

# Solvability of orbit-finite systems of linear equations

Arka Ghosh\*  
University of Warsaw  
Poland

Piotr Hofman†  
University of Warsaw  
Poland

Sławomir Lasota‡  
University of Warsaw  
Poland

## Abstract

We study orbit-finite systems of linear equations, in the setting of sets with atoms. Our principal contribution is a decision procedure for solvability of such systems. The procedure works for every field (and even commutative ring) under mild effectiveness assumptions, and reduces a given orbit-finite system to a number of finite ones: exponentially many in general, but polynomially many when the atom dimension of input systems is fixed. Towards obtaining the procedure we push further the theory of vector spaces generated by orbit-finite sets, and show that each such vector space admits an orbit-finite basis. This fundamental property is a key tool in our development, but should be also of wider interest.

**CCS Concepts:** • Theory of computation → Concurrency; Logic and verification; Verification by model checking.

**Keywords:** linear equations, sets with atoms, orbit-finite sets

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## 1 Introduction

Applications of linear algebra, and in particular of systems of linear equations, are ubiquitous in computer science (see e.g. [6, 7, 29]). In this paper, motivated by recent and potential future applications to analysis of data-enriched models [3, 11, 13, 15], we augment systems of linear equations with *atoms* [1, 28] (also called data values) thus shifting from finite to orbit-finite systems. The infinite sets that we study

are constructed using atoms which can only be accessed in a very limited way, namely can only be tested for equality.

Fix a countably infinite set  $\text{ATOMS} = \{1, 2, 3, \dots\}$ , whose elements are called *atoms*, assuming that the only operations on atoms are (dis)equality tests. As an example, consider pairs of distinct atoms  $C = \{\alpha\beta \in \text{ATOMS}^2 \mid \alpha \neq \beta\}$  as unknowns (for succinctness, here and in the sequel we write ordered pairs  $(\alpha, \beta)$  of atoms as  $\alpha\beta$ , and likewise for triples), and the infinite system of equations

$$\alpha\beta - 2\beta\gamma + \gamma\alpha = 1 \quad (\alpha, \beta, \gamma \in \text{ATOMS}, \alpha \neq \beta \neq \gamma \neq \alpha).$$

The system is finitely described by the above formula using only (dis)equalities between atoms, and therefore is invariant under all permutations of atoms. Furthermore, up to permutation of atoms the system consists of just one equation – it is one *orbit*; in the sequel we consider orbit-finite systems (finite unions of orbits). Each unknown  $\alpha\beta \in C$  is determined (*supported*) by 2 atoms (its *atom dimension* is 2) while each equation by 3 atoms, therefore the atom dimension of the whole example system is 3. The example equations are finite, but need not to be so in general. Our primary goal is to algorithmically test if such a system has a solution, that is a rational assignment  $\mathbf{x} : C \rightarrow \mathbb{Q}$ , or maybe an integer assignment  $\mathbf{x} : C \rightarrow \mathbb{Z}$ , that satisfies all the equations, i.e.,

$$\mathbf{x}(\alpha\beta) - 2\mathbf{x}(\beta\gamma) + \mathbf{x}(\gamma\alpha) = 1$$

for every  $\alpha\beta\gamma \in \text{ATOMS}^3$  such that  $\alpha \neq \beta \neq \gamma \neq \alpha$ .

We use the language of linear algebra. For instance, a solution is a vector over  $C$  (belongs to the vector space generated by  $C$ ), and the above system may be presented as an infinite matrix plus the infinite right-hand side vector:

$$\begin{array}{cccccccc} & 12 & 13 & 23 & 34 & 31 & 41 & 42 & \dots \\ 123 & \left[ \begin{array}{cccccccc} 1 & 0 & -2 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & -2 & 0 & 0 & 1 & \dots \\ 0 & 1 & 0 & -2 & 0 & 1 & 0 & \dots \\ -2 & 0 & 1 & 0 & 1 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \right] & \left[ \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{array} \right] \\ 234 & & & & & & & & \\ 134 & & & & & & & & \\ 312 & & & & & & & & \\ \vdots & & & & & & & & \end{array}$$

The columns of the matrix are indexed by pairs  $\alpha\beta \in C$ , and rows by triples  $\alpha\beta\gamma \in \text{ATOMS}^3$  where  $\alpha \neq \beta \neq \gamma \neq \alpha$ .

**Contribution.** As the main contribution, we provide an algorithm for solvability of orbit-finite systems of linear equations. More formally, our algorithm accepts as input a system consisting of an orbit-finite matrix  $\mathbf{A}$  and a right-hand side vector  $\mathbf{t}$ , both *finitely-supported* (i.e., definable using finitely many fixed atoms, hence finitely presentable). The

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algorithm checks whether the given system admits a solution which is also finitely-supported (hence also finitely presentable).

The coefficients in  $A$  and  $t$ , as well as in the solutions, are assumed to come from an arbitrary fixed commutative ring  $(\mathbb{K}, 0, 1, +, \cdot)$  which is assumed to be *effective*: its elements are finitely representable; equality is decidable for these representations; ring operations (addition, subtraction, multiplication) are computable using the representations; and solvability of finite systems over  $\mathbb{K}$  is decidable. Examples abound: the rational field  $\mathbb{Q}$ ; the integer ring  $\mathbb{Z}$ ; finite commutative rings; the field of algebraic numbers; the field of complex numbers.

In brief, the algorithm computes a number of finite systems of linear equations over  $\mathbb{K}$  and answers positively exactly when all these systems are solvable. The number of finite systems and their sizes are exponential in general; however, once the atom dimension of the input system is fixed, the algorithm computes only polynomially many finite systems of polynomial size. In particular, for fixed atom dimension we obtain polynomial time procedures for solvability over  $\mathbb{Q}$  or  $\mathbb{Z}$ . On the way we also provide an algorithm for *finitary* solvability where one only seeks solutions which assign zero to almost all unknowns.

On the mathematical level, we push further the theory of orbit-finitely generated vector spaces initiated in [3], in order to obtain a key tool for our algorithmic considerations: we show that each orbit-finitely generated vector space admits an orbit-finite basis. We believe that this finding is of independent wider interest.

**Outline.** After preliminaries on sets with atoms, in Section 3 we introduce orbit-finitely generated vector spaces and Orbit-finite Basis Theorem, and in Section 4 we introduce orbit-finite systems of linear equations and formulate the main result. The remaining sections contain the proofs. Some missing parts thereof are to be found in the full version [10].

**Motivations.** The main motivation for this work comes from past and potential future applications in analysis of computation models enriched with data, including different kinds of automata over infinite alphabets [2, 9, 27]. For example, while studying Parikh images [12] of register automata [9] or register context-free grammars [1, 2, 5], one works with nonnegative integer vectors of the form  $\Sigma \rightarrow \mathbb{N}$ , where  $\Sigma$  is an infinite alphabet. Another potential application of orbit-finite systems of linear equations is the recently proposed algorithm for equivalence of weighted register automata, including unambiguous register automata [3].

Numerous applications arise in data-enriched Petri nets [20, 22] (or vector addition systems [14]), an extension of classical Petri nets [29] where tokens carry atoms (data values) that are compared by transitions. In case when tokens are restricted to carry single data values (atom dimension 1) one obtains a well structured transition system and hence

standard decision problems like coverability or boundedness are decidable [14, 20, 22, 23]. Status of the reachability problem is unknown; since integer linear equations form a crucial component in a decision procedure for reachability of classical Petri nets [18, 19, 25, 26], lifting the procedure to data-enriched setting would require solving orbit-finite systems of integer linear equations. In case when tokens may carry tuples of atoms (arbitrary atom dimension) all the standard problems are undecidable [20]. Decidability may be regained by resorting to relaxations: continuous semantics [11] allowing for fractional executions of transitions, or so-called integer semantics [15] dropping non-negativeness restriction on configurations. Both these results have been obtained by reduction to solving certain systems of linear equations.

**State of the art.** Our results generalise, or are closely related to, some earlier partial results [15–17].

Systems of linear equations in [15] have row indexes of atom dimension 1 in which case finitary solvability is in P over  $\mathbb{Z}$  or  $\mathbb{Q}$ , and in NP over  $\mathbb{N}$ . In a more general but still restricted case studied in [16], where in particular all row indexes are assumed to have the same atom dimension, finitary solvability is still in P over  $\mathbb{Z}$  or  $\mathbb{Q}$ , but in EXPTIME over  $\mathbb{N}$ , both for fixed atom dimension. Columns of a matrix are assumed to be finitary in [15, 16]. Systems in another related work [17] are over a finite field, contain only finite equations, and are studied as a special case of orbit-finite constraint satisfaction problems; furthermore, solutions sought are not restricted to be finitely-supported.

Additionally, the work [13] investigates system of linear equations, in atom dimension 1, over ordered atoms: solvability is in P over  $\mathbb{Z}$  or  $\mathbb{Q}$ , but equivalent to VAS reachability (and hence ACKERMANN-complete [8, 21, 24]) over  $\mathbb{N}$ .

Our Orbit-Finite Basis Theorem is a follow-up and strengthening of Theorem VI.4 in [3]: each orbit-finitely generated vector space has an orbit-finite *spanning* set.

## 2 Preliminaries on sets with atoms

Our definitions rely on basic notions and results of the theory of *sets with atoms* [1], also known as nominal sets [28]. We only work with *equality atoms* which have no additional structure except for the equality.

We fix a countably infinite set  $\text{ATOMS} = \{1, 2, 3, \dots\}$ , whose elements we call *atoms*. We reserve Greek letters  $\alpha, \beta, \gamma, \dots$  to range over atoms. Informally speaking, a set with atoms is a set that can have atoms, or other sets with atoms, as elements. Formally, we define the universe of sets with atoms by a suitably adapted cumulative hierarchy of sets, by transfinite induction: the only set of rank 0 is the empty set; and for a cardinal  $i$ , a set of rank  $i$  may contain, as elements, sets of rank smaller than  $i$  as well as atoms. In particular, nonempty subsets  $X \subseteq \text{ATOMS}$  have rank 1.

The group  $\text{AUT}$  of all permutations of  $\text{ATOMS}$ , called in this paper *atom automorphisms*, acts on sets with atoms by

consistently renaming all atoms in a given set. Formally, by another transfinite induction, for  $\pi \in \text{AUT}$  we define  $\pi(X) = \{\pi(x) \mid x \in X\}$ . Via standard set-theoretic encodings of pairs or finite sequences we obtain, in particular, the pointwise action on pairs  $\pi(x, y) = (\pi(x), \pi(y))$ , and likewise on finite sequences. Relations and functions from  $X$  to  $Y$  are considered as subsets of  $X \times Y$ .

We restrict to sets with atoms that only depend on finitely many atoms, in the following sense. For  $S \subseteq \text{ATOMS}$ , let  $\text{AUT}_S = \{\pi \in \text{AUT} \mid \pi(\alpha) = \alpha \text{ for every } \alpha \in S\}$  be the set of atom automorphisms that *fix*  $S$ . We call elements of  $\text{AUT}_S$  *S-atom automorphisms*. A *support* of  $x$  is any finite set  $S \subseteq_{\text{fin}} \text{ATOMS}$  (we use the symbol  $\subseteq_{\text{fin}}$  for finite subsets) such that for all  $\pi \in \text{AUT}_S$  it holds  $\pi(x) = x$ . In this case we also say that  $x$  is *S-supported*. As a special case, a function  $f$  is supported by  $S$  if  $f(\pi(x)) = \pi(f(x))$  for every argument  $x$  and  $\pi \in \text{AUT}_S$ . An  $S$ -supported set is also  $S'$ -supported, as long as  $S \subseteq S'$ . An element (or set)  $x$  is *finitely supported* if it has some finite support; in this case  $x$  has *the least support*, denoted  $\text{sup}(x)$ , called *the support* of  $x$  (cf. [1, Sect. 6]). Sets supported by  $\emptyset$  we call *equivariant*.

For instance, given  $\alpha, \beta \in \text{ATOMS}$ , the support of the set  $\text{ATOMS} \setminus \{\alpha, \beta\}$  is  $\{\alpha, \beta\}$ ; in general, a set is  $S$ -supported if and only if it is invariant under all  $S$ -atom automorphisms. The set  $\text{ATOMS}^2$  and the projection function  $\pi_1 : \text{ATOMS}^2 \rightarrow \text{ATOMS} : (\alpha, \beta) \mapsto \alpha$  are both equivariant; and the support of a tuple  $(\alpha_1, \dots, \alpha_n) \in \text{ATOMS}^n$ , encoded as a set in a standard way, is the set of atoms  $\{\alpha_1, \dots, \alpha_n\}$  appearing in it.

From now on, we shall only consider sets that are hereditarily finitely supported, i.e., ones that have a finite support, whose every element has some finite support, and so on.

**Orbit-finite sets.** Let  $S \subseteq_{\text{fin}} \text{ATOMS}$ . Two atoms or sets with atoms  $x, y$  are *in the same S-orbit* if  $\pi(x) = y$  for some  $\pi \in \text{AUT}_S$ . This equivalence relation splits all atoms and sets with atoms into equivalence classes, which we call *S-orbits*;  $\emptyset$ -orbits we call *equivariant orbits*. By the very definition, every  $S$ -orbit  $O$  is  $S$ -supported:  $\text{sup}(O) \subseteq S$  and, even if the inclusion is strict (which may happen only for singleton orbits),  $O$  is also a  $\text{sup}(O)$ -orbit. When the set  $S$  is irrelevant, we simply speak of an *orbit*, meaning an  $S$ -orbit for some  $S \subseteq_{\text{fin}} \text{ATOMS}$ .

Every  $S$ -supported set is a union of (necessarily disjoint)  $S$ -orbits; the set is *orbit-finite* if this union is finite. Orbit-finiteness is stable under orbit-refinement: if  $S \subseteq S'$ , a finite union of  $S$ -orbits is also a finite union of  $S'$ -orbits (but the number of orbits may increase). Examples of orbit-finite sets are:  $\text{ATOMS}$  (1 orbit);  $\text{ATOMS} - \{\alpha\}$  for some  $\alpha \in \text{ATOMS}$  (1 orbit);  $\text{ATOMS}^2$  (2 orbits: diagonal and non-diagonal);  $\text{ATOMS}^3$  (5 orbits, corresponding to equality types of triples); non-repeating  $n$ -tuples of atoms (1 orbit)

$$\text{ATOMS}^{(n)} = \{(\alpha_1, \dots, \alpha_n) \in \text{ATOMS}^n \mid \alpha_i \neq \alpha_j \text{ for all } i \neq j\};$$

$$n\text{-sets of atoms } \binom{\text{ATOMS}}{n} = \{X \subseteq \text{ATOMS} \mid |X| = n\} \text{ (1 orbit).}$$

The set  $\mathcal{P}_{\text{fin}}(\text{ATOMS})$  of all finite subsets of atoms is orbit-finite as cardinality is an invariant of each orbit.

**Orbit representation.** For a positive integer  $k > 0$ , denote by  $S_k$  the symmetric group on  $\{1, \dots, k\}$ . Given a subgroup  $G \leq S_k$  of the symmetric group  $S_k$ , we denote by  $\text{ATOMS}^{(k)}/G$  the set of non-repeating  $k$ -tuples of atom modulo coordinate permutations from the group  $G$ . More formally, we define an equivalence in  $\text{ATOMS}^{(k)}$ , where a tuple  $a = (\alpha_1, \dots, \alpha_k) \in \text{ATOMS}^{(k)}$  is equivalent to every tuple  $a \circ \sigma := (\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(k)})$ , where  $\sigma \in G$ . The equivalence classes are thus finite. Then we define a canonical quotient  $\pi_G : \text{ATOMS}^{(k)} \rightarrow \text{ATOMS}^{(k)}/G$  mapping a tuple  $a \in \text{ATOMS}^{(k)}$  to its equivalence class.

**Example 2.1.** Let  $k = 3$  and  $G \leq S_3$  be generated by the cyclic shift  $\sigma$  to the right:  $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ . The quotient  $\pi_G : \text{ATOMS}^{(3)} \rightarrow \text{ATOMS}^{(3)}/G$  maps each triple  $(\alpha, \beta, \gamma)$  to  $\{(\alpha, \beta, \gamma), (\gamma, \alpha, \beta), (\beta, \gamma, \alpha)\}$ .  $\triangleleft$

**Lemma 2.2** ([1], Thm. 6.3). *Every equivariant orbit is in equivariant bijection with  $\text{ATOMS}^{(k)}/G$  for some  $k \in \mathbb{N}$  and some subgroup  $G \leq S_k$ .*

### 3 Orbit-Finite Basis Theorem

**Proviso.** Throughout the paper we fix a countable commutative ring  $(\mathbb{K}, 0, 1, +, \cdot)$  with multiplicative unit 1, and assume that the ring is *effective*: its elements are finitely representable and solvability of finite systems of linear equations is decidable. As a direct consequence, equality is decidable for the element representations, and the ring operations (addition, subtraction, multiplication) are computable using the representations. The most prominent examples are rationals  $\mathbb{Q}$  and integers  $\mathbb{Z}$ .

**Vectors.** We are investigating vector spaces<sup>1</sup> generated by an orbit-finite set. Let  $B$  be a fixed orbit-finite set.

**Definition 3.1.** By a *vector* over  $B$  we mean any finitely-supported function  $\mathbf{v}$  from  $B$  to  $\mathbb{K}$ , written  $\mathbf{v} : B \rightarrow_{\text{fs}} \mathbb{K}$  (vectors are written using boldface).

The set of all vectors over  $B$  we denote by  $\text{LIN}(B) = B \rightarrow_{\text{fs}} \mathbb{K}$ . It is a vector space, with pointwise addition and scalar multiplication: for  $\mathbf{v}, \mathbf{v}' \in \text{LIN}(B)$ ,  $b \in B$  and  $q \in \mathbb{K}$ , we have  $(\mathbf{v} + \mathbf{v}')(b) = \mathbf{v}(b) + \mathbf{v}'(b)$  and  $(q \cdot \mathbf{v})(b) = q \cdot \mathbf{v}(b)$ . The space  $\text{LIN}(B)$  may be considered as the vector space *generated* by  $B$ , and  $B$  as its *dimension*<sup>2</sup>. We define the *domain* of a vector  $\mathbf{v} \in \text{LIN}(B)$  as  $\text{dom}(\mathbf{v}) = \{b \in B \mid \mathbf{v}(b) \neq 0\}$ . A vector  $\mathbf{v}$  over  $B$  is *finitary*, written  $\mathbf{v} : B \rightarrow_{\text{fin}} \mathbb{K}$ , if  $\mathbf{v}(b) = 0$  for all except finitely many  $b \in B$  (i.e.,  $\text{dom}(\mathbf{v})$  is finite). A finitary vector  $\mathbf{v}$  with domain  $\text{dom}(\mathbf{v}) = \{b_1, \dots, b_k\}$  such that

<sup>1</sup>Formally, in case when  $\mathbb{K}$  is not a field, we should use the term *module*. Since modules/vector spaces studied in this paper are of particularly simple kind, we prefer to stick to a widely known term *vector space*.

<sup>2</sup>Not to be confused with *atom dimension* introduced in Section 5.

$\mathbf{v}(b_1) = q_1, \dots, \mathbf{v}(b_k) = q_k$ , may be identified with a formal linear combination of elements of  $B$ :

$$\mathbf{v} = q_1 \cdot b_1 + \dots + q_k \cdot b_k. \quad (1)$$

The subspace of  $\text{LIN}(B)$  consisting of all finitary vectors we denote by  $\text{FIN-LIN}(B) = B \rightarrow_{\text{fin}} \mathbb{K}$ . For finite  $B$  of size  $|B| = n$ ,  $\text{LIN}(B) = \text{FIN-LIN}(B)$  is isomorphic to  $\mathbb{K}^n$ .

For a subset  $X \subseteq B$ , we denote by  $\mathbf{1}_X \in \text{LIN}(B)$  the characteristic function of  $X$ , i.e., the vector that maps each element of  $X$  to 1 and all elements of  $B \setminus X$  to 0:

$$\mathbf{1}_X : b \mapsto \begin{cases} 1 & \text{if } b \in X \\ 0 & \text{otherwise.} \end{cases}$$

We write  $\mathbf{1}_b$  instead of  $\mathbf{1}_{\{b\}}$ , and  $\mathbf{1}$  instead of  $\mathbf{1}_B$ . We sometimes want to treat  $B$  itself as a subset of  $\text{FIN-LIN}(B)$ , identifying every  $b \in B$  with the vector  $\mathbf{1}_b$ , or equivalently with the trivial linear combination  $\mathbf{1} \cdot b$  as in (1).

**Lemma 3.2.** *Consider  $S \subseteq_{\text{fin}} \text{ATOMS}$  and an  $S$ -supported  $\mathbf{v} \in \text{LIN}(B)$ . Then*

- (i)  $\mathbf{v}$  is constant, restricted to every  $S$ -orbit  $O \subseteq B$ ;
- (ii)  $\mathbf{v}$  is a linear combination of characteristic vectors  $\mathbf{1}_O$  of  $S$ -orbits  $O \subseteq B$ .

*Proof.* The first part follows immediately as  $S$  supports  $\mathbf{v}$ . This allows us to write  $\mathbf{v}(O) \in \mathbb{K}$  in place of  $\mathbf{v}(x)$  for  $x \in O$ . As required in the second part, we have:

$$\mathbf{v} = \sum_O \mathbf{v}(O) \cdot \mathbf{1}_O, \quad (2)$$

where  $O$  ranges over finitely many  $S$ -orbits  $O \subseteq B$ .  $\square$

**Orbit-finite bases.** The set  $\{\mathbf{1}_b \mid b \in B\}$  is, by the very definition, a basis of  $\text{FIN-LIN}(B)$ . As our first result we prove that whenever  $B$  is orbit-finite, this set can be extended to an orbit-finite basis of the larger space  $\text{LIN}(B)$ :

**Theorem 3.3** (Orbit-Finite Basis Theorem). *For every orbit-finite set  $B$ , the space  $\text{LIN}(B)$  has an orbit-finite basis.*

The result constitutes a useful tool in our subsequent considerations of solvability of systems of linear equations. The proof is delegated to Section 5.

**Remark 1.** Theorem 3.3, as well as our subsequent results, are all effective. Indeed, the transformation from  $B$  to  $\widehat{B}$  is equivariant, and the set  $\widehat{B}$  as well as the transformation from  $\mathbf{v} \in \text{LIN}(B)$  to its basis representation in  $\text{FIN-LIN}(\widehat{B})$  are supported by  $\text{sup}(B)$ , and therefore all are subject to the general rule of thumb: (hereditarily) orbit-finite sets are finitely representable, and all finitely-supported transformations between these sets are effectively computable (for a detailed presentation we refer to [4] or [1, Sect. 4.8,9]).  $\triangleleft$

**Example 3.4.** Let  $B = \text{ATOMS}^{(2)}$ . For  $\gamma \in \text{ATOMS}$ , let  $\gamma_- = \{\gamma\alpha \mid \alpha \in \text{ATOMS} \setminus \{\gamma\}\} \subseteq B$ ; and symmetrically let  $\gamma_+ = \{\alpha\gamma \mid \alpha \in \text{ATOMS} \setminus \{\gamma\}\} \subseteq B$ . One obtains a basis  $\widehat{B} \subseteq \text{LIN}(B)$  by extending  $\{\mathbf{1}_{\alpha\beta} \mid \alpha\beta \in B\}$  with the constant vector  $\mathbf{1}$  that

maps every pair  $\alpha\beta \in B$  to 1, and also, for every  $\gamma \in \text{ATOMS}$ , with the characteristic vector  $\mathbf{1}_{\gamma_-}$  that maps all pairs in  $\gamma_-$  to 1 and all others to 0, and the characteristic vector  $\mathbf{1}_{\gamma_+}$  that maps all pairs in  $\gamma_+$  to 1 and all others to 0.

Towards seeing that this is indeed a base, consider any vector  $\mathbf{v} \in \text{LIN}(\text{ATOMS}^{(2)})$ . Let  $S = \text{sup}(\mathbf{v})$ . Let  $O_{\bullet\bullet} = (\text{ATOMS} \setminus S)^{(2)}$ ; for  $\alpha \in S$ , let  $O_{\alpha\bullet} = \{\alpha\} \times (\text{ATOMS} \setminus S)$  and  $O_{\bullet\alpha} = (\text{ATOMS} \setminus S) \times \{\alpha\}$ . Note that all these are  $S$ -orbits. The decomposition (2) of  $\mathbf{v}$  may be rewritten into:

$$\begin{aligned} \mathbf{v} &= \mathbf{v}(O_{\bullet\bullet}) \cdot \mathbf{1} + \sum_{\alpha \in S} (\mathbf{v}(O_{\alpha\bullet}) - \mathbf{v}(O_{\bullet\bullet})) \cdot \mathbf{1}_{\alpha_-} + \\ &\quad \sum_{\beta \in S} (\mathbf{v}(O_{\bullet\beta}) - \mathbf{v}(O_{\bullet\bullet})) \cdot \mathbf{1}_{\beta_+} + \\ &\quad \sum_{\alpha\beta \in S^{(2)}} (\mathbf{v}(\alpha\beta) - \mathbf{v}(O_{\alpha\bullet}) - \mathbf{v}(O_{\bullet\beta}) + \mathbf{v}(O_{\bullet\bullet})) \cdot \mathbf{1}_{\alpha\beta}. \end{aligned}$$

This yields a representation of  $\mathbf{v}$  in the base  $\widehat{B}$ , and the representation is unique.  $\triangleleft$

## 4 Solving linear equations

We note that the inner product of two vectors  $\mathbf{x}, \mathbf{y} \in \text{LIN}(B)$ , defined as

$$\mathbf{x} \cdot \mathbf{y} = \sum_{b \in B} \mathbf{x}(b) \mathbf{y}(b),$$

is not always well-defined. We consider the right-hand side sum as well-defined when there are only finitely many  $b \in B$  for which both  $\mathbf{x}(b)$  and  $\mathbf{y}(b)$  are non-zero (equivalently, the intersection  $\text{dom}(\mathbf{x}) \cap \text{dom}(\mathbf{y})$  is finite). In particular, the inner product  $\mathbf{x} \cdot \mathbf{y}$  is always well-defined when one of  $\mathbf{x}, \mathbf{y}$  is finitary.

**Remark 2.** Consider  $\mathbb{K} = \mathbb{Q}$ . Since vectors are finitely supported and hence (c.f. Lemma 3.2) contain only finitely many different numbers,  $\text{dom}(\mathbf{x}) \cap \text{dom}(\mathbf{y})$  is finite exactly when the right-hand side sum is *unconditionally convergent*, i.e., convergent to the same value irrespectively of the order in which the elements  $b \in B$  are enumerated<sup>3</sup>.  $\triangleleft$

**Systems of linear equations.** Fix an orbit-finite set  $C$  (one can think of  $C$  as an indexing set of columns of a matrix). By a linear equation over  $C$  we mean a pair  $e = (\mathbf{a}, t)$  where  $\mathbf{a} \in \text{LIN}(C)$  is a vector of left-hand side coefficients and  $t \in \mathbb{K}$  is a right-hand side target value. A solution of  $e$  is any vector  $\mathbf{x} \in \text{LIN}(C)$  such that the inner product  $\mathbf{a} \cdot \mathbf{x}$  is well-defined and equals  $t$ . We may consider constrained solutions, e.g., *finitary* ones.

A system of linear equations is just an indexed set of equations over the same set  $C$ . Formally, an orbit-finite system of linear equations (over  $C$ ) is any finitely-supported function  $B \rightarrow_{\text{fs}} \text{LIN}(C) \times \mathbb{K}$  from some orbit-finite indexing set  $B$  (one may think of  $B$  as an indexing set of rows of a matrix). By projecting to the first component we get a function  $\mathbf{A} : B \rightarrow_{\text{fs}} \text{LIN}(C)$  which we call the *matrix* of the system;

<sup>3</sup>We are grateful to Szymon Toruńczyk for attracting our attention to unconditional convergence.

by projecting to the second component (the target) we get a finitely-supported function  $\mathbf{t} : B \rightarrow_{\text{fs}} \mathbb{K}$ , i.e., a vector in  $\text{LIN}(B)$ , which we call the *target* of the system. The representation  $B \rightarrow_{\text{fs}} \text{LIN}(C)$  of the matrix may be equivalently written as a finitely-supported function  $\mathbf{A} : B \times C \rightarrow_{\text{fs}} \mathbb{K}$  (thus  $\mathbf{A} \in \text{LIN}(B \times C)$  and hence it deserves boldface).

Systems of linear equations, when input to algorithms, are assumed in the sequel to be given by a matrix-target pair  $(\mathbf{A}, \mathbf{t})$ :

$$\begin{array}{c} \vdots \\ b \left[ \begin{array}{ccc} \cdots & c & \cdots \\ \vdots & \mathbf{A}(b, c) & \vdots \\ \vdots & \vdots & \vdots \end{array} \right] \left[ \begin{array}{c} \vdots \\ \mathbf{t}(b) \\ \vdots \end{array} \right] \end{array}$$

A solution of a system of equations is any vector  $\mathbf{x} \in \text{LIN}(C)$  which is a solution of all equations in the system. Note that  $C$  can be seen as the indexing set of unknowns of the system.

For  $b \in B$  we denote by  $\mathbf{A}(b, \_)$   $\in \text{LIN}(C)$  the row vector indexed by  $b$ , and symmetrically, for  $c \in C$  we denote by  $\mathbf{A}(\_, c) \in \text{LIN}(B)$  the column vector indexed by  $c$ . One can also consider the *augmented* matrix  $\mathbf{A} | \mathbf{t} : B \times (C \uplus \{*\}) \rightarrow_{\text{fs}} \mathbb{K}$ .

In all the examples below let  $\mathbb{K} = \mathbb{Q}$ .

**Example 4.1.** Let columns be indexed by  $C = \text{ATOMS}^{(2)}$  and rows by  $B = \binom{\text{ATOMS}}{2}$ . Consider the system of equations containing, for every  $\{\alpha, \beta\} \in B$ , the equation  $(\mathbf{1}_{\alpha\beta} + \mathbf{1}_{\beta\alpha}, 1)$ . Using the formal-sum notation as in (1) it may be written as  $(\alpha\beta + \beta\alpha, 1)$  or, identifying column indexes  $\alpha\beta \in C$  with unknowns, as:

$$\alpha\beta + \beta\alpha = 1 \quad (\alpha, \beta \in \text{ATOMS}, \alpha \neq \beta).$$

All the equations are thus finitary, and the target is  $\mathbf{t} = \mathbf{1}_B$ . The constant vector  $\mathbf{x} = \frac{1}{2} : (\alpha, \beta) \mapsto \frac{1}{2}$  is a solution. The system has no finitary solution, as such a solution is in contradiction with the infinitary target  $\mathbf{t} = \mathbf{1}_B$ . Furthermore, the system has no integer (infinitary) solution either, as any such solution  $\mathbf{x}$  would necessarily satisfy, for every distinct atoms  $\alpha, \beta \in \text{ATOMS} \setminus \text{sup}(\mathbf{x})$ , the equality  $\mathbf{x}(\alpha\beta) = \mathbf{x}(\beta\alpha)$ , which is in contradiction with  $\mathbf{x}(\alpha\beta) + \mathbf{x}(\beta\alpha) = 1$ .  $\triangleleft$

**Example 4.2.** Let  $C = \text{ATOMS}^{(2)}$ ,  $B = \text{ATOMS}$ , and consider the system of equations containing, for every  $\alpha \in \text{ATOMS}$ , the equation  $(\mathbf{1}_{\alpha, \_}, 1)$ . As before, identifying column indexes  $\alpha\beta \in C$  with unknowns, the system may be written as:

$$\sum_{\beta \in \text{ATOMS} \setminus \{\alpha\}} \alpha\beta = 1 \quad (\alpha \in \text{ATOMS}).$$

All the equations are thus infinitary. The system has an integer solution. Take any two fixed atoms  $\gamma, \delta \in \text{ATOMS}$  and consider the vector

$$\mathbf{x} = \mathbf{1}_{-\gamma} + \mathbf{1}_{\gamma\delta}.$$

Indeed, for  $\alpha \neq \gamma$  we have  $\mathbf{1}_{\alpha, \_} \cdot \mathbf{x} = \mathbf{1}_{\alpha\gamma} \cdot \mathbf{1}_{\alpha\gamma} = 1$  as required. Furthermore, for  $\alpha = \gamma$  we have  $\mathbf{1}_{\alpha, \_} \cdot \mathbf{x} = \mathbf{1}_{\gamma\delta} \cdot \mathbf{1}_{\gamma\delta} = 1$  as required. The system has no finitary solution (essentially for

the same reason as in the previous example), and no equivariant one (as the only equivariant vectors over  $C$  are constant ones  $q \cdot \mathbf{1}$ , and the inner product  $\mathbf{1}_{\alpha, \_} \cdot \mathbf{1}$  is ill-defined for every  $\alpha \in \text{ATOMS}$  as long as  $q \neq 0$ ).  $\triangleleft$

The two above examples show that the solvability problem is sensitive to additional restrictions on solutions: the answer changes if solutions are additionally required to be equivariant, finitary, or integer. The next example shows that our implicit restriction to finitely-supported solutions also matters:

**Example 4.3.** Let  $C = \binom{\text{ATOMS}}{2}$ ,  $B = \text{ATOMS}$ , and consider the system of equations containing, for every  $\alpha \in B$ , the equation  $(\mathbf{1}_{\{\alpha, \_ \}}, 1)$ , where

$$\{\alpha, \_ \} = \{ \{\alpha, \gamma\} \mid \alpha \neq \gamma \in \text{ATOMS} \}$$

is the set of all 2-sets containing  $\alpha$ . We argue that the system has no (finitely supported) solution (despite the apparent similarity to the system in Example 4.2). Towards contradiction suppose it has a solution  $\mathbf{x}$ , supported by some  $S \subseteq_{\text{fin}} \text{ATOMS}$ . Thus it is constant on every  $S$ -orbit in  $\binom{\text{ATOMS}}{2}$ . An infinite  $S$ -orbit in  $\binom{\text{ATOMS}}{2}$  is either the set  $\binom{\text{ATOMS} \setminus S}{2}$  of all 2-sets disjoint from  $S$  or, for some fixed  $\alpha \in S$ , the set of all 2-sets with one element  $\alpha$  and the other element not in  $S$ :

$$\{ \{\alpha, \gamma\} \mid \gamma \in \text{ATOMS} \setminus S \}$$

Therefore each infinite  $S$ -orbit in  $\binom{\text{ATOMS}}{2}$  intersects infinitely with  $\{\alpha, \_ \}$  for some  $\alpha \in \text{ATOMS}$ . In consequence,  $\mathbf{x}$  is necessarily 0 when restricted to any infinite  $S$ -orbit in  $\binom{\text{ATOMS}}{2}$  as otherwise  $\mathbf{1}_{\{\alpha, \_ \}} \cdot \mathbf{x}$  would be ill-defined for some  $\alpha \in \text{ATOMS}$ . Therefore  $\mathbf{x}$  is forcedly finitary, and the argument of the previous examples applies.

On the other hand the system would have an integer solution if we drop the implicit finite-support constraint. For instance, taking any enumeration  $\text{ATOMS} = \{\alpha_0, \alpha_1, \alpha_2, \dots\}$  of atoms, the function  $\mathbf{x} : C \rightarrow \mathbb{K}$  that maps each set  $\{\alpha_{2n}, \alpha_{2n+1}\}$  to 1, for  $n = 0, 1, \dots$ , and all other sets to 0, satisfies all equations. Note that  $\mathbf{x}$  is not finitely supported, i.e., there is no finite  $S \subseteq \text{ATOMS}$  such that  $\pi(\mathbf{x}) = \mathbf{x}$  for all  $\pi \in \text{AUT}_S$ .  $\triangleleft$

**Solvability of linear equations.** We investigate the following type of solvability problems:

**SOLV**( $\mathbb{K}$ ):

**Input:** an orbit-finite system of linear equations.

**Question:** does it have a solution?

As our main result we prove:

**Theorem 4.4.** **SOLV**( $\mathbb{K}$ ) is decidable for every fixed effective commutative ring  $\mathbb{K}$ .

The proof, occupying the whole Sections 6 and 7, is by a reduction to solvability of finite systems of linear equations, and the transformation suffers from a singly-exponential blowup. As an intermediate step we also consider a variant of the problem where solutions are constrained to be finitary, called **FIN-SOLV**( $\mathbb{K}$ ).

**Remark 3.** In case  $\mathbb{K} = \mathbb{Q}$ , when coefficients in the input system are rational and we seek for rational solutions, as a corollary of the proof we deduce that the answer does not change if solutions are relaxed to real ones.  $\triangleleft$

**Spans.** For a subset  $P \subseteq \text{LIN}(B)$ , we define  $\text{FIN-SPAN}(P) \subseteq \text{LIN}(B)$  as the set of all linear combinations of vectors from  $P$ , forming a subspace of  $\text{LIN}(B)$ :

$$\text{FIN-SPAN}(P) = \{ q_1 \cdot \mathbf{p}_1 + \dots + q_k \cdot \mathbf{p}_k \mid k \geq 0, \\ q_1, \dots, q_k \in \mathbb{K}, \mathbf{p}_1, \dots, \mathbf{p}_k \in P \}.$$

Given a matrix  $\mathbf{A} \in \text{LIN}(B \times C)$  with rows  $B$  and columns  $C$ , we can define a partial operation of multiplication of  $\mathbf{A}$  by a vector  $\mathbf{v} \in \text{LIN}(C)$  in an expected way:

$$(\mathbf{A} \cdot \mathbf{v})(b) = \mathbf{A}(b, \_ ) \cdot \mathbf{v}$$

for every  $b \in B$ . The result  $\mathbf{A} \cdot \mathbf{v} \in \text{LIN}(B)$  is well-defined if  $\mathbf{A}(b, \_ ) \cdot \mathbf{v}$  is well-defined for all  $b \in B$ . The multiplication  $\mathbf{A} \cdot \mathbf{v}$  can be also seen as an orbit-finite linear combination of column vectors  $\mathbf{A}(\_, c)$ , for  $c \in C$ , with coefficients given by  $\mathbf{v}$ . This allows us to define the span of  $\mathbf{A}$  seen as a  $C$ -indexed orbit-finite set of vectors  $\mathbf{A}(\_, c) \in \text{LIN}(B)$ :

$$\text{SPAN}(\mathbf{A}) := \{ \mathbf{A} \cdot \mathbf{v} \mid \mathbf{v} \in \text{LIN}(C), \mathbf{A} \cdot \mathbf{v} \text{ well-defined} \}. \quad (3)$$

The solvability problem for a system of equations  $(\mathbf{A}, \mathbf{t})$  amounts thus to deciding if  $\mathbf{t} \in \text{SPAN}(\mathbf{A})$ . When  $\mathbf{v}$  is finitary, well-definedness is vacuous, and we may define:

$$\text{FIN-SPAN}(\mathbf{A}) := \{ \mathbf{A} \cdot \mathbf{v} \mid \mathbf{v} \in \text{FIN-LIN}(C) \} = \text{FIN-SPAN}(P),$$

for  $P = \{ \mathbf{A}(\_, c) \mid c \in C \}$  the set of column vectors of  $\mathbf{A}$ .

**Outline.** Concerning the proofs, we proceed in three steps. We start by proving the Orbit-Finite Basis Theorem in Section 5, a crucial technical tool for subsequent steps. As a key novelty, we introduce here the concept of *tight* orbits. Then we prove decidability of  $\text{FIN-SOLV}(\mathbb{K})$  in Section 6, by reducing it to solvability of classical finite systems of linear equations. This step relies on a generalisation of *cogs* introduced in [3, 16]. Finally, in Section 7 we reduce  $\text{SOLV}(\mathbb{K})$  to  $\text{FIN-SOLV}(\mathbb{K})$ , thus completing the proof of Theorem 4.4. This part strongly relies again on the technology developed in Section 5.

## 5 Proof of the Orbit-Finite Basis Theorem

In this section we prove Theorem 3.3, i.e., provide a construction of an orbit-finite basis in  $\text{LIN}(B)$ , where  $B$  is an arbitrary orbit-finite set.

**Preliminaries.** The mapping  $x \mapsto \text{sup}(x)$  is equivariant:

**Claim 1.**  $\text{sup}(\pi(x)) = \pi(\text{sup}(x))$  for every element  $x$  and  $\pi \in \text{AUT}$ .

We rely on the following basic properties of orbits:

**Claim 2.** Let  $S \subseteq_{\text{fin}} \text{ATOMS}$ . Each equivariant orbit  $O$  contains at most  $|S|!$  many elements  $x$  with  $\text{sup}(x) = S$ .

**Claim 3.** Every orbit is either a singleton or an infinite set.

**Definition 5.1.** Let  $S \subseteq_{\text{fin}} \text{ATOMS}$ . We define the  $S$ -atom dimension of an  $S$ -orbit  $O$ , written  $S\text{-dim}(O)$ , as the size of  $\text{sup}(x)$  for some (every) element  $x \in O$ , but not counting elements of  $S$ :

$$S\text{-dim}(O) := |\text{sup}(x) \setminus S|.$$

The choice of  $x$  is irrelevant due to Claim 1. When  $S$  is clear from the context we omit  $S$  and speak of *atom dimension*.

**Reduction to single-orbit  $B$ .** We claim that we can assume, w.l.o.g., that  $B$  is a single orbit. Indeed, let  $T = \text{sup}(B)$  and let  $B = B_1 \uplus \dots \uplus B_n$  be the partition into  $T$ -orbits. Then  $\text{LIN}(B)$  is isomorphic to the Cartesian product  $\text{LIN}(B_1) \times \dots \times \text{LIN}(B_n)$ . Denote by  $\iota_i : \text{LIN}(B_i) \rightarrow \text{LIN}(B)$  the natural embedding that extends a vector  $\mathbf{v} : B_i \rightarrow_{\text{fs}} \mathbb{K}$  by 0 for all  $b \in B \setminus B_i$ :

$$\iota_i(\mathbf{v})(b) := \begin{cases} \mathbf{v}(b) & \text{if } b \in B_i \\ 0 & \text{otherwise.} \end{cases}$$

Supposing we have orbit-finite bases  $\widehat{B}_1, \dots, \widehat{B}_n$  of the vector spaces  $\text{LIN}(B_1), \dots, \text{LIN}(B_n)$ , respectively, we get the basis  $\widehat{B}$  of  $\text{LIN}(B)$  as the union of embeddings of  $\widehat{B}_1, \dots, \widehat{B}_n$ :

$$\iota_1(\widehat{B}_1) \cup \dots \cup \iota_n(\widehat{B}_n).$$

We thus assume w.l.o.g. that  $B$  is a single  $T$ -orbit.

As the support of a function is also a support (but not necessarily the support) of its domain, we note:

**Claim 4.**  $T \subseteq \text{sup}(\mathbf{v})$  for every vector  $\mathbf{v} \in \text{LIN}(B)$ .

**Tight orbits.** A key role is played in the proof by the concept of *tight* orbits.

**Definition 5.2.** Let  $S \subseteq_{\text{fin}} \text{ATOMS}$ . An  $S$ -orbit  $O$  is called *tight* if  $S \subseteq \text{sup}(x)$  for every  $x \in O$ .

In particular, every singleton is a tight orbit.

**Example 5.3.** Recall Example 3.4. In case of  $B = \text{ATOMS}^{(2)}$ , the tight orbits  $O \subseteq B$  are the following ones:

$$B \quad \alpha\_ \quad \_ \beta \quad \{ \alpha \beta \}$$

where  $\alpha, \beta$  range over atoms and  $\alpha \neq \beta$ . The set  $B$  is an equivariant orbit,  $\alpha\_$  is an  $\{ \alpha \}$ -orbit,  $\_ \beta$  is a  $\{ \beta \}$ -orbit, and  $\{ \alpha, \beta \}$  is an  $\{ \alpha, \beta \}$ -orbit. Contrarily, for two fixed and distinct  $\alpha, \beta \in \text{ATOMS}$ , the  $\{ \alpha, \beta \}$ -orbit

$$\neq \alpha \beta = \{ \gamma \beta \mid \gamma \notin \{ \alpha, \beta \} \},$$

is not tight.  $\triangleleft$

W.l.o.g. we can assume that  $B$  is tight, i.e.,  $T \subseteq \text{sup}(b)$  for every  $b \in B$ . Indeed, it is sufficient to continue with  $B' := B \times \{ T \}$ . Then  $B'$  and  $B$  are related by a  $T$ -supported bijection. For future use we state:

**Claim 5.** Let  $S \subseteq_{\text{fin}} \text{ATOMS}$ . Every  $S$ -orbit  $O$  is in an  $S$ -supported bijection with a tight  $S$ -orbit.

For every tight  $S$ -orbit  $O \subseteq B$ , the size of  $S$  is at most the size of the support of elements of  $B$ . Furthermore, by Claim 2, for every fixed  $S \subseteq_{\text{fin}} \text{ATOMS}$  there are only finitely many  $S$ -orbits inside  $B$ . In consequence we deduce that the set of all tight orbits in  $B$  is orbit-finite:

**Claim 6.** *The set  $\{O \mid O \subseteq B \text{ a tight orbit}\}$  is orbit-finite.*

In the sequel we order tight orbits in  $B$  with respect to inclusion.

**Definition of the basis.** We define  $\widehat{B}$  as the set of characteristic vectors of all tight orbits  $O \subseteq B$ :

$$\widehat{B} := \{ \mathbf{1}_O \mid O \subseteq B \text{ a tight orbit} \}.$$

Once  $B$  is fixed, the set  $\widehat{B}$  is orbit-finite due to Claim 6. Since every singleton is a tight orbit,  $\mathbf{1}_b \in \widehat{B}$  for every  $b \in B$ ; informally speaking,  $\widehat{B}$  extends  $B$ .

**Example 5.4.** Continuing Example 5.3, where  $B = \text{ATOMS}^{(2)}$ , the basis vectors are the following ones:

$$\mathbf{1} \quad \mathbf{1}_{\alpha_-} \quad \mathbf{1}_{\beta} \quad \mathbf{1}_{\alpha\beta},$$

for any non-equal  $\alpha, \beta \in \text{ATOMS}$ .  $\triangleleft$

It now remains to argue that  $\widehat{B}$  spans the whole space  $\text{LIN}(B)$ , and that it is linearly independent.

**Spanning.** Given a subset  $S \subseteq_{\text{fin}} \text{ATOMS}$  such that  $T \subseteq S$ , we distinguish the set of all tight  $S'$ -orbits for  $T \subseteq S' \subseteq S$ :

$$\text{TO}(T, S) := \{O \mid O \subseteq B \text{ a tight } S'\text{-orbit}, T \subseteq S' \subseteq S\}.$$

For every fixed  $S$  the set  $\text{TO}(T, S)$  is finite since, due to Claim 2,  $B$  includes only finitely many  $S'$ -orbits for every fixed  $S' \subseteq_{\text{fin}} \text{ATOMS}$ .

We prove that  $\widehat{B}$  spans the whole space, i.e., each vector is a linear combination of vectors from  $\widehat{B}$ . To this aim we fix a finite subset  $S \subseteq \text{ATOMS}$  such that  $T \subseteq S$  and prove that every  $S$ -supported vector  $\mathbf{v}$  is a linear combination of vectors from

$$\widehat{B}_S = \{ \mathbf{1}_O \mid O \in \text{TO}(T, S) \} \subseteq \widehat{B}.$$

For every fixed  $S$  the set  $\widehat{B}_S$  is finite, as  $\text{TO}(T, S)$  is so.

**Lemma 5.5 (Spanning).** *Let  $S \subseteq_{\text{fin}} \text{ATOMS}$  such that  $T \subseteq S$ . Each  $S$ -supported vector  $\mathbf{v} \in \text{LIN}(B)$  is a linear combination of vectors from  $\widehat{B}_S$ .*

*Proof.* Let  $\mathbf{v} \in \text{LIN}(B)$  and  $S \subseteq_{\text{fin}} \text{ATOMS}$  such that  $\text{sup}(\mathbf{v}) \subseteq S$ . By Lemma 3.2(i),  $\mathbf{v}$  is constant when restricted to every  $S$ -orbit  $O$ ; we may thus write  $\mathbf{v}(O)$  to denote this constant value. We naturally define the  $S$ -orbit-domain of  $\mathbf{v}$  as follows:

$$S\text{-orbit-dom}(\mathbf{v}) := \{O \mid O \subseteq B \text{ an } S\text{-orbit}, \mathbf{v}(O) \neq 0\}.$$

For two  $S$ -supported vectors  $\mathbf{w}, \mathbf{w}' \in \text{LIN}(B)$ , we write  $\mathbf{w} < \mathbf{w}'$  if  $S\text{-orbit-dom}(\mathbf{w})$  is obtained from  $S\text{-orbit-dom}(\mathbf{w}')$  by removing one  $S$ -orbit and replacing it by arbitrarily many  $S$ -orbits of strictly smaller  $S$ -atom dimension.

We define a representation of  $\mathbf{v}$  in basis  $\widehat{B}$  by structural induction with respect to the transitive closure of  $<$ . Concerning the induction base, if  $S\text{-orbit-dom}(\mathbf{v})$  is empty then  $\mathbf{v}$  is the zero vector and the claim holds vacuously. Otherwise, suppose the claim holds for all strictly smaller vectors  $\mathbf{w}$ . Take an  $S$ -orbit  $O \in S\text{-orbit-dom}(\mathbf{v})$  of maximal  $S$ -atom dimension. Let

$$S' := \text{sup}(x) \cap S \quad (4)$$

for some (every)  $x \in O$ . Note that  $T \subseteq S'$  as  $T \subseteq \text{sup}(x)$  (since  $B$  is tight) and  $T \subseteq S$  (by Claim 4). We define the  $S'$ -orbit  $O'$  as  $S'$ -closure of  $O$ :

$$O' := \{ \pi(x) \mid x \in O, \pi \in \text{AUT}_{S'} \}.$$

By definition,  $S'$  is included in the support of every element of  $O'$ , therefore the orbit  $O'$  is tight, and hence  $\mathbf{1}_{O'} \in \widehat{B}_S$ . As  $S' \subseteq S$ , every  $S$ -orbit in  $B$  is either included in  $O'$  or disjoint from it, and hence  $O'$  is a finite union of  $S$ -orbits. We claim that  $O$  has the largest  $S$ -atom dimension among all  $S$ -orbits included in  $O'$ :

**Claim 7.** *For every  $S$ -orbit  $M$  included in  $O'$  but different than  $O$ , we have  $S\text{-dim}(M) < S\text{-dim}(O)$ .*

*Proof.* Recall that  $S' \subseteq S \cap \text{sup}(x)$  for every  $x \in O'$ .

Consider the subset  $N \subseteq O'$  containing those elements  $x \in O'$  for which  $S' = \text{sup}(x) \cap S$ . By the definition of  $S'$  (4) we have  $O \subseteq N$ . We prove  $N \subseteq O$ , by showing that every element  $y \in N$  is related by an  $S$ -atom automorphism to some element of  $x \in O$ . Indeed, consider any  $x \in O$  and  $y = \pi'(x)$  for any  $\pi' \in \text{AUT}_{S'}$  such that  $y \in N$ . We have

$$S' = \text{sup}(x) \cap S = \text{sup}(\pi'(x)) \cap S$$

and hence there is some  $\pi \in \text{AUT}_S$ , possibly different than  $\pi'$ , that coincides with  $\pi'$  on  $\text{sup}(x)$ , which implies  $y = \pi(x)$ , as required. The two inclusions imply  $N = O$ .

Finally, for all  $x \in O' \setminus N = O' \setminus O$  we have  $S' \subsetneq \text{sup}(x) \cap S$ , which implies that each  $S$ -orbit  $M \subseteq O'$  different than  $O$  has strictly smaller  $S$ -atom dimension than  $O$ .  $\square$

Consider the vector

$$\mathbf{w} := \mathbf{v} - \mathbf{v}(O) \cdot \mathbf{1}_{O'}. \quad (5)$$

Note that  $\mathbf{w}$  is supported by  $S$  as both  $\mathbf{v}$  and  $\mathbf{1}_{O'}$  are so, and  $\mathbf{w}(O) = 0$ . By Claim 7 we infer that  $\mathbf{w} < \mathbf{v}$  and therefore by the induction assumption  $\mathbf{w}$  is a linear combination of vectors from  $\widehat{B}_S$ . By (5) we deduce the same for  $\mathbf{v}$ . This completes the proof of Lemma 5.5.  $\square$

**Linear independence.** We rely on the following property of tight orbits (not true for arbitrary orbits):

**Claim 8.** *If orbits  $O, O_1, \dots, O_n$  are tight and  $O \subseteq O_1 \cup \dots \cup O_n$  then  $O \subseteq O_i$  for some  $i = 1, \dots, n$ .*

*Proof.* If  $O$  is a singleton then the claim holds vacuously. Relying on Claim 3 we may thus assume that  $O$  is infinite.

Suppose  $O \subseteq O_1 \cup \dots \cup O_n$  for a tight  $S$ -orbit  $O$  and arbitrary tight orbits  $O_1, \dots, O_n$ . Take any  $x \in O$  and let  $R := \text{sup}(x) \setminus S$ . Consider elements  $\pi(x) \in O$  for all  $S$ -atom automorphisms  $\pi$ , thus ranging over all elements of the orbit  $O$ . At least one of the orbits  $O_1, \dots, O_m$ , say the  $S_1$ -orbit  $O_1$ , necessarily contains  $\pi(x)$  and  $\pi'(x)$ , for some two  $S$ -atoms automorphisms  $\pi, \pi'$ , such that the sets  $\pi(R)$  and  $\pi'(R)$  are disjoint. By tightness of  $O_1$  (and relying on Claim 1) we get  $S_1 \subseteq \text{sup}(\pi(x)) = S \cup \pi(R)$  and  $S_1 \subseteq \text{sup}(\pi'(x)) = S \cup \pi'(R)$ , and hence  $S_1 \subseteq S$ , which implies  $\pi(x) \in O_1$  for all  $S$ -atom automorphisms  $\pi$ , i.e.,  $O \subseteq O_1$ .  $\square$

We now argue that the set  $\widehat{B}$  is linearly independent. Towards contradiction, suppose that the zero vector is obtainable as a linear combination of basis vectors

$$q_1 \cdot \mathbf{1}_{O_1} + \dots + q_n \cdot \mathbf{1}_{O_n} = \mathbf{0}, \quad (6)$$

for some tight pairwise-different orbits  $O_1, \dots, O_n \subseteq B$  and  $q_1, \dots, q_n \in \mathbb{K} \setminus \{0\}$ . Take any inclusion-maximal orbit among  $O_1, \dots, O_n$ , say  $O_1$ . We distinguish two cases.

1. If  $O_1 \subseteq O_2 \cup \dots \cup O_n$  then using Claim 8 we arrive at a contradiction with the inclusion-maximality of  $O_1$ .

2. Otherwise  $O_1 \not\subseteq O_2 \cup \dots \cup O_n$ . Taking any  $x \in O_1 \setminus (O_2 \cup \dots \cup O_n)$  we derive a contradiction, as the value of the left-hand side of (6) on  $x$  is non-zero:

$$(q_1 \cdot \mathbf{1}_{O_1} + \dots + q_n \cdot \mathbf{1}_{O_n})(x) = q_1 \neq 0,$$

while the value of the right-hand side is  $\mathbf{0}(x) = 0$ .

## 6 Decidability of finitary solvability

In this section we prove decidability of the finitary solvability problem.

FIN-SOLV( $\mathbb{K}$ ):

**Input:** an orbit-finite system of linear equations.

**Question:** does it have a finitary solution?

**Theorem 6.1.** FIN-SOLV( $\mathbb{K}$ ) is decidable for every fixed effective commutative ring  $\mathbb{K}$ .

Let  $\mathbf{A} \in \text{LIN}(B \times C)$  and  $\mathbf{t} \in \text{LIN}(B)$  be the input. We need to check if  $\mathbf{t} \in \text{FIN-SPAN}(\mathbf{A})$ , or equivalently  $\mathbf{t} \in \text{FIN-SPAN}(P)$ , where  $P = \{\mathbf{A}(\_, c) \mid c \in C\}$  is an orbit-finite set of vectors from  $\text{LIN}(B)$ . As  $P$  can be computed from  $\mathbf{A}$ , from now on we assume we are given  $P$  and  $\mathbf{t}$ .

**Simplifying assumptions.** First, for simplicity of presentation we assume that  $P$  (but not  $\mathbf{t}$ ) is equivariant; hence also  $B$  is forcedly so.

We further assume w.l.o.g. that all vectors are finitary:  $P \subseteq \text{FIN-LIN}(B)$  and  $\mathbf{t} \in \text{FIN-LIN}(B)$ . Indeed, according to Remark 1 we may compute an orbit-finite basis  $\widehat{B}$  of  $\text{LIN}(B)$ , and then compute the representations  $P' \subseteq \text{FIN-LIN}(\widehat{B})$  and  $\mathbf{t}' \in \text{FIN-LIN}(\widehat{B})$  of  $P$  and  $\mathbf{t}$  in this basis. As  $\widehat{B}$  is a basis, the representation preserves solvability:  $\mathbf{t} \in \text{FIN-SPAN}(P)$  if, and only if  $\mathbf{t}' \in \text{FIN-SPAN}(P')$ .

Finally, we assume w.l.o.g. that  $B$  is *straight*, by which we mean that each of its orbits is in equivariant bijection with  $\text{ATOMS}^{(k)}$  for some  $k \in \mathbb{N}$ . By Lemma 2.2, each (equivariant) orbit in  $B$  is in equivariant bijection with  $\text{ATOMS}^{(k)}/G$  for some  $k \in \mathbb{N}$  and some subgroup  $G \leq S_k$ . The vector space  $\text{LIN}(\text{ATOMS}^{(k)}/G)$  is, in turn, in equivariant bijection with the subspace of all  $G$ -invariant vectors in  $\text{LIN}(\text{ATOMS}^{(k)})$ , i.e. vectors  $v : \text{ATOMS}^{(k)} \rightarrow \mathbb{K}$  satisfying  $v(a \circ \sigma) = v(a)$  for every  $a \in \text{ATOMS}^{(k)}$  and  $\sigma \in G$ . This yields the embedding

$$\iota : \text{LIN}(\text{ATOMS}^{(k)}/G) \rightarrow \text{LIN}(\text{ATOMS}^{(k)})$$

given by pre-composing with the canonical quotient  $\pi_G : \text{ATOMS}^{(k)} \rightarrow \text{ATOMS}^{(k)}/G$ ,

$$\mathbf{v} \mapsto \iota(\mathbf{v}) = \mathbf{v} \circ \pi_G. \quad (7)$$

The embedding  $\iota$  extends to  $\text{LIN}(B) \rightarrow \text{LIN}(B')$ , where  $B'$  is the disjoint union of straight orbits corresponding to orbits of  $B$ . The embedding is efficiently computable, and preserves linear combinations and finitariness. By the latter property we may restrict  $\iota$  to finitary vectors, namely  $\iota : \text{FIN-LIN}(B) \rightarrow \text{FIN-LIN}(B')$ . Therefore, writing  $P'$  and  $\mathbf{t}'$  for  $\iota(P)$  and  $\iota(\mathbf{t})$ , respectively, we deduce that  $\mathbf{t} \in \text{FIN-SPAN}(P)$  if and only if  $\mathbf{t}' \in \text{FIN-SPAN}(P')$ .

Summing up, by an *instance* of the problem we mean a triple  $(V, P, \mathbf{t})$  consisting of a vector space  $V = \text{FIN-LIN}(B)$  generated by an equivariant straight orbit-finite set  $B$ , an equivariant orbit-finite subset  $P \subseteq V$ , and a vector  $\mathbf{t} \in V$ . The instance is *solvable* if  $\mathbf{t} \in \text{FIN-SPAN}(P)$ .

**Canonical form.** Recall that the atom dimension of the orbit  $\text{ATOMS}^{(k)}$  is  $k$ . Up to an equivariant bijection, we may present  $B$  as a disjoint union  $B = B_1 \uplus \dots \uplus B_n$  where  $B_i = \text{ATOMS}^{(p_i)}$  for some  $p_i \in \mathbb{N}$ , for  $i = 1, \dots, n$ . Therefore the vector space  $\text{FIN-LIN}(B)$  is equivariantly isomorphic to

$$(\text{ATOMS}^{(p_1)} \rightarrow_{\text{fin}} \mathbb{K}) \times \dots \times (\text{ATOMS}^{(p_n)} \rightarrow_{\text{fin}} \mathbb{K}).$$

For convenience we prefer to work with vector spaces in the following canonical form, where all orbits  $\text{ATOMS}^{(p)}$  of the same atom dimension  $p$  are grouped together:

$$V = (\text{ATOMS}^{(k_1)} \rightarrow_{\text{fin}} \mathbb{K}^{\ell_1}) \times \dots \times (\text{ATOMS}^{(k_m)} \rightarrow_{\text{fin}} \mathbb{K}^{\ell_m}), \quad (8)$$

where  $k_1, \dots, k_m$  are pairwise different nonnegative integers, and  $\ell_1, \dots, \ell_m$  are arbitrary positive integers. A vector space  $V$  in canonical form (8) is thus the Cartesian product of  $m$  *components*. The definition of domain naturally extended to vectors of the form  $\mathbf{v} : \text{ATOMS}^{(k)} \rightarrow_{\text{fs}} \mathbb{K}^{\ell}$  as follows:

$$\text{dom}(\mathbf{v}) = \left\{ a \in \text{ATOMS}^{(k)} \mid \mathbf{v}(a) \neq (0, \dots, 0) \in \mathbb{K}^{\ell} \right\}.$$

**Definition 6.2.** By the *atom dimension* of a vector space  $V$  in canonical form (8) we mean the maximum among atom dimensions of orbits  $\text{ATOMS}^{(k_i)}$ , i.e.,  $\max(k_1, \dots, k_m)$ .



The component  $V_i = \text{ATOMS}^{(k_i)} \rightarrow_{\text{fs}} \mathbb{K}^{\ell_i}$  of largest atom dimension we call the *main component* of  $V$  and denote as  $\tilde{V}$ . Assuming w.l.o.g.  $i = 1$  (the main component is the first one) we may write

$$V = \tilde{V} \times V'$$

where  $V'$  is the Cartesian product of all non-main components. Thus every vector  $\mathbf{v} \in V$  decomposes as a pair

$$\mathbf{v} = (\tilde{\mathbf{v}}, \mathbf{v}') \in \tilde{V} \times V'. \quad (9)$$

Furthermore,  $\tilde{V}$  embeds into  $V$  as the subspace  $\tilde{V} \times \{\mathbf{0}\} \times \dots \times \{\mathbf{0}\}$ , where  $\mathbf{0} : \text{ATOMS}^{(k_i)} \rightarrow_{\text{fs}} \mathbb{K}^{\ell_i}$  maps every tuple  $a \in \text{ATOMS}^{(k_i)}$  to  $(0, \dots, 0) \in \mathbb{K}^{\ell_i}$ , and likewise  $V'$  embeds into  $V$ . Using the embeddings implicitly, we may write

$$\mathbf{v} = \tilde{\mathbf{v}} + \mathbf{v}' \quad (10)$$

in place of (9).

Summing up, instances  $(V, P, \mathbf{t})$  are assumed from now on to consist of a vector space  $V$  in canonical form (8).

**Locally solvable instances.** We distinguish *locally solvable* instances  $(V, P, \mathbf{t})$ , defined as follows. Let  $\tilde{V} = \text{ATOMS}^{(k)} \rightarrow_{\text{fs}} \mathbb{K}^{\ell}$  be the main component (for succinctness of notation we write  $k, \ell$  instead of  $k_1, \ell_1$ ). We use the restriction operation: for  $X \subseteq \text{ATOMS}^{(k)}$  and  $\mathbf{w} : \text{ATOMS}^{(k)} \rightarrow_{\text{fs}} \mathbb{K}^{\ell}$  we define

$$\mathbf{w} \upharpoonright_X(a) = \begin{cases} \mathbf{w}(a) & \text{if } a \in X \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Given a  $k$ -set  $A \in \binom{\text{ATOMS}}{k}$ , we may consider the  $A$ -restriction  $(\tilde{V}, P', \mathbf{t}')$  of the instance, where

$$P' = \{\tilde{\mathbf{v}} \upharpoonright_{A^{(k)}} \mid \mathbf{v} \in P\} \quad \mathbf{t}' = \tilde{\mathbf{t}} \upharpoonright_{A^{(k)}}.$$

Thus the  $A$ -restriction is essentially a finite system of at most  $|A^{(k)}| = k!$  equations. Any restriction of a solvable instance is solvable too. An instance is called *locally solvable* if each of its  $A$ -restrictions is solvable, for every  $A \in \binom{\text{ATOMS}}{k}$ . Clearly, each solvable instance is locally solvable, but the opposite implication is not true in general (one of the reasons is that local solvability only refers to the main component).

**Claim 9.** *Local solvability is decidable.*

We later make use of the fact that for any two different (but not necessarily disjoint)  $k$ -sets  $A, A' \in \binom{\text{ATOMS}}{k}$ , the sets  $A^{(k)}$  and  $(A')^{(k)}$  are always disjoint.

**Reduction of atom dimension.** The following lemma is the core of the proof of Theorem 6.1:

**Lemma 6.3.** *Given a locally solvable instance  $(V, P, \mathbf{t})$  as above, one may construct another instance  $(\tilde{V}, \tilde{P}, \tilde{\mathbf{t}})$  where atom dimension of  $\tilde{V}$  is strictly smaller than that of  $V$ , and such that  $\mathbf{t} \in \text{FIN-SPAN}(P)$  if and only if  $\tilde{\mathbf{t}} \in \text{FIN-SPAN}(\tilde{P})$ .*

*Proof of Theorem 6.1.* Using the lemma we prove that the finitary spanning problem reduces to solvability of finite systems of linear equations, which implies decidability. First,

local solvability of an instance is a necessary condition for solvability, and is decidable by Claim 9. The algorithm thus checks if the input instance is locally solvable: if it is not so it answers negatively, and if it is so the algorithm applies the construction of Lemma 6.3 to produce an instance of strictly smaller atom dimension. Continuing so iteratively, the algorithm finally arrives at  $V$  of atom dimension equal to 0, i.e., at a finitely dimensional vector space  $V$ . In this case the set  $P$ , being an orbit-finite subset of  $V$ , is necessarily finite too, and the problem amounts to solving a finite system of linear equations.  $\square$

We thus concentrate from now on on proving Lemma 6.3.

**Cogs.** We rely on a generalisation of cogs in [3] and of simple hypergraphs in [16]. Let  $A, S \in \binom{\text{ATOMS}}{k}$  be two disjoint subsets of atoms of size  $k$ , and let  $\sigma : A \rightarrow S$  be a bijection. For every  $I \subseteq A$ , we define an injective mapping

$$\sigma_I : A \rightarrow A \cup S \quad \sigma_I(\alpha) = \begin{cases} \alpha & \text{if } \alpha \notin I \\ \sigma(\alpha) & \text{if } \alpha \in I. \end{cases} \quad (11)$$

Intuitively, the set  $I$  specifies those elements  $\alpha \in A$  that should be replaced by  $\sigma(\alpha)$ . In particular,  $\sigma_{\emptyset}$  is the identity on  $A$  and  $\sigma_A = \sigma$ . Let  $\mathbf{w} : \text{ATOMS}^{(k)} \rightarrow_{\text{fin}} \mathbb{K}^{\ell}$  be a vector satisfying  $\text{dom}(\mathbf{w}) \subseteq A^{(k)}$ . In (12) below we implicitly extend  $\sigma_I$ , in an arbitrary way, to an atom automorphism  $\text{ATOMS} \rightarrow \text{ATOMS}$ . A *cog* of  $\mathbf{w}$  via  $\sigma$  is the vector  $[\sigma](\mathbf{w}) : \text{ATOMS}^{(k)} \rightarrow_{\text{fin}} \mathbb{K}^{\ell}$  defined as:

$$[\sigma](\mathbf{w}) = \sum_{I \subseteq A} (-1)^{|I|} \cdot \sigma_I(\mathbf{w}). \quad (12)$$

Thus the domain of  $[\sigma](\mathbf{w})$  is a finite set of size at most  $k! \cdot 2^k$ .

**Example 6.4.** Let  $k = 2, \ell = 1, A = \{\alpha, \beta\} \subseteq \text{ATOMS}$ , and

$$\mathbf{w} = \alpha\beta + 2 \cdot \beta\alpha \in \text{ATOMS}^{(2)} \rightarrow_{\text{fin}} \mathbb{K}.$$

Let  $\sigma : \{\alpha, \beta\} \rightarrow \{\gamma, \delta\}$  be defined by  $\sigma(\alpha) = \gamma$  and  $\sigma(\beta) = \delta$ . Then we have

$$\begin{aligned} [\sigma](\mathbf{w}) = & \alpha\beta + 2 \cdot \beta\alpha - \gamma\beta - 2 \cdot \beta\gamma \\ & + \gamma\delta + 2 \cdot \delta\gamma - \alpha\delta - 2 \cdot \delta\alpha. \end{aligned}$$

**Claim 10.** *Let  $\mathbf{w} : \text{ATOMS}^{(k)} \rightarrow \mathbb{K}^{\ell}$  such that  $\text{dom}(\mathbf{w}) \subseteq A^{(k)}$ . Then  $([\sigma](\mathbf{w})) \upharpoonright_{A^{(k)}} = \mathbf{w}$ .*  $\triangleleft$

*Proof of Lemma 6.3.* Consider some locally solvable instance  $(V, P, \mathbf{t})$  with  $V$  in canonical form (8).

We start by restricting the set  $P$  to a subset  $P' \subseteq P$  while preserving solvability. Let  $S \in \binom{\text{ATOMS}}{k}$  be an arbitrary fixed subset of atoms of size  $k$  disjoint from  $T = \text{sup}(\mathbf{t})$ . For any  $p, q \in \mathbb{N}$  and  $X \subseteq \text{ATOMS}^{(p)}$  let  $X \rightarrow \mathbb{K}^q$  denote the subspace

$$X \rightarrow \mathbb{K}^q := \left\{ \mathbf{w} : \text{ATOMS}^{(p)} \rightarrow_{\text{fin}} \mathbb{K}^q \mid \text{dom}(\mathbf{w}) \subseteq X \right\}.$$

Furthermore, let

$$\text{ATOMS}_{-S}^p = (\text{ATOMS} \setminus S)^{(p)}$$

denote the set of non-repeating  $p$ -tuples containing *no* element of  $S$ , and define the subspace  $V_S$  of  $V$ :

$$V_S = (\text{ATOMS}_{\neg S}^{k_1} \rightarrow \mathbb{K}^{\ell_1}) \times \dots \times (\text{ATOMS}_{\neg S}^{k_m} \rightarrow \mathbb{K}^{\ell_m}). \quad (13)$$

Thus  $V_S$  contains only those vectors in  $V$  whose support is disjoint from  $S$ . Consider an instance  $(V, P', \mathbf{t})$  where

$$P' := P \cap V_S = \{ \mathbf{v} \in P \mid \text{sup}(\mathbf{v}) \cap S = \emptyset \}.$$

We observe that any finitary solution

$$q_1 \cdot \mathbf{v}_1 + \dots + q_m \cdot \mathbf{v}_m = \mathbf{t}$$

of  $(V, P, \mathbf{t})$ , where  $q_1, \dots, q_m \in \mathbb{K}$  and  $\mathbf{v}_1, \dots, \mathbf{v}_m \in P$ , may be renamed, using a  $T$ -atom automorphism  $\pi$ , to a solution involving only vectors  $\pi(\mathbf{v}_1), \dots, \pi(\mathbf{v}_m) \in P$  with support disjoint from  $S$ . We have thus argued that:

**Claim 11.** *For every  $S \subseteq_{\text{fin}} \text{ATOMS}$  disjoint from  $\text{sup}(\mathbf{t})$ , the instance  $(V, P, \mathbf{t})$  is solvable if and only if  $(V, P \cap V_S, \mathbf{t})$  is so.*

The instance  $(V, P', \mathbf{t})$  is forcedly locally solvable, and computable from  $(V, P, \mathbf{t})$ .

**The instance  $(\bar{V}, \bar{P}, \bar{\mathbf{t}})$ .** Let  $\tilde{V} = \text{ATOMS}^{(k_1)} \rightarrow_{\text{fs}} \mathbb{K}^{\ell_1}$  be the main component of  $V$  (for succinctness of notation we write  $k, \ell$  in place of  $k_1, \ell_1$ ). For any  $p, q \in \mathbb{N}$ , let

$$\text{ATOMS}_S^p = \text{ATOMS}^{(p)} \setminus \text{ATOMS}_{\neg S}^p$$

denote the set of non-repeating  $p$ -tuples containing *at least* one element of  $S$ . We define  $\bar{V}$  as the subspace of  $V$  where the domain in the main (first) component is included in  $\text{ATOMS}_S^{k_1} = \text{ATOMS}_S^k$ , and in all other components in  $\text{ATOMS}_{\neg S}^{k_i}$ , for  $i > 1$ :

$$\begin{aligned} \bar{V} = & (\text{ATOMS}_S^k \rightarrow \mathbb{K}^{\ell}) \times (\text{ATOMS}_{\neg S}^{k_2} \rightarrow \mathbb{K}^{\ell_2}) \\ & \times \dots \times (\text{ATOMS}_{\neg S}^{k_m} \rightarrow \mathbb{K}^{\ell_m}). \end{aligned} \quad (14)$$

Formally speaking, the space  $\bar{V}$  is not in canonical form and it is not even clear how its atom dimension would be defined. The canonical form may be easily recovered by "eliminating" atoms from  $S$ . This is tackled formally below.

We now proceed to defining  $\bar{P}$  and  $\bar{\mathbf{t}}$ . Note that for every vector  $\mathbf{v} \in P'$ , the domain of its main component  $\tilde{\mathbf{v}}$  is included in  $\text{ATOMS}_{\neg S}^k$ . Our aim is to replace every vector  $\mathbf{v} \in P'$  by a finite set of vectors  $\bar{\mathbf{v}}$  whose domain, after projecting to the main component, is disjoint from  $\text{ATOMS}_{\neg S}^k$ . Likewise we aim at replacing  $\mathbf{t}$  by a vector  $\bar{\mathbf{t}}$ , while preserving solvability.

In the sequel we fix an arbitrary total order on  $S$ . Let  $\mathcal{O}$  denote the set of all total orders  $<$  on  $\text{ATOMS} \setminus S$ . Given an order  $<$  in  $\mathcal{O}$ , for every  $k$ -set  $A \subseteq \text{ATOMS} \setminus S$  the restriction of  $<$  to  $A$  uniquely induces an (order preserving) bijection  $\sigma_A^< : A \rightarrow S$ . For a finitary vector  $\mathbf{w} \in \tilde{V} = \text{ATOMS}^{(k)} \rightarrow_{\text{fs}} \mathbb{K}^{\ell}$  we define a finitary vector  $\Delta^< \mathbf{w} \in \tilde{V}$  as follows:

$$\Delta^< \mathbf{w} = \sum_{A \subseteq \text{ATOMS} \setminus S, |A|=k} [\sigma_A^<](\mathbf{w} \upharpoonright_{A^{(k)}}). \quad (15)$$

The sum is infinite but well-defined for finitary vectors  $\mathbf{w} \in \bar{V}$ , as only finitely many cogs  $[\sigma_A^<](\mathbf{w} \upharpoonright_{A^{(k)}})$  are non-zero,

namely only when  $A^{(k)} \cap \text{dom}(\mathbf{w}) \neq \emptyset$ . For every  $< \in \mathcal{O}$ , the function  $\mathbf{w} \mapsto \Delta^< \mathbf{w}$  is a linear mapping (from  $\tilde{V}$  to  $\tilde{V}$ ), and in consequence so is the function  $\mathbf{v} \mapsto \mathbf{v} - \Delta^< \tilde{\mathbf{v}}$ :

**Claim 12.** *For every  $< \in \mathcal{O}$ , the function  $\mathbf{v} \mapsto \mathbf{v} - \Delta^< \tilde{\mathbf{v}}$  is a linear mapping from  $V$  to  $\bar{V}$ .*

Using Claim 10 we observe that  $\Delta^< \mathbf{w}(a) = \mathbf{w}(a)$  for every  $a \in \text{ATOMS}_{\neg S}^k$ . We define

$$\bar{\mathbf{v}} = \{ \mathbf{v} - \Delta^< \tilde{\mathbf{v}} \mid < \in \mathcal{O} \} \quad (16)$$

and derive, using the above observation:

**Claim 13.** *For every  $\mathbf{v} \in P'$  we have  $\bar{\mathbf{v}} \subseteq \bar{V}$ .*

Since all vectors  $\mathbf{v} \in P'$  are finitary, the set  $\bar{\mathbf{v}}$  is finite for every  $\mathbf{v} \in P'$ , even if  $<$  ranges in (16) over all uncountably many total orders  $< \in \mathcal{O}$ .

We define  $\bar{P} := \bigcup_{\mathbf{v} \in P'} \bar{\mathbf{v}}$  and derive  $\bar{P} \subseteq \bar{V}$  by the last claim. We also define  $\bar{\mathbf{t}} := \mathbf{t} - \Delta^{<_0} \tilde{\mathbf{t}}$  for some fixed arbitrarily chosen total order  $<_0 \in \mathcal{O}$ . We observe that the mapping  $\mathbf{v} \mapsto \bar{\mathbf{v}}$  is supported by  $S$ , since the set  $\mathcal{O}$  of total orders is supported by  $S$ . In consequence,  $\bar{P}$  is supported by  $\text{sup}(P') = S \cup \text{sup}(P)$ . As an orbit-finite union of orbit-finite sets is always orbit-finite [1, Exercise 62, Sect. 3], so is also an orbit-finite union of finite sets, and we have:

**Claim 14.**  *$\bar{P}$  is orbit-finite.*

Referring to Remark 1 we may state:

**Claim 15.**  *$(\bar{V}, \bar{P}, \bar{\mathbf{t}})$  is computable from  $(V, P, \mathbf{t})$ .*

**Correctness.** Before proving correctness, we need to state and prove two key technical facts: cogs appearing in (15) are spanned by vectors from  $P'$ , and so is also the vector  $\Delta^{<_0} \tilde{\mathbf{t}}$ . Our notation below relies on the implicit embedding of  $\tilde{V}$  into  $V = \tilde{V} \times V'$ , cf. (10), which allows us to consider every vector  $\mathbf{w} \in \tilde{V}$ , in particular every cog, as a vector in  $V$ .

**Claim 16.** *For every  $< \in \mathcal{O}$ , vector  $\mathbf{w} \in (\bar{P}')$  and a  $k$ -set  $A \subseteq \text{ATOMS} \setminus S$ ,*

$$[\sigma_A^<](\mathbf{w} \upharpoonright_{A^{(k)}}) \in \text{FIN-SPAN}(P').$$

*Proof.* Let  $\mathbf{v} \in P'$  be any vector such that  $\tilde{\mathbf{v}} = \mathbf{w}$ . Thus  $\text{sup}(\mathbf{v}) \cap S = \emptyset$ . For every  $I \subseteq A$ , we extend  $(\sigma_A^<)_I : A \rightarrow A \cup S$  to an atom automorphism  $\sigma_I \in \text{AUT}$  that acts as identity on  $\text{sup}(\mathbf{v}) \setminus A$ . We are going to show that  $[\sigma_A^<](\mathbf{w} \upharpoonright_{A^{(k)}})$  is equal to the following linear combination of vectors from  $P'$  (cf. the definition (12) of cogs):

$$[\sigma_A^<](\mathbf{w} \upharpoonright_{A^{(k)}}) = \sum_{I \subseteq A} (-1)^{|I|} \cdot \sigma_I(\mathbf{v}). \quad (17)$$

Recalling the implicit embedding of  $\tilde{V}$  and  $V'$  into  $V = \tilde{V} \times V'$ , we present  $\mathbf{v}$  as the sum  $\mathbf{v} = \mathbf{w} + \mathbf{v}'$  (recall (10)), where  $\mathbf{v}'$  is the projection to all non-main components. Furthermore,

we decompose  $\mathbf{w}$  into  $\mathbf{w} = \mathbf{w} \upharpoonright_{A^{(k)}} + \mathbf{w}'$ . Thus the right-hand side in (17) decomposes into three summands:

$$\sum_{I \subseteq A} (-1)^{|I|} \cdot \sigma_I(\mathbf{v}') + \sum_{I \subseteq A} (-1)^{|I|} \cdot \sigma_I(\mathbf{w}') + \sum_{I \subseteq A} (-1)^{|I|} \cdot \sigma_I(\mathbf{w} \upharpoonright_{A^{(k)}}).$$

The last one is equal to the left-hand side in (17) and hence it is sufficient to show that the first two summands are zero vectors. Denote the first two summands as  $\mathbf{s}_1$  and  $\mathbf{s}_2$ , respectively. Recall that, given a tuple of atoms  $b$  in the domain of  $\mathbf{s}_1$  or  $\mathbf{s}_2$ , respectively, we have

$$\begin{aligned} \mathbf{s}_1(b) &= \sum_{I \subseteq A} (-1)^{|I|} \cdot \mathbf{v}'(\sigma_I^{-1}(b)) \\ \mathbf{s}_2(b) &= \sum_{I \subseteq A} (-1)^{|I|} \cdot \mathbf{w}'(\sigma_I^{-1}(b)). \end{aligned} \quad (18)$$

In each of the two summands, every tuple  $b = (b_1, \dots, b_{k'})$  in the domain contains less than  $k$  elements of  $A \cup S$ :

$$|\{b_1, \dots, b_{k'}\} \cap (A \cup S)| < k. \quad (19)$$

In case of  $\mathbf{s}_1$  the reason is that the domain of every non-main component contains tuples  $b \in \text{ATOMS}^{k'}$  of atoms of length  $k' < k = |A|$ . In case of  $\mathbf{s}_2$ , while  $b \in \text{ATOMS}^{(k)}$ , the reason is twofold: first,  $\sigma_I^{-1}(b) \notin A^{(k)}$  which implies  $|\{b_1, \dots, b_{k'}\} \cap A| < k$ ; second,  $S \cap \text{sup}(\mathbf{w}) = \emptyset$  which implies  $|\{b_1, \dots, b_{k'}\} \cap S| = 0$ . Due to the property (19), for every tuple  $b$  in the domain of a respective vector  $\mathbf{s}_1$  or  $\mathbf{s}_2$ , when  $I$  ranges over all subsets of  $A$ , each tuple  $\sigma_I^{-1}(b)$  appears as many times for  $I$  of odd size as for  $I$  of even size. In consequence all these appearances cancel out and, whatever the vectors  $\mathbf{v}'$  and  $\mathbf{w}'$  and tuple  $b$  are, the right-hand sides in the two equalities (18) are necessarily zero vectors. This completes the proof of Claim 16.  $\square$

Using Claim 16 one further shows:

**Claim 17.**  $\Delta^0 \tilde{\mathbf{t}} \in \text{FIN-SPAN}(P')$ .

It remains to show:

**Claim 18.**  $\mathbf{t} \in \text{FIN-SPAN}(P')$  if and only if  $\tilde{\mathbf{t}} \in \text{FIN-SPAN}(\bar{P})$ .

*Proof.* The ‘only if’ direction is immediate due to Claim 12: if  $\mathbf{t} \in \text{FIN-SPAN}(P')$ , i.e., for some  $\ell \in \mathbb{N}$  and  $q_1, \dots, q_\ell$  and  $\mathbf{v}_1, \dots, \mathbf{v}_\ell \in P'$  we have:

$$\mathbf{t} = q_1 \cdot \mathbf{v}_1 + \dots + q_\ell \cdot \mathbf{v}_\ell$$

then by Claim 12 applied to the total order  $<_0 \in \mathcal{O}$  we also have:

$$\tilde{\mathbf{t}} = q_1 \cdot (\mathbf{v}_1 - \Delta^0 \tilde{\mathbf{v}}_1) + \dots + q_\ell \cdot (\mathbf{v}_\ell - \Delta^0 \tilde{\mathbf{v}}_\ell). \quad (20)$$

Therefore  $\tilde{\mathbf{t}} \in \text{FIN-SPAN}(\bar{P})$ .

For ‘if’ direction we assume  $\tilde{\mathbf{t}} \in \text{FIN-SPAN}(\bar{P})$ , i.e., for some  $\ell \in \mathbb{N}$  and  $q_1, \dots, q_\ell, \mathbf{v}_1, \dots, \mathbf{v}_\ell \in P'$ , and  $<_1, \dots, <_\ell \in \mathcal{O}$ , we have:

$$\mathbf{t} - \Delta^0 \tilde{\mathbf{t}} = q_1 \cdot (\mathbf{v}_1 - \Delta^1 \tilde{\mathbf{v}}_1) + \dots + q_\ell \cdot (\mathbf{v}_\ell - \Delta^\ell \tilde{\mathbf{v}}_\ell).$$

By Claim 16 we know that  $\Delta^k \mathbf{w} \in \text{FIN-SPAN}(P')$  for every  $\mathbf{w} \in \tilde{P}'$  and  $< \in \mathcal{O}$  (since the sum in (15) is essentially finite), and hence the right-hand side is in  $\text{FIN-SPAN}(P')$ . By Claim 17 we get  $\mathbf{t} \in \text{FIN-SPAN}(P')$ , as required.  $\square$

**Canonical form.**  $(\bar{V}, \bar{P}, \tilde{\mathbf{t}})$  is not a formally correct instance as the space  $\bar{V}$  (14) is not in canonical form, and furthermore neither  $\bar{V}$  nor  $\bar{P}$  are equivariant.  $(\bar{V}, \bar{P}, \tilde{\mathbf{t}})$  may be however easily transformed further into a formally correct instance as follows. Consider the partition of  $\text{ATOMS}_S^k$  into  $S$ -orbits:

$$\text{ATOMS}_S^k = O_1 \uplus \dots \uplus O_{m'},$$

for  $m' \in \mathbb{N}$ , an observe that the main component  $\text{ATOMS}_S^k \rightarrow \mathbb{K}^\ell$  of  $\bar{V}$  is isomorphic to

$$(O_1 \rightarrow \mathbb{K}^\ell) \times \dots \times (O_{m'} \rightarrow \mathbb{K}^\ell). \quad (21)$$

In case of all other components  $i = 2, \dots, m'$ , the set  $\text{ATOMS}_{-S}^{k_i}$  is a single  $S$ -orbit. For  $i = 1, \dots, m'$ , let  $r_i > 0$  denote the  $S$ -atom dimension of  $O_i$ . As each  $O_i$  is related by an  $S$ -supported isomorphism to  $\text{ATOMS}_{-S}^{r_i}$ , the vector space (21) is related by an  $S$ -supported isomorphism to

$$(\text{ATOMS}_{-S}^{r_1} \rightarrow \mathbb{K}^\ell) \times \dots \times (\text{ATOMS}_{-S}^{r_{m'}} \rightarrow \mathbb{K}^\ell).$$

We group together orbits with the same atom dimension  $r_i$ : there are some pairwise different  $r'_1, \dots, r'_p \in \mathbb{N}$ , and some positive  $\ell'_1, \dots, \ell'_p \in \mathbb{N}$ , such that the main component of  $\bar{V}$  is related by an  $S$ -supported isomorphism to the subspace

$$(\text{ATOMS}_{-S}^{r'_1} \rightarrow \mathbb{K}^{\ell'_1}) \times \dots \times (\text{ATOMS}_{-S}^{r'_p} \rightarrow \mathbb{K}^{\ell'_p}) \quad (22)$$

of the vector space in canonical form

$$(\text{ATOMS}^{(r'_1)} \rightarrow_{\text{fin}} \mathbb{K}^{\ell'_1}) \times \dots \times (\text{ATOMS}^{(r'_p)} \rightarrow_{\text{fin}} \mathbb{K}^{\ell'_p}). \quad (23)$$

Relying on (14) we deduce that the whole vector space  $\bar{V}$  is also related by an  $S$ -supported isomorphism to the subspace (22) of some vector space of similar form (23). Denote the latter vector space by  $\bar{V}'$ , and observe that the subspace (22) is exactly  $\bar{V}'_S$  (as defined in (13)).

Applying the above  $S$ -supported isomorphism also to  $\bar{P}$  and  $\tilde{\mathbf{t}}$ , we get an instance  $(\bar{V}', \bar{P}', \tilde{\mathbf{t}}')$  equisolvable with  $(\bar{V}, \bar{P}, \tilde{\mathbf{t}})$ . Finally, we replace  $\bar{P}'$  by its equivariant closure

$$\bar{P}'' = \left\{ \pi(\mathbf{v}) \mid \pi \in \text{AUT}, \mathbf{v} \in \bar{P}' \right\}$$

(therefore  $\bar{P}' = \bar{P}'' \cap \bar{V}'_S$ ) and deduce using Claim 11 that the so obtained instance  $(\bar{V}', \bar{P}'', \tilde{\mathbf{t}}')$  is equisolvable with  $(\bar{V}', \bar{P}', \tilde{\mathbf{t}}')$ . The transformation from  $(\bar{V}, \bar{P}, \tilde{\mathbf{t}})$  to  $(\bar{V}', \bar{P}'', \tilde{\mathbf{t}}')$  is effective (cf. Remark 1). Finally, each  $r'_i$  in (23) is strictly smaller than  $k$  and hence the atom dimension of  $\bar{V}'$  is smaller than that of  $V$ . We have thus shown:

**Claim 19.**  $(\bar{V}', \bar{P}'', \tilde{\mathbf{t}}')$  in canonical form is computable from  $(\bar{V}, \bar{P}, \tilde{\mathbf{t}})$ , it is equisolvable with  $(\bar{V}, \bar{P}, \tilde{\mathbf{t}})$ , and  $\bar{V}'$  has smaller atom dimension than  $V$ .

Claims 15, 18 and 19 conclude the proof of Lemma 6.3.  $\square$

## 7 Solvability reduces to finitary solvability

In this section we reduce solvability to finitary solvability:

**Theorem 7.1.**  $\text{SOLV}(\mathbb{K})$  reduces to  $\text{FIN-SOLV}(\mathbb{K})$ .

Let  $\mathbf{A} \in \text{LIN}(B \times C)$  and  $\mathbf{t} \in \text{LIN}(B)$  be the input system. In terms of spans, the solvability problem amounts to deciding if  $\mathbf{t} \in \text{SPAN}(\mathbf{A})$ . We will prove the result by effectively constructing a matrix  $\tilde{\mathbf{A}}$  with the same row-indexing set  $B$  as  $\mathbf{A}$ , such that  $\text{SPAN}(\mathbf{A}) = \text{FIN-SPAN}(\tilde{\mathbf{A}})$ .

**Well-definedness and exactness.** Let  $\mathbf{x} \in \text{LIN}(C)$  a vector. We start by a characterisation of vectors  $\mathbf{x} \in \text{LIN}(C)$  for which the product  $\mathbf{y} = \mathbf{A} \cdot \mathbf{x}$  is well-defined. Recall that  $\mathbf{y}(b)$  is well-defined if and only if there are only finitely many  $c \in C$  such that  $\mathbf{A}(b, c) \neq 0$  and  $\mathbf{x}(c) \neq 0$ . Let  $S = \text{sup}(\mathbf{A}) \cup \text{sup}(\mathbf{x})$ ; in other words,  $S$  is the support of the pair  $(\mathbf{A}, \mathbf{x})$ . We say that the pair  $(\mathbf{A}, \mathbf{x})$  is *exact* if for every  $b \in B$  and  $c \in C$  such that  $\mathbf{A}(b, c) \neq 0$  and  $\mathbf{x}(c) \neq 0$  it holds

$$\text{sup}(c) \subseteq \text{sup}(b) \cup S. \quad (24)$$

**Lemma 7.2.**  $\mathbf{A} \cdot \mathbf{x}$  is well-defined if and only if  $(\mathbf{A}, \mathbf{x})$  is exact.

*Proof.* For the if direction, suppose  $(\mathbf{A}, \mathbf{x})$  is exact, and consider an arbitrary fixed  $b \in B$ . Let  $T = \text{sup}(b) \cup S$ . By (24) the support of every  $c$  satisfying  $\mathbf{A}(b, c) \neq 0$  and  $\mathbf{x}(c) \neq 0$  is included in  $T$ . By Claim 2 in Section 5, for every fixed set  $T' \subseteq T$ , every orbit  $O \subseteq C$  contains at most  $|T'|!$  elements  $c \in O$  such that  $\text{sup}(c) = T'$ , and since  $C$  is orbit-finite, there are only finitely many  $c \in C$  satisfying  $\mathbf{A}(b, c) \neq 0$  and  $\mathbf{x}(c) \neq 0$ . The product  $\mathbf{A} \cdot \mathbf{x}$  is thus well-defined, as required.

For the opposite direction, suppose  $(\mathbf{A}, \mathbf{x})$  is not exact, i.e., for some  $b \in B$  and  $c \in C$  we have:

$$\mathbf{A}(b, c) \neq 0, \quad \mathbf{x}(c) \neq 0, \quad \text{sup}(c) \not\subseteq \text{sup}(b) \cup S.$$

According to the latter condition, some atom  $\alpha \in \text{ATOMS}$  satisfies  $\alpha \in \text{sup}(c)$  and  $\alpha \notin T = \text{sup}(b) \cup S$ . Note that every  $T$ -atom automorphism preserves  $b$  and  $\mathbf{A}$ , and hence also preserves the row vector  $\mathbf{A}(b, \_)$ . Consider an infinite family of  $T$ -automorphisms  $\pi$  that map  $\alpha$  to different atoms  $\pi(\alpha) \notin T$ . For every such  $\pi$  we have  $\pi(c) \neq c$ , but  $\mathbf{A}(b, \pi(c)) = \mathbf{A}(b, c) \neq 0$ . Furthermore, every such  $\pi$  preserves  $\mathbf{x}$ , and hence we have  $\mathbf{x}(\pi(c)) = \mathbf{x}(c) \neq 0$ . In consequence, there are infinitely many  $c \in C$  such that  $\mathbf{A}(b, c) \neq 0$  and  $\mathbf{x}(c) \neq 0$ , i.e., the product  $\mathbf{A} \cdot \mathbf{x}$  is not well-defined on  $b$ . This completes the proof.  $\square$

The following lemma is a crucial tool in our proof:

**Lemma 7.3.** Let  $B$  be an orbit-finite set,  $T \subseteq_{\text{fin}} \text{ATOMS}$ , and  $C$  a  $T$ -orbit. Let  $\mathbf{A} \in \text{LIN}(B \times C)$  be a  $T$ -supported matrix and  $\mathbf{v} \in \text{LIN}(C)$  a vector. If  $\mathbf{A} \cdot \mathbf{v}$  is well-defined and  $\mathbf{1}_{O'} \in \widehat{C}$  appears in the basis representation of  $\mathbf{v}$  then  $\mathbf{A} \cdot \mathbf{1}_{O'}$  is well-defined too.

*Proof.* By Claim 5 in Section 5 assume w.l.o.g. that  $C$  is a tight  $T$ -orbit. Let  $\mathbf{v} \in \text{LIN}(C)$ , and let  $S = \text{sup}(\mathbf{v}) \cup T$ . We follow the definition of the basis representation of  $S$ -supported

vector  $\mathbf{v}$  by structural induction with respect to the transitive closure of  $<$ , as in the proof of Lemma 5.5. If  $S\text{-orbit-dom}(\mathbf{v})$  is empty then  $\mathbf{v}$  is the zero vector and the claim holds vacuously. Otherwise, suppose the claim holds for all strictly smaller  $S$ -supported vectors  $\mathbf{w}$ . As in the proof of Lemma 5.5, take an  $S$ -orbit  $O \in S\text{-orbit-dom}(\mathbf{v})$  of maximal  $S$ -dimension. Let

$$S' := \text{sup}(c) \cap S \quad (25)$$

for some (every)  $c \in O$ . Since  $T \subseteq S$  and  $T \subseteq \text{sup}(c)$  (as  $C$  is tight), we deduce  $T \subseteq S'$ . We define the  $S'$ -orbit  $O'$  as  $S'$ -closure of  $O$ :

$$O' := \{ \pi(c) \mid c \in O, \pi \in \text{AUT}_{S'} \}.$$

By definition,  $S'$  is included in the support of every element of  $O'$ , therefore the orbit  $O'$  is tight, and hence  $\mathbf{1}_{O'} \in \widehat{C}$ . According to the proof of Lemma 5.5, the vector  $\mathbf{1}_{O'}$  appears in the basis representation of  $\mathbf{v}$ , together with the vectors appearing in the basis representation of the vector

$$\mathbf{w} := \mathbf{v} - \mathbf{v}(O) \cdot \mathbf{1}_{O'}. \quad (26)$$

Note that  $\mathbf{w}$  is supported by  $S$  as both  $\mathbf{v}$  and  $\mathbf{1}_{O'}$  are so, and  $\mathbf{w}(O) = 0$ . By Claim 7 we infer that  $\mathbf{w} < \mathbf{v}$  and therefore, relying on the induction assumption  $\mathbf{w}$ , it is sufficient to show that  $\mathbf{A} \cdot \mathbf{1}_{O'}$  is well-defined.

According to the assumption and Lemma 7.2 we know that  $(\mathbf{A}, \mathbf{v})$  is exact. Using Lemma 7.2 again, it is sufficient to show that  $(\mathbf{A}, \mathbf{1}_{O'})$  is exact too.

Choose an arbitrary element  $c \in O'$  and  $b \in B$  such that  $\mathbf{A}(b, c) \neq 0$ , and an arbitrary  $S'$ -atom automorphism  $\pi$  such that  $\pi(c) \in O$ .  $\mathbf{A}$  is  $T$ -supported so it is also  $S'$ -supported (since  $T \subseteq S'$ ). Hence  $\mathbf{A}(\pi(b), \pi(c)) \neq 0$ . As  $(\mathbf{A}, \mathbf{v})$  is exact and  $\mathbf{v}(O) \neq 0$ , we have:

$$\text{sup}(\pi(c)) \subseteq \text{sup}(\pi(b)) \cup S. \quad (27)$$

By definition (25) of  $S'$ , as  $\pi(c) \in O$ , we have  $S' = \text{sup}(\pi(c)) \cap S$ , and thus the inclusion (27) can be strengthened to

$$\text{sup}(\pi(c)) \subseteq \text{sup}(\pi(b)) \cup S'.$$

Application of  $\pi^{-1}$  to both sides yields  $\text{sup}(c) \subseteq \text{sup}(b) \cup S'$ . As  $b$  and  $c$  were chosen arbitrarily, we conclude that  $(\mathbf{A}, \mathbf{1}_{O'})$  is exact, as required.  $\square$

**Proof of Theorem 7.1.** Consider a system of equations  $(\mathbf{A}, \mathbf{t})$  where  $\mathbf{A} \in \text{LIN}(B \times C)$  is a matrix and  $\mathbf{t} \in \text{LIN}(B)$ . Let  $T = \text{sup}(\mathbf{A})$ . Thus  $B$  and  $C$  are supported by  $T$  as well.

We are going to construct effectively a matrix  $\tilde{\mathbf{A}}$  with the same row-indexing set  $B$  as  $\mathbf{A}$ , which satisfies  $\text{SPAN}(\mathbf{A}) = \text{FIN-SPAN}(\tilde{\mathbf{A}})$ . We claim that it is enough to consider the special case when  $C$  is a single  $T$ -orbit. Indeed, split the matrix  $\mathbf{A}$  into  $m$  matrices

$$\mathbf{A} = [\mathbf{A}_1 \mid \dots \mid \mathbf{A}_m]$$

each corresponding to one  $T$ -orbit  $C_i \subseteq C$ . Assuming matrices  $\tilde{\mathbf{A}}_i$  such that  $\text{SPAN}(\mathbf{A}_i) = \text{FIN-SPAN}(\tilde{\mathbf{A}}_i)$  for  $i = 1, \dots, m$ , we construct a matrix  $\tilde{\mathbf{A}}$  as

$$\tilde{\mathbf{A}} = [\tilde{\mathbf{A}}_1 \mid \dots \mid \tilde{\mathbf{A}}_m]$$

and claim that  $\text{SPAN}(\mathbf{A}) = \text{FIN-SPAN}(\tilde{\mathbf{A}})$  as well. Indeed,  $\mathbf{v} \in \text{SPAN}(\mathbf{A})$  if and only if  $(*) \mathbf{v} = \mathbf{v}_1 + \dots + \mathbf{v}_m$  where  $\mathbf{v}_i \in \text{SPAN}(\mathbf{A}_i)$  for  $i = 1, \dots, m$ ; replacing  $\mathbf{v}_i \in \text{SPAN}(\mathbf{A}_i)$  by equivalent  $\mathbf{v}_i \in \text{FIN-SPAN}(\tilde{\mathbf{A}}_i)$  for every  $i = 1, \dots, m$ , the claim  $(*)$  is equivalent to  $\mathbf{v} \in \text{FIN-SPAN}(\tilde{\mathbf{A}})$ . We thus proceed under the assumption that  $C$  is a single  $T$ -orbit. Therefore  $\mathbf{A}$  satisfies the assumptions of Lemma 7.3.

As the indexing set of  $\tilde{\mathbf{A}}$  we take those basis vectors  $\mathbf{w} \in \widehat{C}$  for which  $\mathbf{A} \cdot \mathbf{w}$  is well defined:

$$\widetilde{C} = \{ \mathbf{w} \in \widehat{C} \mid \mathbf{A} \cdot \mathbf{w} \text{ is well-defined} \}.$$

The new indexing set  $\widetilde{C}$  is orbit-finite as  $\widehat{C}$  is so, and is  $T$ -supported since both  $\mathbf{A}$  and  $\widehat{C}$  are  $T$ -supported. We define the new matrix  $\tilde{\mathbf{A}} : B \times \widetilde{C} \rightarrow_{\text{fs}} \mathbb{K}$  as follows

$$\tilde{\mathbf{A}}(\_, \mathbf{w}) = \mathbf{A} \cdot \mathbf{w}.$$

Note the injection  $c \mapsto \mathbf{1}_c$  of  $C$  into  $\widetilde{C}$ , as  $\mathbf{A} \cdot \mathbf{1}_c = \mathbf{A}(\_, c)$  is always well-defined. Therefore  $\tilde{\mathbf{A}}$  extends  $\mathbf{A}$ , as  $\tilde{\mathbf{A}}(\_, \mathbf{1}_c) = \mathbf{A} \cdot \mathbf{1}_c = \mathbf{A}(\_, c)$ . It is now sufficient to prove:

**Claim 20.**  $\text{SPAN}(\mathbf{A}) = \text{FIN-SPAN}(\tilde{\mathbf{A}})$ .

*Proof.* W.l.o.g. we assume that  $\mathbf{A}$  contains non-zero column vectors only (otherwise, since  $C$  is a single orbit, all column vectors in  $\mathbf{A}$  are zero vectors and the claim holds vacuously). In one direction, consider any vector  $\mathbf{v} \in \text{FIN-SPAN}(\tilde{\mathbf{A}})$ , i.e.,

$$\mathbf{v} = q_1 \cdot (\mathbf{A} \cdot \mathbf{w}_1) + \dots + q_n \cdot (\mathbf{A} \cdot \mathbf{w}_n)$$

for  $q_1, \dots, q_n \in \mathbb{K}$  and  $\mathbf{w}_1, \dots, \mathbf{w}_n \in \widetilde{C}$ , which immediately yields the required membership in  $\text{SPAN}(\mathbf{A})$ :

$$\mathbf{v} = \mathbf{A} \cdot (q_1 \cdot \mathbf{w}_1 + \dots + q_n \cdot \mathbf{w}_n) \in \text{SPAN}(\mathbf{A}).$$

In the opposite direction, let  $\mathbf{v} = \mathbf{A} \cdot \mathbf{x}$  be well-defined for some  $\mathbf{x} \in \text{LIN}(C)$ . We are going to prove that  $\mathbf{v} \in \text{FIN-SPAN}(\tilde{\mathbf{A}})$ . Consider the representation of  $\mathbf{x}$  in the basis  $\widehat{C}$ :

$$\mathbf{x} = q_1 \cdot \mathbf{w}_1 + \dots + q_\ell \cdot \mathbf{w}_\ell.$$

Due to Lemma 7.3 we know that  $\mathbf{A} \cdot \mathbf{w}_i$  is well-defined and hence  $\mathbf{w}_i \in \widetilde{C}$  for all  $i = 1, \dots, \ell$ . Therefore

$$\begin{aligned} \mathbf{v} &= \mathbf{A} \cdot (q_1 \cdot \mathbf{w}_1 + \dots + q_\ell \cdot \mathbf{w}_\ell) = \\ &= q_1 \cdot (\mathbf{A} \cdot \mathbf{w}_1) + \dots + q_\ell \cdot (\mathbf{A} \cdot \mathbf{w}_\ell) = \\ &= q_1 \cdot \tilde{\mathbf{A}}(\_, \mathbf{w}_1) + \dots + q_\ell \cdot \tilde{\mathbf{A}}(\_, \mathbf{w}_\ell) \in \text{FIN-SPAN}(\tilde{\mathbf{A}}), \end{aligned}$$

as required.  $\square$

As discussed in Remark 1, the transformation from  $\mathbf{A}$  to  $\tilde{\mathbf{A}}$  is effective. This completes the proof of Theorem 7.1.  $\square$

**Complexity.** We conclude with a rough estimation of complexity with respect to the number of orbits in  $B$  and  $C$ , and the atom dimension of the input system  $(\mathbf{A}, \mathbf{t})$  defined as the largest atom dimension of each of its orbits, plus the size of its support.

The blow-up of reduction of Theorem 7.1 is exponential in the atom dimension of input, but polynomial in the number of orbits in  $B$  and  $C$ . Likewise is the number and size of finite systems of equations that are produced in the procedure of Theorem 6.1. Summing up, the combined algorithm for  $\text{SOLV}(\mathbb{K})$  produces exponentially many finite systems of exponential size (polynomially many finite systems of polynomial size, when atom dimension of input is fixed), and answers positively exactly when all these systems are solvable.

In the two most significant special cases, namely  $\mathbb{K} = \mathbb{Q}$  or  $\mathbb{K} = \mathbb{Z}$ , finite systems are solvable in P. Therefore, the problems  $\text{SOLV}(\mathbb{Q})$  and  $\text{SOLV}(\mathbb{Z})$  are in EXP TIME, and likewise are  $\text{FIN-SOLV}(\mathbb{Q})$  and  $\text{FIN-SOLV}(\mathbb{Z})$ . When atom dimension of input is fixed, all these problems are in P.

## 8 Final remarks

We have shown decidability of solvability of orbit-finite systems of linear equations over an arbitrary effective commutative ring. We expect applicability of this general result in various corners; as a first example, combining our result with the insight of [16] leads to decidability of reachability in integer-relaxation of data-enriched Petri nets.

We leave a lot of questions for further research—here we list the most important ones. First, the immediate next step is to compute the whole solution sets represented, for instance, as a (coset of) an orbit-finitely spanned vector subspace. Second, an intriguing open question is whether solvability is still decidable if the finite-support restriction on solutions is dropped (like in [17])? Furthermore, an important restriction on solutions is *nonnegativity*, as it allows to model systems of inequalities. According to our preliminary results  $\text{FIN-SOLV}(\mathbb{Q})$ ,  $\text{FIN-SOLV}(\mathbb{Z})$  and  $\text{SOLV}(\mathbb{Q})$  are decidable under the nonnegativity restriction, but we don't know the status of  $\text{SOLV}(\mathbb{Z})$ . Finally, in this paper we have exclusively considered equality atoms and are very curious about other richer structures. For instance, concerning ordered atoms, the results of [13] indicate a huge increase of complexity of solvability, compared to equality atoms.

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## A Proofs missing in Sections 5–6

**Claim 2.** *Let  $S \subseteq_{fin} \text{ATOMS}$ . Each equivariant orbit  $O$  contains at most  $|S|!$  many elements  $x$  with  $\text{sup}(x) = S$ .*

*Proof.* Fix some element  $x \in O$  with  $\text{sup}(x) = S$  and consider  $\pi(x)$  for all  $\pi \in \text{AUT}$ , thus ranging over all elements of  $O$ . By the definition of support, if  $\pi$  and  $\pi'$  agree on  $S$  then  $\pi(x) = \pi'(x)$ . Under the condition  $\text{sup}(\pi(x)) = S$ , i.e.  $\pi(S) = S$  (by Claim 1), there are only  $|S|!$  different possibilities for  $\pi$  restricted to  $S$ , and hence at most that many different elements  $\pi(x)$ .  $\square$

**Claim 3.** *Every orbit is either a singleton or an infinite set.*

*Proof.* Consider an  $S$ -orbit  $O$  and some element  $x \in O$ . If  $\text{sup}(x) \subseteq S$  then every  $S$ -atom automorphism  $\pi \in \text{AUT}_S$  preserves  $x$ ,  $\pi(x) = x$ , and hence  $O = \{x\}$ . Otherwise, choose any  $\alpha \in \text{sup}(x) \setminus S$  and consider, for each  $\beta \in \text{ATOMS} \setminus (\text{sup}(x) \cup S)$ , some arbitrary  $S$ -atom automorphisms  $\pi_\beta$  that map  $\alpha$  to  $\beta$  and preserves  $\text{sup}(x) \setminus \{\alpha\}$ . By Claim 1,  $\text{sup}(\pi_\beta(x)) \neq \text{sup}(\pi_\gamma(x))$  for  $\beta \neq \gamma$ , which implies  $\pi_\beta(x) \neq \pi_\gamma(x)$  for  $\beta \neq \gamma$ . Therefore  $O$  is infinite.  $\square$

**Claim 5.** *Let  $S \subseteq_{fin} \text{ATOMS}$ . Every  $S$ -orbit  $O$  is in an  $S$ -supported bijection with a tight  $S$ -orbit.*

*Proof.* Given an  $S$ -orbit  $O$ , the mapping  $x \mapsto (x, S)$  is the required  $S$ -supported bijection between  $O$  and the tight  $S$ -orbit  $\{(x, S) \mid x \in O\}$ .  $\square$

**Claim 9.** *Local solvability is decidable.*

*Proof.* Consider an instance  $(V, P, \mathbf{t})$ . Let  $k$  be the atom dimension of  $V$  and let  $T = \text{sup}(P) \cup \text{sup}(\mathbf{t})$ . The set of all  $k$ -sets  $A \in \binom{\text{ATOMS}}{k}$  splits into finitely many  $T$ -orbits (exponentially many with respect to  $k$ ), and for two such  $k$ -sets in the same  $T$ -orbit the resulting restrictions are also in the same  $T$ -orbit. Therefore the set of  $A$ -restrictions of the instance, for all  $k$ -sets  $A$ , splits also into finitely many  $T$ -orbits.

To check local solvability it is enough to checking solvability of a representative of each  $T$ -orbit, i.e., solvability of a finite number of finite systems of linear equations.  $\square$

**Claim 17.**  $\Delta^{\leq 0} \tilde{\mathbf{t}} \in \text{FIN-SPAN}(P')$ .

*Proof.* Let  $\mathbf{u} = \tilde{\mathbf{t}} \in \tilde{V}$ . We use local solvability of the instance  $(V, P', \mathbf{t})$ : for every  $k$ -set  $A \subseteq \text{ATOMS}$ ,

$$\mathbf{u} \upharpoonright_{A^{(k)}} \in \text{FIN-SPAN}(\{ \mathbf{w} \upharpoonright_{A^{(k)}} \mid \mathbf{w} \in \widetilde{(P')} \}). \quad (28)$$

We consider below only these finitely many subsets  $A$  for which  $A^{(k)} \cap \text{dom}(\mathbf{u}) \neq \emptyset$ . In consequence of (28), and because the mapping  $\mathbf{w} \upharpoonright_{A^{(k)}} \mapsto [\sigma_A^{\leq 0}](\mathbf{w} \upharpoonright_{A^{(k)}})$  is linear, for

every such  $A$  we have:

$$[\sigma_A^{\leq 0}](\mathbf{u} \upharpoonright_{A^{(k)}}) \in \text{FIN-SPAN}(\{ [\sigma_A^{\leq 0}](\mathbf{w} \upharpoonright_{A^{(k)}}) \mid \mathbf{w} \in \widetilde{(P')} \}).$$

As  $S \cap \text{sup}(\mathbf{t}) = \emptyset$ , we have  $S \cap \text{sup}(\mathbf{u}) = \emptyset$  and hence we know that all considered subsets  $A$  satisfy  $A \subseteq \text{ATOMS} \setminus S$ . We can thus apply Claim 16 to all  $\mathbf{w}$  involved in the linear combination above, thus obtaining:

$$[\sigma_A^{\leq 0}](\mathbf{u} \upharpoonright_{A^{(k)}}) \in \text{FIN-SPAN}(P').$$

Finally, the vector  $\Delta^{\leq 0} \mathbf{u}$ , being a finite sum of cogs of the form  $[\sigma_A^{\leq 0}](\mathbf{u} \upharpoonright_{A^{(k)}})$ , for finitely many subsets  $A$  for which  $A^{(k)} \cap \text{dom}(\mathbf{u}) \neq \emptyset$ , is also in  $\text{FIN-SPAN}(P')$ , as required.  $\square$