On the lattice of flats of a boolean representable simplicial complex

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ABSTRACT

It is shown that the lattices of flats of boolean representable simplicial complexes are always atomistic, but semimodular if and only if the complex is a matroid. A canonical construction is introduced for arbitrary finite atomistic lattices, providing a characterization of the lattices of flats of boolean representable simplicial complexes and a decidability condition. We remark that every finite lattice occurs as the lattice of flats of some simplicial complex.

1 Introduction

In a series of three papers [3, 4, 5], Izhakian and Rhodes introduced the concept of boolean representation for various algebraic and combinatorial structures. These ideas were inspired by previous work by Izhakian and Rowen on supertropical matrices (see e.g. [2, 6, 7, 8]), and were subsequently developed by Rhodes and Silva in a recent monograph, devoted to boolean representable simplicial complexes [12].

The original approach is to consider matrix representations over the superboolean semiring $\mathbb{SB}$, using appropriate notions of vector independence and rank. Writing $\mathbb{N} = \{0, 1, 2, \ldots\}$, we can define $\mathbb{SB}$ as the quotient of $(\mathbb{N}, +, \cdot)$ (usual operations) by the congruence which identifies all integers ≥ 2. In this context, boolean representation refers to matrices using only 0 and 1 as entries.

In this paper, we view (finite) simplicial complexes in their abstract form, as hereditary collections. However, boolean representations have also provided results of a more geometric and topological nature, namely on the homotopy of (geometric) simplicial complexes (see [12, Chapter 7], [9]).

As an alternative, boolean representability can be characterized by means of the lattice of flats. The lattice of flats plays a fundamental role in matroid theory but is not usually considered for arbitrary simplicial complexes, probably due to the fact that, unlike in the matroid case, the structure of a simplicial complex cannot in general be recovered from its lattice of flats. However, this is precisely what happens with boolean representable simplicial complexes (BRSCs). If $\mathcal{H} = (V, H)$ is a simplicial complex and $\mathrm{Fl}\mathcal{H}$ denotes its lattice of flats, then $\mathcal{H}$ is boolean representable if and only if H equals the set of transversals of the successive differences for chains in $\mathrm{Fl}\mathcal{H}$. This implies in particular that all (finite) matroids are boolean representable. If the BRSC is simple, it can be characterized by the isomorphism class of its lattice of flats, plus a bijection from its vertex set onto the set of atoms of the lattice.

Therefore it is a natural question to inquire about the nature of the lattices of flats of BRSCs. We note that in the matroid case the lattices of flats are precisely the geometric lattices (atomistic and semimodular) [10, Theorem 1.7.5].

We note that it is known that any finite lattice embeds in a finite partition lattice [11], which is a geometric lattice, so corresponds to a matroid. Therefore every finite lattice is isomorphic to some full sublattice of the lattice of flats of some $\mathcal{H} \in \mathcal{BR}$.

The paper is organized as follows. In Section 2 we present all the basic notions and results needed in the paper. In Section 3 we show that every finite lattice occurs as the lattice of flats of some simplicial complex. In Section 4 we show that lattices of flats of BRSCs are always atomistic, but not necessarily semimodular. In Section 5 we construct a simple BRSC $\mathcal{T}_L$ for every finite atomistic lattice L and show that $L \cong \mathrm{Fl}\mathcal{H}$ for some BRSC $\mathcal{H}$ if and only if $L \cong \mathrm{Fl}\mathcal{T}_L$. Moreover, in this case $\mathcal{T}_L$ is isomorphic to some restriction of $\mathcal{H}$, and if $\mathcal{H}$ is simple, then $\mathcal{T}_L$ is isomorphic to $\mathcal{H}$. This provides an easy way of deciding whether or not a finite lattice is isomorphic to the lattice of flats of some BRSC, and the complexity is polynomial for fixed height. In Section 6, we prove a graph-theoretical characterization for lattices of height 3, also decidable in polynomial time.

2 Preliminaries

All lattices and simplicial complexes in this paper are assumed to be finite. The reader is assumed to have some familiarity with basic notions of lattice theory, being referred to [1].

Given a set V and $n \geq 0$, we denote by $P_n(V)$ (respectively $P_{\leq n}(V)$) the set of all subsets of V with precisely (respectively at most) n elements. To simplify notation, we shall often represent sets $\{a_1, a_2, \ldots, a_n\}$ in the form $a_1 a_2 \ldots a_n$.

A (finite) simplicial complex is a structure of the form $\mathcal{H} = (V, H)$, where V is a finite nonempty set and $H \subseteq 2^V$ is nonempty and closed under taking subsets. Simplicial complexes, in this abstract viewpoint, are also known as *hereditary collections*.

Two simplicial complexes (V, H) and (V', H') are *isomorphic* if there exists a bijection $\varphi : V \to V'$ such that

$$X \in H \text{ if and only if } X\varphi \in H'$$

holds for every $X \subseteq V$.

If $\mathcal{H} = (V, H)$ is a simplicial complex and $W \subseteq V$ is nonempty, we call

$$\mathcal{H}|_W = (W, H \cap 2^W)$$

the *restriction* of $\mathcal{H}$ to W. It is obvious that $\mathcal{H}|_W$ is still a simplicial complex.

We say that $X \subseteq V$ is a *flat* of $\mathcal{H}$ if

$$\forall I \in H \cap 2^X \;\; \forall p \in V \setminus X \quad I \cup \{p\} \in H.$$

The set of all flats of $\mathcal{H}$ is denoted by $\mathrm{Fl}\mathcal{H}$.

Clearly, the intersection of any set of flats (including $V = \cap\emptyset$) is still a flat. If we order $\mathrm{Fl}\mathcal{H}$ by inclusion, it is then a $\wedge$-semilattice, and therefore a lattice with

$$(X \vee Y) = \cap\{F \in \mathrm{Fl}\mathcal{H} \mid X \cup Y \subseteq F\}$$

for all $X, Y \in \mathrm{Fl}\mathcal{H}$. We call $\mathrm{Fl}\mathcal{H}$ the *lattice of flats* of $\mathcal{H}$. The lattice of flats induces a closure operator on 2^V defined by

$$\overline{X} = \cap\{F \in \mathrm{Fl}\mathcal{H} \mid X \subseteq F\}$$

for every $X \subseteq V$.

We say that X is a *transversal of the successive differences* for a chain of subsets

$$A_0 \subset A_1 \subset \ldots \subset A_k$$

if X admits an enumeration $x_1, \ldots, x_k$ such that $x_i \in A_i \setminus A_{i-1}$ for $i = 1, \ldots, k$.

Let $\mathcal{H} = (V, H)$ be a simplicial complex. If $X \subseteq V$ is a *transversal of the successive differences* for a chain

$$F_0 \subset F_1 \subset \ldots \subset F_k$$

in $\mathrm{Fl}\mathcal{H}$, it follows easily by induction that $\{x_1, x_2, \ldots x_i\} \in H$ for $i = 0, \ldots, k$. In particular, $X \in H$.

We say that $\mathcal{H}$ is *boolean representable* if every $X \in H$ is a transversal of the successive differences for a chain in $\mathrm{Fl}\mathcal{H}$. We denote by $\mathcal{BR}$ the class of all (finite) boolean representable simplicial complexes (BRSCs).

A simplicial complex $\mathcal{H} = (V, H)$ is called a *matroid* if it satisfies the *exchange property*:

(EP) For all $I, J \in H$ with $|I| = |J| + 1$, there exists some $i \in I \setminus J$ such that $J \cup \{i\} \in H$.

All matroids are boolean representable, but the converse is not true (see [12, Example 5.2.11]).

3 Arbitrary complexes

In the literature, the lattice of flats is defined only for matroids, but there is no reason to prevent considering it for arbitary simplicial complexes. It turns out that every finite lattice can be constructed this way.

Theorem 3.1 *Every finite lattice is isomorphic to the lattice of flats of some simplicial complex.*

Proof. Let L be a finite lattice. If L is trivial, then $L \cong \mathrm{Fl}\mathcal{H}$ for $\mathcal{H} = (V, \{\emptyset\})$, hence we assume that L is nontrivial. Let B and T denote the bottom and the top elements of L. Let

$$V = \{a^{(i)} \mid a \in L \setminus \{B\},\ i = 1, 2, 3\}$$

and let $\pi : V \to L \setminus \{B\}$ be the canonical mapping. Let

$$J = \{X \subset V \mid \pi|_X \text{ is injective}\}.$$

For every $P \subseteq L \setminus \{B\}$, let

$$P\alpha = \{a \in L \setminus \{B\} \mid p \not\leq a \text{ for every } p \in P \text{ and } (a \vee p) \neq q \text{ for all distinct } p, q \in P\}.$$

We define

$$H = J \cup \{X \cup \{a^{(1)}, a^{(2)}\} \mid X \in J,\ a \in X\pi\alpha\}.$$

It is easy to check that $\mathcal{H} = (V, H)$ is a (finite) simplicial complex. This follows from J being closed under taking subsets and

$$X \cup \{a^{(i)}\} \in J \text{ whenever } X \in J,\ a \in X\pi\alpha \text{ and } i \in \{1, 2\}.$$

For every $a \in L$, let

$$a\beta = \{x \in L \setminus \{B\} \mid x \leq a\}.$$

Since $P_1(V) \subseteq H$, we get

$$\overline{\emptyset} = \emptyset = B\beta\pi^{-1}. \tag{1}$$

Next let $a \in L \setminus \{B\}$ and $i \in \{1, 2, 3\}$. We show that

$$\overline{a^{(i)}} = a\beta\pi^{-1}. \tag{2}$$

Since $a^{(i)} \in H$ and $\{a^{(1)}, a^{(3)}\}, \{a^{(2)}, a^{(3)}\} \notin H$, we get $a^{(3)} \in \overline{a^{(i)}}$ and thus $a\pi^{-1} \subseteq \overline{a^{(i)}}$. Now let $B \neq x < a$ in L and let $j \in \{1, 2, 3\}$. Then $\{a^{(1)}, a^{(2)}, x^{(j)}\} \notin H$ because $a \notin x\alpha$. Since $\{a^{(1)}, a^{(2)}\} \in H$, we get $x^{(j)} \in \overline{a^{(i)}}$ and so $a\beta\pi^{-1} \subseteq \overline{a^{(i)}}$.

Thus it is enough to prove that $a\beta\pi^{-1} \in \mathrm{Fl}\mathcal{H}$. Let $I \subseteq a\beta\pi^{-1}$ be such that $I \in H$ and let $y^{(j)} \in V \setminus a\beta\pi^{-1}$. If $I \in J$, then $y^{(j)} \notin a\beta\pi^{-1}$ together with $I \subseteq a\beta\pi^{-1}$ yield $I \cup \{y^{(j)}\} \in J \subseteq H$, hence we may assume that $I = X \cup \{a^{(1)}, a^{(2)}\}$ for some $X \in J$ and $a \in X\pi\alpha$.

Suppose that $x^{(k)} \in X$. Then $x^{(k)} \in I \subseteq a\beta\pi^{-1}$ and so $x \leq a$. On the other hand, $a \in X\pi\alpha$ yields $x \not\leq a$, a contradiction. Thus $X = \emptyset$ and so $I = \{a^{(1)}, a^{(2)}\}$. It follows that

$$I \cup \{y^{(j)}\} = \{y^{(j)}\} \cup \{a^{(1)}, a^{(2)}\}.$$

Clearly, $\{y^{(j)}\} \in J$ and $y^{(j)} \notin a\beta\pi^{-1}$ yields $a \in y^{(j)}\pi\alpha$. Therefore $I \cup \{y^{(j)}\} \in H$. It follows that $a\beta\pi^{-1} \in \mathrm{Fl}\mathcal{H}$ and so (2) holds.

Next we show that

$$\overline{\{a^{(i)}, b^{(j)}\}} = (a \vee b)\beta\pi^{-1} \tag{3}$$

holds for all $a, b \in L \setminus \{B\}$.

Let $c = (a \vee b)$. In view of (2), we may assume that $c > a, b$. It is easy to check that $\{a^{(1)}, a^{(2)}, b^{(1)}\} \in H$ (because $b \not\leq a$) and $\{a^{(1)}, a^{(2)}, b^{(1)}, c^{(1)}\} \notin H$ (because $c = (a \vee b)$). Since $\{a^{(1)}, a^{(2)}, b^{(1)}\} \subseteq \overline{\{a^{(i)}, b^{(j)}\}}$ by (2), we get $c^{(1)} \in \overline{\{a^{(i)}, b^{(j)}\}}$ and so $c\beta\pi^{-1} \subseteq \overline{\{a^{(i)}, b^{(j)}\}}$ by (2). Now the opposite inclusion follows from $\{a^{(i)}, b^{(j)}\} \subseteq (a \vee b)\beta\pi^{-1}$, therefore (3) holds.

Now we claim that

$$\mathrm{Fl}\mathcal{H} = \{a\beta\pi^{-1} \mid a \in L\}. \tag{4}$$

The opposite inclusion follows from (1) and (2). Take $F \in \mathrm{Fl}\mathcal{H}$. In view of (1), we may assume that $F \neq \emptyset$. It follows from (3) that $F\pi$ has a maximum $a \neq B$, and (2) yields $F = a\beta\pi^{-1}$. Thus (4) holds and it follows easily that

$$\begin{array}{rcl} L & \to & \mathrm{Fl}\mathcal{H} \\ a & \mapsto & a\beta\pi^{-1} \end{array}$$

is an isomorphism of posets and therefore a lattice isomorphism. □

4 A necessary condition

In this section, we start the discussion on the lattices of flats of BRSCs.

The main theorem of the section proves a necessary condition for a lattice to be isomorphic to such a lattice of flats. This implies that we cannot assume the simplicial complexes to be boolean representable in Theorem 3.1. But first we prove two simple lemmas.

Lemma 4.1 *Let $\mathcal{H} = (V,H)$ be a simplicial complex and let $W \subseteq V$. Let $F \in \mathrm{Fl}\mathcal{H}$. Then $F \cap W \in \mathrm{Fl}(\mathcal{H}|_W)$.*

Proof. Let $H' = H \cap 2^W$. Assume that $I \in H' \cap 2^{F \cap W}$ and $p \in W \setminus (F \cap W)$. Then $I \in H \cap 2^F$ and $p \in V \setminus F$, hence $F \in \mathrm{Fl}\mathcal{H}$ yields $I \cup \{p\} \in H$. Since $I \cup \{p\} \subseteq W$, we get $I \cup \{p\} \in H'$. Thus $F \cap W \in \mathrm{Fl}(\mathcal{H}|_W)$. □

Lemma 4.2 *Let $\mathcal{H} = (V,H)$ be a simplicial complex and let $V' = \{p \in V \mid \{p\} \in H\}$. Then $\mathrm{Fl}\mathcal{H} \cong \mathrm{Fl}(\mathcal{H}|_{V'})$.*

Proof. Write $\mathcal{H}' = \mathcal{H}|_{V'}$ and $H' = H \cap 2^{V'}$. Let $\alpha : \mathrm{Fl}\mathcal{H} \to \mathrm{Fl}\mathcal{H}'$ and $\beta : \mathrm{Fl}\mathcal{H}' \to \mathrm{Fl}\mathcal{H}$ be the mappings defined by

$$F\alpha = F \cap V', \quad F'\beta = F' \cup (V \setminus V').$$

By Lemma 4.1, α is well defined. Now let $F' \in \mathrm{Fl}\mathcal{H}'$. Assume that $I \in H \cap 2^{F'\beta}$ and $p \in V \setminus F'\beta$. By definition of V', we have $I \subseteq V'$, hence $I \in H' \cap 2^{F'}$. Since also $p \in V' \setminus F'$, then $F' \in \mathrm{Fl}\mathcal{H}'$ yields $I \cup \{p\} \in H' \subseteq H$. Thus $F'\beta \in \mathrm{Fl}\mathcal{H}$ and β is well defined.

Next we show that the mappings α and β are mutually inverse. Let $F \in \mathrm{Fl}\mathcal{H}$. Then

$$F\alpha\beta = (F \cap V') \cup (V \setminus V')$$

and so $F \subseteq F\alpha\beta$. Suppose that $p \in F\alpha\beta \setminus F$. Then $p \in V \setminus V'$. Since $\emptyset \in H \cap 2^F$ and $\{p\} \notin H$, then $F \in \mathrm{Fl}\mathcal{H}$ yields $p \in F$, a contradiction. Thus $F\alpha\beta = F$ and so $\alpha\beta = 1$.

On the other hand, for every $F' \in \mathrm{Fl}\mathcal{H}'$ we have

$$F'\beta\alpha = (F' \cup (V \setminus V')) \cap V' = F',$$

thus $\beta\alpha = 1$. Therefore α and β are mutually inverse.

Since α and β are both order-preserving, they are poset isomorphisms and therefore lattice isomorphisms. □

Let L be a lattice. Given $a, b \in L$, we say that b *covers* a if $a < b$ and there is no $c \in L$ such that $a < b < c$. An *atom* of L is a element covering the bottom element B. We denote by $\mathrm{At}(L)$ the set of atoms of L. The lattice L is *atomistic* if every element of L is a join of atoms. We show next that being atomistic is a necessary condition for a lattice to be isomorphic to the lattice of flats of a BRSC.

Theorem 4.3 *Let $\mathcal{H} = (V, H) \in \mathcal{BR}$. Then:*

(i) if $P_1(V) \subseteq H$, then V is the union of the atoms of $\mathrm{Fl}\mathcal{H}$*;*

(ii) $\mathrm{Fl}\mathcal{H}$ *is atomistic.*

Proof. (i) In [9], we introduced the equivalence relation on V defined by

$$p \eta q \quad \text{if } \overline{p} = \overline{q}$$

and the *simplification* $\mathcal{H}_S = (V/\eta, H/\eta)$. By [9, Proposition 4.2](iii), we have $\mathrm{Fl}\mathcal{H} \cong \mathrm{Fl}\mathcal{H}_S$. On the other hand, if $\varphi : V \to V/\eta$ denotes the canonical projection, it follows from [9, Proposition 4.2](ii) that the atoms of $\mathrm{Fl}\mathcal{H}$ are of the form $A\varphi^{-1}$, where A is an atom of $\mathcal{H}_S$. But the the atoms of $\mathrm{Fl}\mathcal{H}_S$ are the singleton sets, thus we are done.

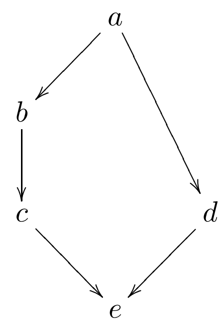
(ii) By Lemma 4.2, we may assume that $P_1(V) \subseteq H$, and by [9, Proposition 4.2](iii) we may assume that $\mathcal{H}$ is simple. Hence the atoms of $\mathcal{H}$ are the singleton sets $\{p\}$ with $p \in V$. Now the claim becomes obvious. Note that $\emptyset$ is the bottom element of $\mathrm{Fl}\mathcal{H}$, which is the join of the empty set of atoms. $\square$

Corollary 4.4 *The 3-point chain*

is the smallest lattice not isomorphic to the lattice of flats of some $\mathcal{H} \in \mathcal{BR}$.

Proof. In fact, this chain happens to be the smallest non atomistic lattice. On the other hand, the trivial lattice and the 2-point lattice occur as the lattices of flats for $(V, \{\emptyset\})$ and $(V, P_{\leq 1}(V))$, respectively. Now the claim follows from Theorem 4.3(ii). $\square$

A lattice L is said to be (upper) *semimodular* if has no sublattice of the form

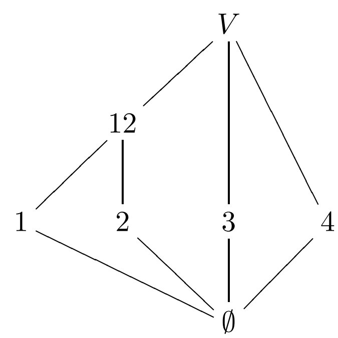


with d covering e. A *geometric* lattice is a lattice which is both semimodular and atomistic. It is well known that a finite lattice is geometric if and only if it is isomorphic to the lattice of flats of some (finite) matroid [10, Theorem 1.7.5].

The next example shows that the lattice of flats of a BRSC is not necessarily semimodular (we must take a BRSC which is not a matroid).

Example 4.5 *Let* $V = \{1,2,3,4\}$ *and* $H = P_{\leq 2}(V) \cup \{123, 124\}$. *Then* $\mathcal{H} = (V,H) \in \mathcal{BR}$ *but* $Fl\mathcal{H}$ *is not semimodular.*

Indeed, it is easy to check that $\mathrm{Fl}\mathcal{H} = P_{\leq 1}(V) \cup \{V, 12\}$, and can therefore be described as



and is therefore boolean representable (see [12, Example 5.2.11]).

By considering the sublattice $\{V, 12, 2, 4, \emptyset\}$, we deduce that $\mathrm{Fl}\mathcal{H}$ is not semimodular.

5 The minimal simplicial complex on a lattice of flats

We show in this section that, whenever a lattice is isomorphic to the lattice of flats of some $\mathcal{H} \in \mathcal{BR}$, there exists a minimal simplicial complex satisfying this condition. To prove this claim, we introduce the following construction.

Let L be an atomistic lattice. For every $x \in L$, let

$$x\xi = \{a \in \mathrm{At}(L) \mid a \leq x\}.$$

We say that $A \subseteq \mathrm{At}(L)$ is a *transversal of a chain* in L if there exists an enumeration $a_1, \ldots, a_m$ of the elements of A and a chain

$$x_0 < x_1 < \ldots < x_m$$

in L such that $a_i \in x_i\xi \setminus x_{i-1}\xi$ for $i = 1, \ldots, m$.

Let $T_L \subseteq 2^{\mathrm{At}(L)}$ consist of all the transversals of some chain in L and write $\mathcal{T}_L = (\mathrm{At}(L), T_L)$. It is immediate that $\mathcal{T}_L$ is a simplicial complex. We prove the following lemma.

Lemma 5.1 *Let L be an atomistic lattice. Then:*

(i) $x\xi \in \mathrm{Fl}\mathcal{T}_L$ for every $x \in L$;

(ii) ξ is a lattice isomorphism of L onto $L\xi \subseteq \mathrm{Fl}\mathcal{T}_L$;

(iii) $\mathcal{T}_L \in \mathcal{BR}$;

(iv) $\mathcal{T}_L$ is simple.

Proof. (i) Let $x \in L$. If $x = T$, then the conclusion is clear, so assume that $x \neq T$. Assume that $A \in T_L \cap 2^{x\xi}$ and $p \in \mathrm{At}(L) \setminus x\xi$. Since $A \in T_L$, there exists an enumeration $a_1, \ldots, a_m$ of A and a chain $x_0 < \ldots < x_m$ in L such that $a_i \in x_i\xi \setminus x_{i-1}\xi$ for $i = 1, \ldots, m$. Now

$$(x_0 \wedge x) \leq (x_1 \wedge x) \leq \ldots \leq (x_m \wedge x). \tag{5}$$

Since $A \subseteq x\xi$, we get $a_i \in (x_i \wedge x)\xi \setminus (x_{i-1} \wedge x)\xi$ for $i = 1, \ldots, m$, hence the ordering in (5) must be strict. Since $(x_m \wedge x)\xi \subseteq x\xi$, then $p \in T\xi \setminus (x_m \wedge x)\xi$ and so $A \cup \{p\}$ is a transversal of the chain

$$(x_0 \wedge x) < (x_1 \wedge x) < \ldots < (x_m \wedge x) < T$$

in K. Thus $A \cup \{p\} \in T_L$ and so $x\xi \in \mathrm{Fl}\mathcal{T}_L$.

(ii) Since L is atomistic, we have

$$x = \vee(x\xi) \text{ for every } x \in L, \tag{6}$$

hence ξ is injective.

Let $x, y \in L$. It is immediate that $x \leq y$ implies $x\xi \subseteq y\xi$, hence ξ is order-preserving. Finally, in view of (6), $x\xi \subseteq y\xi$ yields

$$x = \vee(x\xi) \leq \vee(y\xi) = y,$$

hence $\xi : L \to L\xi$ is a poset isomorphism and therefore a lattice isomorphism (preserving top and bottom).

(iii) Let $A \in T_L$. Then there exists an enumeration $a_1, \ldots, a_m$ of the elements of A and a chain

$$x_0 < x_1 < \ldots < x_m$$

in L such that $a_i \in x_i\xi \setminus x_{i-1}\xi$ for $i = 1, \ldots, m$. Since $x_i\xi \in \mathrm{Fl}\mathcal{T}_L$ for $i = 0, \ldots, m$, it follows that A is a transversal of the successive differences for a chain in $\mathrm{Fl}\mathcal{T}_L$. Therefore $\mathcal{T}_L$ is boolean representable.

(iv) It is immediate that $P_2(\mathrm{At}(L)) \subseteq T_L$. $\square$

The following theorem asserts the canonical role played by $\mathcal{T}_L$.

Theorem 5.2 *Let L be an atomistic lattice. Then the following conditions are equivalent.*

(i) $L \cong \mathrm{Fl}\mathcal{H}$ *for some* $\mathcal{H} \in \mathcal{BR}$*;*

(ii) $L \cong \mathrm{Fl}\mathcal{T}_L$*;*

(iii) $\xi : L \to \mathrm{Fl}\mathcal{T}_L$ *is onto;*

(iv) $(\vee F)\xi \subseteq F$ *for every* $F \in \mathrm{Fl}\mathcal{T}_L$*.*

Moreover, in this case $\mathcal{T}_L$ is isomorphic to some restriction of $\mathcal{H}$, and for $\mathcal{H}$ simple we get an isomorphism.

Proof. (i) $\Rightarrow$ (ii). Let $\mathcal{H} = (V, H)$. By Lemma 4.2, we may assume that $P_1(V) \subseteq H$, and by [9, Proposition 4.2](iii) we may assume that $\mathcal{H}$ is simple (replacing $\mathcal{H}$ by its simplification $\mathcal{H}_S$, isomorphic to some restriction of $\mathcal{H}$).

Let $\varphi : L \to \mathrm{Fl}\mathcal{H}$ be a lattice isomorphism. Since $\mathrm{At}(\mathrm{Fl}\mathcal{H}) = \{\{p\} \mid p \in V\}$, φ induces a bijection $\varphi' : \mathrm{At}(L) \to V$ defined by $a\varphi = \{a\varphi'\}$. We claim that

$$\mathcal{T}_L \cong \mathcal{H}. \tag{7}$$

Indeed, let $X \subseteq \mathrm{At}(L)$. Then $X \in T_L$ if and only if there exists an enumeration $a_1, \ldots, a_k$ of the elements of X and a chain $\ell_0 < \ell_1 < \ldots < \ell_k$ in L such that $a_i \in \ell_i\xi \setminus \ell_{i-1}\xi$ for $i = 1, \ldots, k$. It is easy to check that $\ell_0\varphi < \ell_1\varphi < \ldots < \ell_k\varphi$ is a chain in $\mathrm{Fl}\mathcal{H}$ such that $a_i\varphi' \in \ell_i\varphi \setminus \ell_{i-1}\varphi$ for $i = 1, \ldots, k$. Hence $X\varphi' \in H$. Similarly, $X\varphi' \in H$ implies $X \in T_L$, therefore φ' defines an isomorphism between $\mathcal{T}_L$ and $\mathcal{H}$, and so (7) holds.

Note that, for arbitrary $\mathcal{H}$, the reductions performed above through Lemma 4.2 and [9, Proposition 4.2](iii) replace the original BRSC $\mathcal{H}$ by one of its restrictions, so $\mathcal{T}_L$ is indeed isomorphic to some restriction of $\mathcal{H}$.

(ii) $\Rightarrow$ (i). By Lemma 5.1(i).

(ii) $\Leftrightarrow$ (iii). In view of Lemma 5.1(ii), and since L is finite.

(iii) $\Rightarrow$ (iv). Let $F \in \mathrm{Fl}\mathcal{T}_L$. By condition (iii), we have $F = x\xi$ for some $x \in L$. Hence $a \leq x$ for every $a \in F$ and so $\vee F \leq x$. Thus $(\vee F)\xi \subseteq x\xi = F$.

(iv) $\Rightarrow$ (iii). Let $F \in \mathrm{Fl}\mathcal{T}_L$ and let $x = \vee F$. Since $a \in F$ implies $a \leq x$, we have $F \subseteq x\xi$. On the other hand, condition (iv) yields $x\xi \subseteq F$, hence $\xi : L \to \mathrm{Fl}\mathcal{T}_L$ is onto. $\square$

Corollary 5.3 *Let L be a lattice. Then it is decidable whether or not $L \cong \mathrm{Fl}\mathcal{H}$ for some $\mathcal{H} \in \mathcal{BR}$.*

Proof. By Theorem 4.3(ii), L being atomistic is a necessary condition. Since being atomistic is certainly decidable, we may assume that L is atomistic.

Since we can successively compute $\mathcal{T}_L$ and $\mathrm{Fl}(\mathcal{T}_L)$, the claim now follows from Theorem 5.2. $\square$

Predictably, being atomistic is not a sufficient condition, as we show in the next result. We recall that the *height* of a lattice L, denoted by $\mathrm{ht}(L)$, is the maximal length n of a chain $x_0 < x_1 < \ldots < x_n$ in L. Given a set X, let $(2^X, \subseteq)$ be the lattice of all subsets of X with respect to inclusion.

Proposition 5.4 *Let L be a lattice with $\mathrm{ht}(L) = |\mathrm{At}(L)|$. Then the following conditions are equivalent:*

(i) $L \cong \mathrm{Fl}\mathcal{H}$ for some $\mathcal{H} \in \mathcal{BR}$;

(ii) $L \cong (2^{\mathrm{At}(L)}, \subseteq)$.

Proof. (i) $\Rightarrow$ (ii). By Theorem 4.3(ii), L is atomistic. Let

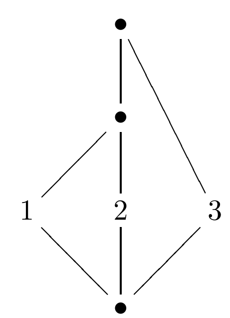
$$x_0 < x_1 < \ldots < x_n$$

be a chain of maximal length in L. Since L is atomistic, for every $i \in \{1, \ldots, n\}$ there exists some $a_i \in x_i\xi \setminus x_{i-1}\xi$. It is easy to see that the a_i are all distinct and so $\mathrm{ht}(L) = |\mathrm{At}(L)|$ yields $\mathrm{At}(L) = \{a_1, \ldots, a_n\}$. Hence $\mathrm{At}(L) \in T_L$ and so $T_L = 2^{\mathrm{At}(L)}$. Thus $2^{\mathrm{At}(L)} = \mathrm{Fl}\mathcal{T}_L$ is a lattice (with height $|\mathrm{At}(L)|$) with respect to inclusion. Now the claim follows from Theorem 5.2.

(ii) $\Rightarrow$ (i). Let $\mathcal{H} = (\mathrm{At}(L), 2^{\mathrm{At}(L)})$. Then $\mathrm{Fl}\mathcal{H} = 2^{\mathrm{At}(L)}$ and so $\mathcal{H} \in \mathcal{BR}$. Since $L \cong \mathrm{Fl}\mathcal{H}$, we are done. $\square$

We can now produce the equivalent of Corollary 4.4 for atomistic lattices.

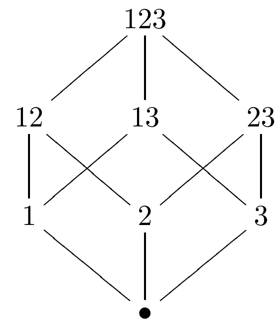
Corollary 5.5 *The lattice L depicted by*



is the smallest atomistic lattice not isomorphic to the lattice of flats of some $\mathcal{H} \in \mathcal{BR}$.

Proof. Since the above lattice, which is clearly atomistic, has 3 atoms, height 3 and 6 elements, it is not isomorphic to the lattice of flats of some $\mathcal{H} \in \mathcal{BR}$ by Proposition 5.4.

Indeed, it is easy to check that $\mathcal{T}_L = (123, P_{\leq 3}(123))$ and so $\mathrm{Fl}\mathcal{T}_L$ is the lattice



clearly not isomorphic to L.

It is easy to check that any other atomistic lattice with less than 7 elements must have height ≤ 2 and is therefore geometric. It follows that it must be isomorphic to the lattice of flats of a matroid, and matroids are boolean representable. $\square$

We end this section by showing that the complexity of the algorithm outlined in Corollary 5.3 is polynomial for fixed height.

We recall the O notation from complexity theory. Let P be an algorithm defined for instances depending on parameters $n_1, \ldots, n_k$. If $\varphi : \mathbb{N}^k \to \mathbb{N}$ is a function, we write $P \in O((n_1, \ldots, n_k)\varphi)$ if there exist constants $K, L > 0$ such that P processes each instance of type $(n_1, \ldots, n_k)$ in time $\leq K((n_1, \ldots, n_k)\varphi) + L$ (where time is measured as the number of elementary operations performed).

Proposition 5.6 *It is decidable in time $O(n^{3h})$ whether an atomistic lattice of height h with n atoms is isomorphic to* $\mathrm{Fl}\mathcal{H}$ *for some $\mathcal{H} \in \mathcal{BR}$.*

Proof. Let $M = (m_{\ell a})$ be the $L \times \mathrm{At}(L)$ defined by

$$m_{\ell a} = \begin{cases} 0 & \text{if } x \geq a \\ 1 & \text{otherwise} \end{cases}$$

It follows from the results in [12, Section 3.5] that M is a boolean matrix representation of $\mathcal{T}_L$.

We may assume $h \geq 3$. By [9, Theorem 7.4], it is possible to compute in time $O(n^{3d+3})$ the list of flats of a simplicial complex of dimension d defined by a reduced boolean matrix with n columns. Since L has height h, the dimension of $\mathcal{T}_L$ (corresponding to the maximum size of an element of T_L minus 1) is $h-1$. Moreover, M is reduced (all rows are distinct) since L is atomistic. Thus we can enumerate in time $O(n^{3h})$ the list of flats of $\mathcal{T}_L$ and compute $|\mathrm{Fl}\mathcal{T}_L|$.

Now, in view of Lemma 5.1(ii) and Theorem 5.2, we have $L \cong \mathrm{Fl}\mathcal{H}$ for some $\mathcal{H} \in \mathcal{BR}$ if and only if $|\mathrm{Fl}\mathcal{T}_L| = |L|$. Thus we obtain the claimed complexity. $\square$

6 Height 3

If L is an atomistic lattice of height 2, then $L \cong \mathrm{Fl}\mathcal{T}_L$ since $T_L = P_{\leq 2}(\mathrm{At}(L))$ and so $\mathrm{Fl}\mathcal{T}_L = P_{\leq 1}(\mathrm{At}(L)) \cup \{\mathrm{At}(L)\}$.

We provide in this section a necessary and sufficient graph-theoretical condition for the case of (atomistic) lattices of height 3 to be lattices of flats of a BRSC. Note that lattices of height 3 are important since every simplicial complex of dimension 2 has a lattice of flats of height ≤ 3, and dimension 2 is a broad universe. For instance, any finitely presented group can occur as the fundamental group of a simplicial complex of dimension 2 (see [13, Theorem 7.45]).

Let $\Gamma = (V, E)$ be a finite (undirected) graph. A *clique* of Γ is a subset W of V inducing a complete subgraph of Γ. The clique W is nontrivial if $|W| \geq 2$. A nontrivial clique W is a *superclique* if, for all $a, b \in W$ distinct, every $c \in V \setminus W$ is not adjacent to either a or b. In particular, every superclique is a maximal clique.

Given an atomistic lattice L of height 3 (with top element T), we define a graph $\Gamma_L = (\mathrm{At}(L), E_L)$ by

$$E_L = \{\{a, b\} \mid a, b \in \mathrm{At}(L),\ (a \vee b) = T\}.$$

We remark that, if L is the lattice of flats of a BRSC $\mathcal{H}$, then Γ_L ia actually the complement graph of $\Gamma\mathrm{Fl}\mathcal{H}$, the *graph of flats* of $\mathcal{H}$, introduced in [12, Section 6.4].

Theorem 6.1 *Let L be a lattice of height 3. Then the following conditions are equivalent.*

(i) $L \cong \mathrm{Fl}\mathcal{H}$ for some $\mathcal{H} \in \mathcal{BR}$;

(ii) L is atomistic and Γ_L has no supercliques.

Proof. (i) $\Rightarrow$ (ii). By Theorem 4.3, L is atomistic. By Theorem 5.2, we may assume that $\mathcal{H} = \mathcal{T}_L = (\mathrm{At}(L), T_L)$.

Suppose that $W \subseteq \mathrm{At}(L)$ is a superclique of Γ_L. Since L has height 3, we have $W \subset \mathrm{At}(L)$. We claim that $W \in \mathrm{Fl}\mathcal{T}_L$. Indeed, let $I \in T_L \cap 2^W$ be nonempty and $p \in \mathrm{At}(L) \setminus W$. Then I admits an enumeration $a_1, \ldots, a_m$ such that

$$a_1 < (a_1 \vee a_2) < \ldots < (a_1 \vee \ldots \vee a_m).$$

Since $a_1 — a_2$ is an edge of Γ_L, we get $(a_1 \vee a_2) = T$ and so $|I| \leq 2$. By Lemma 5.1(iv), we may assume that $|I| = 2$. Since W is a superclique of Γ_L, there exists some $a \in W$ such that $\{a, p\} \notin E_L$. Hence $(a \vee p) < T$. Writing $I = \{a, b\}$, we have $(a \vee b) = T$, hence $a < (a \vee p) < T = (a \vee b \vee p)$ and so $I \cup \{p\} \in T_L$. Thus $W \in \mathrm{Fl}\mathcal{T}_L$.

It follows from Theorem 5.2 that $\mathrm{At}(L) = T\xi = (\vee W)\xi \subseteq W$, contradicting $W \subset \mathrm{At}(L)$. Therefore Γ_L has no supercliques.

(ii) $\Rightarrow$ (i). Suppose that (i) fails. By Theorem 5.2, there exists some $F \in \mathrm{Fl}\mathcal{T}_L$ such that $(\vee F)\xi \not\subseteq F$. It follows that $|F| \geq 2$.

Suppose that $(a \vee b) < T$ for some distinct $a, b \in F$. Let $p \in (a \vee b)\xi \setminus F$. By Lemma 5.1(iv), we get $abp \in T_L$. Since $(a \vee b \vee p) = (a \vee b) < T$, this contradicts $\mathrm{ht}(L) = 3$. Thus $(a \vee b) = T$ for all distinct $a, b \in F$ and so F is a clique of Γ_L.

Let $a, b \in F$ be distinct and let $p \in \mathrm{At}(L) \setminus F$. By Lemma 5.1(iv), we get $abp \in T_L$. Now $(a \vee b) = T$ implies that either $(a \vee p) < T$ or $(b \vee p) < T$, hence $\{\{a,p\},\{b,p\}\} \not\subseteq E_L$ and so F is a superclique of Γ_L, a contradiction. Therefore (ii) holds as required. $\square$

We have remarked before that the lattices of flats of matroids are precisely the geometric lattices. Since every matroid is boolean representable, it follows that Γ_L has no supercliques when L is a geometric lattice of height 3. Indeed, in this case the graph Γ_L has no edges at all. Suppose that $a, b \in at(L)$ satisfy $(a \vee b) = T$. Since a geometric lattice must satisfy the Jordan-Dedekind condition (all maximal chains have the same length), there exists some $c \in L$ such that $a < c < T$. But then

$$\begin{tikzcd}
& T \arrow[dl] \arrow[ddr] & \\
c \arrow[d] & & \\
a \arrow[dr] & & b \arrow[dl] \\
& B &
\end{tikzcd}$$

is a subsemilattice of L, contradicting semimodularity.

Note also that, for the lattice L of Corollary 5.5, Γ_L is the graph

$$1 — 3 — 2$$

and has therefore supercliques (13 and 23). Therefore Theorem 6.1 provides an alternative way of showing that L is not isomorphic to the lattice of flats of some $\mathcal{H} \in \mathcal{BR}$.

We can show that the algorithm implicit in Theorem 6.1 has polynomial complexity.

Proposition 6.2 *It is decidable in time $O(n^5)$ whether a graph with n vertices has no supercliques.*

Proof. Let $\Gamma = (V, E)$ be a graph with n vertices. Given an edge $a — b$ in Γ, we define a sequence of sets of vertices
$$ab = W_0 \subset W_1 \subset \ldots \subset W_k \subseteq V \tag{8}$$
as follows.

Assume that W_i is defined. If there exists some $v \in V \setminus W_i$ adjacent to at least two vertices in W_i, let $W_{i+1} = W_i \cup \{v\}$. Otherwise, the sequence terminates at W_i.

Note that this procedure is nondeterministic, but it turns out to be confluent. Indeed, let $ab = W'_0 \subset \ldots \subset W'_\ell$ be an alternative sequence. Write $W'_j \setminus W'_{j-1} = \{w'_j\}$. Let $j \in \{1, \ldots, \ell\}$ and assume that $\{w'_1, \ldots, w'_{j-1}\} \subseteq W_k$. Then w'_j is adjacent to at least two vertices of W_k. Since (8) terminates at W_k, it follows that $w'_j \in W_k$. By induction, we get $W'_\ell \subseteq W_k$ and so $W'_\ell = W_k$ by symmetry. Therefore the procedure is confluent and we may define $C(ab) = W_k$.

We show that
$$\text{every superclique of } \Gamma \text{ is of the form } C(ab) \text{ for some } ab \in E. \tag{9}$$

Suppose that $W \subseteq V$ is a superclique of Γ. Since $|W| \geq 2$, there exists an edge $ab \in E \cap P_2(W)$. We show that $W = C(ab)$.

Consider the sequence (8) with $W_k = C(ab)$. Straightforward induction shows that $W_i \subseteq W$ for $i = 0, \ldots, k$. Thus $C(ab) \subseteq W$ and so $C(ab)$ is a clique of Γ, actually a superclique since every $v \in V \setminus C(ab)$ is adjacent to at most one vertex of $C(ab)$. Since every superclique is a maximal clique, it follows that $W = C(ab)$ and so (9) holds.

It is easy to see that

$$C(ab) \text{ is a superclique of } \Gamma \text{ if and only if it is a clique.} \tag{10}$$

Now Γ has $O(n^2)$ edges. For each one of these edges, say ab, we can compute $C(ab)$ in time $O(n^3)$. Indeed, $k \leq n-2$, and we can go from W_{i-1} to W_i in time $O(n^2)$. This can be argued by considering the adjacency matrix of Γ (an $n \times n$ boolean matrix): assuming that the rows and columns corresponding to W_{i-1} are marked, it suffices to go through the entries of the matrix once to search for the vertex of $W_i \setminus W_{i-1}$. Using the adjacency matrix, we can also check if $C(ab)$ is a clique in time $O(n^2)$.

In view of (9) and (10), we can compute in time $O(n^5)$ any possible supercliques of Γ. $\square$

This complexity bound can most probably be improved.

It is clear that the construction of Γ_L from L can be performed in time at most $O(n^5)$, so we get the polynomial complexity claim on the lattice (in $O(n^5)$). Note that the general algorithm from Proposition 5.6 yields only a complexity in $O(n^9)$.

The following question follows naturally from the results in this paper: is there a more efficient algorithm to decide if a lattice is the lattice of flats of a BRSC?

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