N-valid trees in wavelet theory on Vilenkin groups [1]

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Abstract

We consider a class of (N, M)-elementary step functions on the p-adic Vilenkin group. We prove that (N, M)-elementary step function generates a MRA on p-adic Vilenkin group iff it is generated by a special N-valid rooted tree on the set of vertices $\{0, 1, \dots p-1\}$ with the vector $(0, ..., 0) \in \mathbb{Z}^N$ as a root. Bibliography: 15 titles.

keywords: zero-dimensional group, Vilenkin group, multiresolution analysis, wavelet bases, tree.

1 Introduction

In articles [1]-[4] first examples of orthogonal wavelets on the dyadic Cantor group ($p = 2$) are constructed and their properties are studied. Yu.Farkov [5]-[7] found necessary and sufficient conditions for a refinable function to generate an orthogonal MRA in the $L_2(\mathfrak{G})$ -spaces on the p-adic Vilenkin group $\mathfrak{G}$. These conditions use the Strang-Fix and the modified Cohen properties.

In [7] this construction is given in a concrete fashion for p = 3. In [8], some algorithms for constructing orthogonal and biorthogonal compactly supported wavelets on Vilenkin groups are proposed. In [5]-[8] two types of orthogonal wavelet examples are constructed: step functions and sums of Vilenkin series.

Khrennikov, Shelkovich, and Skopina [10],[11] introduced the concept of a p-adic MRA with orthogonal refinable function, and described a general pattern for their construction. This method was developed for an orthogonal refinable function φ with condition $\mathrm{supp}\widehat{\varphi} \subset B_0(0)$, where $B_0(0) = \{x : |x|_p \le 1\}$ is the unit ball in the field $\mathbb{Q}_p$. Similar results were obtained for an arbitrary zero-dimensional group [13]. The condition $\mathrm{supp}\widehat{\varphi} \subset B_0(0)$ is very important. S. Albeverio, S. Evdokimov, M. Skopina

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[12] proved that if a refinable step function φ generates an orthogonal p-adic MRA, then $\mathrm{supp}\hat{\varphi}(\chi) \subset B_0(0)$.

On the other hand on Vilenkin groups Yu.A.Farkov constructs examples of step refinable functions φ, which generate an orthogonal MRA with $\mathrm{supp}\widehat{\varphi} \subset G_1^{\perp}$. In the author's work [14] a necessary condition for a support of orthogonal refinable step function are found: if step refinable $(1, M)$-elementary function φ generates an orthogonal MRA on p-adic Vilenkin group, then $\mathrm{supp}\widehat{\varphi} \subset G_{p-2}^{\perp}$. In [15] some trees was used to construct refinable function .

In this work we consider more general situation and study a structure of the set $\mathrm{supp}\widehat{\varphi}$. We define a concept of N-valid tree and prove that (N, M)-elementary function φ generates an orthogonal MRA on p-adic Vilenkin group iff the function φ is generated by means of some N-valid tree. For any N-valid tree we give an algorithm for constructing corresponding refinable function and orthogonal wavelets.

The paper is organized as follows. We consider p-adic Vilenkin group $\mathfrak{G}$ as a zero-dimensional group $(G, \dot{+})$ with condition pg_n=0. Therefore, in section 2, we recall some concepts and facts from the theory of zero-dimensional group. We will systematically use the notation and the results from [13],[14].

In section 3 and the following sections we consider MRA on p-adic Vilenkin group $\mathfrak{G}$. In section 3 we study refinable step-functions which generate the orthogonal MRA. We define a class of (N, M)-elementary set and prove that the shifts system $\varphi(x \dot{-} h)_{h \in H_0}$ is orthonormal if $\mathrm{supp}\widehat{\varphi}$ is (N, M)-elementary set.

In section 4 we introduce such concepts as "a set generated by a tree" and "a refinable step function generated by a tree" and prove, that any rooted N-valid tree generates a refinable step function that generates an orthogonal MRA on Vilenkin group.

In section 5 we give an algorithm for constructing orthogonal wavelets according to the tree.

2 Preliminaries

We will consider the Vilenkin group as a locally compact zero-dimensional Abelian group with additional condition $p_n g_n = 0$. Therefore we start with some basic notions and facts related to analysis on zero-dimensional groups. One may find more information on the topic in [12]–[14].

Let $(G, \dot{+})$ be a locally compact zero-dimensional Abelian group with the topology generated by a countable system of open subgroups

$$\cdots \supset G_{-n} \supset \cdots \supset G_{-1} \supset G_0 \supset G_1 \supset \cdots \supset G_n \supset \cdots$$

where

$$\bigcup_{n=-\infty}^{+\infty} G_n = G, \qquad \bigcap_{n=-\infty}^{+\infty} G_n = \{0\},$$

p_n is an order of quotient group G_n/G_{n+1}. We will always assume that all p_n are prime numbers. We will name such chain as *basic chain.* In this case, a base of the topology is formed by all possible cosets $G_n \dot{+} g$, $g \in G$.

We further define the numbers $(\mathfrak{m}_n)_{n=-\infty}^{+\infty}$ as follows:

$$\mathfrak{m}_0 = 1, \qquad \mathfrak{m}_{n+1} = \mathfrak{m}_n \cdot p_n.$$

Let μ be a Haar measure on G, we know that $\mu G_n = \frac{1}{\mathfrak{m}_n}$. Further, let $\int_G f(x)\, d\mu(x)$ be the absolutely convergent integral of the measure μ.

Given $n \in \mathbb{Z}$, consider an element $g_n \in G_n \setminus G_{n+1}$ and fix it. Then any $x \in G$ has a unique representation in the form

$$x = \sum_{n=-\infty}^{+\infty} a_n g_n, \qquad a_n = \overline{0, p_n - 1}. \tag{2.1}$$

The sum (2.1) contain finite number of terms with negative subscripts, that is,

$$x = \sum_{n=m}^{+\infty} a_n g_n, \qquad a_n = \overline{0, p_n - 1}, \quad a_m \neq 0. \tag{2.2}$$

We will name system $(g_n)_{n\in\mathbb{Z}}$ as *a basic system.*

Classical examples of zero-dimensional groups are Vilenkin groups and groups of p-adic numbers (see [12, Ch. 1, § 2]). A direct sum of cyclic groups $Z(p_k)$ of order p_k, $k \in \mathbb{Z}$, is called a *Vilenkin group.* This means that the elements of a Vilenkin group are infinite sequences $x = (x_k)_{k=-\infty}^{+\infty}$ such that:

1) $x_k = \overline{0, p_k - 1}$;

2) only a finite number of x_k with negative subscripts are different from zero;

3) the group operation $\dot{+}$ is the coordinate-wise addition modulo p_k, that is,

$$x\dot{+}y = (x_k\dot{+}y_k), \qquad x_k\dot{+}y_k = (x_k + y_k) \mod p_k.$$

A topology on such group is generated by the chain of subgroups

$$G_n = \big\{x \in G : x = (\dots, 0, 0, \dots, 0, x_n, x_{n+1}, \dots),\ x_\nu = \overline{0, p_\nu - 1},\ \nu \geq n\big\}.$$

The elements $g_n = (\dots, 0, 0, 1, 0, 0, \dots)$ form a basic system. From definition of the operation $\dot{+}$ we have $p_n g_n = 0$. Therefore we will name a zero-dimensional group $(G, \dot{+})$ with the condition $p_n g_n = 0$ as Vilenkin group.

By X we denote the collection of the characters of a group $(G, \dot{+})$; it is a group with respect to multiplication, too. Also let $G_n^\perp = \{\chi \in X : \forall\, x \in G_n\ , \chi(x) = 1\}$ be the annihilator of the group G_n. Each annihilator $G_n^\perp$ is a group with respect to multiplication, and the subgroups $G_n^\perp$ form an increasing sequence

$$\dots \subset G_{-n}^\perp \subset \dots \subset G_0^\perp \subset G_1^\perp \subset \dots \subset G_n^\perp \subset \dots \tag{2.3}$$

with

$$\bigcup_{n=-\infty}^{+\infty} G_n^\perp = X \quad \text{and} \quad \bigcap_{n=-\infty}^{+\infty} G_n^\perp = \{1\},$$

the quotient group $G_{n+1}^\perp / G_n^\perp$ having order p_n. The group of characters X is a zero-dimensional group with a basic chain (2.3). The group may be supplied with the topology using the chain of subgroups (2.3), the family of the cosets $G_n^\perp \cdot \chi$, $\chi \in X$, being taken as a base of the topology. The collection of such cosets, along with the empty set, forms the semiring $\mathscr{X}$. Given a coset $G_n^\perp \cdot \chi$, we define a measure ν on it by $\nu(G_n^\perp \cdot \chi) = \nu(G_n^\perp) = \mathfrak{m}_n$ (so that always $\mu(G_n)\nu(G_n^\perp) = 1$). The measure ν can be extended onto the σ-algebra of measurable sets in the standard way. One then forms the absolutely convergent integral $\displaystyle\int_X F(\chi)\, d\nu(\chi)$ using this measure.

The value $\chi(g)$ of the character χ at an element $g \in G$ will be denoted by (χ, g). The Fourier transform $\widehat{f}$ of an $f \in L_2(G)$ is defined as follows

$$\widehat{f}(\chi) = \int_G f(x)\overline{(\chi, x)}\, d\mu(x) = \lim_{n \to +\infty} \int_{G_{-n}} f(x)\overline{(\chi, x)}\, d\mu(x),$$

with the limit being in the norm of $L_2(X)$. For any $f \in L_2(G)$, the inversion formula is valid

$$f(x) = \int_X \widehat{f}(\chi)(\chi, x)\, d\nu(\chi) = \lim_{n\to+\infty} \int_{G_n^\perp} \widehat{f}(\chi)(\chi, x)\, d\nu(\chi);$$

here the limit also signifies the convergence in the norm of $L_2(G)$. If $f, g \in L_2(G)$ then the Plancherel formula is valid

$$\int_G f(x)\overline{g(x)}\, d\mu(x) = \int_X \widehat{f}(\chi)\overline{\widehat{g}(\chi)}\, d\nu(\chi).$$

Provided with this topology, the group of characters X is a zero-dimensional locally compact group; there is, however, a dual situation: every element $x \in G$ is a character of the group X, and G_n is the annihilator of the group $G_n^\perp$. The union of disjoint sets E_j we will denote by $\bigsqcup E_j$.

For any $n \in \mathbb{Z}$ we choose a character $r_n \in G_{n+1}^\perp \backslash G_n^\perp$ and fixed it. $(r_n)_{n\in\mathbb{Z}}$ is called a Rademacher system. Let us denote

$$H_0 = \{h \in G : h = a_{-1}g_{-1} \dot{+} a_{-2}g_{-2} \dot{+} \dots \dot{+} a_{-s}g_{-s}, s \in \mathbb{N},\ a_j = \overline{0, p-1}\},$$

$$H_0^{(s)} = \{h \in G : h = a_{-1}g_{-1} \dot{+} a_{-2}g_{-2} \dot{+} \dots \dot{+} a_{-s}g_{-s},\ a_j = \overline{0, p-1}\}, s \in \mathbb{N}.$$

The set H_0 is an analog of the set $\mathbb{N}_0 = \mathbb{N}\bigsqcup\{0\}$.

If in the zero-dimensional group G $p_n = p$ for any $n \in \mathbb{Z}$ then we can define the mapping $\mathcal{A}\colon G \to G$ by $\mathcal{A}x := \sum_{n=-\infty}^{+\infty} a_n g_{n-1}$, where $x = \sum_{n=-\infty}^{+\infty} a_n g_n \in G$. The mapping $\mathcal{A}$ is called a dilation operator if $\mathcal{A}(x \dot{+} y) = \mathcal{A}x \dot{+} \mathcal{A}y$ for all $x, y \in G$. By definition, put $(\chi\mathcal{A}, x) = (\chi, \mathcal{A}x)$.

Lemma 2.1 ([14]) *For any zero-dimensional group*
1) $\int\limits_{G_0^\perp} (\chi, x)\, d\nu(\chi) = \mathbf{1}_{G_0}(x)$, *2)* $\int\limits_{G_0} (\chi, x)\, d\mu(x) = \mathbf{1}_{G_0^\perp}(\chi)$.

Lemma 2.2 ([14]) *If $p_n = p$ for any $n \in \mathbb{Z}$ and the mapping $\mathcal{A}$ is additive then*
1) $\int\limits_{G_n^\perp} (\chi, x)\, d\nu(\chi) = p^n \mathbf{1}_{G_n}(x)$,
2) $\int\limits_{G_n} (\chi, x)\, d\mu(x) = \frac{1}{p^n} \mathbf{1}_{G_n^\perp}(\chi)$.

Lemma 2.3 ([14]) *Let $\chi_{n,s} = r_n^{\alpha_n} r_{n+1}^{\alpha_{n+1}} \dots r_{n+s}^{\alpha_{n+s}}$ be a character which does not belong to $G_n^\perp$. Then*

$$\int\limits_{G_n^\perp \chi_{n,s}} (\chi, x)\, d\nu(\chi) = p^n (\chi_{n,s}, x) \mathbf{1}_{G_n}(x).$$

Lemma 2.4 ([14]) *Let* $h_{n,s} = a_{n-1}g_{n-1}\dot{+}a_{n-2}g_{n-2}\dot{+}\ldots\dot{+}a_{n-s}g_{n-s} \notin G_n$. *Then*

$$\int\limits_{G_n\dot{+}h_{n,s}} (\chi, x)\, d\mu(x) = \frac{1}{p^n}(\chi, h_{n,s})\mathbf{1}_{G_n^\perp}(\chi).$$

Definition 2.1 ([14]) *Let* $M, N \in \mathbb{N}$. *We denote by* $\mathfrak{D}_M(G_{-N})$ *the set of step-functions* $f \in L_2(G)$ *such that 1)* $\operatorname{supp} f \subset G_{-N}$, *and 2)* f *is constant on cosets* $G_M\dot{+}g$. $\mathfrak{D}_{-N}(G_M^\perp)$ *is defined similarly.*

Lemma 2.5 ([14]) *Let* $M, N \in \mathbb{N}$. $f \in \mathfrak{D}_M(G_{-N})$ *if and only if* $\hat{f} \in \mathfrak{D}_{-N}(G_M^\perp)$.

3 MRA and refinable function on Vilenkin groups

In what follows we will consider groups G for which $p_n = p$ and $pg_n = 0$ for any $n \in \mathbb{Z}$. We know that it is a Vilenkin group. We will denote a Vilenkin group as $\mathfrak{G}$.

In this group we can choose Rademacher functions in various ways. We define Rademacher functions with the equation

$$\left(r_n, \sum_{k\in\mathbb{Z}} a_k g_k\right) = \exp\left(\frac{2\pi i}{p} a_n\right).$$

In this case

$$(r_n, g_k) = \exp\left(\frac{2\pi i}{p}\delta_{nk}\right).$$

Our main objective is to find a simple algorithm to get a refinable step-function that generates an orthogonal MRA on Vilenkin group.

Definition 3.1 *A family of closed subspaces* V_n, $n \in \mathbb{Z}$, *is said to be a multiresolution analysis of* $L_2(\mathfrak{G})$ *if the following axioms are satisfied:*

A1) $V_n \subset V_{n+1}$;

A2) $\overline{\bigcup_{n\in\mathbb{Z}} V_n} = L_2(\mathfrak{G})$ *and* $\bigcap_{n\in\mathbb{Z}} V_n = \{0\}$;

A3) $f(x) \in V_n \iff f(\mathcal{A}x) \in V_{n+1}$ *($\mathcal{A}$ is a dilation operator);*

A4) $f(x) \in V_0 \implies f(x\dot{-}h) \in V_0$ *for all* $h \in H_0$; *(H_0 is analog of $\mathbb{Z}$).*

A5) there exists a function $\varphi \in L_2(\mathfrak{G})$ *such that the system* $(\varphi(x\dot{-}h))_{h\in H_0}$ *is an orthonormal basis for* V_0.

A function φ occurring in axiom A5 is called a scaling function.

Next we will follow the conventional approach. Let $\varphi(x) \in L_2(\mathfrak{G})$, and assume that $(\varphi(x \dot{-} h))_{h\in H_0}$ is an orthonormal system in $L_2(\mathfrak{G})$. With the function φ and the dilation operator $\mathcal{A}$, we define the linear subspaces $L_n = (\varphi(\mathcal{A}^n x \dot{-} h))_{h\in H_0}$ and closed subspaces $V_n = \overline{L_n}$. It is evident that the functions $p^{\frac{n}{2}}\varphi(\mathcal{A}^{\backslash} x \dot{-} h)_{h\in H_0}$ form an orthonormal basis for V_n, $n \in \mathbb{Z}$. If subspaces V_n form a MRA, then the function φ is said to *generate* an MRA in $L_2(\mathfrak{G})$. If a function φ generates an MRA, then we obtain from the axiom A1

$$\varphi(x) = \sum_{h\in H_0} \beta_h \varphi(\mathcal{A}x \dot{-} h) \quad \left(\sum |\beta_h|^2 < +\infty\right). \tag{3.1}$$

Therefore we will look up a function $\varphi \in L_2(\mathfrak{G})$, which generates an MRA in $L_2(\mathfrak{G})$, as a solution of the refinement equation (3.1), A solution of refinement equation (3.1) is called a *refinable function.*

Lemma 3.1 ([14]) *Let $\varphi \in \mathfrak{D}_M(\mathfrak{G}_{-N})$ be a solution of (3.1). Then*

$$\varphi(x) = \sum_{h\in H_0^{(N+1)}} \beta_h \varphi(\mathcal{A}x \dot{-} h) \tag{3.2}$$

The refinement equation (3.2) may be written in the form

$$\hat{\varphi}(\chi) = m_0(\chi)\hat{\varphi}(\chi\mathcal{A}^{-1}), \tag{3.3}$$

where

$$m_0(\chi) = \frac{1}{p} \sum_{h\in H_0^{(N+1)}} \beta_h \overline{(\chi\mathcal{A}^{-1}, h)} \tag{3.4}$$

is a mask of the equation (3.3).

Lemma 3.2 ([14]) *Let $\varphi \in \mathfrak{D}_M(\mathfrak{G}_{-N})$. Then the mask $m_0(\chi)$ is constant on cosets $\mathfrak{G}_{-N}^{\perp}\zeta$. If $\hat{\varphi}(\mathfrak{G}_{-N}^{\perp}) \neq 0$ then $m_0(\mathfrak{G}_{-N}^{\perp}) = 1$.*

Lemma 3.3 ([14]) *The mask $m_0(\chi)$ is a periodic function with any period $r_1^{\alpha_1} r_2^{\alpha_2} \dots r_s^{\alpha_s}$ ($s \in \mathbb{N}$, $\alpha_j = \overline{0, p-1}$, $j = \overline{1, s}$).*

So, if $m_0(\chi)$ is a mask of (3.3) then
T1) $m_0(\chi)$ is constant on cosets $\mathfrak{G}_{-N}^{\perp}\zeta$,
T2) $m_0(\chi)$ is periodic with any period $r_1^{\alpha_1} r_2^{\alpha_2} \dots r_s^{\alpha_s}$, $\alpha_j = \overline{0, p-1}$,
T3) $m_0(\mathfrak{G}_{-N}^{\perp}) = 1$.
Therefore we will assume that m_0 satisfies these conditions.

Theorem 3.1 **([14])** *$m_0(\chi)$ is a mask of equation (3.3) on the class $\mathfrak{D}_{-N}(\mathfrak{G}_M^\perp)$ if and only if*

$$m_0(\chi)m_0(\chi\mathcal{A}^{-1})\dots m_0(\chi\mathcal{A}^{-M-N})=0 \qquad (3.5)$$

on $\mathfrak{G}_{M+1}^\perp\setminus\mathfrak{G}_M^\perp$. If, in addition, the system $\varphi(x\dot{-}h)_{h\in H_0}$ is orthonormal, then $\varphi(x)$ generate an orthogonal MRA.

So, to find a refinable function that generates orthogonal MRA, we need to take a function $m_0(\chi)$ that satisfies conditions T1, T2, T3, (3.5), construct the function

$$\hat{\varphi}(\chi)=\prod_{k=0}^{\infty}m_0(\chi\mathcal{A}^{-k})\in\mathfrak{D}_{-N}(\mathfrak{G}_M^\perp)$$

and check that the system $\varphi(x\dot{-}h)_{h\in H_0}$ is orthonormal.

For any zero-dimensional group G the shifts system $(\varphi(x\dot{-}h))_{h\in H_0}$ is orthonormal if the condition $|\hat{\varphi}(\chi)|=\mathbf{1}_{G_0^\perp}(\chi)$ is valid [14]. For Vilenkin group $\mathfrak{G}$ we can give another condition.

Definition 3.2 *Let $N,M\in\mathbb{N}$. A set $E\subset X$ is called (N,M)-elementary if E is disjoint union of p^N cosets*

$$\mathfrak{G}_{-N}^\perp\zeta_j=\mathfrak{G}_{-N}^\perp\underbrace{r_{-N}^{\alpha_{-N}}r_{-N+1}^{\alpha_{-N+1}}\dots r_{-1}^{\alpha_{-1}}}_{\xi_j}\underbrace{r_0^{\alpha_0}\dots r_{M-1}^{\alpha_{M-1}}}_{\eta_j}=\mathfrak{G}_{-N}^\perp\xi_j\eta_j,$$

$j=0,1,...,p^N-1, j=\alpha_{-N}+\alpha_{-N+1}p+\dots+\alpha_{-1}p^{N-1}$ ($\alpha_\nu=\overline{0,p-1}$) such that

1) $\bigsqcup\limits_{j=0}^{p^N-1}\mathfrak{G}_{-N}^\perp\xi_j=\mathfrak{G}_0^\perp$, $\mathfrak{G}_{-N}^\perp\zeta_0=\mathfrak{G}_{-N}^\perp$,

2) for any $l=\overline{0,M+N-1}$ the intersection $(\mathfrak{G}_{-N+l+1}^\perp\setminus\mathfrak{G}_{-N+l}^\perp)\bigcap E\neq\emptyset$.

Lemma 3.4 *The set $H_0\subset\mathfrak{G}$ is an orthonormal system on any (N,M)-elementary set $E\subset X$.*

Proof. Using the definition of (N,M)-elementary set we have

$$\int\limits_E(\chi,h)\overline{(\chi,g)}\,d\nu(x)=\sum_{j=0}^{p^N-1}\int\limits_{\mathfrak{G}_{-N}^\perp\zeta_j}(\chi,h)\overline{(\chi,g)}\,d\nu(x)=$$

$$=\sum_{j=0}^{p^N-1}\int\limits_X\mathbf{1}_{\mathfrak{G}_{-N}^\perp\zeta_j}(\chi)(\chi,h)\overline{(\chi,g)}\,d\nu(x)=$$

$$= \sum_{j=0}^{p^N-1} \int\limits_X \mathbf{1}_{\mathfrak{G}_{-N}^{\perp}\zeta_j}(\chi\eta_j)(\chi\eta_j, h)\overline{(\chi\eta_j, g)}\, d\nu(x) =$$

$$= \sum_{j=0}^{p^N-1} \int\limits_X \mathbf{1}_{\mathfrak{G}_{-N}^{\perp}\xi_j}(\chi)(\chi, h)\overline{(\chi, g)}(\eta_j, h)\overline{(\eta_j, g)}\, d\nu(x).$$

Since

$$(\eta_j, h) = (r_0^{\alpha_0} r_1^{\alpha_1} \dots r_{M-1}^{\alpha_{M-1}}, a_{-1}g_{-1}\dot{+}a_{-2}g_{-2}\dot{+}\dots\dot{+}a_{-s}g_{-s}) = 1,$$

$$(\eta_j, g) = (r_0^{\alpha_0} r_1^{\alpha_1} \dots r_{M-1}^{\alpha_{M-1}}, b_{-1}g_{-1}\dot{+}b_{-2}g_{-2}\dot{+}\dots\dot{+}b_{-s}g_{-s}) = 1,$$

then

$$\int\limits_E (\chi, h)\overline{(\chi, g)}\, d\nu(x) = \sum_{j=0}^{p^N-1} \int\limits_{\mathfrak{G}_{-N}^{\perp}\xi_j} (\chi, h)\overline{(\chi, g)}\, d\nu(x) = \int\limits_{\mathfrak{G}_0^{\perp}} (\chi, h)\overline{(\chi, g)}\, d\nu(x) =$$

$= \delta_{h,g}$. □

Theorem 3.2 *Let $(\mathfrak{G}, \dot{+})$ be a p-adic Vilenkin group, $E \subset \mathfrak{G}_M^{\perp}$ – an (N, M)-elementary set. If $|\hat{\varphi}(\chi)| = \mathbf{1}_E(\chi)$ on X then the system of shifts $(\varphi(x\dot{-}h))_{h\in H_0}$ is an orthonormal system on $\mathfrak{G}$.*

Proof. Let $\tilde{H}_0 \subset H_0$ be a finite set. Using the Plansherel equation we have

$$\int\limits_{\mathfrak{G}} \varphi(x\dot{-}g)\overline{\varphi(x\dot{-}g)}\, d\mu(x) = \int\limits_X |\hat{\varphi}(\chi)|^2\overline{(\chi, g)}(\chi, h) d\nu(\chi) = \int\limits_E (\chi, h)\overline{(\chi, g)} d\nu(\chi) =$$

$$= \sum_{j=0}^{p^N-1} \int\limits_{\mathfrak{G}_{-N}^{\perp}\zeta_j} (\chi, h)\overline{(\chi, g)}\, d\nu(\chi).$$

Transform the inner integral

$$\int\limits_{\mathfrak{G}_{-N}^{\perp}\zeta_j} (\chi, h)\overline{(\chi, g)}\, d\nu(\chi) = \int\limits_X \mathbf{1}_{\mathfrak{G}_{-N}^{\perp}\zeta_j}(\chi)(\chi, h)\overline{(\chi, g)}\, d\nu(\chi) =$$

$$= \int\limits_X \mathbf{1}_{\mathfrak{G}_{-N}^{\perp}\zeta_j}(\chi\eta_j)(\chi\eta_j, h\dot{-}g)\, d\nu(\chi) = \int\limits_X \mathbf{1}_{\mathfrak{G}_{-N}^{\perp}\xi_j}(\chi)(\chi\eta_j, h\dot{-}g)\, d\nu(\chi) =$$

$$= \int\limits_{\mathfrak{G}_{-N}^{\perp}\xi_j} (\chi\eta_j, h\dot{-}g)\, d\nu(\chi).$$

Repeating the arguments of lemma 3.4 we obtain

$$\int\limits_{\mathfrak{G}} \varphi(x\dot{-}h)\overline{\varphi(x\dot{-}g)}\, d\mu(x) = \delta_{h,g}. \ \square$$

Theorem 3.3 ([14]) *Let $\varphi(x) \in \mathfrak{D}_M(\mathfrak{G}_{-N})$. A shifts system $(\varphi(x\dot{-}h))_{h\in H_0}$ will be orthonormal if and only if for any $\alpha_{-N}, \alpha_{-N+1}, \dots, \alpha_{-1} = \overline{(0, p-1)}$*

$$\sum_{\alpha_0,\alpha_1,\dots,\alpha_{M-1}=0}^{p-1} |\hat{\varphi}(\mathfrak{G}_{-N}^{\perp} r_{-N}^{\alpha_{-N}} \dots r_0^{\alpha_0} \dots r_{M-1}^{\alpha_{M-1}})|^2 = 1. \tag{3.6}$$

Lemma 3.5 ([14]) *Let $\hat{\varphi} \in \mathfrak{D}_{-N}(\mathfrak{G}_M^{\perp})$ be a solution of the refinement equation*

$$\hat{\varphi}(\chi) = m_0(\chi)\hat{\varphi}(\chi\mathcal{A}^{-1})$$

and $(\varphi(x\dot{-}h))_{h\in H_0}$ be an orthonormal system.
Then for any $\alpha_{-N}, \alpha_{-N+1}, \dots, \alpha_{-1} = \overline{0, p-1}$

$$\sum_{\alpha_0=0}^{p-1} |m_0(\mathfrak{G}_{-N}^{\perp} r_{-N}^{\alpha_{-N}} r_{-N+1}^{\alpha_{-N+1}} \dots r_{-1}^{\alpha_{-1}} r_0^{\alpha_0})|^2 = 1. \tag{3.7}$$

4 Trees and refinable functions

In this section we reduce the problem of construction of step refinable function to construction of some tree.

We will consider some special class of refinable functions $\varphi(\chi)$ for which $|\hat{\varphi}(\chi)|$ is a characteristic function of a set. Define this class.

Definition 4.1 *A mask $m_0(\chi)$ is called N-elementary ($N \in \mathbb{N}_0$) if $m_0(\chi)$ is constant on cosets $\mathfrak{G}_{-N}^{\perp}\chi$, its absolute value $|m_0(\chi)|$ has two values only: 0 and 1, and $m_0(\mathfrak{G}_{-N}^{\perp}) = 1$. The refinable function $\varphi(x)$ with Fourier transform*

$$\hat{\varphi}(\chi) = \prod_{n=0}^{\infty} m_0(\chi\mathcal{A}^{-n})$$

is called N-elementary too. N-elementary function φ is called (N, M)-elementary if $\hat{\varphi}(\chi) \in \mathfrak{D}_{-N}(\mathfrak{G}_M^{\perp})$. In this case we will call the Fourier transform $\hat{\varphi}(\chi)$ (N, M)-elementary, also.

Definition 4.2 *Let* $\tilde{E} = \bigsqcup_{\alpha_{-N},\dots,\alpha_{-1},\alpha_0} \mathfrak{G}_{-N}^{\perp} r_{-N}^{\alpha_{-N}} \dots r_{-1}^{\alpha_{-1}} r_0^{\alpha_0} \subset \mathfrak{G}_1^{\perp}$ *be an* $(N,1)$*-elementary set. We say that the set* $\tilde{E}_X$ *is a periodic extension of* $\tilde{E}$ *if*

$$\tilde{E}_X = \bigcup_{s=1}^{\infty} \bigsqcup_{\alpha_1,\dots,\alpha_s=0}^{p-1} \tilde{E} r_1^{\alpha_1} r_2^{\alpha_2} \dots r_s^{\alpha_s}.$$

We say that $\tilde{E}$ *generates an* (N,M) *elementary set* E*, if* $\bigcap_{n=0}^{\infty} \tilde{E}_X \mathcal{A}^n = E$*.*

Since $\tilde{E}_X \supset \mathfrak{G}_{-N}^{\perp}$ then $\bigcap_{n=0}^{M+1} \tilde{E}_X \mathcal{A}^n = E$ and $\left(\bigcap_{n=0}^{M+1} \tilde{E}_X \mathcal{A}^n\right) \bigcap (\mathfrak{G}_{M+1}^{\perp} \setminus \mathfrak{G}_M^{\perp}) = \emptyset$. The converse is also true. Since

$$\left(\bigcap_{n=0}^{M+1} \tilde{E}_X \mathcal{A}^n\right) \bigcap (\mathfrak{G}_{M+1}^{\perp} \setminus \mathfrak{G}_M^{\perp}) = \emptyset.$$

Then we have

$$\left(\bigcap_{n=0}^{M+2} \tilde{E}_X \mathcal{A}^n\right) \bigcap (\mathfrak{G}_{M+2}^{\perp} \setminus \mathfrak{G}_{M+1}^{\perp}) = \tilde{E}_X \bigcap \left(\bigcap_{n=0}^{M+1} \tilde{E}_X \mathcal{A}^n \bigcap (\mathfrak{G}_{M+1}^{\perp} \setminus \mathfrak{G}_M^{\perp})\right) \mathcal{A} =$$

$$= \tilde{E}_X \bigcap \emptyset = \emptyset.$$

Let N be a natural number. Denote $V = \{0, 1. \dots, p-1\}$ and construct a tree $T(V)$ in the following way:

1) The root of this tree and its vertices of level $1, 2, \dots, N-1$ are equal to zero.

2) Any path $(\alpha_k \to \alpha_{k+1} \to \dots \to \alpha_{k+N-1})$ of length N is present in the tree $T(V)$ exactly 1 time.

Such tree we will call N-valid.

For example for $p = 3, N = 2$ we can construct the tree

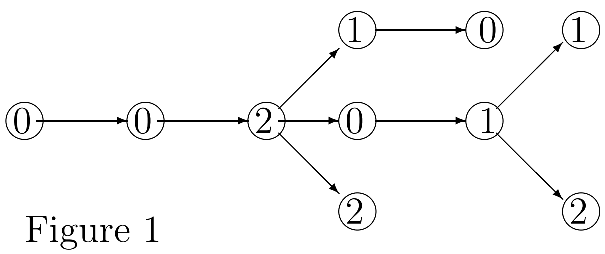

Figure 1

This tree contains any edge

$$(0,0), (0,1), (0,1), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2)$$

exactly 1 time and height $T(V) = 6$.

Using the tree $T(V)$ we will construct the family of cosets in the following way:
For any a path

$$(\alpha_s \to \alpha_{s-1} \to \cdots \to \alpha_{s-N+1} \to \alpha_{s-N} \to \alpha_{s-N-1} \to \cdots \to \alpha_{-N+1} \to \alpha_{-N})$$

in which $\alpha_s = \alpha_{s-1} = \cdots = \alpha_{s-N+1} = 0$.
we construct cosets

$$G^\perp_{-N} r_{-N}^{\alpha_{-N}} r_{-N+1}^{\alpha_{-N+1}} \ldots r_0^{\alpha_0}, G^\perp_{-N} r_{-N}^{\alpha_{-N+1}} r_{-N+1}^{\alpha_{-N+2}} \ldots r_0^{\alpha_1}, \ldots, G^\perp_{-N} r_{-N}^{\alpha_{s-N}} r_{-N+1}^{\alpha_{s-N+1}} \ldots r_0^{\alpha_s}, \tag{4.1}$$

$$G^\perp_{-N} r_{-N}^{\alpha_{s-N+1}} \ldots r_{-1}^{\alpha_s}, G^\perp_{-N} r_{-N}^{\alpha_{s-N+2}} \ldots r_{-2}^{\alpha_s}, \ldots, G^\perp_{-N} r_{-N}^{\alpha_s}. \tag{4.2}$$

The union of all such cosets we denote as $\tilde{E}$. It is clear that $\tilde{E} \subset G_1^\perp$.

Definition 4.3 *Let $\tilde{E}_X$ be a periodic extension of $\tilde{E}$. We say that the tree $T(V)$ generates a set E, if $E = \bigcap\limits_{n=0}^{\infty} \tilde{E}_X \mathcal{A}^n$.*

Lemma 4.1 *Let $T(V)$ be a N-valid tree. Let $E \subset X$ be a set generated by the tree $T(V)$, H – height of $T(V)$. Then E is an $(N, H-2N)$-elementary set.*

Proof. Let us denote

$$m(\chi) = \mathbf{1}_{\tilde{E}_X}(\chi), \;\; M(\chi) = \prod_{n=0}^{\infty} m(\chi \mathcal{A}^{-n}).$$

First we note that $M(\chi) = \mathbf{1}_E(\chi)$. Indeed

$$\mathbf{1}_E(\chi) = 1 \Leftrightarrow \chi \in E \Leftrightarrow \forall\, n,\; \chi \mathcal{A}^{-n} \in \tilde{E}_X \Leftrightarrow \forall\, n,\; \mathbf{1}_{\tilde{E}_X}(\chi \mathcal{A}^{-n}) = 1 \Leftrightarrow$$

$$\forall\, n,\; m(\chi \mathcal{A}^{-n}) = 1 \Leftrightarrow \prod_{n=0}^{\infty} m(\chi \mathcal{A}^{-n}) = 1 \Leftrightarrow M(\chi) = 1.$$

It means that $M(\chi) = \mathbf{1}_E(\chi)$.
Now we will prove, that $\mathbf{1}_E(\chi) = 0$ for $\chi \in \mathfrak{G}^\perp_{H-2N+1} \setminus \mathfrak{G}^\perp_{H-2N}$. Since $\tilde{E}_X \supset \mathfrak{G}^\perp_{-N}$ it follows that $\mathbf{1}_{\tilde{E}_X}(\mathfrak{G}^\perp_{H-2N} \mathcal{A}^{-H+N}) = \mathbf{1}_{\tilde{E}_X}(\mathfrak{G}^\perp_{-N}) = 1$. Consequently

$$\prod_{n=0}^{\infty} \mathbf{1}_{\tilde{E}_X}(\chi \mathcal{A}^{-n}) = \prod_{n=0}^{H-N-1} \mathbf{1}_{\tilde{E}_X}(\chi \mathcal{A}^{-n})$$

if $\chi \in \mathfrak{G}_{H-2N+1}^{\perp} \setminus \mathfrak{G}_{H-2N}^{\perp}$. Let us denote

$$m(\mathfrak{G}_{-N}^{\perp} r_{-N}^{\alpha_{-N}} r_{-N+1}^{\alpha_{-N+1}} \dots r_0^{\alpha_0}) = \lambda_{\alpha_{-N},\alpha_{-N+1},\dots,\alpha_0}.$$

By the definition of cosets (4.1), (4.2) $m(\mathfrak{G}_{-N}^{\perp} r_{-N}^{\alpha_{-N}} r_{-N+1}^{\alpha_{-N+1}} \dots r_0^{\alpha_0}) \neq 0 \Leftrightarrow$ the vector $(\alpha_0, \alpha_1, ..., \alpha_{-N+1}, \alpha_{-N})$ is a path $(\alpha_0 \to \alpha_1 \to ... \to \alpha_{-N+1} \to \alpha_{-N})$ of the tree $T(V)$.

We need to prove that

$$\mathbf{1}_E(\mathfrak{G}_{-N}^{\perp} r_{-N}^{\alpha_{-N}} r_{-N+1}^{\alpha_{-N+1}} \dots r_{H-2N}^{\alpha_{H-2N}}) = 0$$

for $\alpha_{H-2N} \neq 0$. Since $\tilde{E}_X$ is a periodic extension of $\tilde{E}$ it follows that the function $m(\chi) = \mathbf{1}_{\tilde{E}_X}(\chi)$ is periodic with any period $r_1^{\alpha_1} r_2^{\alpha_2} \dots r_s^{\alpha_s}$, $s \in \mathbb{N}$, i.e. $m(\chi r_1^{\alpha_1} r_2^{\alpha_2} \dots r_s^{\alpha_s}) = m(\chi)$ when $\chi \in \mathfrak{G}_1^{\perp}$. Using this fact we can write $M(\chi)$ for $\chi \in \mathfrak{G}_{H-2N+1}^{\perp} \setminus \mathfrak{G}_{H-2N}^{\perp}$ in the form

$$\begin{gathered} M(\mathfrak{G}_{-N}^{\perp}\zeta) = M(\mathfrak{G}_{-N}^{\perp} r_{-N}^{\alpha_{-N}} r_{-N+1}^{\alpha_{-N+1}} \dots r_{H-2N}^{\alpha_{H-2N}}) = \\ = m(\mathfrak{G}_{-N}^{\perp} r_{-N}^{\alpha_{-N}} r_{-N+1}^{\alpha_{-N+1}} \dots r_0^{\alpha_0}) m(\mathfrak{G}_{-N}^{\perp} r_{-N}^{\alpha_{-N+1}} r_{-N+1}^{\alpha_{-N+2}} \dots r_0^{\alpha_1}) \dots \\ m(\mathfrak{G}_{-N}^{\perp} r_{-N}^{\alpha_{H-3N}} r_{-N+1}^{\alpha_{H-3N+1}} \dots r_{-1}^{\alpha_{H-2N-1}} r_0^{\alpha_{H-2N}}) \qquad (4.3) \\ m(\mathfrak{G}_{-N}^{\perp} r_{-N}^{\alpha_{H-3N+1}} r_{-N+1}^{\alpha_{H-3N+1}} \dots r_{-1}^{\alpha_{H-2N}}) \dots m(\mathfrak{G}_{-N}^{\perp} r_{-N}^{\alpha_{H-2N-1}} r_{-N+1}^{\alpha_{H-2N}}) m(\mathfrak{G}_{-N}^{\perp} r_{-N}^{\alpha_{H-2N}}). \end{gathered}$$

Assume that $M(\mathfrak{G}_{-N}^{\perp}\zeta) \neq 0$. Then all factors in (4.3) are nonzero. So we have the path

$$0 \to \dots \to 0 \to \alpha_{H-2N} \neq 0 \to \alpha_{H-2N-1} \to \dots \to \alpha_0 \to \dots \to \alpha_{-N+1} \to \alpha_{-N},$$

where there are N zeroes at the beginning of the path. The length of such path is $H+1$, which contradicts the condition $height(T) = H$.

Now we prove that E is $(1, H-2N)$ elementary set. Since the tree T(V) is N-valid, it has all possible combinations of N numbers $\alpha_i = \overline{0, p-1}$ as its paths, and we have the first property of elementary sets satisfied. Also, since $height(T) = H$, there exists a path

$$\alpha_1 = 0 \to \dots \to \alpha_N = 0 \to \alpha_{N+1} \neq 0 \to \alpha_{N+2} \to \dots \to \alpha_H$$

of length H. Such path generates the set $\mathfrak{G}_{-N}^{\perp} r_{-N}^{\alpha_{N+1}} \subset \mathfrak{G}_{-N+1} \setminus \mathfrak{G}_{-N}$, since $\alpha_{N+1} \neq 0$. Also, the same path generates the set $\mathfrak{G}_{-N}^{\perp} r_{-N}^{\alpha_{N+2}} r_{-N+1}^{\alpha_{N+1}} \subset \mathfrak{G}_{-N+2} \setminus \mathfrak{G}_{-N+1}$. Continuing this process we will obtain all sets

$$\forall l = \overline{0, H-N-1}, \mathfrak{G}_{-N}^{\perp} \prod_{n=0}^{l} r_{-N+n}^{\alpha_{N+1+n}} \subset \mathfrak{G}_{-N+l+1} \setminus \mathfrak{G}_{-N+l},$$

which means the second property of elementary sets is also satisfied. Thus we can conclude that E is $(1, H-2N)$-elementary set and the lemma is proved. $\square$.

Theorem 4.1 *Let $M, p \in \mathbb{N}$, $p \geq 3$. Let $E \subset \mathfrak{G}_M^{\perp}$ be an (N,M)-elementary set, $\hat{\varphi} \in \mathfrak{D}_{-N}(\mathfrak{G}_M^{\perp})$, $|\hat{\varphi}(\chi)| = \mathbf{1}_E(\chi)$, $\hat{\varphi}(\chi)$ the solution of the equation*

$$\hat{\varphi}(\chi) = m_0(\chi)\hat{\varphi}(\chi\mathcal{A}^{-1}), \tag{4.4}$$

where $m_0(\chi)$ is an N-elementary mask. Then there exists a rooted tree $T(V)$ with $\text{height}(T) = M + 2N$ *that generates the set E.*

Prof. Since the set E is (N,M)-elementary set and $|\hat{\varphi}(\chi)| = \mathbf{1}_E(\chi)$, it follows from theorem 3.2 that the system $(\varphi(x \dot{-} h))_{h \in H_0}$ is an orthonormal system in $L_2(\mathfrak{G})$. Using the theorem 3.3 we obtain that $\forall \alpha_{-N}, \ldots, \alpha_{-1} = \overline{0, p-1}$

$$\sum_{\alpha_0, \alpha_1, \ldots, \alpha_{M-1}=0}^{p-1} |\hat{\varphi}(\mathfrak{G}_{-N}^{\perp} r_{-N}^{\alpha_{-N}} \ldots r_{-1}^{\alpha_{-1}} r_0^{\alpha_0} \ldots r_{M-1}^{\alpha_{M-1}})|^2 = 1.$$

Since $\hat{\varphi}$ is a solution of refinement equation (4.4) it follows from lemma 3.5 that $\forall \alpha_{-N}, \ldots, \alpha_{-1} = \overline{0, p-1}$

$$\sum_{\alpha_0=0}^{p-1} |m_0(\mathfrak{G}_{-N}^{\perp} r_{-N}^{\alpha_{-N}} r_{-1}^{\alpha_{-1}} r_0^{\alpha_0})|^2 = 1. \tag{4.5}$$

Let as denote $\lambda_{\alpha_{-N}, \ldots, \alpha_{-1}, \alpha_0} := m_0(\mathfrak{G}_{-N}^{\perp} r_{-N}^{\alpha_{-N}} \ldots r_{-1}^{\alpha_{-1}} r_0^{\alpha_0})$. Then we write (4.5) in the form

$$\sum_{\alpha_0=0}^{p-1} |\lambda_{\alpha_{-N}, \ldots, \alpha_{-1}, \alpha_0}|^2 = 1. \tag{4.6}$$

Since the mask $m_0(\chi)$ is N-elementary it follows that $|\lambda_{\alpha_{-N}, \ldots, \alpha_{-1}, \alpha_0}|$ takes one of two values only: 0 or 1.

Now we will construct the tree T. We will begin with the path of N zeros

$$0_1 \to 0_2 \to \cdots \to 0_N,$$

where 0_1 is the root of the tree.

Let $\mathfrak{U}$ be a family of cosets $\mathfrak{G}_{-N}^{\perp}\zeta \subset \mathfrak{G}_M^{\perp}$ such that $\hat{\varphi}(\mathfrak{G}_{-N}^{\perp}\zeta) \neq 0$ and $\mathfrak{G}_{-N}^{\perp} \notin \mathfrak{U}$. We can write a coset $\mathfrak{G}_{-N}^{\perp}\zeta \in \mathfrak{U}$ in the form

$$\mathfrak{G}_{-N}^{\perp}\zeta = \mathfrak{G}_{-N}^{\perp} r_{-N}^{\alpha_{-N}} \ldots r_{-1}^{\alpha_{-1}} r_0^{\alpha_0} \ldots r_s^{\alpha_s}, \ \alpha_s \neq 0. \tag{4.7}$$

Here $s \le M-1$ since each coset in $\mathfrak{U}$ is a subset of $\mathfrak{G}_M^\perp$, and there exists at least one coset with $s = M-1$ since function is (N,M)-elementary. If $s = M-1$ and $\alpha_{s+1} + \dots + \alpha_{s+l} \neq 0$ then coset

$$\mathfrak{G}_{-N}^\perp r_{-N}^{\alpha_{-N}} \dots r_{-1}^{\alpha_{-1}} r_0^{\alpha_0} \dots r_s^{\alpha_s} r_{s+1}^{\alpha_{s+1}} \dots r_{s+l}^{\alpha_{s+l}} \notin \mathfrak{U}$$

0)Initially, we take a coset

$$\mathfrak{G}_{-N}^\perp \zeta_1 = \mathfrak{G}_{-N}^\perp r_{-N}^{\alpha_{-N}^{(1)}} \dots r_{-1}^{\alpha_{-1}^{(1)}} r_0^{\alpha_0^{(1)}} \dots r_{s_1}^{\alpha_{s_1}^{(1)}} \in \mathfrak{U},\ \alpha_{s_1} \neq 0$$

and connect the path

$$p^{(1)} = \alpha_{s_1}^{(1)} \to \dots \to \alpha_0^{(1)} \to \alpha_{-1}^{(1)} \to \dots \to \alpha_{-N}^{(1)}$$

to the 0_N vertex. We obtain the tree $T^{(0)}$ that contains unique branch

$$T^{(0)} = (0_1 \to 0_2 \to \dots \to 0_N \to \alpha_{s_1} \to \dots \to \alpha_0 \to \alpha_{-1} \to \dots \to \alpha_{-N}).$$

1) On the **first step**, take another coset

$$\mathfrak{G}_{-N}^\perp \zeta_2 = \mathfrak{G}_{-N}^\perp r_{-N}^{\alpha_{-N}^{(2)}} \dots r_{-1}^{\alpha_{-1}^{(2)}} r_0^{\alpha_0^{(2)}} \dots r_{s_2}^{\alpha_{s_2}^{(2)}} \in \mathfrak{U} \setminus \mathfrak{G}_{-N}^\perp \zeta_1, \alpha_{s_2}^{(2)} \neq 0$$

which generates the path

$$p^{(2)} = (\alpha_{s_2}^{(2)} \to \dots \to \alpha_0^{(2)} \to \alpha_{-1}^{(2)} \to \dots \to \alpha_{-N}^{(2)})$$

Let us add the path $0_1 \to 0_2 \to \dots \to 0_N$ to the beginning of the path $p^{(2)}$ and denote it as $\tilde{p}^{(2)}$, i.e.

$$\tilde{p}^{(2)} = (0_1 \to 0_2 \to \dots \to 0_N \to \alpha_{s_2}^{(2)} \to \dots \to \alpha_0^{(2)} \to \alpha_{-1}^{(2)} \to \dots \to \alpha_{-N}^{(2)})$$

Now we will include this path into our tree $T^{(0)}$. To include it we will compare the path $\tilde{p}^{(2)}$ with the tree $T^{(0)}$.

There are 3 possible cases:
1)The path $p^{(0)}$ is shorter than $p^{(1)}$ and

$$\alpha_{s_0}^{(0)} = \alpha_{s_1}^{(1)}, \alpha_{s_0-1}^{(0)} = \alpha_{s_1-1}^{(1)}, ..., \alpha_{-N}^{(0)} = \alpha_{s_1-(s_0+N)}^{(1)}.$$

In this case we connect the tail

$$\alpha_{s_1-(s_0+N+1)}^{(1)} \to \alpha_{s_1-(s_0+N+2)}^{(1)} \to \dots \to \alpha_{-N}^{(1)}$$

of the path $p^{(1)}$ to the vertex $\alpha^{(0)}_{-N}$.
2)The path $p^{(0)}$ is longer than $p^{(1)}$ and

$$\alpha^{(0)}_{s_0} = \alpha^{(1)}_{s_1}, \alpha^{(0)}_{s_0-1} = \alpha^{(1)}_{s_1-1}, ..., \alpha^{(0)}_{s_0-(s_1+N)} = \alpha^{(1)}_{-N}.$$

In this case the path $\tilde{p}^{(1)}$ is already a path of the tree $T^{(0)}$ and we leave the tree $T^{(0)}$ unchanged.
3)There exists an integer l such that $\alpha^{(1)}_{s_1-l} \neq \alpha^{(0)}_{s_0-l}$ and $\forall k < l, \alpha^{(1)}_{s_1-k} = \alpha^{(0)}_{s_0-k}$. If $l = -1$ then we get $\alpha^{(1)}_{s_1-l} = 0_N$. When l is calculated we connect the path

$$\alpha^{(1)}_{s_1-l} \to \alpha^{(1)}_{s_1-l-1} \to \cdots \to \alpha^{(1)}_{-N}$$

to the vertex $\alpha^{(0)}_{s_1-l+1}$ and obtain the tree

$$\begin{array}{ccccccccccc} & & & & & & \alpha^{(1)}_{s_1-l} \longrightarrow \ldots \longrightarrow \alpha^{(1)}_{-1} \longrightarrow \ldots \longrightarrow \alpha^{(1)}_{-N} \\ & & & & & \nearrow & \\ 0_1 \longrightarrow \ldots \longrightarrow 0_N \longrightarrow \alpha^{(0)}_{s_0} \ldots \longrightarrow \alpha^{(0)}_{s_0-l+1} & & & & & & \longrightarrow \alpha^{(0)}_{s_0-l} \longrightarrow \ldots \longrightarrow \alpha^{(0)}_{-1} \longrightarrow \ldots \longrightarrow \alpha^{(0)}_{-N}. \end{array}$$

Figure 2

This is the end of first step.

Consider n steps fulfilled, i.e. paths $p^{(0)}, p^{(1)}, ..., p^{(n)}$ are chosen and the correspondent tree $T^{(n)}$ is constructed. Now we will perform the $(n+1)$-th step. Let us take a coset

$$\mathfrak{G}^{\perp}_{-N}\zeta_{n+1} = \mathfrak{G}^{\perp}_{-N} r_{-N}^{\alpha^{(n+1)}_{-N}} \ldots r_{-1}^{\alpha^{(n+1)}_{-1}} r_0^{\alpha^{(n+1)}_0} \ldots r_{s_{n+1}}^{\alpha^{(n+1)}_{s_{n+1}}} \in \mathfrak{U} \backslash \bigcup_{k=1}^{n} \mathfrak{G}^{\perp}_{-N}\zeta_k,\ \alpha^{(n+1)}_{s_{n+1}} \neq 0,$$

which generates a path

$$p^{(n+1)} = (\alpha^{(n+1)}_{s_{n+1}} \to \cdots \to \alpha^{(n+1)}_0 \to \alpha^{(n+1)}_{-1} \to \cdots \to \alpha^{(n+1)}_{-N}).$$

and denote

$$\tilde{p}^{(n+1)} = (0_1 \to \cdots \to 0_N \to \alpha^{(n+1)}_{s_{n+1}} \to \cdots \to \alpha^{(n+1)}_0 \to \alpha^{(n+1)}_{-1} \to \cdots \to \alpha^{(n+1)}_{-N}).$$

Now we will include the path $\tilde{p}^{(n+1)}$ into the tree $T^{(n)}$. To do it, we will be looking for a path in the tree $T^{(n)}$ such that it has the longest starting sequence matching with the beginning of $\tilde{p}^{(n+1)}$.
Step $n + 1.1$. If $\alpha^{(n+1)}_{s_{n+1}}$ is not equal to any vertex of level $N + 1$ of the tree $T^{(n)}$ then we connect the path $p^{(n+1)}$ to the vertex 0_N, obtain the new tree $T^{(n)}$ and finish the step.

Step $n+1.2$. Otherwise there exists such (always unique) vertex of the level $N+1$, which we will denote by $\alpha_{(N+1),i}$, equal to $\alpha^{(n+1)}_{s_{n+1}}$ we consider all vertices of level $N+2$ connected to it. If there are no vertices connected or there are no such vertices matching $\alpha^{(n+1)}_{s_{n+1}-1}$ then we connect the tail of $p^{(n+1)}$ starting from the element $\alpha^{(n+1)}_{s_{n+1}-1}$ to the vertex $\alpha_{(N+1),i_j}$, obtain new tree and finish the step. Otherwise, if there exists vertex of level $N+2$ $\alpha_{(N+2),i}$ equal to $\alpha^{(n+1)}_{s_{n+1}-1}$, we continue the process of including the path $p^{(n+1)}$ into the tree until either there are no more elements in the path $p^{(n+1)}$ or at some level there are no vertices equal to corresponding element of the path $p^{(n+1)}$. In the first case the tree is left unchanged at this step. In the second case the tail of $p^{(n+1)}$ is added to the tree somewhere. Obviously, since the path $p^{(n+1)}$ has finite number of elements the process will also be finite.

The description of the $(n+1)$-th step is finished and there are only few final remarks left.

1)The resulting graph is a tree, since we produce no cycles at each step.

2)The process of constructing such tree is finite, i.e. contains finite number of steps since during each step we use different coset of $\mathfrak{U}$ and there is a finite number of such cosets.

So, at this point we have obtained a tree. Let us prove that this tree T is N-valid. To prove it, we must show, that each path of N elements is unique in our tree. Firstly, let us prove that the path of N zeros appears only once in our tree – and it is the path starting from its root. Indeed, let us assume that the path exists somewhere else in the tree T and that it is a part of some path

$$0_1 \to \dots \to 0_N \to \alpha_s \to \dots \to \alpha_k \to 0_1 \to \dots \to 0_N \to \dots \to \alpha_{-N}$$

from root to leaf. Since $\alpha_s \neq 0$ there exists at least one nonzero element between two instances of the path $0_1 \to \dots \to 0_N$. Without the loss of generality we can consider $\alpha_k \neq 0$.

Using the same technique as in (4.3), we can conclude, that

$$|\hat{\varphi}(\mathfrak{G}^{\perp}_{-N} r_{-N}^{\alpha_{-N}} \dots r_{k-2N-1}^{0_N} \dots r_{k-N-1}^{0_1} r_{k-N}^{\alpha_k} \dots r_s^{\alpha_s})| =$$

$$= |\lambda_{\alpha_{-N},\dots,\alpha_{-1},\alpha_0}| \dots |\lambda_{0_N,\dots,0_1,\alpha_k}| \dots |\lambda_{\alpha_s,0,\dots,0}| = 1,$$

which in particular means that $|\lambda_{0,\dots,0,\alpha_k}| = 1$. Also, by the properties of the mask $\lambda_{0,\dots,0,0} = 1$. These equalities contradict (4.6).

Now, let us assume that the arbitrary path $\gamma_{-1} \to \dots \to \gamma_{-N}$ appears twice. Thus, it is a subpath of 2 different paths from root to leaf

$$0_1 \to \dots 0_N \to \alpha_s \to \dots \to \alpha_k \to \gamma_{-1} \to \dots \to \gamma_{-N} \to \dots \to \alpha_{-N},\ k < s,$$

$$0_1 \to \dots 0_N \to \beta_{s'} \to \dots \to \beta_{k'} \to \gamma_{-1} \to \dots \to \gamma_{-N} \to \dots \to \beta_{-N},\ k' < s'.$$

Let us denote $0_i = \alpha_{s+N-i+1} = \beta_{s'+N-i+1}$. Now, let us prove, that $\exists j \geqslant 0 : \alpha_{k+j} \neq \beta_{k'+j}$.

We assume that the length of α subpath is less than the length of β subpath, i.e. $s - k < s' - k'$. Firstly, let us check if $\alpha_k \neq \beta_{k'}$. If they are equal, let's check if $\alpha_{k+1} \neq \beta_{k'+1}$. If we haven't encountered nonequal pair before $0_N = \alpha_{s+1}$ and $\beta_{k'-k+s+1}$, we check if they are nonequal. If not (i.e they are equal), we check all the remaining pairs. If next $N-1$ elements of β subpath are equal to elements 0_i of the α subpath, it contradicts the fact that there is only one subpath of N zeros in our tree. Thus in this case $\exists j \geqslant 0 : \alpha_{k+j} \neq \beta_{k'+j}$.

Now, let us assume, that both subpaths are of the same length. If $\forall j \geqslant 0 : \alpha_{k+j} = \beta_{k'+j}$ then, by construction of the tree T these two paths actually correspond to the same vertices from 0_1 to γ_{-N}, which means subpath γ does not appear twice in our tree. It contradicts the initial assumption that it does appear twice. Thus in this case $\exists j \geqslant 0 : \alpha_{k+j} \neq \beta_{k'+j}$, too.

Let us assume, without loss of generality, that $\alpha_k \neq \beta_{k'}$. Using the same technique as in (4.3), we can conclude, that

$$|\hat{\varphi}(\mathfrak{G}_{-N}^{\perp} r_{-N}^{\alpha_{-N}} \dots r_{k-2N-1}^{\gamma_{-N}} \dots r_{k-N-1}^{\gamma_{-1}} r_{k-N}^{\alpha_k} \dots r_s^{\alpha_s})| =$$

$$= |\lambda_{\alpha_{-N},\dots,\alpha_{-1},\alpha_0}| \dots |\lambda_{\gamma_{-N},\dots,\gamma_{-1},\alpha_k}| \dots |\lambda_{\alpha_s,0,\dots,0}| = 1,$$

$$|\hat{\varphi}(\mathfrak{G}_{-N}^{\perp} r_{-N}^{\beta_{-N}} \dots r_{k'-2N-1}^{\gamma_{-N}} \dots r_{k'-N-1}^{\gamma_{-1}} r_{k'-N}^{\beta_{k'}} \dots r_{s'}^{\beta_{s'}})| =$$

$$= |\lambda_{\beta_{-N},\dots,\beta_{-1},\beta_0}| \dots |\lambda_{\gamma_{-N},\dots,\gamma_{-1},\beta_{k'}}| \dots |\lambda_{\beta_{s'},0,\dots,0}| = 1.$$

That means, in particular, that $|\lambda_{\gamma_{-N},\dots,\gamma_{-1},\beta_{k'}}| = |\lambda_{\gamma_{-N},\dots,\gamma_{-1},\alpha_k}| = 1$, which contradicts (4.6). Thus our tree is N-valid.

It is evident that this tree generates refinable function $\hat{\varphi}$ with a mask m_0. Let's show that height$(T) = M + 2N$. Indeed, since $\hat{\varphi} \in \mathfrak{D}_{-N}(\mathfrak{G}_M^{\perp})$ it follows that there exists a coset $\mathfrak{G}_{-N}^{\perp} r_{-N}^{\alpha_{-N}} \dots r_{-1}^{\alpha_{-1}} r_0^{\alpha_0} \dots r_{M-1}^{\alpha_{M-1}}$, $\alpha_{M-1} \neq 0$ for which $|\hat{\varphi}(\mathfrak{G}_{-N}^{\perp} r_{-N}^{\alpha_{-N}} \dots r_{-1}^{\alpha_{-1}} r_0^{\alpha_0} \dots r_{M-1}^{\alpha_{M-1}})| = 1$. This coset generates a path

$$0_1 \to \dots \to 0_N \to \alpha_{M-1} \to \dots \to \alpha_0 \to \alpha_{-1} \to \dots \to \alpha_{-N}$$

of T. This path contain $M + 2N$ vertices. It means that height$(T) \geq M + 2N$. On the other hand there is no coset $\mathfrak{G}_{-N}^{\perp}\zeta \subset \mathfrak{G}_{M+1}^{\perp} \setminus \mathfrak{G}_M^{\perp}$, consequently there is no path with $L > M+2N$. So height$(T) = M+2N$. The theorem is proved. $\Box$

Definition 4.4 *Let $T(V)$ be an N-valid tree, H = height(T). Using cosets (4.1) we define the mask $m_0(\chi)$ in the subgroup $\mathfrak{G}_1^\perp$ as follows: $m_0(\mathfrak{G}_{-N}^\perp) = 1, m_0(\mathfrak{G}_{-N}^\perp r_{-N}^{\alpha_{-N}} \dots r_{-1}^{\alpha_{-1}} r_0^{\alpha_0}) = \lambda_{\alpha_{-N},\dots,\alpha_{-1},\alpha_0}$, $|\lambda_{\alpha_{-N},\dots,\alpha_{-1},\alpha_0}| = 1$ when $\mathfrak{G}_{-N}^\perp r_{-N}^{\alpha_{-N}} \dots r_{-1}^{\alpha_{-1}} r_0^{\alpha_0} \subset \tilde{E}$, $m_0(\mathfrak{G}_{-N}^\perp r_{-N}^{\alpha_{-N}} \dots r_{-1}^{\alpha_{-1}} r_0^{\alpha_0}) = \lambda_{\alpha_{-N},\dots,\alpha_{-1},\alpha_0} = 0$ when $\mathfrak{G}_{-N}^\perp r_{-N}^{\alpha_{-N}} \dots r_{-1}^{\alpha_{-1}} r_0^{\alpha_0} \subset \mathfrak{G}_1^\perp \backslash \tilde{E}$. Let us extend the mask $m_0(\chi)$ on the $X \setminus \mathfrak{G}_1^\perp$ periodically, i.e. $m_0(\chi r_1^{\alpha_1} r_2^{\alpha_2} \dots r_s^{\alpha_s}) = m_0(\chi)$. Then we say that the tree $T(V)$ generates the mask $m_0(\chi)$. Set $\hat{\varphi}(\chi) = \prod_{n=0}^{\infty} m_0(\chi \mathcal{A}^{-n})$. It follows from lemma 4.1 that*
1) $\operatorname{supp} \hat{\varphi}(\chi) \subset \mathfrak{G}_{H-2N}^\perp$,
2) $\hat{\varphi}(\chi)$ is $(N, H-2N)$ elementary function,
3) $(\varphi(x \dot{-} h))_{h \in H_0}$ is an orthonormal system.
In this case we say that the tree $T(V)$ generates the refinable function $\varphi(x)$.

Theorem 4.2 *Let $p \geq 3$ be a prime number, $T(V)$ an N-valid tree. Let H be are height of $T(V)$. By $\varphi(x)$ denote the function generated by the $T(V)$. Then $\varphi(x)$ generates an orthogonal MRA on p-adic Vilenkin group.*

Proof. Since $T(V)$ generates the the function φ then 1)$\hat{\varphi} \in \mathfrak{D}_{-N}(\mathfrak{G}_M^\perp)$, 2)$\hat{\varphi}(\chi)$ is $(N, H-2N)$- elementary function, 3)$\hat{\varphi}(\chi)$ is a solution of refinable equation (3.3), 4)$(\varphi(x \dot{-} h))_{h \in H_0}$ is an orthonormal system. From the theorem 3.1 it follows that $\varphi(x)$ generates an orthogonal MRA. $\Box$

5 Construction of wavelet bases

In [6] and [7] Yu.A.Farkov reduces the problem of p-wavelet decomposition into a problem of matrix extension. We will use another method [13].

As usual, W_n stands for the orthogonal complement of V_n in V_{n+1}: that is $V_{n+1} = V_n \oplus W_n$ and $V_n \perp W_n$ ($n \in \mathbb{Z}$, and $\oplus$ denotes the direct sum).
It is readily seen that
1) $f \in W_n \Leftrightarrow f(\mathcal{A}x) \in W_{n+1}$,
2) $W_n \perp W_k$ for $k \neq n$,
3) $\oplus W_n = L_2(\mathfrak{G})$, $n \in \mathbb{Z}$.

From theorems 4.1, 4.2 we derive an algorithm for constructing wavelet bases.
Step 1. Choose an arbitrary tree T - N-valid. Let H be a height of the

tree T.
Step 2. Choose a finite sequence $(\lambda_{\alpha_{-N},\dots,\alpha_0})_{\alpha_{-N},\dots,\alpha_0=0}^{p-1}$ such that $\lambda_{0,0,\dots,0} = 1$, $|\lambda_{\alpha_{-N},\dots,\alpha_0}| = 1$ if there exists subpath $\alpha_{-N} \to \cdots \to \alpha_0$ in the tree T, $|\lambda_{\alpha_{-N},\dots,\alpha_0}| = 0$ otherwise.
Step 3. Construct the mask $m_0(\chi)$ and Fourier transform $\hat{\varphi}(\chi)$ using definition 4.4. It is clear that $E = \mathrm{supp}(\hat{\varphi}(\chi))$ is $(N, H-2N)$-elementary set.
Step 4. Find coefficients β_h for which

$$m_0(\chi) = \frac{1}{p} \sum_{h \in H_0^{(N+1)}} \beta_h \overline{(\chi \mathcal{A}^{-1}, h)}. \tag{5.1}$$

To find coefficients β_h, we write this equation in the form

$$m_0(\chi_k) = \frac{1}{p} \sum_{j=0}^{p^{N+1}-1} \beta_j \overline{(\chi_k, \mathcal{A}^{-1} h_j)} \tag{5.2}$$

where

$$\begin{array}{ll} h_j = a_{-1}g_{-1} \dot{+} a_{-2}g_{-2} \dot{+} \dots \dot{+} a_{-N-1}g_{-N-1}, & \chi_k \in \mathfrak{G}_{-N}^{\perp} r_{-N}^{\alpha_{-N}} \dots r_{-1}^{\alpha_{-1}} r_0^{\alpha_0}, \\ j = a_{-1} + a_{-2}p + \cdots + a_{-N-1}p^N, & k = \alpha_{-N} + \cdots + \alpha_{-1}p^{N-1} + \alpha_0 p^N, \\ a_{-1}, a_{-2}, \dots, a_{-N} = \overline{0, p-1}, & \alpha_{-N-1}, \dots, \alpha_{-1}, \alpha_0 = \overline{0, p-1}. \end{array}$$

Since the matrix $\frac{1}{p}\overline{(\chi_k, \mathcal{A}^{-1} h_j)}$ of this system is unitary it follows that the system (5.2) has a unique solution.
Step 5. We set $m_l(\chi) = m_0(\chi r_0^{-l})$, $l = \overline{1, p-1}$, $X_0 = \{\chi : |m_0(\chi)| = 1\}$. Clearly, $m_l(\chi)$ may be written as

$$m_l(\chi) = \frac{1}{p} \sum_{h \in H_0^{(N+1)}} \beta_h \overline{(\chi r_0^{-l}, \mathcal{A}^{-1} h)} = \frac{1}{p} \sum_{h \in H_0^{(N+1)}} \beta_h^{(l)} \overline{(\chi, \mathcal{A}^{-1} h)}$$

where $\beta_h^{(l)} = \beta_h (r_0^l, \mathcal{A}^{-1} h)$. By the construction of $m_l(\chi)$ we have $|m_l(X_0 r_0^l)| = 1$. From the necessary condition (3.7) it follows that $|m_l(X_0 r_0^{\nu})| = 0$ for $\nu \neq l$, $m_l(\chi) m_k(\chi) = 0$ when $k \neq l$.
Step 6. Define the functions

$$\psi_l(x) = \sum_{h \in H_0^{(N+1)}} \beta_h^{(l)} \varphi(\mathcal{A}x \dot{-} h).$$

Theorem 5.1 *The functions $\psi_l(x \dot{-} h)$, where $l = \overline{1, p-1}$, $h \in H_0$, form an orthonormal basis for W_0.*

Proof. a) We claim that $(\varphi(\cdot\dot{-}g^{(1)}), \psi_l(\cdot\dot{-}g^{(2)})) = 0$ for any $g^{(1)}$, $g^{(2)} \in H_0$. Since

$$\hat{\varphi}_{\cdot\dot{-}h}(\chi) = \overline{(\chi, h)}\hat{\varphi}(\chi), \ \ \hat{\varphi}_{\mathcal{A}\cdot\dot{-}g}(\chi) = \frac{1}{p}\overline{(\chi, \mathcal{A}^{-1}g)}\hat{\varphi}(\chi\mathcal{A}^{-1}),$$

it follows that

$$(\varphi(\cdot\dot{-}g^{(1)}), \psi_l(\cdot\dot{-}g^{(2)})) = \int\limits_X \hat{\varphi}(\chi)\overline{\hat{\varphi}(\chi\mathcal{A}^{-1})}\overline{(\chi, g^{(1)})}(\chi, g^{(2)})\overline{m_l(\chi)}\, d\nu(\chi) = 0$$

because $\operatorname{supp}\hat{\varphi}(\chi) = E$ and $m_l(E) = 0$, $l = \overline{1, p-1}$.
b) By analogy

$$(\psi_k(\cdot\dot{-}g^{(1)}), \psi_l(\cdot\dot{-}g^{(2)})) = \int\limits_X |\hat{\varphi}(\chi\mathcal{A}^{-1})|^2(\chi, g^{(2)}\dot{-}g^{(1)})m_k(\chi)\overline{m_l(\chi)}\, d\nu(\chi) = 0$$

when $k \neq l$.
c) We verify that $(\psi_l(\cdot\dot{-}g^{(1)}), \psi_l(\cdot\dot{-}g^{(2)})) = 0$, provided that $g^{(1)}, g^{(2)} \in H_0$ and $g^{(1)} \neq g^{(2)}$. Write this scalar product in the form

$$(\psi_l(\cdot\dot{-}g^{(1)}), \psi_l(\cdot\dot{-}g^{(2)})) = \int\limits_X |\hat{\varphi}(\chi\mathcal{A}^{-1})|^2(\chi, g^{(2)}\dot{-}g^{(1)})|m_l(\chi)|^2\, d\nu(\chi) =$$

$$= \int\limits_{E\mathcal{A}\bigcap X_0 r_0^l} (\chi, g^{(2)}\dot{-}g^{(1)})\, d\nu(\chi).$$

Show that $E\mathcal{A}\bigcap X_0 r_0^l$ is an $(N, H-2N)$-elementary set. By the definition

$$E = \bigsqcup_{\overline{\alpha}\in T(V)} \mathfrak{G}_{-N}^{\perp} r_{-N}^{\alpha_{-N}} ... r_0^{\alpha_0} \dots r_s^{\alpha_s} r_{s+1}^0 ... r_{s+N}^0 \ \ (s \leq H - 2N - 1) \qquad (5.3)$$

where the union is taken over all paths

$$\overline{\alpha} = (0, ..., 0, \alpha_s, \alpha_{s-1}, \dots, \alpha_0, \alpha_{-1}, ..., \alpha_{-N})$$

of the tree T. It means that for any vector $(\alpha_{-1}, ..., \alpha_{-N})$, $\alpha_j = \overline{0, p-1}$ the union (5.3) contains unique coset $\mathfrak{G}_{-N}^{\perp} r_{-N}^{\alpha_{-N}} ... r_{-1}^{\alpha_{-1}} r_0^{\alpha_0} \dots r_s^{\alpha_s} r_{s+1}^0 ... r_{s+N}^0$.
Consequently $E\mathcal{A} =$

$$\bigsqcup_{\overline{\alpha}\in T(V)} \mathfrak{G}_{-N+1}^{\perp} r_{-N+1}^{\alpha_{-N}} ... r_0^{\alpha_{-1}} \dots r_{s+1}^{\alpha_s} r_{s+2}^0 ... r_{s+N+1}^0 =$$

$$\bigsqcup_{\alpha_{-N-1}=0}^{p-1} \bigsqcup_{\overline{\alpha}\in T(V)} \mathfrak{G}_{-N}^{\perp} r_{-N}^{\alpha_{-N-1}} r_{-N+1}^{\alpha_{-N}} ... r_0^{\alpha_{-1}} \dots r_{s+1}^{\alpha_s} r_{s+2}^0 ... r_{s+N+1}^0.$$

On the other hand

$$X_0 r_0^l = \bigcup_{j\in\mathbb{N}} \bigsqcup_{(\gamma_{-N},...,\gamma_{-1},\gamma_0)\in T(V)} \bigsqcup_{b_1,b_2,...,b_j=0}^{p-1} \mathfrak{G}_{-N}^{\perp} r_{-N}^{\gamma_{-N}} ... r_{-1}^{\gamma_{-1}} r_0^{\gamma_0+l} r_1^{b_1} \dots r_j^{b_j}.$$

Therefore $E\mathcal{A}\bigcap X_0 r_0^l$ consists of all cosets

$$\mathfrak{G}_{-N}^{\perp} r_{-N}^{\gamma_{-N}} ... r_{-1}^{\gamma_{-1}} r_0^{\alpha_{-1}} r_1^{\alpha_0} \dots r_{s+1}^{\alpha_s} r_{s+2}^0 ... r_{s+N+1}^0$$

where

$$(0, ..., 0, \alpha_s, \alpha_{s-1}, \dots, \alpha_{-1} = \gamma_0 + l, \gamma_{-1}, ..., \gamma_{-N}) \in T$$

Since the tree T is N-valid it follows that $E\mathcal{A}\bigcap X_0 r_0^l$ is $(N, H-2N+1)$-elementary set. By lemma 3.4 it follows that

$$\int\limits_{E\mathcal{A}\bigcap X_0 r_0^l} (\chi, g^{(2)} \dot{-} g^{(1)})\, d\nu(\chi) = 0.$$

d) We claim that any function $f \in W_0$ can be expanded uniquely in a series in terms of $(\psi_l(x\dot{-}g))_{l=\overline{1,p-1},g\in H_0}$. The proof of this fact may be found in [13], theorem 5.1. □

Step 7. Since the subspaces $(V_j)_{j\in\mathbb{Z}}$ form an MRA in $L_2(\mathfrak{G})$, it follows that the functions

$$(\psi_l(\mathcal{A}^n x\dot{-}h)) \;\; l = \overline{1, p-1}, n \in \mathbb{Z}, h \in H_0$$

form a complete orthogonal system in $L_2(\mathfrak{G})$.

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