

On Discrete Preferences and Coordination *

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Abstract

An active line of research has considered games played on networks in which payoffs depend on both a player’s individual decision and also the decisions of his or her neighbors. Such games have been used to model issues including the formation of opinions (in which people wish to express views consistent with those of their friends) and the adoption of technology (in which people or firms seek compatibility with their network neighbors).

A basic question that has remained largely open in this area is to consider games where the strategies available to the players come from a fixed, discrete set, and where players may have different intrinsic preferences among the possible strategies. It is natural to model the tension among these different preferences by positing a distance function on the strategy set that determines a notion of “similarity” among strategies; a player’s payoff is determined by the distance from her chosen strategy to her preferred strategy and to the strategies chosen by her network neighbors. Even when there are only two strategies available, this framework already leads to natural open questions about a version of the classical Battle of the Sexes problem played on a graph; such questions generalize issues in the study of network coordination games.

We develop a set of techniques for analyzing this class of games, which we refer to as *discrete preference games*. We parametrize the games by the relative extent to which a player takes into account the effect of her preferred strategy and the effect of her neighbors’ strategies, allowing us to interpolate between network coordination games and unilateral decision-making. When these two effects are balanced, we show that the price of stability is equal to 1 for any discrete preference game in which the distance function on the strategies is a tree metric; as a special case, this includes the Battle of the Sexes on a graph. We also show that trees essentially form the maximal family of metrics for which the price of stability is 1, and produce a collection of metrics on which the price of stability converges to an asymptotically tight bound of 2.

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1 Introduction

People often make decisions in settings where the outcome depends not only on their own choices, but also on the choices of the people they interact with. A natural model for such situations is to consider a game played on a graph that represents an underlying social network, where the nodes are the players. Each node’s personal decision corresponds to selecting a strategy, and the node’s payoff depends on the strategies chosen by itself and its neighbors in the graph [2, 4, 11].

Coordination and Internal Preferences A fundamental class of such games involves payoffs based on the interplay between *coordination* — each player has an incentive to match the strategies of his or her neighbors — and *internal preferences* — each player also has an intrinsic preference for certain strategies over others, independent of the desire to match what others are doing. Trade-offs of this type come up in a very broad collection of situations, and it is worth mentioning several that motivate our work here.

- In the context of opinion formation, a group of people or organizations might each possess different internal views, but they are willing to express or endorse a “compromise” opinion so as to be in closer alignment with their network neighbors.
- Questions involving technological compatibility among firms tend to have this trade-off as a fundamental component: firms seek to coordinate on shared standards despite having internal cost structures that favor different solutions.
- Related to the previous example, a similar issue comes up in cooperative facility location problems, where firms have preferences for where to locate, but each firm also wants to locate near the firms with which it interacts.

A line of work beginning in the mathematical social sciences has considered versions of this question — often motivated by the first class of examples above, concerned with opinion formation — where the possible strategies correspond to a continuous space such as \mathbb{R}^d [5, 9]. This makes it possible for players to adopt arbitrarily fine-grained “average” strategies from among any set of options, and most of the dynamics and equilibrium properties of such models are driven by this type of averaging. In particular, dynamics based on repeated averaging have been shown in early work to exhibit nice convergence properties [5], and recent work including by two of the authors has developed bounds on the relationship between equilibria and social optima [1].

Discrete Preferences In many settings that exhibit a tension between coordination and individual preferences, however, there is no natural way to average among the available options. Instead, the alternatives are drawn from a fixed discrete set — for example, there is only a given set of available technologies for firms to choose among, or a fixed set of political candidates to endorse or vote for. On a much longer time scale, there is always the possibility that additional options could be created to interpolate between what’s available, but on the time scale over which the strategic interaction takes place, the players must choose from among the discrete set of alternatives that is available.

Among a small fixed set of players, coordination with discrete preferences is at the heart of a long line of games in the economic theory literature — perhaps the most primitive example is the classic *Battle of the Sexes* game, based on a pedagogical story in which one member of a couple wants to see movie *A* while the other wants to see movie *B*, but both want to go to a movie together. This provides a very concrete illustration of a set of payoffs in which the (two) players have (i)

conflicting internal preferences (A and B respectively), (ii) an incentive to arrive at a compromise, and (iii) no way to “average” between the available options.

But essentially nothing is known about the properties of the games that arise when we consider such a payoff structure in a network context. Even the direct generalization of Battle of the Sexes (BoS) to a graph is more or less unexplored in this sense — each node plays a copy of BoS on each of its incident edges, choosing a single strategy A or B for use in all copies, incurring a cost from miscoordination with neighbors and an additional fixed cost when the node’s choice differs from its inherent preference. Indeed, as some evidence of the complexity of even this formulation, note that the version in which each node has an intrinsic preference for A is equivalent to the standard network coordination game, which already exhibits rich graph-theoretic structure [11]. And beyond this, of course, lies the prospect of such games with larger and more involved strategy sets.

Formalizing Discrete Preference Games In this paper, we develop a set of techniques for analyzing this type of discrete preference games on a network, and we establish tight bounds on the price of stability for several important families of such games.

To formulate a general model for this type of game, we start with an undirected graph $G = (V, E)$ representing the network on the players, and an underlying finite set L of strategies. Each player $i \in V$ has a *preferred strategy* $s_i \in L$, which is what i would choose in the absence of any other players. Finally, there is a metric $d(\cdot, \cdot)$ on the strategy set L — that is, a distance function satisfying (i) $d(i, i) = 0$ for all i , (ii) $d(i, j) = d(j, i)$ for all i, j , and (iii) $d(i, j) \leq d(i, k) + d(k, j)$ for all i, j and k . For $i, j \in L$, the distance $d(i, j)$ intuitively measures how “different” i and j are as choices; players want to avoid choosing strategies that are at large distance from either their own internal preference or from the strategies chosen by their neighbors.

Each player’s objective is to minimize her cost (think of this as the negative of her payoff): for a fixed parameter $\alpha \in [0, 1]$, the cost to player i when players choose the strategy vector $z = \langle z_j : j \in V \rangle$ is

$$c_i(z) = \alpha \cdot d(s_i, z_i) + \sum_{j \in N(i)} (1 - \alpha) \cdot d(z_i, z_j),$$

where $N(i)$ is the set of neighbors of i in G . The parameter α essentially controls the extent to which players are more concerned with their preferred strategies or their network neighbors; we will see that the behavior of the game can undergo qualitative changes as we vary α .

We say that the above formulation defines a *discrete preference game*. Note that the network version of Battle of the Sexes described earlier is essentially the special case in which $|L| = 2$, and network coordination games are the special case in which $|L| = 2$ and $\alpha = 0$, since then players are only concerned with matching their neighbors. The case in which $d(\cdot, \cdot)$ is the distance metric among nodes on a path is also interesting to focus on, since it is the discrete analogue of the one-dimensional space of real-valued opinions from continuous averaging models [1, 5] — consider for example the natural scenario in which a finite number of discrete alternatives in an election are arranged along a one-dimensional political spectrum.

We also note that discrete preference games belong to the well-known framework of graphical games, which essentially consist of games in which the utility of every player depends only on the actions of its neighbors in a network. The interested reader is referred to the relevant chapter in [12] and the references within. In this context, Gottlob et al. proposed a generalization of Battle of the Sexes (BoS) to a graphical setting [8], but their formulation was much more complex than our starting point, with their questions correspondingly focused on existence and computational complexity, rather than on the types of performance guarantees we will be seeking.

For any discrete preference game, we will see that it is possible to define an exact potential function, and hence these games possess pure Nash equilibria.

Price of Stability in Discrete Preference Games We can also ask about the *social cost* of a strategy vector $z = \langle z_j : j \in V \rangle$, defined as the sum of all players' costs:

$$c(z) = \sum_{i \in V} \alpha \cdot d(s_i, z_i) + 2 \sum_{(i,j) \in E} (1 - \alpha) \cdot d(z_i, z_j).$$

We note that the problem of minimizing the social cost is an instance of the *metric labeling problem*, in which we want to assign labels to nodes in order to minimize a sum of per-node costs and edge separation costs [3, 10].

Since an underlying motivation for studying this class of games is the tension between preferred strategies and agreement on edges, it is natural to study its consequences on the social cost via the price of anarchy and/or the price of stability. The price of anarchy is in fact too severe a measure for this class of games; indeed, as we discuss in the next section, it is already unbounded for the well-studied class of network coordination games that our model contains as a special case.

We therefore consider the price of stability, which turns out to impart a rich structure to the problem. The price of stability is also natural in terms of the underlying examples discussed earlier as motivation; in most of these settings, it makes sense to propose a solution — for example, a compromise option in a political setting or a proposed set of technology choices for a set of interacting firms — and then to see if it is stable with respect to equilibrium.

Overview of Results As a starting point for reference, observe that network coordination games (where players are not concerned with their preferred strategies) clearly have a price of stability of 1: the players can all choose the same strategy and achieve a cost of 0. But even for a general discrete preference game with two strategies — i.e. Battle of the Sexes on a network — the price of stability is already more subtle, since the social optimum may have a more complex structure (as a two-label metric labeling problem, and hence a minimum cut problem).

We begin by giving tight bounds on the maximum possible price of stability in the two-strategy case as a function of the parameter α . The dependence on α has a complex non-monotonic character; in particular, the price of stability is equal to 1 for all instances if and only if $\alpha \leq 1/2$ or $\alpha = 2/3$, and more generally the price of stability as a function of α displays a type of “saw-tooth” behavior with infinitely many local minima in the interval $[0, 1]$. Our analysis uses a careful scheduling of the best-response dynamics so as to track the updates of players toward a solution with low social cost.

Above we also mentioned the distance metric of a path as a case of interest in opinion formation. We show that when $\alpha \leq 1/2$, the price of stability for instances based on such metrics is always 1, by proving the stronger statement that in fact the price of stability is always 1 for any discrete preference game based on a tree metric. Our analysis for tree metrics involves considering how players' best responses lie at the medians of their neighbors' strategies in the metric, and then developing combinatorial techniques for reasoning about the arrangement of these collections of medians on the underlying tree.

Like path metrics, tree metrics are also relevant to motivating scenarios in terms of opinion formation, when individuals classify the space of possible opinions according to a hierarchical structure rather than a linear one. To take one example of this, consider students choosing a major in college, where each student has an internal preference and an interest in picking a major that is similar to the choices of her friends. The different subjects roughly follow a hierarchy —

on top we might have science, engineering, and humanities; under science we can have for example biology, physics, and other areas; and under biology we can have subjects including genetics and plant breeding. This setting fits our model since each person has some internal inclination for a major, but still it is arguably the case that a math major has more in common in her educational experience with her computer science friends than with her friends in comparative literature.

The two families of instances described above (two strategies and tree metrics) both have price of stability equal to 1 when $\alpha \leq 1/2$. But the price of stability can be greater than 1 for more general metrics when $\alpha \leq 1/2$. It is not hard to show (as we do in the next section) that the price of stability is always at most 2 for all α , and we match this bound by constructing and analyzing examples, based on perturbations of uniform metrics, showing that the price of stability can be arbitrarily close to 2 when $\alpha = 1/2$.

Finally, we consider a generalization of our model of discrete preference games, which we term an *anchored preference game*. Suppose that nodes are partitioned into two types: there are *fixed nodes* i that have a preferred strategy anchored on a particular value s_i , and there are *strategic nodes* that have no preferred strategy, so the cost of such a node i is purely the term $\sum_{j \in N(i)} d(z_i, z_j)$. Only the strategic nodes choose strategies, and only they take part in the definition of the equilibrium.

Anchored preference games are also of interest in their own right, and not just as a generalization of discrete preference games, since they can model settings where certain parts of the network represent unmodifiable constraints — for example, how technological compatibility might have to take into account certain unchangeable factors, or how individuals adapting their opinions to each other might also be taking into account agents such as media sources that are in effect outside the immediate strategic environment. We generalize our results for tree metrics to the case of anchored preference games, parametrizing the price of stability by the maximum number of fixed nodes among the neighbors of any strategic node.

2 Preliminaries

Recall that in a discrete preference game played on a graph $G = (V, E)$ with strategy set L , each player $i \in V$ has a preferred strategy $s_i \in L$. The cost incurred by player i when all players choose strategies $z = \langle z_j : j \in V \rangle$ is

$$c_i(z) = \alpha \cdot d(s_i, z_i) + \sum_{j \in N(i)} (1 - \alpha) \cdot d(z_i, z_j).$$

The social cost of the game is the sum of all the players' costs:

$$c(z) = \sum_{i \in V} \alpha \cdot d(s_i, z_i) + 2 \sum_{(i,j) \in E} (1 - \alpha) \cdot d(z_i, z_j).$$

Another quantity that is useful to define is the contribution of player i to the social cost — by this we quantify not only the cost player i is exhibiting but also the cost it is inflicting on its neighbors:

$$sc_i(z) = \alpha \cdot d(s_i, z_i) + 2 \sum_{j \in N(i)} (1 - \alpha) \cdot d(z_i, z_j).$$

As is standard, we denote by z_{-i} the strategy vector z without the i^{th} coordinate.

We first show that this class of games includes instances for which the price of anarchy (PoA) is unbounded. A simple instance for which the PoA is unbounded is one in which the preferred strategy of all the players is the same — thus the cost of the optimal solution is 0, and it is also an

equilibrium for all the players to play some other strategy and incur a positive cost.¹ In the next claim, we use this idea to construct, for every value of $\alpha < 1$, an instance for which the PoA is unbounded.

Claim 2.1 *For any $\alpha < 1$ there exists an instance for which the price of anarchy is unbounded.*

Proof: Assume the strategy space contains two strategies A and B , such that $d(A, B) = 1$. For any $0 < \alpha < 1$ we consider a clique of size $\lceil \frac{\alpha}{1-\alpha} \rceil + 1$ in which all players' preferred strategy is A and show it is an equilibrium for all the players to play strategy B . To see why, observe that if the rest of the players play strategy B , then player i 's cost for playing strategy A is $(1 - \alpha) \cdot \lceil \frac{\alpha}{1-\alpha} \rceil$ which is at least α . Since the cost of player i for playing strategy B is α we have that it is an equilibrium for all players to play strategy B . The PoA of such an instance is unbounded as the cost of the equilibrium in which all players play strategy B is strictly positive but the cost of the optimal solution is 0.

To show that the PoA can be unbounded for $\alpha = 0$, a slightly different instance is required, which will be familiar from the literature of network coordination games. When players do not have a preference the optimal solution is clearly for all players to play the same strategy, as such a solution has a cost of 0. However, Figure 1 depicts an instance for which there exists a Nash equilibrium in which not all the players play the same strategy and hence the cost of this equilibrium is strictly positive. \square

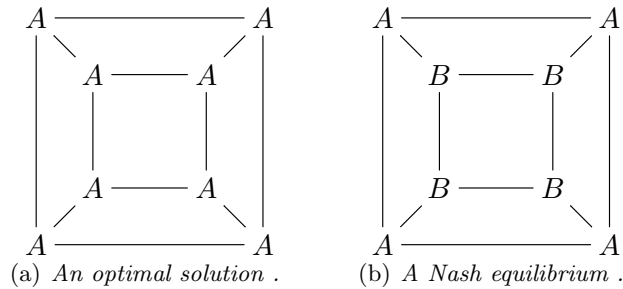


Figure 1: An instance illustrating that the PoA can be unbounded even when the players do not have a preferred strategy (i.e., $\alpha = 0$).

We note that the worst equilibrium in the previous instances is not a strong equilibrium. Thus, if all the players could coordinate a joint deviation to strategy A they can all benefit. A natural question is what happens if we restrict ourselves to the worst strong Nash equilibrium (strong PoA), in which a simultaneous deviation by a set of players is allowed. Unfortunately, the strong PoA can still be quite high (linear in the number of players). Take for example $\alpha < \frac{1}{2}$ and consider a clique of size n in which all but one of the players prefer strategy A . In this case it is not hard to verify that the equilibrium in which all players play strategy B is strong. The reason is that the player that prefers strategy B cannot gain from any deviation, and the cost of any other player would increase by at least $1 - \alpha - \alpha > 0$ for any deviation. The cost of such an equilibrium is $\alpha(n - 1)$ in comparison to the optimal solution which has a cost of α .

¹This type of equilibrium, in addition to simply producing an unbounded PoA, has a natural interpretation in our motivating contexts. In technology adoption, it corresponds to convergence on a standard that no firm individually wants, but which is hard to move away from once it has become the consensus. In opinion formation, it corresponds essentially to a kind of “superstitious” belief that is universally expressed, and hence is hard for people to outwardly disavow even though they prefer an alternate opinion.

As we just showed both the PoA and the strong PoA can be very high, and hence for the remainder of the paper we focus on the qualities of the best Nash equilibrium, trying to bound the price of stability (PoS). We begin by showing that the price of stability is bounded by 2. This is done by a potential function argument which also proves that a Nash equilibrium always exists.

Claim 2.2 *The price of stability is bounded by 2.*

Proof: We first prove that the following function is an exact potential function:

$$\phi(z) = \alpha \sum_{i \in V} d(z_i, s_i) + (1 - \alpha) \sum_{(i,j) \in E} d(z_i, z_j).$$

To see why, note that: $\phi(z_i, z_{-i}) - \phi(z'_i, z_{-i}) =$

$$\begin{aligned} & \alpha \cdot d(z_i, s_i) + (1 - \alpha) \sum_{j \in N(i)} d(z_i, z_j) - \left(\alpha \cdot d(z'_i, s_i) + (1 - \alpha) \sum_{j \in N(i)} d(z'_i, z_j) \right) \\ & = c_i(z_i, z_{-i}) - c_i(z'_i, z_{-i}). \end{aligned}$$

Denote by x the global minimizer of the potential function and by y the optimal solution. By definition x is an equilibrium and it provides a 2-approximation to the optimal social cost since $c(x) \leq 2\phi(x) \leq 2\phi(y) \leq 2c(y)$. \square

3 The Case of Two Strategies: Battle of the Sexes on a Network

We begin by considering the subclass of instances in which the players only have two different strategies A and B . Without loss of generality we assume that $d(A, B) = 1$. We denote by $N_j(i)$ the set of i 's neighbors using strategy j and by \bar{s}_i the strategy opposite to s_i . When the strategy space contains only two strategies, a player's best response is to pick a strategy which is the weighted majority of its own preferred strategy and the strategies played by its neighbors. The next two observations formalize this statement and a similar statement regarding a player's strategy minimizing the social cost:

Observation 3.1 *The strategy s_i minimizes player i 's cost ($c_i(z)$) if:*

$$(1 - \alpha)N_{\bar{s}_i}(i) \leq \alpha + (1 - \alpha)N_{s_i}(i) \implies N_{\bar{s}_i}(i) \leq \frac{\alpha}{1 - \alpha} + N_{s_i}(i).$$

Observation 3.2 *The strategy s_i minimizes the social cost ($sc_i(z)$) if:*

$$2(1 - \alpha)N_{\bar{s}_i}(i) \leq \alpha + 2(1 - \alpha)N_{s_i}(i) \implies N_{\bar{s}_i}(i) \leq \frac{\alpha}{2(1 - \alpha)} + N_{s_i}(i).$$

We present a simple best response order that results in a Nash equilibrium after a linear number of best responses. We will later see how this order can be used to bound the PoS.

Lemma 3.3 *Starting from some initial strategy vector, the following best response order results in a Nash equilibrium:*

1. *While there exists a player that can reduce its cost by changing its strategy to A , let it do a best response. If there is no such player continue to the second step.*

2. While there exists a player that can reduce its cost by changing its strategy to B , let it do a best response.

Proof: To see why the resulting strategy vector is a Nash equilibrium, observe that after the first step, all nodes are either satisfied with their current strategy choice, or can benefit from changing their strategy to B . This property remains true after some of the nodes change their strategy to B since the fact that a node has more neighbors using strategy B can only reduce the attractiveness of switching to strategy A . Thus, at the end of the second step all nodes are satisfied. \square

Next, we characterize the values of α for which the price of stability is 1.

Claim 3.4 *If $\alpha \leq \frac{1}{2}$ or $\alpha = \frac{2}{3}$, then in any instance there exists an optimal solution which is also a Nash equilibrium.*

Proof: Let y be an optimal solution minimizing the potential function $\phi(\cdot)$. Assume towards a contradiction that it is not a Nash equilibrium. Let player i be a player that prefers to switch to a different strategy than y_i . Denote player i 's best response by x_i . By Observations 3.1 and 3.2, it is easy to see that if $y_i = s_i$, then the strategy minimizing player i 's cost is also s_i . Thus, we have that $y_i \neq s_i$ and $x_i = s_i$. If s_i is a minimizer of the social cost function then (s_i, y_{-i}) is also an optimal solution. This contradicts the assumption that y is a minimizer of the potential function $\phi(\cdot)$ since $\phi(s_i, y_{-i}) < \phi(y)$. Therefore by Observations 3.1 and 3.2 we have that $N_{\bar{s}_i}(i) < \frac{\alpha}{1-\alpha} + N_{s_i}(i)$ and $\frac{\alpha}{2(1-\alpha)} + N_{s_i}(i) < N_{\bar{s}_i}(i)$. By combining the two inequalities we get that:

$$\frac{\alpha}{2(1-\alpha)} + N_{s_i}(i) < N_{\bar{s}_i}(i) < \frac{\alpha}{1-\alpha} + N_{s_i}(i).$$

Since $N_{\bar{s}_i}(i)$ and $N_{s_i}(i)$ are both integers, this implies that there exists an integer k such that $\frac{\alpha}{2(1-\alpha)} < k < \frac{\alpha}{1-\alpha}$. This holds for $k = 1$ if $\frac{\alpha}{2(1-\alpha)} < 1 < \frac{\alpha}{1-\alpha}$ (implying $\frac{1}{2} < \alpha < \frac{2}{3}$) or for some $k > 1$ if $\frac{\alpha}{1-\alpha} - \frac{\alpha}{2(1-\alpha)} > 1$ (implying $\alpha > \frac{2}{3}$). Thus, for $\alpha \leq \frac{1}{2}$ or $\alpha = \frac{2}{3}$ any instance admits an optimal solution which is also a Nash equilibrium. \square

It is natural to ask what is the PoS for the values of α for which we know the optimal solution is not a Nash equilibrium. The following theorem provides an answer to this question by computing the ratio between the optimal solution and a Nash equilibrium obtained by performing the sequence of best responses Lemma 3.3 prescribes.

Theorem 3.5 *For $\frac{1}{2} < \alpha < 1$, $PoS \leq 2 \left\lceil \frac{\alpha}{1-\alpha} - 1 \right\rceil \cdot \frac{1-\alpha}{\alpha}$.*

Proof: Let x be the equilibrium achieved by the sequence described in Lemma 3.3 starting from an optimal solution y . Denote the strategy vector at the end of the first step by x_1 and at the end of the sequence by $x_2 = x$. We assume that a player performs a best response only when it can strictly decrease the cost by doing so, thus we only reason about the case where the player's best response is unique. In the following Lemma we bound the increase in the social cost inflicted by the players' unique best responses (the proof can be found below):

Lemma 3.6 *Let player i 's unique best response when the rest of the players play z_{-i} be x_i then:*

1. If $x_i = \bar{s}_i$ then $c(\bar{s}_i, z_{-i}) - c(s_i, z_{-i}) \leq \alpha - 2(1-\alpha) \left\lceil \frac{\alpha}{1-\alpha} + 1 \right\rceil$.

2. If $x_i = s_i$ then $c(s_i, z_{-i}) - c(\bar{s}_i, z_{-i}) \leq -\alpha + 2(1 - \alpha) \left\lceil \frac{\alpha}{1 - \alpha} - 1 \right\rceil$.

Notice that by statement (1) of Lemma 3.6 a node that changes its strategy to a strategy different than its preferred strategy can only reduce the social cost. Also, note that if a node changes its strategy in the first step to A and in the second step back to B its total contribution to the social cost is non-positive. The reason is that in one of these changes the player changed its strategy from s_i to \bar{s}_i and in the other from \bar{s}_i to s_i . The effect of these two changes on the social cost sums up to $2(1 - \alpha) \left(\left\lceil \frac{\alpha}{1 - \alpha} \right\rceil - \left\lfloor \frac{\alpha}{1 - \alpha} \right\rfloor - 2 \right) \leq 0$. Thus we can ignore such changes as well.

The only nodes that are capable of increasing the social cost by performing a best response are ones that play in the optimal solution a different strategy than their preferred strategy ($y_i \neq s_i$). By definition their number equals exactly $\sum_i d(y_i, s_i)$ as $d(y_i, s_i) = 1$ if $y_i \neq s_i$ and 0 otherwise. Statement (2) of Lemma 3.6 guarantees us that each of these nodes can increase the social cost by at most $-\alpha + 2(1 - \alpha) \left\lceil \frac{\alpha}{1 - \alpha} - 1 \right\rceil$. Thus, we get the following bound:

$$\begin{aligned} c(x) &\leq c(y) + \left(-\alpha + 2(1 - \alpha) \left\lceil \frac{\alpha}{1 - \alpha} - 1 \right\rceil \right) \sum_{i \in V} d(y_i, s_i) \\ &= 2(1 - \alpha) \sum_{(i,j) \in E} d(y_i, y_j) + 2(1 - \alpha) \left\lceil \frac{\alpha}{1 - \alpha} - 1 \right\rceil \sum_{i \in V} d(y_i, s_i). \end{aligned}$$

We are now ready to compute the bound on the PoS:

$$\begin{aligned} PoS &\leq \frac{2 \left\lceil \frac{\alpha}{1 - \alpha} - 1 \right\rceil \cdot (1 - \alpha) \sum_{i \in V} d(y_i, s_i) + 2(1 - \alpha) \sum_{(i,j) \in E} d(y_i, y_j)}{\alpha \sum_{i \in V} d(y_i, s_i) + 2(1 - \alpha) \sum_{(i,j) \in E} d(y_i, y_j)} \\ &\leq \frac{2 \left\lceil \frac{\alpha}{1 - \alpha} - 1 \right\rceil \cdot \frac{1 - \alpha}{\alpha} \cdot \left(\alpha \sum_{i \in V} d(y_i, s_i) + 2(1 - \alpha) \sum_{(i,j) \in E} d(y_i, y_j) \right)}{\alpha \sum_{i \in V} d(y_i, s_i) + 2(1 - \alpha) \sum_{(i,j) \in E} d(y_i, y_j)} \\ &\leq 2 \left\lceil \frac{\alpha}{1 - \alpha} - 1 \right\rceil \cdot \frac{1 - \alpha}{\alpha}. \end{aligned}$$

□

Proof of Lemma 3.6: Notice we are only considering the effect player i 's strategy has on the social cost, thus we have that: $c(\bar{s}_i, z_{-i}) - c(s_i, z_{-i}) = sc_i(\bar{s}_i, z_{-i}) - sc_i(s_i, z_{-i})$. Implying that $c(\bar{s}_i, z_{-i}) - c(s_i, z_{-i}) = \alpha + 2(1 - \alpha)(N_{s_i}(i) - N_{\bar{s}_i}(i))$. For proving statement (1) we observe that since \bar{s}_i is player i 's unique best response then: $N_{\bar{s}_i}(i) > \frac{\alpha}{1 - \alpha} + N_{s_i}(i)$ as player i has a strictly smaller cost for playing strategy \bar{s}_i than for playing strategy s_i . Since, $N_{\bar{s}_i}(i)$ and $N_{s_i}(i)$ are integers this implies that $N_{\bar{s}_i}(i) - N_{s_i}(i) \geq \left\lfloor \frac{\alpha}{1 - \alpha} + 1 \right\rfloor$ and the bound is achieved.

For proving statement (2), observe that $c(s_i, z_{-i}) - c(\bar{s}_i, z_{-i}) = -\alpha + 2(1 - \alpha)(N_{\bar{s}_i}(i) - N_{s_i}(i))$. Now since s_i is player i 's best response we have that: $N_{\bar{s}_i}(i) < \frac{\alpha}{1 - \alpha} + N_{s_i}(i)$ as player i 's best response is to use strategy s_i . Which similarly to the previous bound implies that $N_{\bar{s}_i}(i) - N_{s_i}(i) \leq \left\lceil \frac{\alpha}{1 - \alpha} - 1 \right\rceil$ as required. □

It is interesting to take a closer look at the upper bound on the PoS we computed (as we will see later this bound is tight). In Figure 2 we plot the upper bound on the PoS as a function of α . We can see that as α approaches 1 the PoS approaches 2 and also that for any $k \geq 2$, as ϵ approaches

0, the PoS of $\alpha = \frac{k-1}{k} + \epsilon$ also approaches to 2. This uncharacteristic saw-like behavior of the PoS originates from the fact that for every value of α the maximal PoS is achieved by a star graph. This is proved in the following claim.

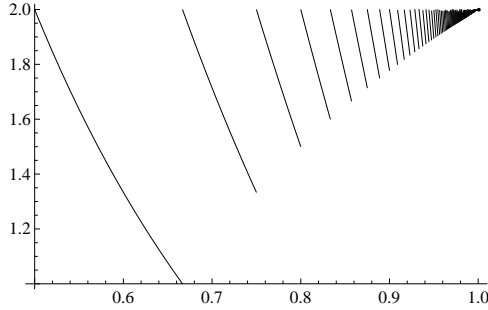


Figure 2: The tight upper bound on the PoS for two strategies as a function of α for the range $\frac{1}{2} < \alpha < 1$.

Claim 3.7 For any $\alpha > 1/2$, $\alpha \neq 2/3$ there exists an instance achieving a price of stability of $2 \left\lceil \frac{\alpha}{1-\alpha} - 1 \right\rceil \cdot \frac{1-\alpha}{\alpha}$.

Proof: Consider a star consisting of $\left\lceil \frac{\alpha}{1-\alpha} - 1 \right\rceil$ peripheral nodes that prefer strategy A and a central node that prefers strategy B . In the optimal solution the central node plays strategy A for a cost of α . However, this is not a Nash equilibrium since for playing strategy B it exhibits a cost of $(1-\alpha) \cdot \left\lceil \frac{\alpha}{1-\alpha} - 1 \right\rceil < \alpha$. Thus, the central node prefers to play its preferred strategy. \square

Corollary 3.8 As n goes to infinity the PoS for $\alpha = \frac{n-1}{n}$ approaches 2.

4 Richer Strategy Spaces

In the previous section we have seen that even when there are only two strategies in the game (the Battle of the Sexes on a network), for at least some values of $\alpha > \frac{1}{2}$, the PoS can be quite close to 2.

These bounds carry over to larger strategy spaces since an instance can always use only two strategies from the strategy space. However, for $\alpha \leq \frac{1}{2}$ the PoS for the Battle of the Sexes on a network is 1, so a natural question is how bad the PoS can be once we have more strategies in the space. This is the question we deal with for the rest of the paper.

4.1 Tree Metrics

We begin by considering the case in which the distance function on the strategy set is a tree metric, defined as the shortest-path metric among the nodes in a tree. (As such, tree metrics are a special case of graphic metrics, in which there is a graph on the elements of the space and the distance between every two elements is defined to be the length of the shortest path between them in the graph.) We show that if the distance function is a tree metric then the price of stability is 1 for any rational $\alpha \leq \frac{1}{2}$.

Denote by $C_i(z)$ and $SC_i(z)$ the strategies of player i that minimize $c_i(z) = \alpha \cdot d(z_i, s_i) + (1 - \alpha) \sum_{j \in N(i)} d(z_i, z_j)$ and $sc_i(z) = \alpha \cdot d(z_i, s_i) + 2(1 - \alpha) \sum_{j \in N(i)} d(z_i, z_j)$ respectively. We show that if for every player i the intersection of the two sets $C_i(z)$ and $SC_i(z)$ is always non-empty then the price of stability is 1:

Claim 4.1 *If for every player i and strategy vector z , $SC_i(z) \cap C_i(z) \neq \emptyset$, then $PoS = 1$.*

Proof: First, recall the potential function $\phi(\cdot)$ used in the proof of Claim 2.2, and consider an optimal solution y minimizing this potential function $\phi(\cdot)$. If y is also a Nash equilibrium then we are done. Else, there exists a node i that can strictly reduce its cost by performing a best response. By our assumption node i can do this by choosing a strategy $x_i \in SC_i(y) \cap C_i(y)$. The fact that $x_i \in SC_i(y)$ implies that the change in strategy of player i does not affect the social cost. Therefore, (x_i, y_{-i}) is also an optimal solution and $\phi(y) > \phi(x_i, y_{-i})$, in contradiction to the assumption that y is the optimal solution minimizing $\phi(\cdot)$. \square

Our goal now is to show that the conditions of Claim 4.1 hold for a tree metric. Our first step is relating the strategies that consist of a player's best response (or social cost minimizer) and the set of medians of a node-weighted tree.

Definition 4.2 (medians of a tree) *Given a tree T where the weight of node v is denoted $w(v)$, the set of T 's medians is $M(T) = \arg \min_{u \in V} \{\sum_{v \in V} w(v) \cdot d(u, v)\}$.*

Definition 4.3 *Given a network G , a tree metric T , a strategy vector z , a player i and non-negative integers q and r , we denote by $T_{i,z}(q, r)$ the tree T with the following node weights:*

$$w(v) = \begin{cases} q + r \cdot |\{j \in N(i) | z_j = v\}| & \text{for } v = s_i \\ r \cdot |\{j \in N(i) | z_j = v\}| & \text{for } v \neq s_i \end{cases}$$

Next we show that for $\alpha = \frac{a}{a+b}$, every player i and strategy vector z , it holds that $M(T_{i,z}(a, b)) = C_i(z)$. To see why, observe that by construction we have $M(T_{i,z}(a, b)) =$

$$\begin{aligned} & \arg \min_{u \in V} \{ (a + b \cdot |\{j \in N(i) | z_j = s_i\}|) \cdot d(u, s_i) + \sum_{v \neq s_i \in V} b \cdot |\{j \in N(i) | z_j = v\}| \cdot d(u, v) \} \\ & = \arg \min_{u \in V} \{ a \cdot d(u, s_i) + b \sum_{j \in N(i)} d(u, z_j) \} = C_i(z). \end{aligned}$$

Similarly, it is easy to show that $M(T_{i,z}(a, 2b)) = SC_i(z)$. Thus, to show that $SC_i(z) \cap C_i(z) \neq \emptyset$ it is sufficient to show that $T_{i,z}(a, b)$ and $T_{i,z}(a, 2b)$ share a median. This is done by using the following proposition:

Proposition 4.4 *Let T_1 and T_2 be two node-weighted trees with the same edges and nodes, then:*

- *If there exists a node v , such that for every $u \neq v \in V$, we have $w_1(u) = w_2(u)$ and for v we have $|w_1(v) - w_2(v)| = 1$, then T_1 and T_2 share a median.*
- *If T_1 and T_2 share a median, then it is also a median of their union $T_1 \cup T_2$. Where the union of T_1 and T_2 is a tree with the same nodes and edges where the weight of node v is $w_{1+2}(v) = w_1(v) + w_2(v)$.*

The proof of this proposition is based on combinatorial claims from Section 4.2 showing that a tree's medians and separators (defined below) coincide and establishing connections between the separators of different trees. Given Proposition 4.4 we can now show that $T_{i,z}(a, b)$ and $T_{i,z}(a, 2b)$ share a median:

Lemma 4.5 *For $\alpha = \frac{a}{a+b} \leq \frac{1}{2}$, every player i and strategy vector z , $M(T_{i,z}(a, b)) \cap M(T_{i,z}(a, 2b)) \neq \emptyset$.*

Proof: First, observe that by Proposition 4.4 we have that $T_{i,z}(0, 1)$ and $T_{i,z}(1, 1)$ share a median. As medians are invariant to scaling this implies that $T_{i,z}(0, b-a)$ and $T_{i,z}(a, a)$ also share a median. Next, by the second statement of Proposition 4.4 we have that any median they share is also a median of their union $T_{i,z}(a, b)$; let us denote this median by u . Since u is a median of $T_{i,z}(0, b)$ and $T_{i,z}(a, b)$, it is also a median of $T_i(a, 2b)$ by applying Proposition 4.4 again. Thus we have that u is a median of both $T_{i,z}(a, b)$ and $T_{i,z}(a, 2b)$ which concludes the proof. \square

Hence we have proven the following theorem:

Theorem 4.6 *If the distance metric is a tree metric then for rational $\alpha \leq \frac{1}{2}$, there exists an optimal solution which is also a Nash equilibrium (PoS=1).*

4.2 Combinatorial Properties of Medians and Separators in Trees

We now state and prove the combinatorial facts about medians and separators in trees that we used for the analysis above. This builds on the highly tractable structure of medians in trees developed in early work; see [6, 7] and the references therein.

Consider a tree where all nodes have integer weights, and denote the weight of the tree by $w(V) = \sum_{v \in V} w(v)$.

Definition 4.7 *A separator of a tree T is a node v such that the weight of each connected component of $T - v$ is at most $w(V)/2$.*

Claim 4.8 *A node u is a median of a tree T if and only if it is a separator of T .*

Proof: Let u be a median of a tree T , and assume towards a contradiction that it is not a separator; that is, there exists a component of $T - u$ of weight strictly greater than $w(V)/2$. Let v be the neighbor of u in this component. Consider locating the median at v . This reduces the distance to a total node weight of at least $w(V)/2 + 1/2$ by 1, and increases the distance to less than a total node weight of $w(V)/2$ by 1. Hence the sum of all distances decreases, and this contradicts the fact that u is a median. Thus, every median of a tree is also a separator.

To show that any separator is also a median, we show that for any two separators u_1 and u_2 it holds that $\sum_{v \in V} w(v) \cdot d(u_1, v) = \sum_{v \in V} w(v) \cdot d(u_2, v)$. Since, we know that there exists a median which is a separator this will imply that any separator is a median.

Denote by C the connected component of the graph $T - u_1 - u_2$ that includes the nodes on the path between u_1 and u_2 . If u_1 and u_2 are adjacent let $C = \emptyset$. Denote the connected component of $T - u_1 - C$ that includes u_2 by C_2 and the connected component of $T - u_2 - C$ that includes u_1 by C_1 . Note that by construction C, C_1, C_2 are disjoint and the union of their nodes equals V . Since u_1 is a separator it holds that $w(C_2) + w(C) \leq w(V)/2$. This in turn implies that $w(C_1) \geq w(V)/2$. Similarly, since u_2 is a separator it holds that $w(C_1) + w(C) \leq w(V)/2$. This in turn implies that

$w(C_2) \geq w(V)/2$. Therefore, it has to be the case that $w(C) = 0$, $w(C_1) = w(C_2) = w(V)/2$. We next show this implies that $\sum_{v \in V} w(v) \cdot d(u_1, v) = \sum_{v \in V} w(v) \cdot d(u_2, v)$. Observe that:

$$\begin{aligned} \sum_{v \in V} w(v) \cdot d(u_1, v) &= \sum_{v \in C_1} w(v) \cdot d(u_1, v) + \sum_{v \in C_2} w(v) \cdot (d(u_1, u_2) + d(u_2, v)) \\ &= \sum_{v \in C_1} w(v) \cdot d(u_1, v) + \sum_{v \in C_2} w(v) \cdot d(u_2, v) + w(C_2) \cdot d(u_1, u_2). \end{aligned}$$

and similarly that:

$$\sum_{v \in V} w(v) \cdot d(u_2, v) = \sum_{v \in C_2} w(v) \cdot d(u_2, v) + \sum_{v \in C_1} w(v) \cdot d(u_1, v) + w(C_1) \cdot d(u_1, u_2).$$

The claim follows as we have shown that $w(C_2) = w(C_1)$. \square

Next, we prove two claims relating the separators of different trees. We first show that if two trees differ only in the weight of a single node and the difference in weight of this node in the two trees is 1 – then they share a separator:

Claim 4.9 *Consider two trees T_1 and T_2 with the same set of edges and nodes that differ only in the weight of a single node v – such that $w_2(v) = w_1(v) + 1$. Then, T_1 and T_2 share a separator.*

Proof: We first handle the case where $w_1(V)$ is odd. Let u be a separator of T_1 , then, in this case the size of each component in $T_1 - u$ is at most $w_1(V)/2 - 1/2$. Thus for the same separator u in T_2 the size of each component is at most $w_1(V) + 1/2 = w_2(V)/2$. Therefore u is still a separator. Assume that $w_1(V)$ is even. This implies that $w_1(V) + 1$ is odd. Consider a separator u' of T_2 . Then the size of each connected component in $T_2 - u'$ is at most $w_2(V)/2 + 1/2$, since $w_1(V)$ is even, this implies that the weight of each connected component is bounded by $w_1(V)/2$ and therefore u' is also a separator of T_1 . \square

We show that if u is a separator of both T_1 and T_2 it is also the separator of their union:

Claim 4.10 *Every separator that T_1 and T_2 share is also a separator of $T_1 \cup T_2$.*

Proof: Let u be a separator of both T_1 and T_2 . This implies that in T_1 the weight of every connected component in $T_1 - u$ is at most $w_1(T_1)/2$ and in T_2 the weight of every connected component in $T_2 - u$ is at most $w_2(T_2)/2$. Hence, in $T_1 \cup T_2$ the weight of every connected component in $T_1 \cup T_2 - u$ is at most $w_1(T_1)/2 + w_2(T_2)/2 = w_{1+2}(T_1 \cup T_2)/2$. Thus, u is also a separator of $T_1 \cup T_2$. \square

The previous three claims establish the proof of Proposition 4.4. First Claim 4.8 shows that the set of medians and separators coincide. Then, Claim 4.8 Claim 4.10 prove the two statements of the proposition respectively.

4.3 Lower Bounds in Non-Tree Metrics

In some sense tree metrics are the largest class of metrics for which the optimal solution is always a Nash equilibrium for $\alpha \leq \frac{1}{2}$. The next example demonstrates that even when the distance metric is a simple cycle the PoS can be as high as $\frac{4}{3}$ for $\alpha = \frac{1}{2}$. In the following section, we give a family of more involved constructions that converge to the asymptotically tight lower bound of 2 on the price of stability.

Example 4.11 Consider a metric which is a cycle of size $3k+1$ for some integer $k \geq 1$. Let A, B, C be three strategies in this strategy space such that $d(A, B) = k$, $d(A, C) = k$ and $d(B, C) = k + 1$. Consider an instance where a node with a preferred strategy A is connected to a node with preferred strategy B and to another node with a preferred strategy of C . Also, assume that the node with preferred strategy B is part of a clique of size $3k$ in which all nodes prefer strategy B . Similarly, the node with preferred strategy C is part of a clique of size $3k$ in which all nodes prefer strategy C .

Consider the following equilibrium in which the nodes in both cliques play their preferred strategies. Then, the central node should play its preferred strategy. To see why, note that for playing strategy A its cost is $(1/2)2k = k$. On the other hand, its cost for playing any other strategy x which is between A and B (including B) on the cycle is

$$\frac{1}{2}(d(x, A) + d(x, B) + d(x, C)) = \frac{1}{2}\left(d(A, B) + \min\{d(x, A) + d(A, C), d(x, B) + d(B, C)\}\right).$$

which is greater than k . Similarly one can show that the central player prefers strategy A over any strategy x . The cost of the Nash equilibrium is $2k$. Note that this is the best Nash equilibrium since the cost of any solution in which some of the nodes in a clique play a strategy different than their preferred strategy is at least $3k$. In the optimal solution the central node should play strategy B (or C) for a total cost of $(1/2)k + 2 \cdot (1/2)(k + 1) = (3/2)k + 1$. Thus we have that the price of stability approaches $4/3$ as k approaches infinity.

Note that this lower bound of $4/3$ is achieved on an instance in which the lowest-cost Nash equilibrium and the socially optimal solution differ only in the strategy choice of a single player. We now show that in such cases, where the difference between these two solutions consists of the decision of just a single player, $4/3$ is the maximum possible price of stability for $\alpha = \frac{1}{2}$. More generally we show that $\frac{2}{2-\alpha}$ is the maximum possible price of stability for $\alpha < \frac{1}{2}$.

By the definition of the model, a player's strategy only affects its cost and the cost of its neighbors. Recall that we denote this part of the social cost by $sc_i(z)$: $sc_i(z) = \alpha \cdot d(s_i, z_i) + 2(1 - \alpha) \cdot \sum_{j \in N(i)} d(z_i, z_j)$. We now prove the following claim.

Claim 4.12 Let $\alpha \leq \frac{1}{2}$. Fix an optimal solution y which is not a Nash equilibrium and let player i be a player that can reduce its cost by playing x_i . Then $\frac{sc_i(x_i, y_{-i})}{sc_i(y)} < \frac{2}{2-\alpha}$.

Proof: Since x_i is player i 's best response, then $\alpha \cdot d(s_i, x_i) + (1 - \alpha) \sum_{j \in N(i)} d(x_i, y_j) < c_i(y)$. By rearranging the terms we get that $(1 - \alpha) \sum_{j \in N(i)} d(x_i, y_j) < c_i(y) - \alpha \cdot d(s_i, x_i)$. This in turn implies that

$$\begin{aligned} sc_i(x_i, y_{-i}) &= \alpha \cdot d(s_i, x_i) + 2(1 - \alpha) \sum_{j \in N(i)} d(x_i, y_j) < \alpha \cdot d(s_i, x_i) + 2(c_i(y) - \alpha \cdot d(s_i, x_i)) \\ &= 2c_i(y) - \alpha \cdot d(s_i, x_i). \end{aligned}$$

Thus, we have that

$$\frac{sc_i(x_i, y_{-i})}{sc_i(y)} < \frac{2c_i(y) - \alpha \cdot d(s_i, x_i)}{c_i(y) + (1 - \alpha) \cdot \sum_{j \in N(i)} d(y_i, y_j)}.$$

If $\sum_{j \in N(i)} d(y_i, y_j) \geq c_i(y)$ then $sc_i(y) \geq c_i(y) + (1 - \alpha)c_i(y) = (2 - \alpha)c_i(y)$ and the claim follows. Else, we show that $d(s_i, x_i) > d(s_i, y_i) - \sum_{j \in N(i)} d(y_i, y_j)$ in Lemma 4.13 below; this in turn implies

that

$$\begin{aligned} sc_i(x_i, y_{-i}) &< 2c_i(y) - \alpha \cdot d(s_i, x_i) \leq 2c_i(y) - \alpha(d(s_i, y_i) - \sum_{j \in N(i)} d(y_i, y_j)) \\ &= c_i(y) + \sum_{j \in N(i)} d(y_i, y_j). \end{aligned}$$

This brings us to the following bound:

$$\frac{sc_i(x_i, y_{-i})}{sc_i(y)} < \frac{c_i(y) + \sum_{j \in N(i)} d(y_i, y_j)}{c_i(y) + (1 - \alpha) \cdot \sum_{j \in N(i)} d(y_i, y_j)} = 1 + \frac{\alpha \sum_{j \in N(i)} d(y_i, y_j)}{c_i(y) + (1 - \alpha) \cdot \sum_{j \in N(i)} d(y_i, y_j)}.$$

Recall that by our assumption $c_i(y) > \sum_{j \in N(i)} d(y_i, y_j)$, this implies that $c_i(y) + (1 - \alpha) \cdot \sum_{j \in N(i)} d(y_i, y_j) > (2 - \alpha) \sum_{j \in N(i)} d(y_i, y_j)$ and the claim follows. \square

Lemma 4.13 *Let $\alpha \leq \frac{1}{2}$. Fix an optimal solution y which is not a Nash equilibrium and let player i be a player that can reduce its cost by playing x_i . Then: $d(s_i, x_i) > d(s_i, y_i) - \sum_{j \in N(i)} d(y_i, y_j)$.*

Proof: Note the following: by the triangle inequality for any player j it holds that: $d(x_i, y_j) \geq d(s_i, y_j) - d(s_i, x_i)$ and $d(s_i, y_j) \geq d(s_i, y_i) - d(y_i, y_j)$. By combining the two together we have that $d(x_i, y_j) \geq d(s_i, y_i) - d(y_i, y_j) - d(s_i, x_i)$. This gives us the following lower bound on $c_i(x_i, y_{-i})$:

$$\begin{aligned} c_i(x_i, y_{-i}) &= \alpha \cdot d(s_i, x_i) + (1 - \alpha) \sum_{j \in N(i)} d(x_i, y_j) \\ &\geq \alpha \cdot d(s_i, x_i) + (1 - \alpha) \sum_{j \in N(i)} \left(d(s_i, y_i) - d(y_i, y_j) - d(s_i, x_i) \right) \\ &= \alpha \cdot d(s_i, x_i) + (1 - \alpha) \cdot |N(i)| \cdot (d(s_i, y_i) - d(s_i, x_i)) - (1 - \alpha) \sum_{j \in N(i)} d(y_i, y_j). \end{aligned}$$

Since x_i minimizes player i 's cost it has to be the case that: $c_i(x_i, y_{-i}) < c_i(y)$. Thus the following inequality holds:

$$\begin{aligned} \alpha \cdot d(s_i, x_i) + (1 - \alpha)|N(i)| \cdot (d(s_i, y_i) - d(s_i, x_i)) - (1 - \alpha) \sum_{j \in N(i)} d(y_i, y_j) \\ < \alpha \cdot d(s_i, y_i) + (1 - \alpha) \sum_{j \in N(i)} d(y_i, y_j). \end{aligned}$$

After some rearranging we get that:

$$d(s_i, x_i) > d(s_i, y_i) - \frac{2(1 - \alpha)}{(1 - \alpha) \cdot |N(i)| - \alpha} \sum_{j \in N(i)} d(y_i, y_j)$$

which implies that the claim holds whenever $\frac{2(1 - \alpha)}{(1 - \alpha) \cdot |N(i)| - \alpha} \leq 1$. For $\alpha \leq \frac{1}{2}$, this later bound occurs for $|N(i)| \geq 3$.

The case of $|N(i)| \leq 2$ is handled separately and requires we use the assumption that y is an optimal solution. Denote i 's neighbors by j and k . Then:

$$\alpha \cdot d(s_i, y_j) + 2(1 - \alpha) \cdot d(y_j, y_k) \geq \alpha \cdot d(s_i, y_i) + 2(1 - \alpha)(d(y_i, y_j) + d(y_i, y_k)).$$

By the triangle inequality the previous inequality implies that $d(s_i, y_j) \geq d(s_i, y_i)$. When combining this with the fact that $d(x_i, y_j) \geq d(s_i, y_j) - d(s_i, x_i)$ we get that $d(x_i, y_j) \geq d(s_i, y_i) - d(s_i, x_i)$; similarly we get for k that $d(x_i, y_k) \geq d(s_i, y_i) - d(s_i, x_i)$. Therefore,

$$\begin{aligned} c_i(x_i, y_{-i}) &= \alpha \cdot d(s_i, x_i) + (1 - \alpha)(d(x_i, y_j) + d(x_i, y_k)) \\ &\geq \alpha \cdot d(s_i, x_i) + 2(1 - \alpha)(d(s_i, y_i) - d(s_i, x_i)). \end{aligned}$$

and since x_i is player i 's best response it has to be the case that:

$$\alpha \cdot d(s_i, x_i) + 2(1 - \alpha)(d(s_i, y_i) - d(s_i, x_i)) < \alpha \cdot d(s_i, y_i) + (1 - \alpha)(d(y_i, y_j) + d(y_i, y_k)).$$

After some rearranging we get that:

$$(2 - 3\alpha)d(s_i, y_i) - (1 - \alpha)(d(y_i, y_j) + d(y_i, y_k)) < (2 - 3\alpha)d(s_i, x_i).$$

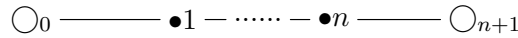
By dividing both sides of the inequality by $(2 - 3\alpha)$ we get that $d(s_i, x_i) > d(s_i, y_i) - \sum_{j \in N(i)} d(y_i, y_j)$ holds whenever $\frac{1 - \alpha}{2 - 3\alpha} \leq 1$. This completes the proof since for $\alpha \leq \frac{1}{2}$ it is always the case that $\frac{1 - \alpha}{2 - 3\alpha} \leq 1$. \square

5 Lower Bounds on the Price of Stability

At the end of the previous section, we saw that even in very simple non-tree metrics, the price of stability can be greater than 1. We now give a set of stronger lower bounds, using a more involved family of constructions. First we give an asymptotically tight lower bound of 2 when $\alpha = \frac{1}{2}$, and then we adapt this construction to give non-trivial lower bounds for all $0 < \alpha < \frac{1}{2}$.

5.1 Price of stability for $\alpha = \frac{1}{2}$

The following example illustrates that the PoS for $\alpha = \frac{1}{2}$ can be arbitrarily close to 2. The network we consider is composed of a path of n nodes and two cliques of size n^2 connected to each of the endpoints of the path. We assume that the preferred strategy of node i on the path is s_i , the preferred strategy of all nodes in the leftmost clique is s_0 , and the preferred strategy of all nodes in the rightmost clique is s_{n+1} . The following is a sketch of the network:



Since all the s_i 's are distinct we use them also as names for the different possible strategies. We define the following distance metric on these strategies: for $i > j$, we have $d(s_i, s_j) = 1 + (i - j - 1)\epsilon$. (When $i < j$, we simply use $d(s_i, s_j) = d(s_j, s_i)$.) In Claim 5.1 below we show that the best Nash equilibrium is the one in which all players play their preferred strategies. The cost of this equilibrium is $c(s) = \frac{1}{2} \cdot 2 \sum_{i=0}^n d(s_i, s_{i+1}) = n + 1$. On the other hand, consider the assignment in which for some node i all the nodes up till node i choose strategy s_0 and all the nodes from node $i + 1$ choose strategy s_{n+1} . The cost of such assignment is $\frac{1}{2}(n + 2 + O(\epsilon))$; therefore as n goes to infinity and ϵ to zero the PoS goes to 2.

Claim 5.1 *In the previously defined instance the best Nash equilibrium is for each player to play its preferred strategy.*

Proof: We show that the cost of any other equilibrium is at least $\frac{1}{2}n^2$. Consider an equilibrium in which there exists at least one node i that plays strategy s_j such that $j < i$. By the following lemma (which we prove below) this implies that node i 's neighbors play strategies s_a and s_b such that $a, b < i$ or $a, b > i$.

Lemma 5.2 *Let s_a and s_b be the strategies played by player i 's neighbors such that $a \leq b$. If $a \leq i \leq b$, then player i 's best response is to play strategy s_i .*

Observe that for i 's best response to be strategy s_j it clearly has to be the case that $a, b < i$. By applying Lemma 5.2 repeatedly, we get that in this equilibrium, all nodes $k > i$ play strategies k' such that $k' < k$. This includes the n^{th} node of the path, implying that its right neighbor which belongs to the right clique plays strategy $s_{k'}$ such that $k' < n + 1$. The cost incurred by the nodes in the clique in any such equilibrium is at least n^2 : indeed, if r nodes play a strategy different than their preferred one, they pay a cost of at least $\frac{1}{2}r$ and the remaining $n^2 - r$ pay a cost of at least $n^2 - r$ for the edges connecting them to one of the r nodes playing a strategy different than its preferred strategy. To complete the proof, one can use an analogous argument for the case in which there exists an equilibrium in which there is a node i playing strategy s_j such that $j > i$. \square

Proof of Lemma 5.2: We first note that by the definition of the metric it is never in i 's best interest to play a strategy s_j such that $j \neq i, a, b$: playing such a strategy has cost $3 + O(\epsilon)$ whereas the cost of playing the preferred strategy is $2 + O(\epsilon)$.

Observe that player i prefers to play strategy s_i over strategy $s_a \neq s_i$ whenever $d(s_i, s_a) + d(s_i, s_b) < d(s_i, s_a) + d(s_a, s_b)$ implying that $d(s_i, s_b) < d(s_a, s_b)$. This condition holds according to our assumptions since $1 + (b - i - 1)\epsilon = d(s_i, s_b) < d(s_a, s_b) = 1 + (b - a - 1)\epsilon$. For the same reason, player i prefers strategy s_i over $s_b \neq s_i$ since $1 + (i - a - 1)\epsilon = d(s_i, s_a) < d(s_a, s_b) = 1 + (b - a - 1)\epsilon$ under the lemma's assumptions. \square

5.2 Extension for $\alpha < \frac{1}{2}$

We extend the construction in Section 5.1 to $0 < \alpha < \frac{1}{2}$ by defining the following metric: for $i > j$, let $d(s_i, s_j) = 1 + (i - j - 1) \left(\frac{1 - 2\alpha}{1 - \alpha} (1 + \epsilon) \right)$. We consider the same family of instances defined in Section 5.1 except for the fact that we increase the size of the cliques to n^2/α .

Next, we show that Lemma 5.2 also holds for this newly defined family of instances. This fact together with the observation that the proof of Claim 5.1 carries over with only minor modifications, imply that in the best Nash equilibrium of the previously defined family of instances all players play their preferred strategies.

Lemma 5.3 *Let s_a and s_b be the strategies played by player i 's neighbors such that $a \leq b$. If $a \leq i \leq b$, then player i 's best response is to play strategy s_i .*

Proof: For proving this claim it will be easier to use an equivalent distance function which is: $d(s_i, s_j) = 1 + (|i - j| - 1) \left(\frac{1 - 2\alpha}{1 - \alpha} (1 + \epsilon) \right)$. Also, we only present the proof for the case that $a < i < b$, the proof for the rest of the cases is very similar. We first show that player i prefers to play strategy s_i over playing any strategy s_j such that $j \neq a, b$. The cost of player i for playing s_i

is:

$$\begin{aligned}
& (1 - \alpha) \left(2 + (|i - a| - 1 + |b - i| - 1) \cdot \frac{1 - 2\alpha}{1 - \alpha} (1 + \epsilon) \right) \\
& = 2(1 - \alpha) + (|b - a| - 2) \cdot (1 - 2\alpha)(1 + \epsilon) \\
& \leq 1 + (|b - a| - 1) \cdot (1 - 2\alpha)(1 + \epsilon).
\end{aligned}$$

The cost of playing strategy s_j such that $j \neq a, b$ is:

$$\begin{aligned}
& \alpha(1 + (|i - j| - 1) \frac{1 - 2\alpha}{1 - \alpha} (1 + \epsilon)) + (1 - \alpha) \left(2 + (|j - a| - 1 + |j - b| - 1) \cdot \frac{1 - 2\alpha}{1 - \alpha} (1 + \epsilon) \right) \\
& \geq 2 - \alpha + (|b - a| - 2) \cdot (1 - 2\alpha)(1 + \epsilon).
\end{aligned}$$

The last transition is due to the fact that $|j - a| + |j - b| \geq |b - a|$. Thus we conclude that player i 's best response can only be s_i, s_a or s_b . Next we consider strategies s_a and s_b . By writing the cost of playing each one of these strategies it is easy to see that these costs are greater than the costs for playing s_i .

The cost of playing strategy $s_a \neq s_i$ is:

$$\begin{aligned}
& \alpha(1 + (|i - a| - 1) \frac{1 - 2\alpha}{1 - \alpha} (1 + \epsilon)) + (1 - \alpha) \left(1 + (|b - a| - 1) \cdot \frac{1 - 2\alpha}{1 - \alpha} (1 + \epsilon) \right) \\
& = 1 + \left(\frac{\alpha}{1 - \alpha} (|i - a| - 1) + |b - a| - 1 \right) \cdot (1 - 2\alpha)(1 + \epsilon).
\end{aligned}$$

The cost of playing strategy $s_b \neq s_i$ is:

$$\begin{aligned}
& \alpha(1 + (|b - i| - 1) \frac{1 - 2\alpha}{1 - \alpha} (1 + \epsilon)) + (1 - \alpha) \left(1 + (|b - a| - 1) \cdot \frac{1 - 2\alpha}{1 - \alpha} (1 + \epsilon) \right) \\
& = 1 + \left(\frac{\alpha}{1 - \alpha} (|b - i| - 1) + |b - a| - 1 \right) \cdot (1 - 2\alpha)(1 + \epsilon).
\end{aligned}$$

This concludes the proof as we have shown that under the assumptions of the claim, player i 's best response is to play its preferred strategy. \square

To get a lower bound on the PoS, we would like to simulate the technique we used in the proof for $\alpha = \frac{1}{2}$ to compare between the cost of two solutions: (i) a solution in which there is exactly one edge such that its two endpoints play different strategies, and (ii) the best Nash equilibrium. We refer to the first solution as a *bi-consensus solution*. The cost of the best bi-consensus solution is an upper bound on the optimal solution, and hence computing the ratio between the best Nash equilibrium and best bi-consensus solution gives a lower bound on the PoS achieved by instances defined above.

Observe that in the best bi-consensus solution nodes $i \in [1 \dots \lfloor n/2 \rfloor]$ play strategy s_0 and nodes $i \in [\lfloor n/2 \rfloor + 1 \dots n]$ play strategy s_{n+1} . The cost of this solution b is the following:

$$\begin{aligned}
c(b) &= \alpha \left(\sum_{i=1}^{\lfloor n/2 \rfloor} (1 + (i - 1) \left(\frac{1 - 2\alpha}{1 - \alpha} (1 + \epsilon) \right)) + \sum_{i=\lfloor n/2 \rfloor + 1}^n (1 + (n - i) \left(\frac{1 - 2\alpha}{1 - \alpha} (1 + \epsilon) \right)) \right) \\
&\quad + 2(1 - \alpha)(1 + (n + 1 - 0 - 1) \left(\frac{1 - 2\alpha}{1 - \alpha} (1 + \epsilon) \right)) \\
&\leq \alpha \cdot n + \alpha \frac{1}{4} (n - 1)^2 \cdot \frac{1 - 2\alpha}{1 - \alpha} (1 + \epsilon) + 2(1 - \alpha) + 2n(1 - 2\alpha)(1 + \epsilon).
\end{aligned}$$

Where the last transition is due to the fact that:

$$\begin{aligned} \sum_{i=1}^{\lfloor n/2 \rfloor} (i-1) + \sum_{i=\lfloor n/2 \rfloor+1}^n (n-i) &\leq \sum_{i=1}^{\lfloor n/2 \rfloor-1} i + \sum_{i=1}^{\lfloor n/2 \rfloor} i = (\lfloor n/2 \rfloor - 1) \cdot \lfloor n/2 \rfloor + \lfloor n/2 \rfloor \\ &= \lfloor n/2 \rfloor^2 = \frac{1}{4}(n-1)^2. \end{aligned}$$

The cost of the best Nash equilibrium x is simply $c(x) = 2(1-\alpha)(n+1)$. Interestingly, once we pick $\alpha < \frac{1}{2}$ the maximum PoS for this example is obtained for an intermediate value of n . By taking the first derivative of the function $\frac{c(x)}{c(b)}$ with respect to n and comparing it to 0, we get that the maximum PoS is achieved for $n = \lceil \frac{1-2\alpha-2\sqrt{2-7\alpha+6\alpha^2}}{-1+2\alpha} \rceil$ or $n = \lfloor \frac{1-2\alpha-2\sqrt{2-7\alpha+6\alpha^2}}{-1+2\alpha} \rfloor$.

In Figure 3 we plot the lower bound on the PoS that can be achieved by this example (solid line). As one might expect, as α approaches $\frac{1}{2}$ the PoS approaches 2. For comparison, we also plot (via the dashed line) the lower bound of $\frac{2}{2-\alpha}$ on the PoS that we computed in Section 4.3 for instances in which the best Nash equilibrium differs from an optimal solution only in the strategy played by a single player. Interestingly, each of the two constructions offers a better lower bound on the PoS for a different interval of α . We cannot rule out that the maximum of these two constructions could match the best achievable upper bound on the PoS for all α , but there may also be other constructions that can achieve higher lower bounds on the PoS for some ranges of α .

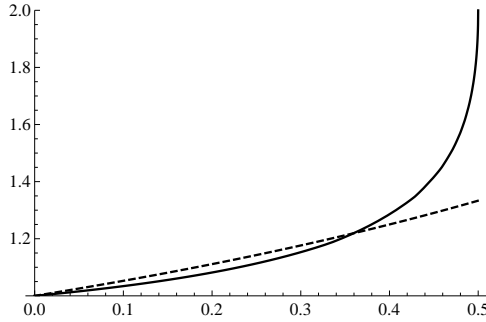


Figure 3: The PoS achievable by a path (solid) and by a single strategic node (dashed).

6 The Anchored Preference Game

In this final section, we consider the following generalization of discrete preference games, with $\alpha = \frac{1}{2}$. We assume that nodes are partitioned into two types: F and S . Nodes in F are *fixed nodes* that always play their preferred strategy. The cost that a node $i \in F$ incurs for playing strategy s_i is 0. The nodes in S are *strategic nodes* that have no preferred strategy, so the cost of a node $i \in S$ is purely the term $\sum_{j \in N(i)} d(z_i, z_j)$. We use $F(i)$ and $S(i)$ to denote the fixed and strategic neighbors of node i respectively. Only the strategic nodes choose strategies, and only they take part in the definition of the equilibrium. The social cost for this model is:

$$c(z) = \sum_{\substack{(i,j) \in E; \\ i \in S; j \in F}} d(z_i, s_j) + 2 \sum_{\substack{(i,j) \in E; \\ i,j \in S}} d(z_i, z_j).$$

As noted in the introduction, this model, which we call an *anchored preference game*, generalizes the model from the previous sections because of the following reduction: given an instance of a

discrete preference game, we can take each node i in the instance and make it a strategic node in the generalized instance by eliminating its preferred strategy s_i , and adding a new fixed node i' to the instance that has preferred strategy s_i and is connected only to node i by an edge (i, i') . In this way i' , which is non-strategic, plays the role of i 's preferred strategy.

Given that discrete preference games are (due to the reduction) a special case of anchored preference games with at most one fixed neighbor per node, it becomes natural to study the price of stability of anchored preference games parametrized by k , the maximum number of fixed nodes adjacent to any strategic nodes. In the next claim we generalize our result on tree metrics by showing that the price of stability for anchored preference games with tree metrics is 1 provided $k \leq 2$. The proof is essentially a generalization of Theorem 4.6.

Claim 6.1 *If the distance function is a tree metric and $k \leq 2$, then the optimal solution is also a Nash equilibrium.*

Proof: Similar to the proof of Theorem 4.6, we define $C_i(z)$ and $SC_i(z)$ to be the strategies of player i that minimize $c_i(z) = \sum_{j \in F(i)} d(z_i, s_j) + \sum_{j \in S(i)} d(z_i, z_j)$ and $sc_i(z) = \sum_{j \in F(i)} d(z_i, s_j) + 2 \sum_{j \in S(i)} d(z_i, z_j)$ respectively. Our goal now is to show that the set of best responses $C(i)$ and the set of local improvements $SC(i)$ always intersect. By Claim 4.1 this implies that the PoS is always 1.

Observe that $C_i(z) \cap SC_i(z) \neq \emptyset$ when the number of fixed neighbors node i has is 0 (by definition) and when the number of fixed neighbors node i has is 1 (by Lemma 4.5). Thus, we can assume that player i has exactly 2 fixed neighbors. We denote their preferred strategies by s_1 and s_2 . Just as in the proof of Theorem 4.6 we show that $C_i(z)$ and $SC_i(z)$ coincide with the set of medians for some trees we define next. For this purpose we define the tree $T_{i,z}(q_1, q_2, r)$:

Definition 6.2 *Given a metric tree T , a strategy vector z and a strategic player i we denote by $T_{i,z}(q_1, q_2, r)$ the tree with the same nodes and edges as T and the following node weights:*

$$w(v) = r \cdot |\{j \in S(i) | z_j = v\}| + \sum_{\{j | j \in \{1,2\}, s_j = v\}} q_j.$$

It is not hard to see that for a player i with two fixed neighbors $C(i) = M(T_{i,z}(1, 1, 1))$ and $SC(i) = M(T_{i,z}(1, 1, 2))$. Recall that by Claim 4.8 the set of medians and separators of trees coincide. Therefore, we should show that the set of separators of $T_{i,z}(1, 1, 1)$ and $T_{i,z}(1, 1, 2)$ intersect. By Claim 4.9 we already have that $T_{i,z}(2, 1, 2)$ and $T_{i,z}(1, 1, 2)$ share a separator u . Since separators are invariant to scaling $T_{i,z}(1, 1, 1)$ and $T_{i,z}(2, 2, 2)$ have the same separators. Thus, to complete the proof we need to show that u is a separator of $T_{i,z}(2, 2, 2)$.

Observe that the sum of weights in $T_{i,z}(2, 1, 2)$ is odd; this implies that the size of every connected component of $T_{i,z}(2, 1, 2) - u$ is at most $w(T_{i,z}(2, 1, 2))/2 - 1/2$. After increasing the weight of a single node by exactly 1 the weight of each component is at most $w(T_{i,z}(2, 1, 2))/2 + 1/2 = w(T_{i,z}(2, 2, 2))/2$. Thus, u is also a separator of $T_{i,z}(2, 2, 2)$ implying that $T_{i,z}(1, 1, 1)$ and $T_{i,z}(1, 1, 2)$ indeed share a separator. \square

We use Claim 6.1 to show that for tree metrics and $k > 2$ the PoS is bounded by $\frac{2(k-1)}{k}$. The proof is again given in the appendix.

Claim 6.3 *If the distance function is a tree metric and each strategic node has at most k fixed neighbors, then $PoS \leq \frac{2(k-1)}{k}$.*

Proof: Let y be the optimal solution for a graph G with a total of n nodes ($|F| + |S| = n$), and let $c_G(y)$ denote the social cost of this solution. By duplicating fixed nodes if necessary, we can assume without loss of generality that each fixed node has at most one strategic neighbor. Denote by x the best Nash equilibrium for this instance.

We construct a new graph G' from G in which each strategic node will have at most two fixed neighbors. For each strategic node i in G , we pick the $k - 2$ fixed neighbors of i with preferred strategies which are closest to y_i in distance. We replace each such fixed neighbor j with a clique of size $\lceil 2c_G(y) \rceil$ consisting of strategic nodes, where the preferred strategy of each node in the clique is s_j . We connect node i to just one of the nodes in the clique. Since j had i as its only strategic neighbor in G , at the end of this process there is a single node connected to each clique.

Let $c_{G'}(\cdot)$ be the social cost function for G' . Since each strategic node has at most two fixed neighbors in G' , Claim 6.1 implies that there exists an optimal solution y' for G' which is also a Nash equilibrium in G' . Observe that in any optimal solution of G' all the nodes in the new cliques we created play their preferred strategies, since the cost of any other strategy vector is at least $\lceil 2c_G(y) \rceil$ and the cost of the optimal solution in G' cannot be larger than twice the cost of the optimal solution in G .

We now define two further solutions. Let y'_G be a strategy vector for G in which the nodes common to G and G' play their strategies in y' , and for each fixed node j replaced by a clique, j plays its preferred strategy. (Above we argued that all the nodes in the clique replacing j will play this same preferred strategy in G' .) We observe that y'_G is a well-defined solution to the anchored preference game on G , since all fixed nodes play their preferred strategy, and y'_G is an equilibrium in G , since y' is an equilibrium in G' . Therefore we have that $c_G(x) \leq c_G(y'_G)$. Now, since $c_G(z) \leq c_{G'}(z)$ and by construction y' and y'_G are practically the same, we have that $c_G(y'_G) \leq c_{G'}(y')$.

Let $y_{G'}$ denote the strategy vector for G' that agrees with y on the nodes of G , and in which the nodes of each clique in G' play the preferred strategy of the fixed node j they replaced. Since y' is an optimal solution for G' we have that $c_{G'}(y') \leq c_{G'}(y_{G'})$. Thus, we have that $c_G(x) \leq c_{G'}(y_{G'})$.

For simplicity of presentation we define $F_i = \sum_{j \in F(i)} d(y_i, s_j)$ to be the cost of player i associated with its fixed neighbors in the graph G . Since in G' the cost of the $k - 2$ fixed neighbors which are closest to y_i was doubled, it is now at most $(1 + \frac{k-2}{k})F_i$.

$$PoS(G) = \frac{c_G(x)}{c_G(y)} \leq \frac{c_{G'}(y_{G'})}{c_{G'}(y')} \leq \frac{\sum_{i \in S} \sum_{j \in S(i)} d(y_i, y_j) + (1 + \frac{k-2}{k}) \sum_{i \in S} F_i}{\sum_{i \in S} \sum_{j \in S(i)} d(y_i, y_j) + \sum_{i \in S} F_i} \leq \frac{2k-2}{k}.$$

□

To see that the bound for $k > 2$ is tight, consider a star in which the central node i is connected to k fixed nodes that prefer strategy A . i is also connected to $k - 1$ strategic nodes that are connected to one another (forming a clique of size $k - 1$) and each one is connected to k fixed nodes that prefer strategy B . Observe that in the best Nash equilibrium node i plays strategy A and the rest of the strategic nodes play strategy B . The social cost of this equilibrium is $2(k - 1)$. However, in the optimal solution node i also plays strategy B which reduces the social cost to k .

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