lemma 27.** *If  $k > 0$  and  $(G_i, H_i)$  is small for  $i = 1, 2, \dots, k$ , then  $(\prod_{i=1}^k G_i, \prod_{i=1}^k H_i)$  is small.*

*Example 28.* Label the vertices of  $P_3 \amalg P_3$  as Figure 5 (a) and let  $\Lambda_0 = (P_3 \amalg P_3)^{\text{opp}}$ . Let us consider subgroups of  $C(\Lambda_0) = \langle a, b, c, e, f, g \rangle$  from now on. In the subgraph of  $\Lambda_0$  induced by  $\{a, b, c\}$ , we have  $\text{Lk}(b) = \emptyset$  and  $\text{Lk}(a) = c$ . Lemma 25 implies that  $(\langle a, b, c \rangle, \langle a, b \rangle)$  is small. By Lemma 27,  $(\langle a, b, c, e, f, g \rangle, \langle a, b, f, g \rangle) = (\langle a, b, c \rangle \times \langle e, f, g \rangle, \langle a, b \rangle \times \langle f, g \rangle)$  is also small.

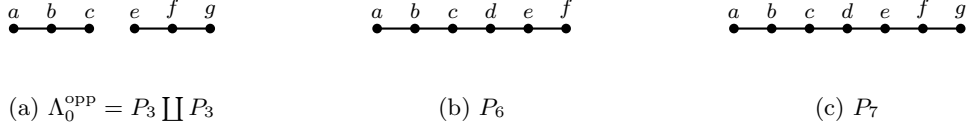


FIGURE 5. Some six and seven vertex graphs

**Lemma 29.** *If the free product of two groups  $A$  and  $B$  amalgamated along a finite subgroup is in  $\mathcal{S}$ , then either  $A$  or  $B$  is in  $\mathcal{S}$ .*

*Proof.* Suppose  $H$  is a hyperbolic surface group and  $C$  is a finite subgroup of both  $A$  and  $B$  such that  $H \leq A *_C B$ . There is a graph of groups decomposition for  $H$  such that each edge group embeds into  $C$ . Since  $H$  is torsion-free and one-ended, this decomposition should essentially have only one vertex group.  $\square$

We denote by  $P_n$  the path on  $n$  vertices. Let us recall from [9] the graphs  $P_1(7)$  and  $P_2(7)$ , whose *opposite graphs* are drawn in Figure 6. The result in [9] implies that when  $\Gamma$  at most seven vertices,  $A(\Gamma) \in \mathcal{S}$  if and only if  $\Gamma$  contains  $C_6^{\text{opp}}$ ,  $P_6^{\text{opp}}$ ,  $P_1(7)$ ,  $P_2(7)$  or  $C_m$  for some  $m \geq 5$  as an induced subgraph.

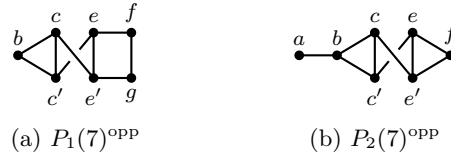


FIGURE 6. Graphs from [9].

We now consider some eight-vertex graphs.

**Lemma 30.** *Let  $\Phi_1, \Phi_2, \dots, \Phi_5$  be the graphs whose opposite graphs are shown in Figure 7 (a) through (e). Then  $C(\Phi_i) \notin \mathcal{S}$  for each  $i = 1, 2, \dots, 5$ .*

*Proof.* We use the vertex labels shown in Figure 7. We also set  $H = \langle a, b, f, g \rangle \leq G = \langle a, b, c, e, f, g \rangle \leq C(P_7^{\text{opp}}) = \langle a, b, c, d, e, f, g \rangle$  as considered in Figure 5 (c). In Example 28, we have seen that  $(G, H)$  is small.

(Case  $\Phi_1$ ) We can write  $C(\Phi_1) = (\langle a, b, c \rangle \times \langle e, f, g \rangle \times \langle t \rangle) *_{\langle a, b \rangle \times \langle f, g \rangle} (\langle a, b \rangle \times \langle d \rangle \times \langle f, g \rangle)$ . By Lemma 27 and Example 28,  $(\langle a, b, c \rangle \times \langle e, f, g \rangle \times \langle t \rangle, \langle a, b \rangle \times \langle f, g \rangle)$  is small. Since  $\langle a, b \rangle \times \langle d \rangle \times \langle f, g \rangle$  is virtually abelian, it contains no  $F_2$  and hence,  $(\langle a, b \rangle \times \langle d \rangle \times \langle f, g \rangle, \langle a, b \rangle \times \langle f, g \rangle)$  is small. Lemma 24 (1) implies that  $C(\Phi_1) \notin \mathcal{S}$ .

(Case  $\Phi_2$ ) We note that  $C(P_7^{\text{opp}}) \leq C(\Phi_1)$  and so,  $C(P_7^{\text{opp}}) \notin \mathcal{S}$ . Moreover, we have  $C(\Phi_1) = C(P_7^{\text{opp}}) *_{\langle t^2 \rangle} G$  where  $t$  is the stable generator. By Lemma 21,  $C(\Phi_2) = \langle a, b, c, d, e, f, g \rangle *_{\langle a, b, c, e, f, g \rangle} \langle a, b, c, d', e, f, g \rangle \leq C(\Phi_1)$ .

(**Case  $\Phi_3$** ) Note that  $C(P_7^{\text{opp}}) = G *_{\langle d \rangle} / \langle\langle d^2 \rangle\rangle$  where  $d$  is the stable generator. Hence,  $C(\Phi_3) = \langle a, b, c, e, f, g \rangle *_{\langle a, b, f, g \rangle} \langle a, b, c', e', f, g \rangle \leq C(P_7^{\text{opp}})$ .

(**Case  $\Phi_4$** ) Let us consider  $\psi: H \rightarrow H$  defined by  $\psi(a) = a, \psi(b) = b, \psi(f) = g$  and  $\psi(g) = f$ . We see that  $G *_{\psi: H \rightarrow H} G = \langle a, b, c, e, f, g \rangle *_{\psi} \langle a, b, c', e', f' = g, g' = f \rangle = C(\Phi_4)$ . Lemma 24 (1) and Example 28 imply that  $C(\Phi_4) \notin \mathcal{S}$ .

(**Case  $\Phi_5$** ) Similarly, define  $\psi: H \rightarrow H$  by  $\psi(a) = b, \psi(b) = a, \psi(f) = g$  and  $\psi(g) = f$ . We again see that  $G *_{\psi: H \rightarrow H} G = \langle a, b, c, e, f, g \rangle *_{\psi} \langle a' = b, b' = a, c', e', f' = g, g' = f \rangle = C(\Phi_5) \notin \mathcal{S}$ .  $\square$

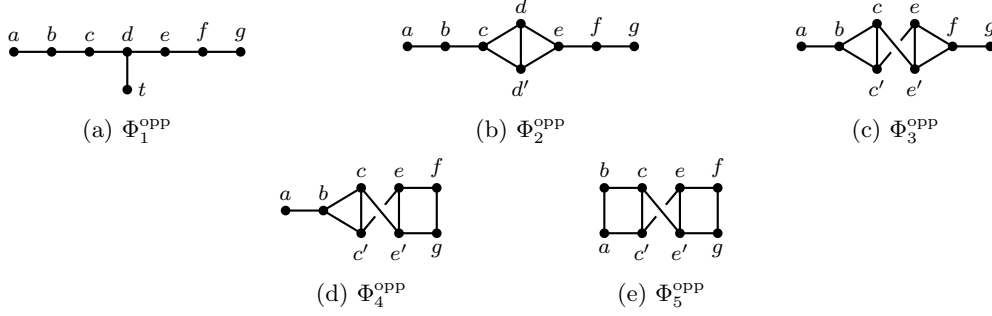


FIGURE 7. Some eight-vertex graphs in Lemma 30.

*Proof of Theorem 8.* We use notations from Lemma 30 and Figure 7. The backward direction is a restatement of Corollaries 6 and 7.

For the forward direction, suppose  $\Gamma$  is weakly chordal and  $C(\Gamma) \in \mathcal{S}$ . Since  $[C(\Gamma), C(\Gamma)] \leq [A(\Gamma), A(\Gamma)]$ , we see that  $A(\Gamma) \in \mathcal{S}$ . By the result in [9, Section 7],  $\Gamma$  contains  $P_6^{\text{opp}}, P_1(7)$  or  $P_2(7)$  as an induced subgraph; see Figure 6. Since  $P_1(7) \leq \Phi_4$  and  $P_2(7) \leq \Phi_3$ , Lemma 30 implies that  $C(P_1(7)) \notin \mathcal{S}$  and  $C(P_2(7)) \notin \mathcal{S}$ . Hence  $P_6^{\text{opp}} \leq \Gamma$ . Moreover,  $|V(\Gamma)| = 7$  since  $C(P_6^{\text{opp}}) \leq C(P_7^{\text{opp}}) \leq C(\Phi_1) \notin \mathcal{S}$ .

We fix the vertex labels of  $P_6 \leq \Gamma^{\text{opp}}$  as in Figure 5 (b) and let  $V(\Gamma) \setminus V(P_6^{\text{opp}}) = \{t\}$ . By  $\text{val}(t)$ , we will mean the valence of  $t$  in  $\Gamma$ .

(**Case  $\text{val}(t) = 0$  or  $1$** ) Note  $C(\Gamma) = C(P_6^{\text{opp}}) * \mathbb{Z}_2$  or  $C(\Gamma) = C(P_6^{\text{opp}}) *_{\mathbb{Z}_2} (\mathbb{Z}_2)^2$ . By applying Lemma 29, we obtain a contradiction that  $C(\Gamma) \notin \mathcal{S}$ .

(**Case  $\text{val}(t) = 2$** ) In  $\Gamma$ , the vertex  $t$  is joined to two vertices, say  $x, y$  of  $P_6^{\text{opp}}$ . We may write  $C(\Gamma) = C(P_6^{\text{opp}}) *_{\langle x, y \rangle} \langle x, y, t \rangle$ . If  $x$  and  $y$  are adjacent in  $P_6^{\text{opp}}$ , then  $C(\Gamma) = C(P_6^{\text{opp}}) *_{(\mathbb{Z}_2)^2} (\mathbb{Z}_2)^3$ ; then, Lemma 29 implies that  $C(\Gamma) \notin \mathcal{S}$ . So,  $x$  and  $y$  are adjacent in  $P_6 \leq \Gamma^{\text{opp}}$ . Since  $\Gamma^{\text{opp}}$  has no induced  $C_5$ , we should have  $\Gamma \cong \Lambda_1$ ; see Figure 8 (a). We have  $C(\Lambda_1) = (\langle a, b \rangle \times \langle d, e, f, t \rangle) *_{\langle a, e, f \rangle} (\langle a \rangle \times \langle c \rangle \times \langle e, f \rangle)$ . By Lemmas 25 and 27,  $(\langle a, b \rangle \times \langle d, e, f, t \rangle, \langle a \rangle \times \langle e, f \rangle)$  is small. Since  $\langle a, c, e, f \rangle$  is virtually abelian, we have  $C(\Lambda_1) \notin \mathcal{S}$  by Lemma 24 (1). This is a contradiction.

(**Case  $\Gamma = \Lambda_2$** ; see Figure 8 (b)) By Lemma 21,  $C(\Lambda_2) = \langle a, b, c, d, e, f \rangle *_{\langle b, c, d, e, f \rangle} \langle t, b, c, d, e, f \rangle$  embeds into  $\langle a, b, c, d, e, f \rangle *_{\langle b, c, d, e, f \rangle} / \langle\langle s^2 \rangle\rangle \cong C(P_7^{\text{opp}}) \notin \mathcal{S}$ , where  $s$  denotes the stable generator.

(**Case  $\Gamma = \Lambda_3$** ; see Figure 8 (c)) Using the vertex labels of  $P_6^{\text{opp}}$  in Figure 5 (b), we see that  $C(\Lambda_3) \cong GP(P_6^{\text{opp}}, \{G_a = \mathbb{Z}_2 \times \mathbb{Z}_2, G_b = G_c = \dots = G_f = \mathbb{Z}_2\})$ . Corollary 15 implies that  $C(\Lambda_3)$  virtually embeds into  $GP(P_6^{\text{opp}}, \{G_a = \mathbb{Z}_2 * \mathbb{Z}_2, G_b = G_c = \dots = G_f = \mathbb{Z}_2\}) \cong C(\Lambda_2) \notin \mathcal{S}$ .

(**Case  $\text{val}(t) = 3$** ) If  $a$  and  $f$  are both adjacent to  $t$  in  $\Gamma^{\text{opp}}$ , then only one of  $b, c, d, e$  are adjacent to  $t$  in  $\Gamma^{\text{opp}}$ . This implies that  $\Gamma^{\text{opp}}$  contains an induced  $C_5$ , and hence a contradiction. So, we may assume  $a$  is not adjacent to  $t$  in  $\Gamma^{\text{opp}}$ . Let us say  $x$  and  $y$  are the other two vertices of  $P_6 \leq \Gamma^{\text{opp}}$  that are non-adjacent to  $t$  in  $\Gamma^{\text{opp}}$ . If  $\{a, x, y\}$  are pairwise non-adjacent in  $P_6$ , then  $C(\Gamma) = C(P_6^{\text{opp}}) *_{\langle x, y, z \rangle} \langle t, x, y, z \rangle = C(P_6^{\text{opp}}) *_{(\mathbb{Z}_2)^3} (\mathbb{Z}_2)^4 \notin \mathcal{S}$ . So there exist at least two vertices in  $\{a, x, y\}$  that are adjacent in  $P_6$ . If  $b \in \{x, y\}$ , then  $\Gamma \cong \Lambda_i$  for  $i = 4, 5, 6, 7$ ; see Figure 8 (d)

through (g). If  $b \notin \{x, y\}$ , then  $x$  and  $y$  must be adjacent in  $P_6$ . Since  $C_5 \not\leq \Gamma^{\text{opp}}$ , we would have a graph isomorphism  $\Gamma \cong \Lambda_5$ .

If  $\Gamma \cong \Lambda_4$ , then  $C(\Gamma) = \langle a, b, c, d, e, f \rangle *_{\langle a, b, c, d, f \rangle} \langle a, b, c, d, t, f \rangle \leq \langle a, b, c, d, e, f \rangle *_{\langle a, b, c, d, f \rangle} / \langle\langle s^2 \rangle\rangle \cong C(\Lambda_3)$  where  $s$  is the stable generator. Similarly if  $\Gamma = \Lambda_5$ , then  $C(\Gamma) = \langle a, b, c, d, e, f \rangle *_{\langle a, b, c, e, f \rangle} \langle a, b, c, t, e, f \rangle \leq \langle a, b, c, d, e, f \rangle *_{\langle a, b, c, e, f \rangle} / \langle\langle s^2 \rangle\rangle \cong C(\Lambda)$  where  $s$  is the stable generator and  $\Lambda^{\text{opp}}$  is the subgraph of  $\Phi_1^{\text{opp}}$  induced by  $\{a, b, c, d, e, f, t\}$ ; see Figure 7 (a).

Suppose  $\Gamma = \Lambda_6$ . Then  $C(\Gamma) = (\langle a, b, c \rangle \times \langle e, f \rangle) *_{\langle a, b \rangle \times \langle f \rangle} (\langle a, b \rangle \times \langle d \rangle \times \langle t, f \rangle)$ . Since  $\langle a, b \rangle \times \langle d \rangle \times \langle t, f \rangle$  is virtually abelian, Example 28 implies that  $C(\Gamma) \notin \mathcal{S}$ .

Consider the case  $\Gamma = \Lambda_7$ . Then  $\Gamma^{\text{opp}}$  is obtained from  $\Phi_4^{\text{opp}}$  in Figure 7 (d) by contracting  $\{c, c'\}$  to a vertex. By Theorem 5,  $C(\Gamma)$  embeds into  $C(\Phi_4)$ .

(Case  $\text{val}(t) = 4$ ) Let  $x$  and  $y$  be the two vertices adjacent to  $t$  in  $\Gamma^{\text{opp}}$ . Since  $C_5 \not\leq \Gamma^{\text{opp}}$ , we see that  $d(x, y) \leq 2$  in  $P_6$ . So  $\Gamma \cong \Lambda_i$  for  $i = 2, 8, 9, 10, 11$ ; see Figure 8.

By Corollary 15,  $C(\Lambda_8)$  and  $C(\Lambda_9)$  virtually embed into  $C(\Lambda_4)$  and  $C(\Lambda_5)$ , respectively. If  $\Gamma = \Lambda_{10}$ , write  $C(\Lambda_{10}) = (\langle a, b, t \rangle \times \langle d, e, f \rangle) *_{\langle a \rangle \times \langle e, f \rangle} (\langle a \rangle \times \langle c \rangle \times \langle e, f \rangle) \notin \mathcal{S}$ . We see that  $\Phi_2^{\text{opp}}$  contracts onto  $\Lambda_{11}^{\text{opp}}$ ; see Figure 7 (b) and Figure 8 (k). So,  $C(\Lambda_{11}) \leq C(\Phi_2)$ .

(Case  $\text{val}(t) = 5$ ) Either  $\Gamma \leq \Phi_1$  or  $\Gamma \cong \Lambda_3$ .

(Case  $\text{val}(t) = 6$ ) Note  $C(\Gamma) = C(P_6^{\text{opp}}) \times \langle t \rangle$ .

□

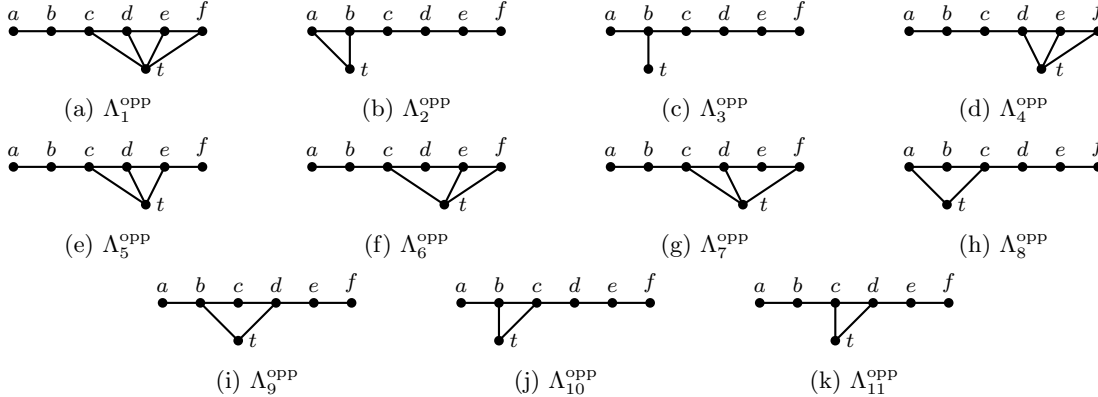


FIGURE 8. Seven-vertex graphs in Theorem 8.

*Remark.* (1) Theorem 8 is not true if  $\Gamma$  has more than seven vertices. That is, there exists a weakly chordal graph  $\Gamma$  such that  $C(\Gamma) \in \mathcal{S}$ . For example, let  $\Gamma$  be the graph whose opposite graph is shown in Figure 9. By [11],  $A(P_6^{\text{opp}})$  is an index-64 subgroup of  $C(\Gamma)$  and so,  $C(\Gamma) \in \mathcal{S}$ .

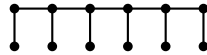


FIGURE 9. The graph  $\Gamma^{\text{opp}}$  in Proposition 5.

(2) There exists a graph  $\Gamma$  such that  $A(\Gamma) \in \mathcal{S}$  and  $C(\Gamma) \notin \mathcal{S}$ . For instance, we may set  $\Gamma$  as one of the graphs  $P_6^{\text{opp}}, P_1(7)$  or  $P_2(7)$ .

**Problem 31.** (1) Does there exist a graph  $\Gamma$  such that  $[A(\Gamma), A(\Gamma)] \notin \mathcal{S}$  while  $A(\Gamma) \in \mathcal{S}$ ?

(2) Does there exist a graph  $\Gamma$  such that  $[A(\Gamma), A(\Gamma)] \in \mathcal{S}$  while  $C(\Gamma) \notin \mathcal{S}$ ?

## 6. CLOSURE UNDER GRAPH PRODUCTS

A group  $G$  is *periodic* if every element of  $G$  has a torsion. The following is well-known.

**Lemma 32** ([16]). *A word-hyperbolic group does not have an infinite periodic subgroup.*

Let us denote by  $\mathcal{X}$  the class of finitely generated groups that are either

- (i) not one-ended, or
- (ii) not word-hyperbolic, or
- (iii) containing hyperbolic surface groups.

An affirmative answer to Question 2 is equivalent to saying that every finitely generated group is in  $\mathcal{X}$ . How large do we know  $\mathcal{X}$  is? We prove that  $\mathcal{X}$  is closed under graph products.

**Theorem 33.** *If  $\mathcal{G} = \{G_v : v \in V(\Gamma)\}$  is a collection of groups in  $\mathcal{X}$ , then  $GP(\Gamma, \mathcal{G})$  is in  $\mathcal{X}$ .*

*Proof.* Suppose that  $G = GP(\Gamma, \mathcal{G})$  is one-ended and word-hyperbolic. We may assume that each  $G_v$  is nontrivial. If  $\Gamma$  contains an induced cycle of length at least five, Corollary 6 implies that  $G \in \mathcal{S}$ . If  $\Gamma$  contains an induced square, whose vertices are denoted as  $a, b, c$  and  $d$  cyclically, then  $G$  would contain  $(G_a * G_c) \times (G_b * G_d) \geq \mathbb{Z} \times \mathbb{Z}$ . So from now on, we will assume that  $C_n \not\leq \Gamma$  for every  $n \geq 4$ ; namely,  $\Gamma$  is a *chordal* graph [17].

Suppose that  $\Gamma$  is complete. Then  $G$  is the direct product of its vertex groups. Since  $G$  is one-ended, at least one vertex group, say  $G_a$ , must be infinite. By Lemma 32, each infinite vertex group of  $G$  contains  $\mathbb{Z}$ . As  $G$  does not contain  $\mathbb{Z} \times \mathbb{Z}$ , exactly one vertex group is infinite. Then  $G$  is virtually  $G_a$ , and so,  $G_a$  is one-ended hyperbolic. Since  $G_a \in \mathcal{X}$ , we have  $G_a \in \mathcal{S}$ .

Now, assume that  $\Gamma$  is not complete. Since  $\Gamma$  is chordal,  $\Gamma$  can be written as  $\Gamma = \Gamma_1 \cup \Gamma_2$  for some induced subgraphs  $\Gamma_1, \Gamma_2$  such that  $\Gamma_0 = \Gamma_1 \cap \Gamma_2$  is complete [13]. We choose a minimal such  $\Gamma_0$ . If all the vertex groups of  $\Gamma_0$  are finite, then  $G$  splits over a finite group, and hence  $G$  has more than one ends. So  $G_a$  is infinite for some  $a \in V(\Gamma_0)$ . By minimality of  $\Gamma_0$ , we can find  $a_i \in \Gamma_i \setminus \Gamma_0$  such that  $a_i$  is adjacent to  $a$  for  $i = 1, 2$ . This implies that  $G$  contains  $G_a \times (G_{a_1} * G_{a_2})$ , and hence,  $\mathbb{Z} \times \mathbb{Z}$ . This is a contradiction.  $\square$

*Remark.* (1) Several other classes of groups are known to be closed under the graph product operation. These classes include residually finite groups [20], semihyperbolic groups [2], automatic groups [26] and diagram groups [22]. Meier characterized exactly when a graph product of word-hyperbolic groups is word-hyperbolic [36].

- (2) Every 3-manifold group is in  $\mathcal{X}$ . To see this, suppose  $M$  is a 3-manifold such that  $\pi_1(M)$  is one-ended and word-hyperbolic. We may assume  $M$  is orientable by taking a double cover if necessary. By the Loop Theorem, either  $M$  has a hyperbolic incompressible boundary component or  $M$  is closed possibly after capping off spherical boundary components. If  $M$  is closed, Perelman's geometrization theorem implies that  $M$  is a closed hyperbolic 3-manifold; then, the work of Kahn and Markovic [29] implies that  $\pi_1(M) \in \mathcal{S}$ .

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