

Exploiting constant trace property in large-scale polynomial optimization

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Abstract

We prove that every semidefinite moment relaxation of a polynomial optimization problem (POP) with a ball constraint can be reformulated as a semidefinite program involving a matrix with constant trace property (CTP). As a result such moment relaxations can be solved efficiently by first-order methods that exploit CTP, e.g., the conditional gradient-based augmented Lagrangian method. We also extend this CTP-exploiting framework to large-scale POPs with different sparsity structures. The efficiency and scalability of our framework are illustrated on second-order moment relaxations for various randomly generated quadratically constrained quadratic programs.

Keywords: polynomial optimization, moment-SOS hierarchy, conditional gradient-based augmented Lagrangian, constant trace property, semidefinite programming

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1 Introduction

This paper is in the line of recent efforts to promote first-order methods as a viable alternative to interior-point methods (IPM) for solving large-scale conic optimization problems, in particular large-scale semidefinite programming (SDP) relaxations of polynomial optimization problems (POPs). We show that a wide class of POPs have a nice property, namely the constant trace property (CTP), and that this property can be exploited in combination with first-order methods to solve large-scale SDP relaxations associated with a POP. So far, this property has been exploited only in a few cases, the most prominent examples being the Shor’s relaxation of Max-Cut [45], in which the authors are able to handle SDP matrices of huge size, and equality constrained POPs on the sphere [28].

Given polynomials f, g_i, h_j , let us consider the following POP with n variables, m inequality constraints and l equality constraints:

$$f^* := \min\{f(\mathbf{x}) : g_i(\mathbf{x}) \geq 0, i \in [m], h_j(\mathbf{x}) = 0, j \in [l]\}, \quad (1.1)$$

Table 1: Complexity comparison of several methods for solving SDP. IP: interior point methods; ADMM: the alternating direction method of multipliers; SBM: spectral bundle methods; CGAL: conditional gradient-based augmented Lagrangian.

Method	Software	SDP type	Convergence rate	The most expensive parts per iteration
IP [15] (second-order)	Mosek [1]	Arbitrary	$\mathcal{O}(\log(1/\varepsilon))$ [32]	System of linear equations solving with $\mathcal{O}((s^{\max})^6)$ [31, Table 1]
ADMM [3] (first-order)	SCS [30], COSMO [9]	Arbitrary	$\mathcal{O}(\varepsilon^{-1})$ [17]	Positive definite system of linear equations solving by LDL^\top -decomposition with $\mathcal{O}((s^{\max})^6)$
SBM [14] (first-order)	ConicBundle [13]	with CTP	$\mathcal{O}(\log(1/\varepsilon)/\varepsilon)$ [7]	Positive definite linear system solving with $\mathcal{O}((s^{\max})^6)$
CGAL [44] (first-order)	SketchyCGAL [45]	with CTP	$\mathcal{O}(\varepsilon^{-1/2})$	Smallest eigenvalue computing by the Arnoldi iteration with $\mathcal{O}(s^{\max})$ [26]

where $[m] := \{1, \dots, m\}$ and $[l] := \{1, \dots, l\}$. In general POP (1.1) is non-convex, NP-hard. It is well known that under some mild condition, the optimal value f^* of POP (1.1) can be approximated as closely as desired by the so-called Moment-Sums of squares (Moment-SOS) hierarchy [22]. There are a lot of important applications of POP (1.1) as well as the Moment-SOS hierarchy; the interested readers are referred to the monograph [16].

Computational cost of moment relaxations. The k -th order moment relaxation for POP (1.1) can be rewritten in compact form as the following standard SDP:

$$\tau = \inf_{\mathbf{X} \in \mathcal{S}^+} \{ \langle \mathbf{C}, \mathbf{X} \rangle : \langle \mathbf{A}_j, \mathbf{X} \rangle = b_j, j \in [\zeta] \}, \quad (1.2)$$

where \mathcal{S}^+ is the set of positive semidefinite (psd) matrices in a block diagonal form: $\mathbf{X} = \text{diag}(\mathbf{X}_1, \dots, \mathbf{X}_\omega)$ with \mathbf{X}_j being a block of size $s^{(j)}$, $j \in [\omega]$ and ζ is the number of affine constraints. We denote the largest block size by $s^{\max} := \max_{j \in [\omega]} s^{(j)}$.

We say that SDP (1.2) has *constant trace property* (CTP) if there exists a positive real number a such that $\text{trace}(\mathbf{X}) = a$, for all feasible solution \mathbf{X} of SDP (1.2). We also say that POP (1.1) has CTP when every moment relaxation of POP (1.1) has CTP.

Table 1 lists several available methods for solving SDP (1.2). In particular, observe that two of them, CGAL and SBM, are first-order methods that exploit CTP. In [45], the authors combined CGAL with the Nyström sketch (named SketchyCGAL), which require dramatically less storage than other methods and is very efficient for solving Shor's relaxation of large-scale MAX-CUT instances.

Note that SDP-relaxation (1.2) of POP (1.1) at step k of the Moment-SOS hierarchy has $\omega = m + 1$ blocks whose largest size is $s^{\max} = \binom{n+k}{n}$ while the number of affine constraints is $\zeta = \mathcal{O}(\binom{n+k}{n}^2)$. Thus the computational cost for solving SDP (1.2) grows very rapidly with k . Fortunately, it is usually possible to reduce the size of this SDP relaxation by exploiting certain structures of POP (1.1). Table 2 lists some of these structures.

Table 2: Several special structures for reducing complexity of the Moment-SOS relaxations.

Structure	Software	POP type
CS [23, 34]	SparsePOP [35]	$f = \sum_{j \in [p]} f_j$ and $f_j, (g_i)_{i \in J_j}, (h_i)_{i \in W_j}$ share the same variables for every $j \in [p]$ and $p > 1$
TS [38, 39]	TSSOS	f, g_i, h_j involve a few of terms
CS-TS [41]	TSSOS	Both CS and TS hold
CTP [28]	SpectralPOP	Equality constrained POPs on a sphere ($m = 0$ and $h_1 := R - \ \mathbf{x}\ _2^2$)

- Correlative sparsity (CS), term sparsity (TS) and their combination (CS-TS) are applied to POPs (1.1) in case that the data f, g_i, h_j are sparse polynomials. The main idea of CS, TS and CS-TS is to break the moment matrices and localizing matrices (which are the psd matrices in the Moment-SOS relaxation) into a lot of blocks according to certain sparsity patterns derived from the POP. If the largest block size is relatively small (say $s^{\max} \leq 100$), then the corresponding SDP can be solved efficiently. But if the largest block size is still large (say $s^{\max} \geq 200$), then the corresponding SDP remains hard to solve.
- In the previous work [28], the first three authors exploited CTP for equality constrained POPs on the sphere and converted the resulting SDP relaxations to spectral minimization problems which could be solved by LMBM efficiently. This method returns approximate optimal values of SDP relaxations involving 2000×2000 matrices for which Mosek encounters memory issues and SketchyCGAL is much less efficient. Importantly, the moment SDP-relaxation of an equality constrained POP has a *single* psd matrix. In contrast, for a POP involving a ball constraint (with possibly other inequality constraints), the resulting moment SDP-relaxations include several psd matrices. Unfortunately for such SDPs, LMBM usually returns inaccurate values even when CTP holds because of ill-conditioning issues. LMBM only updates the dual variables, so it is hard to ensure that the KKT conditions hold. We can overcome the latter ill-conditioning issues by relying on a primal-dual algorithm such as CGAL. It turns out that CGAL (without sketching) is suitable for this type of SDP. For an SDP involving a single matrix, SketchyCGAL stores updated matrices by means of Nyström sketch. In our experimental setting, we rather consider CGAL without sketching, which boils down to relying on implicit updated matrices. It turns out that this strategy is much faster than the one based on Nyström sketch, but does not provide the primal (matrix) solution.

SDP relaxations of non-convex quadratically constrained quadratic programs. A non-convex quadratically constrained quadratic (QCQP) program is a special instance of POP (1.1) for which the degrees of the input polynomials are at most two. Famous instances of non-convex QCQPs include the MAX-CUT problem and the optimal power flow (OPF) problem [19]; in addition we recall that that LCQPs have an equivalent MAX-CUT formulation [25]. They also have applications in deep learning, e.g., the computation of Lipschitz constants [6] and the stability analysis of recurrent neural networks [8]. In practice, non-convex QCQPs usually involve a large number of variables (say $n \geq 1000$) and their associated SDP relaxations (1.2) can be classified in two groups as follows:

- **The first order relaxation:** $k = 1$ (also known as Shor’s relaxation in the literature). In this case the number of affine constraints in SDP (1.2) is typically not larger than the largest block size, i.e., $\zeta \leq s^{\max}$. It can be efficiently solved by most SDP solvers, in particular with SketchyCGAL [45]. Nevertheless the first order relaxation may only provide a lower bound for the optimal value of POP (1.1). In this case, one needs to solve the second and perhaps even higher-order relaxations to obtain tighter bounds or achieve the global optimal value.
- **The second and higher-order relaxations:** $k \geq 2$. In this case the number of affine constraints in SDP (1.2) is typically much larger than the largest block size ($\zeta \gg s^{\max}$). Then unfortunately most SDP solvers cannot handle large-scale SDPs of this form. In our previous work [28], we proposed a remedy for the particular case of second-order SDP relaxations of equality constrained POPs on the sphere, by relying on first-order solvers such as LMBM.

Common issues of solving large-scale SDP relaxations. When solving the second and higher-order SDP relaxations, SDP solvers often encounter the following issues:

- **Storage:** The interior-point methods (IPM) are often chosen by users because of their highly accurate output. These methods are efficient for solving medium-scale SDPs. However they frequently fail due to lack of memory when solving large-scale SDPs (say $s^{\max} > 500$ and $\zeta > 2 \times 10^5$ on a standard laptop). Then first-order methods (e.g., ADMM, SBM, CGAL) provide an alternative to IPM to avoid the memory issue. This is due to the fact that the cost per iteration of first-order methods is much cheaper than that of IPM.

At the price of losing convexity one can also rely on heuristic methods and replace the full matrix \mathbf{X} in SDP (1.2) by a simpler one, in order to save memory. For instance, the Burer-Monteiro method [4] considers a low rank factorization of \mathbf{X} . However, to get correct results the rank cannot be too low [36] and therefore this limitation makes it useless for the second and higher-order relaxations of POPs. Not suffering from such a limitation, CGAL not only maintains the convexity of SDP (1.2) but also possibly runs with implicit matrix \mathbf{X} as described in Remarks A.7 and A.12.

- **Accuracy:** Nevertheless, first-order methods have low convergence rates compared to the interior-point methods. Their performance depends heavily on the problem scaling and conditioning. As a result, in solving large-scale SDPs with first-order methods it is often difficult to obtain results with high accuracy. In contrast the relative gap of the value returned by first-order SDP solvers w.r.t. the exact value is usually expected to be less than 1%.

The goal of this paper is to provide a method which returns the optimal value of the second-order moment SDP-relaxation and which is suitable for a class of large-scale non-convex QCQPs with CTP. Ideally (i) it should avoid the memory issue, and (ii) the resulting relative gap of the approximate value returned by this method w.r.t. the exact value, should be less than 1%.

Contribution. We show that (i) (a large class of) POPs have a very nice *constant trace property* and (ii) that this property can be exploited for solving their associated semidefinite relaxations via appropriate first-order methods. More precisely our contribution is threefold:

1. In Section 3.2 we show that if a positive real number belongs to the interior of every truncated quadratic module associated to the inequality constraints, then the corresponding POP has CTP. Moreover, we prove that this condition always holds when a ball constraint is present.

2. In Section 3.3 we provide a linear programming approach to check whether a POP has CTP. With this approach we prove in Section 3.4 that several special classes of POPs (including POPs on a ball, annulus, simplex) have CTP.
3. Our final contribution is to handle sparse large-scale POPs by integrating sparsity-exploiting techniques into the CTP-exploiting framework.

For practical implementation we have provided a software library called `ctpPOP`. It consists of modeling each moment SDP-relaxation of POPs as a standard SDP with CTP and then solving this SDP by CGAL or the spectral method (SM) with nonsmooth optimization solvers (LMBM or PBM).

In Section 5 we provide extensive numerical experiments to illustrate the efficiency and scalability of `ctpPOP` with the CGAL solver. In all our randomly generated POPs with different sparsity structures, the relative gap of the optimal value provided by CGAL w.r.t. the optimal value provided by Mosek is below 1%. Because of its very cheap cost per iteration, CGAL is more suitable for particularly bulky SDPs (such as moment SDP-relaxations of POPs) than other solvers (e.g. COSMO).

For instance for minimizing a *dense* quadratic polynomial on the unit ball with up to 100 variables, CGAL returns the optimal value of the second-order moment SDP relaxation within 6 hours on a standard laptop while Mosek (considered state-of-the-art IPM SDP solver) runs out of memory. Similarly, for minimizing a *sparse* quadratic polynomial involving thousand variables, with a ball constraint on each clique of variables, CGAL spends around two thousand seconds to solve the second-order moment SDP-relaxation while Mosek runs again out of memory. The largest clique of this POP involves 41 variables.

Classical Optimal Power Flow (OPF) problem without constraints on current magnitudes (as in [10, 18]) can be formulated as a POP with ball and annulus constraints. In many instances Shor’s relaxation usually provides the global optimum. However, for illustration purpose we have compared CGAL and Mosek for solving the second-order CS-TS relaxation for one instance “`case89_pegase_api`” from the PGLib-OPF database¹. The largest block size and the number of equality constraints of this SDP are around 1.7 thousand and 8 million, respectively. While Mosek failed because of memory issue, CGAL still returns the optimal value in 2 days, and with the relative gap w.r.t. a local optimal value less than 0.6%.

2 Notation and preliminary results

With $\mathbf{x} = (x_1, \dots, x_n)$, let $\mathbb{R}[\mathbf{x}]$ stand for the ring of real polynomials and let $\Sigma[\mathbf{x}] \subseteq \mathbb{R}[\mathbf{x}]$ be the subset of sum of squares (SOS) polynomials. Their restrictions to polynomials of degree at most d and $2d$ are denoted by $\mathbb{R}[\mathbf{x}]_d$ and $\Sigma[\mathbf{x}]_d$ respectively. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, let $|\alpha| := \alpha_1 + \dots + \alpha_n$. Let $\mathbb{N}_d^n := \{\alpha \in \mathbb{N}^n : |\alpha| \leq d\}$. Let $(\mathbf{x}^\alpha)_{\alpha \in \mathbb{N}^n}$ be the canonical monomial basis of $\mathbb{R}[\mathbf{x}]$ (sorted w.r.t. the graded lexicographic order) and $\mathbf{v}_d(\mathbf{x})$ be the vector of monomials of degree up to d , with length $s(d, n) := \binom{n+d}{n}$. when it is clear from the context, we also write $s(d)$ instead of $s(d, n)$. A polynomial $p \in \mathbb{R}[\mathbf{x}]_d$ can be written as $p(\mathbf{x}) = \sum_{\alpha \in \mathbb{N}_d^n} p_\alpha \mathbf{x}^\alpha = \mathbf{p}^\top \mathbf{v}_d(\mathbf{x})$, where $\mathbf{p} = (p_\alpha) \in \mathbb{R}^{s(d)}$ is the vector of coefficients in the canonical monomial basis. For $p \in \mathbb{R}[\mathbf{x}]$, let $\lceil p \rceil := \lceil \deg(p)/2 \rceil$. For a positive integer m , let $[m] := \{1, 2, \dots, m\}$. The l_1 -norm of a polynomial p is given by the l_1 -norm of the vector of coefficients \mathbf{p} , that is $\|\mathbf{p}\|_1 := \sum_\alpha |p_\alpha|$. Given $\mathbf{a} \in \mathbb{R}^n$, the l_2 -norm of \mathbf{a} is $\|\mathbf{a}\|_2 := (a_1^2 + \dots + a_n^2)^{1/2}$ and the maximum norm of \mathbf{a} is $\|\mathbf{a}\|_\infty := \max\{|a_j| : j \in [n]\}$. Given a subset \mathcal{S} of real symmetric matrices, let $\mathcal{S}^+ := \{\mathbf{X} \in \mathcal{S} : \mathbf{X} \succeq 0\}$. For $I \subseteq [n]$, let $\mathbf{x}(I) := \{x_j : j \in I\}$ and $\mathbb{N}_d^I := \{\alpha \in \mathbb{N}_d^n : \text{supp}(\alpha) \subseteq I\}$.

¹<https://github.com/power-grid-lib/pglib-opf>

Polynomial optimization problem. A polynomial optimization problem (POP) is defined as

$$f^* := \inf\{f(\mathbf{x}) : \mathbf{x} \in S(g) \cap V(h)\}, \quad (2.3)$$

where $S(g)$ and $V(h)$ are a basic semialgebraic set and a real variety defined respectively by:

$$\begin{aligned} S(g) &:= \{\mathbf{x} \in \mathbb{R}^n : g_i(\mathbf{x}) \geq 0, i \in [m]\} \\ V(h) &:= \{\mathbf{x} \in \mathbb{R}^n : h_j(\mathbf{x}) = 0, i \in [l]\}, \end{aligned} \quad (2.4)$$

for some polynomials $f, g_i, h_j \in \mathbb{R}[\mathbf{x}]$ with $g := \{g_i\}_{i \in [m]}$, $h := \{h_j\}_{j \in [l]}$. We will assume that POP (2.3) has at least one global minimizer.

Riesz linear functional. Given a real-valued sequence $\mathbf{y} = (y_\alpha)_{\alpha \in \mathbb{N}^n}$, define the Riesz linear functional $L_{\mathbf{y}} : \mathbb{R}[\mathbf{x}] \rightarrow \mathbb{R}$, $f \mapsto L_{\mathbf{y}}(f) := \sum_{\alpha} f_{\alpha} y_{\alpha}$. Let d be a positive integer. A real infinite (resp. finite) sequence $(y_{\alpha})_{\alpha \in \mathbb{N}^n}$ (resp. $(y_{\alpha})_{\alpha \in \mathbb{N}_d^n}$) has a *representing measure* if there exists a finite Borel measure μ such that $y_{\alpha} = \int_{\mathbb{R}^n} \mathbf{x}^{\alpha} d\mu(\mathbf{x})$ for every $\alpha \in \mathbb{N}^n$ (resp. $\alpha \in \mathbb{N}_d^n$). In this case, $(y_{\alpha})_{\alpha \in \mathbb{N}^n}$ is called be the moment sequence of μ . We denote by $\text{supp}(\mu)$ the support of a Borel measure μ .

Moment/Localizing matrix. The moment matrix of order d associated with a real-valued sequence $\mathbf{y} = (y_{\alpha})_{\alpha \in \mathbb{N}^n}$ and $d \in \mathbb{N}^{>0}$, is the real symmetric matrix $\mathbf{M}_d(\mathbf{y})$ of size $s(d)$, with entries $(y_{\alpha+\beta})_{\alpha, \beta \in \mathbb{N}_d^n}$. The localizing matrix of order d associated with $\mathbf{y} = (y_{\alpha})_{\alpha \in \mathbb{N}^n}$ and $p = \sum_{\gamma} p_{\gamma} x^{\gamma} \in \mathbb{R}[\mathbf{x}]$, is the real symmetric matrix $\mathbf{M}_d(p\mathbf{y})$ of size $s(d)$ with entries $(\sum_{\gamma} p_{\gamma} y_{\gamma+\alpha+\beta})_{\alpha, \beta \in \mathbb{N}_d^n}$.

Quadratic module. Given $g = \{g_i : i \in [m]\} \subseteq \mathbb{R}[\mathbf{x}]$, the *quadratic module* associated with g is defined by $Q(g) := \{\sigma_0 + \sum_{i \in [m]} \sigma_i g_i : \sigma_0 \in \Sigma[\mathbf{x}], \sigma_i \in \Sigma[\mathbf{x}]\}$, and for a positive integer k , the set $Q_k(g) := \{\sigma_0 + \sum_{i \in [m]} \sigma_i g_i : \sigma_0 \in \Sigma[\mathbf{x}]_k, \sigma_i \in \Sigma[\mathbf{x}]_{k-\lceil g_i \rceil}\}$ is the truncation of $Q(g)$ of order k .

Ideal. Given $h = \{h_i : i \in [l]\} \subseteq \mathbb{R}[\mathbf{x}]$, the set $I(h) := \{\sum_{j \in [l]} \psi_j h_j : \psi_j \in \mathbb{R}[\mathbf{x}]\}$ is the *ideal* generated by h , and the set $I_k(h) := \{\sum_{j \in [l]} \psi_j h_j : \psi_j \in \mathbb{R}[\mathbf{x}]_{2(k-\lceil h_j \rceil)}\}$ is the truncation of $I(h)$ of order k .

Archimedeanity. Assume that there exists $R > 0$ such that $R - \|\mathbf{x}\|_2^2 \in Q(g) + I(h)$. As a consequence, $S(g) \cap V(h) \subseteq \mathcal{B}_R$, where $\mathcal{B}_R := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 \leq \sqrt{R}\}$. In this case, we say that $Q(g) + I(h)$ is *Archimedean* [24].

The Moment-SOS hierarchy [22]. Given a POP (2.3), consider the following associated hierarchy of SOS relaxations indexed by $k \in \mathbb{N}^{\geq k_{\min}}$ with $k_{\min} := \max\{\lceil f \rceil, \{\lceil g_i \rceil\}_{i \in [m]}, \{\lceil h_j \rceil\}_{j \in [l]}\}$:

$$\rho_k := \sup\{\xi \in \mathbb{R} : f - \xi \in Q_k(g) + I_k(h)\}. \quad (2.5)$$

For each $\sigma \in \Sigma[x]_d$, there exists $\mathbf{G} \succeq 0$ such that $\sigma = \mathbf{v}_d^{\top} \mathbf{G} \mathbf{v}_d$. Thus for each $k \in \mathbb{N}^{\geq k_{\min}}$, (2.5) can be rewritten as an SDP:

$$\rho_k = \sup_{\xi, \mathbf{G}_i, \mathbf{u}_j} \left\{ \xi \mid \begin{array}{l} \mathbf{G}_i \succeq 0, f - \xi = \mathbf{v}_k^{\top} \mathbf{G}_0 \mathbf{v}_k \\ \quad + \sum_{i \in [m]} g_i \mathbf{v}_{k-\lceil g_i \rceil}^{\top} \mathbf{G}_i \mathbf{v}_{k-\lceil g_i \rceil} \\ \quad + \sum_{j \in [l]} h_j \mathbf{v}_{2(k-\lceil h_j \rceil)}^{\top} \mathbf{u}_j \end{array} \right\}. \quad (2.6)$$

For every $k \in \mathbb{N}^{\geq k_{\min}}$, the dual of (2.6) reads as

$$\tau_k := \inf_{\mathbf{y} \in \mathbb{R}^{s(2k)}} \left\{ L_{\mathbf{y}}(f) \mid \begin{array}{l} \mathbf{M}_k(\mathbf{y}) \succeq 0, y_0 = 1 \\ \mathbf{M}_{k-\lceil g_i \rceil}(g_i \mathbf{y}) \succeq 0, i \in [m] \\ \mathbf{M}_{k-\lceil h_j \rceil}(h_j \mathbf{y}) = 0, j \in [l] \end{array} \right\}. \quad (2.7)$$

If $Q(g) + V(h)$ is Archimedean, then both $(\rho_k)_{k \in \mathbb{N}^{\geq k_{\min}}}$ and $(\tau_k)_{k \in \mathbb{N}^{\geq k_{\min}}}$ converge to f^* . For details on the Moment-SOS hierarchy and its various applications the interested reader is referred to [24].

3 Exploiting CTP for dense POPs

This section is devoted to developing a framework to exploit CTP for dense POPs. We provide a sufficient condition for a POP to have CTP, as well as a series of linear programs to check whether the sufficient condition holds. In addition we show that several special classes of POPs have CTP.

3.1 CTP for dense POPs

First let us define CTP for a POP. To simplify notation, for every $k \in \mathbb{N}^{\geq k_{\min}}$, denote by \mathcal{S}_k the set of real symmetric matrices

- of size $s_k := s(k) + \sum_{i \in [m]} s(k - \lceil g_i \rceil)$,
- in a block diagonal form $\mathbf{X} = \text{diag}(\mathbf{X}_0, \dots, \mathbf{X}_m)$, and such that
- \mathbf{X}_0 (resp. \mathbf{X}_i) is of size $s(k)$ (resp. $s(k - \lceil g_i \rceil)$) for $i \in [m]$.

Letting $\mathbf{D}_k(\mathbf{y}) := \text{diag}(\mathbf{M}_k(\mathbf{y}), \mathbf{M}_{k-\lceil g_1 \rceil}(g_1 \mathbf{y}), \dots, \mathbf{M}_{k-\lceil g_m \rceil}(g_m \mathbf{y}))$, SDP (2.7) can be rewritten in the form:

$$\tau_k := \inf_{\mathbf{y} \in \mathbb{R}^{s(2k)}} \left\{ L_{\mathbf{y}}(f) \mid \begin{array}{l} \mathbf{D}_k(\mathbf{y}) \in \mathcal{S}_k^+, y_0 = 1, \\ \mathbf{M}_{k-\lceil h_i \rceil}(h_i \mathbf{y}) = 0, i \in [l] \end{array} \right\}. \quad (3.8)$$

Definition 3.1. (CTP for a POP) We say that POP (2.3) has CTP if for every $k \in \mathbb{N}^{\geq k_{\min}}$, there exists $a_k > 0$ and a positive definite matrix $\mathbf{P}_k \in \mathcal{S}_k$ such that for all $\mathbf{y} \in \mathbb{R}^{s(2k)}$,

$$\left. \begin{array}{l} \mathbf{M}_{k-\lceil h_i \rceil}(h_i \mathbf{y}) = 0, i \in [l], \\ y_0 = 1 \end{array} \right\} \Rightarrow \text{trace}(\mathbf{P}_k \mathbf{D}_k(\mathbf{y}) \mathbf{P}_k) = a_k. \quad (3.9)$$

In other words, we say that POP (2.3) has CTP if each moment relaxation (3.8) has an equivalent form involving a psd matrix whose trace is constant. In this case, we call a_k the constant trace and \mathbf{P}_k the basis transformation matrix. In the next subsection, we provide a sufficient condition for POP (2.3) to have CTP.

Example 3.2. (CTP for equality constrained POPs on a sphere [28]) If $g = \emptyset$ and $h_1 = R - \|\mathbf{x}\|_2^2$ for some $R > 0$, then POP (2.3) has CTP with $a_k = (R+1)^k$ and $\mathbf{P}_k := \text{diag}((\theta_{k,\alpha}^{1/2})_{\alpha \in \mathbb{N}_k^n})$, where $(\theta_{k,\alpha})_{\alpha \in \mathbb{N}_k^n} \subseteq \mathbb{R}^{>0}$ satisfies $(1 + \|\mathbf{x}\|_2^2)^k = \sum_{\alpha \in \mathbb{N}_k^n} \theta_{k,\alpha} \mathbf{x}^{2\alpha}$, for all $k \in \mathbb{N}^{\geq k_{\min}}$.

We now provide a general method to solve a POP with CTP. We first convert the k -th order moment relaxation (3.8) of this POP to a standard primal SDP problem with CTP and then leverage appropriate first-order algorithms that exploit CTP to solve the resulting SDP problem.

Suppose POP (2.3) has CTP. For every $k \in \mathbb{N}^{\geq k_{\min}}$, letting $\mathbf{X} = \mathbf{P}_k \mathbf{D}_k(\mathbf{y}) \mathbf{P}_k$, (3.8) can be rewritten as

$$\tau_k = \inf_{\mathbf{X} \in \mathcal{S}_k^+} \{ \langle \mathbf{C}_k, \mathbf{X} \rangle : \mathcal{A}_k \mathbf{X} = \mathbf{b}_k \}, \quad (3.10)$$

where $\mathcal{A}_k : \mathcal{S}_k \rightarrow \mathbb{R}^{\zeta_k}$ is a linear operator such that $\mathcal{A}_k \mathbf{X} = (\langle \mathbf{A}_{k,1}, \mathbf{X} \rangle, \dots, \langle \mathbf{A}_{k,\zeta_k}, \mathbf{X} \rangle)$ with $\mathbf{A}_{k,i} \in \mathcal{S}_k$, $i \in [\zeta_k]$, $\mathbf{C}_k \in \mathcal{S}_k$ and $\mathbf{b}_k \in \mathbb{R}^{\zeta_k}$. Appendix A.4.1 describes how to convert SDP (3.8) to the form (3.10).

The dual of SDP (3.10) reads as

$$\rho_k = \sup_{\mathbf{z} \in \mathbb{R}^{\zeta_k}} \{ \mathbf{b}_k^\top \mathbf{z} : \mathcal{A}_k^\top \mathbf{z} - \mathbf{C}_k \in \mathcal{S}_k^+ \}, \quad (3.11)$$

where $\mathcal{A}_k^\top : \mathbb{R}^{\zeta_k} \rightarrow \mathcal{S}_k$ is the adjoint operator of \mathcal{A}_k , i.e., $\mathcal{A}_k^\top \mathbf{z} = \sum_{i \in [\zeta_k]} z_i \mathbf{A}_{k,i}$.

After replacing $(\mathcal{A}_k, \mathbf{A}_{k,i}, \mathbf{b}_k, \mathbf{C}_k, \mathcal{S}_k, \zeta_k, s_k, \tau_k, \rho_k, a_k)$ by $(\mathcal{A}, \mathbf{A}_i, \mathbf{b}, \mathbf{C}, \mathcal{S}, \zeta, s, \tau, \rho, a)$, the primal-dual (3.10)-(3.11) has an equivalent formulation as the primal-dual (1.36)-(1.37); see also Appendix A.2.1 with $\omega = m + 1$ and $s^{\max} = s(k)$.

Then two first-order algorithms (CGAL and SM) are leveraged for solving the primal-dual (1.36)-(1.37); see Appendix A.2.1 and Appendix A.3.1.

3.2 A sufficient condition for a POP to have CTP

In this section, we provide a sufficient condition for POP (2.3) to have CTP.

For $k \in \mathbb{N}^{\geq k_{\min}}$, let $Q_k^\circ(g)$ be the interior of the truncated quadratic module $Q_k(g)$, i.e., $Q_k^\circ(g) := \{ \mathbf{v}_k^\top \mathbf{G}_0 \mathbf{v}_k + \sum_{i \in [m]} g_i \mathbf{v}_{k-\lceil g_i \rceil}^\top \mathbf{G}_i \mathbf{v}_{k-\lceil g_i \rceil} : \mathbf{G}_i \succ 0, \quad i \in \{0\} \cup [m] \}$.

Theorem 3.3. *The following statements hold:*

1. *If one the following equivalent conditions hold for all $k \in \mathbb{N}^{\geq k_{\min}}$:*

$$\begin{aligned} \mathbb{R}^{>0} \subseteq Q_k^\circ(g) + I_k(h) &\Leftrightarrow \forall \delta > 0, \delta \in Q_k^\circ(g) + I_k(h) \\ &\Leftrightarrow 1 \in Q_k^\circ(g) + I_k(h), \end{aligned} \quad (3.12)$$

then POP (2.3) has CTP, as in Definition 3.1.

2. *Assume that $h = \emptyset$ and $S(g)$ has nonempty interior. Then POP (2.3) has CTP if and only if*

$$\mathbb{R}^{>0} \subseteq Q_k^\circ(g), \forall k \in \mathbb{N}^{\geq k_{\min}}. \quad (3.13)$$

The proof of Theorem 3.3 is postponed to Appendix A.5.

The following lemma will be used later on.

Lemma 3.4. *Let $R > 0$. For all $k \in \mathbb{N}^{\geq 1}$, one has*

$$(R+1)^k = (1 + \|\mathbf{x}\|_2^2)^k + (R - \|\mathbf{x}\|_2^2) \sum_{j=0}^{k-1} (R+1)^j (1 + \|\mathbf{x}\|_2^2)^{k-j-1}. \quad (3.14)$$

Proof. Let $k \in \mathbb{N}^{\geq 1}$. Letting $a = R+1$ and $b = 1 + \|\mathbf{x}\|_2^2$, the desired equality follows from $a^k - b^k = (a-b) \sum_{j=0}^{k-1} a^j b^{k-1-j}$. \square

The next result states that the sufficient condition in Theorem 3.3 holds whenever a ball constraint is present in the POP's description. For a real symmetric matrix \mathbf{A} , denote the largest eigenvalue of \mathbf{A} by $\lambda_{\max}(\mathbf{A})$.

Theorem 3.5. *If $R - \|\mathbf{x}\|_2^2 \in g$ for some $R > 0$ then the inclusions (3.13) hold and therefore POP (2.3) has CTP.*

Proof. Without loss of generality, set $g_m := R - \|\mathbf{x}\|_2^2$ and let $k \in \mathbb{N}^{\geq k_{\min}}$ be fixed. By Lemma 3.4, $(R+1)^k = \Theta + g_m \Lambda$, where $\Theta := (1 + \|\mathbf{x}\|_2^2)^k$ and $\Lambda := \sum_{j=0}^{k-1} (R+1)^j (1 + \|\mathbf{x}\|_2^2)^{k-j-1}$. Note that:

- $\Theta = \sum_{\alpha \in \mathbb{N}_k^n} \theta_\alpha \mathbf{x}^{2\alpha} = \mathbf{v}_k^\top \mathbf{G}_0 \mathbf{v}_k$ for some $(\theta_\alpha)_{\alpha \in \mathbb{N}_k^n} \subseteq \mathbb{R}^{>0}$;
- $\Lambda = \sum_{\alpha \in \mathbb{N}_{k-1}^n} \lambda_\alpha \mathbf{x}^{2\alpha} = \mathbf{v}_{k-1}^\top \mathbf{G}_m \mathbf{v}_{k-1}$ for some $(\lambda_\alpha)_{\alpha \in \mathbb{N}_{k-1}^n} \subseteq \mathbb{R}^{>0}$.

Remark 3.9. One can extend the classes of diagonal matrices $\hat{S}_k, \hat{S}_{k-\lceil g_i \rceil}$ in (3.16) to obtain a smaller constant trace. For instance, one can define $\hat{S}_k, \hat{S}_{k-\lceil g_i \rceil}$ to be the classes of symmetric block diagonal matrices with block size 2. As shown in [38, Lemma 4.3], (3.16) then becomes a second-order cone program (SOCP) which can be also efficiently solved.

3.4 Special classes of POPs with CTP

In this section we identify two classes of POPs whose CTP can be verified by LP (3.16).

3.4.1 POPs with ball or annulus constraints on subsets of variables

Consider the following assumption on the inequality constraints of POP (2.3).

Assumption 3.10. *There exists a nonnegative integer $r \leq m/2$ and*

- $\bar{R}_i > \underline{R}_i > 0, T_i \subseteq [n]$ for $i \in [r]$;
- $\bar{R}_j > 0, T_j \subseteq [n]$ for $j \in [m] \setminus [2r]$

such that

- (1) $(\cup_{i \in [r]} T_i) \cup (\cup_{j \in [m] \setminus [2r]} T_j) = [n]$;
- (2) $g_i := \|\mathbf{x}(T_i)\|_2^2 - \underline{R}_i, g_{i+r} := \bar{R}_i - \|\mathbf{x}(T_i)\|_2^2$ for $i \in [r]$;
- (3) $g_j := \bar{R}_j - \|\mathbf{x}(T_j)\|_2^2$ for $j \in [m] \setminus [2r]$.

Notice that if Assumption 3.10 holds then POP (2.3) has r annulus constraints and $(m-2r)$ ball constraints on subsets of variables. Moreover, $Q(g) + I(h)$ is Archimedean due to (1) in Assumption 3.10.

Example 3.11. *Assumption 3.10 holds in the following cases:*

- (1) $m = 1, r = 0$ and $g_1 := \bar{R}_1 - \|\mathbf{x}\|_2^2$, i.e., $S(g)$ is a ball;
- (2) $m = n, r = 0$ and $g_i := \bar{R}_i - x_i^2$ for $i \in [n]$, i.e., $S(g)$ is a box;
- (3) $m = 2, r = 1$ and $g_1 := \|\mathbf{x}\|_2^2 - \underline{R}_1, g_2 := \bar{R}_1 - \|\mathbf{x}\|_2^2$ ($\bar{R}_1 > \underline{R}_1 > 0$), i.e., $S(g)$ is an annulus.

Proposition 3.12. *If Assumption 3.10 holds then LP (3.16) has a feasible solution for every $k \in \mathbb{N}^{\geq k_{\min}}$, and therefore POP (2.3) has CTP.*

The proof of Proposition 3.12 is postponed to Appendix A.6.

3.4.2 POPs with inequality constraints of equivalent degree

We say that polynomials p_1, \dots, p_t are of equivalent degree if $\lceil p_1 \rceil = \dots = \lceil p_t \rceil$.

Assumption 3.13. *Let $m \geq 3$ and $\{g_j\}_{j \in [m-2]}$ be of equivalent degree. $L > 0$ and $R > 0$ are such that $g_{m-1} = L - \sum_{j \in [m-2]} g_j$ and $g_m = R - \|\mathbf{x}\|_2^2$.*

Proposition 3.14. *If Assumption 3.13 holds then LP (3.16) has a feasible solution for every $k \in \mathbb{N}^{\geq k_{\min}}$, and therefore POP (2.3) has CTP.*

Example 3.15. *Let $R, L > 0$ satisfy $R \geq L^2$ and*

$$m = n + 2, g_i = x_i \text{ for } i \in [n], g_{n+1} = L - \sum_{i \in [n]} x_i \text{ and } g_{n+2} = R - \|\mathbf{x}\|_2^2. \quad (3.17)$$

Then Assumption 3.13 holds and $S(g)$ is a simplex.

When $S(g)$ is compact, we can always reformulate POP (2.3) such that Assumption 3.13 holds. Suppose $S(g) \subseteq \mathcal{B}_R$ for some R . Let $u = \max_{i \in [m]} \lceil g_i \rceil$. Set $\tilde{g}_i := g_i(1 + \|\mathbf{x}\|_2^2)^{u - \lceil g_i \rceil}$ for $i \in [m]$. Let L be a positive number such that $\sum_{i \in [m]} \tilde{g}_i \leq L$ on $S(g)$. Set $\tilde{g}_{m+1} := L - \sum_{i \in [m]} \tilde{g}_i$ and $\tilde{g}_{m+2} := R - \|\mathbf{x}\|_2^2$.

Remark 3.16. For the latter case, one can choose any positive number $L \geq (R + 1)^u \sum_{i \in [m]} \|g_i\|_1$. Indeed, for any $\mathbf{z} \in S(g)$, and since $\|\mathbf{z}\|_2^2 \leq R$:

$$|\mathbf{z}^\alpha| = \prod_{i \in [n]} |z_i|^{\alpha_i} \leq \prod_{i \in [n]} (1 + \|\mathbf{z}\|_2^2)^{\alpha_i/2} = (1 + \|\mathbf{z}\|_2^2)^{|\alpha|/2} \leq (1 + R)^t, \forall \alpha \in \mathbb{N}_{2t}^n.$$

This implies that for every $i \in [m]$,

$$\tilde{g}_i(\mathbf{z}) \leq (1 + R)^{u - \lceil g_i \rceil} \sum_{\alpha \in \mathbb{N}_{2\lceil g_i \rceil}^n} |g_\alpha| |\mathbf{z}^\alpha| \leq (1 + R)^{u - \lceil g_i \rceil} (R + 1)^{\lceil g_i \rceil} \|g_i\|_1 = (1 + R)^u \|g_i\|_1.$$

Thus we have $\sum_{i \in [m]} \tilde{g}_i \leq (1 + R)^u \sum_{i \in [m]} \|g_i\|_1$ on $S(g)$.

Corollary 3.17. With the above notation, $S(g \cup \{\tilde{g}_{m+1}, \tilde{g}_{m+2}\}) = S(g)$ and LP (3.16) has a feasible solution when replacing g by $g \cup \{\tilde{g}_{m+1}, \tilde{g}_{m+2}\}$ for each $k \in \mathbb{N}^{\geq k_{\min}}$. As a result, POP (2.3) is equivalent to the new POP

$$f^* := \inf\{f(\mathbf{x}) : \mathbf{x} \in S(g \cup \{\tilde{g}_{m+1}, \tilde{g}_{m+2}\}) \cap V(h)\} \quad (3.18)$$

which has CTP.

The proof of Corollary 3.17 is postponed to Appendix A.8.

In case where POP (2.3) does not have CTP and $S(g)$ is compact, Corollary 3.17 provides a way to construct an equivalent POP by including two additional redundant constraints. Then CTP of this new POP can be verified by LP.

3.5 Main algorithm

Algorithm 1 below solves POP (2.3) whose CTP can be verified by LP.

Algorithm 1 SpecialPOP-CTP

Input: POP (2.3) and a relaxation order $k \in \mathbb{N}^{\geq k_{\min}}$

Output: The optimal value τ_k of SDP (3.10)

- 1: Solve LP (3.16) with an optimal solution $(\xi_k, \mathbf{G}_{i,k}, \mathbf{u}_{j,k})$;
 - 2: Let $a_k = \xi_k$ and $\mathbf{P}_k = \text{diag}(\mathbf{G}_{0,k}^{1/2}, \dots, \mathbf{G}_{m,k}^{1/2})$;
 - 3: Compute the optimal value τ_k of SDP (3.10) by running an algorithm based on first-order methods, and which exploits CTP ;
-

Examples of algorithms based on first-order methods and which exploit CTP are CGAL (Algorithm 3 in Appendix A.2.1) or SM (Algorithm 5 in Appendix A.3.1).

4 Exploiting CTP for POPs with CS

In this section, we extend the CTP-exploiting framework to POPs with sparsity. For clarity of exposition we only consider *correlative sparsity* (CS). However, in Appendix A.1 we also treat *term sparsity* (TS) [39] as well as *correlative-term sparsity* (CS-TS) [41]. Since the methodology is very similar to that in the dense case described earlier, we omit details and only present the main results.

To begin with, we recall some basic facts on exploiting CS for POP (2.3) initially proposed in [33] by Waki et al.

4.1 POPs with CS

For $\alpha \in \mathbb{N}^n$, let $\text{supp}(\alpha) := \{j \in [n] : \alpha_j > 0\}$. Assume $I \subseteq [n]$. Given $\mathbf{y} = (y_\alpha)_{\alpha \in \mathbb{N}_d^n}$, the moment (resp. localizing) submatrix associated to I of order d is defined by $\mathbf{M}_d(\mathbf{y}, I) := (y_{\alpha+\beta})_{\alpha, \beta \in \mathbb{N}_d^I}$ (resp. $\mathbf{M}_d(q\mathbf{y}, I) := (\sum_\gamma q_\gamma y_{\alpha+\beta+\gamma})_{\alpha, \beta \in \mathbb{N}_d^I}$ for $q \in \mathbb{R}[x(I)]$). Let $\mathbf{v}_d^I := (\mathbf{x}^\alpha)_{\alpha \in \mathbb{N}_d^I}$ with length $s(|I|, d) := \binom{|I|+d}{n}$.

Assume that $\{I_j\}_{j \in [p]}$ (with $n_j := |I_j|$) are the maximal cliques of (a chordal extension of) the correlative sparsity pattern (csp) graph associated to POP (2.3), as defined in [33].

Let $\{J_j\}_{j \in [p]}$ (resp. $\{W_j\}_{j \in [p]}$) be a partition of $[m]$ (resp. $[l]$) such that for all $i \in J_j$, $g_i \in \mathbb{R}[x(I_j)]$ (resp. $i \in W_j$, $h_i \in \mathbb{R}[x(I_j)]$), $j \in [p]$. For each $j \in [p]$, let $m_j := |J_j|$, $l_j := |W_j|$ and $g_{J_j} := \{g_i : i \in J_j\}$, $h_{W_j} := \{h_i : i \in W_j\}$. Then $Q(g_{J_j})$ (resp. $I(h_{W_j})$) is a quadratic module (resp. an ideal) in $\mathbb{R}[x(I_j)]$, for $j \in [p]$.

For each $k \in \mathbb{N}^{\geq k_{\min}}$, consider the following sparse SOS relaxation:

$$\rho_k^{\text{CS}} := \sup \left\{ \xi : f - \xi \in \sum_{j \in [p]} (Q_k(g_{J_j}) + I_k(h_{W_j})) \right\}. \quad (4.19)$$

It is equivalent to the SDP:

$$\rho_k^{\text{CS}} = \sup_{\xi, \mathbf{G}_i^{(j)}, \mathbf{u}_i^{(j)}} \left\{ \xi \left| \begin{array}{l} \mathbf{G}_i^{(j)} \succeq 0, i \in \{0\} \cup J_j, j \in [p], \\ f - \xi = \sum_{j \in [p]} \left((\mathbf{v}_k^{I_j})^\top \mathbf{G}_0^{(j)} \mathbf{v}_k^{I_j} \right. \right. \\ \quad \left. \left. + \sum_{i \in J_j} g_i (\mathbf{v}_{k-\lceil g_i \rceil}^{I_j})^\top \mathbf{G}_i^{(j)} \mathbf{v}_{k-\lceil g_i \rceil}^{I_j} \right. \right. \\ \quad \left. \left. + \sum_{i \in W_j} h_i (\mathbf{v}_{2(k-\lceil h_i \rceil)}^{I_j})^\top \mathbf{u}_i^{(j)} \right) \right. \end{array} \right\}. \quad (4.20)$$

The dual of (4.20) reads:

$$\tau_k^{\text{CS}} := \inf_{\mathbf{y} \in \mathbb{R}^{s(2k)}} \left\{ L_{\mathbf{y}}(f) \left| \begin{array}{l} \mathbf{M}_k(\mathbf{y}, I_j) \succeq 0, j \in [p], \mathbf{y}_0 = 1. \\ \mathbf{M}_{k-\lceil g_i \rceil}(g_i \mathbf{y}, I_j) \succeq 0, i \in J_j, j \in [p], \\ \mathbf{M}_{k-\lceil h_i \rceil}(h_i \mathbf{y}, I_j) = 0, i \in W_j, j \in [p] \end{array} \right. \right\}. \quad (4.21)$$

It is shown in [23, Theorem 3.6] that convergence of the primal-dual (4.20)-(4.21) to f^* is guaranteed if there are additional ball constraints on each clique of variables.

4.2 Exploiting CTP for POPs with CS

Consider POP (2.3) with CS described in Section 4.1. For every $j \in [p]$ and for every $k \in \mathbb{N}^{\geq k_{\min}}$, letting $\mathbf{D}_k(\mathbf{y}, I_j) := \text{diag}(\mathbf{M}_k(\mathbf{y}, I_j), (\mathbf{M}_{k-\lceil g_i \rceil}(g_i \mathbf{y}, I_j))_{i \in J_j})$ for $\mathbf{y} \in \mathbb{R}^{s(2k)}$, SDP (4.21) can be rewritten as

$$\tau_k^{\text{CS}} := \inf_{\mathbf{y} \in \mathbb{R}^{s(2k)}} \left\{ L_{\mathbf{y}}(f) \left| \begin{array}{l} \mathbf{D}_k(\mathbf{y}, I_j) \succeq 0, j \in [p], \mathbf{y}_0 = 1, \\ \mathbf{M}_{k-\lceil h_i \rceil}(h_i \mathbf{y}, I_j) = 0, i \in W_j, j \in [p] \end{array} \right. \right\}. \quad (4.22)$$

We define CTP for POP with CS as follows.

Definition 4.1. (CTP for a POP with CS) We say that POP (2.3) with CS has CTP if for every $k \in \mathbb{N}^{\geq k_{\min}}$ and for every $j \in [p]$, there exists a positive number $a_k^{(j)}$ and a positive definite matrix $\mathbf{P}_k^{(j)} \in \mathcal{S}_k$ such that for all $\mathbf{y} \in \mathbb{R}^{s(2k)}$,

$$\left. \begin{array}{l} \mathbf{M}_{k-\lceil h_i \rceil}(h_i \mathbf{y}, I_j) = 0, i \in W_j, \\ \mathbf{y}_0 = 1 \end{array} \right\} \Rightarrow \text{trace}(\mathbf{P}_k^{(j)} \mathbf{D}_k(\mathbf{y}, I_j) \mathbf{P}_k^{(j)}) = a_k^{(j)}. \quad (4.23)$$

The following result provides a sufficient condition for a POP with CS to have CTP.

Theorem 4.2. *Assume that there is a ball constraint on each clique of variables, i.e.,*

$$\forall j \in [p], R_j - \|\mathbf{x}(I_j)\|_2^2 \in g \text{ for some } R_j > 0. \quad (4.24)$$

Then one has $\mathbb{R}^{>0} \subseteq Q_k^\circ(g_{J_j})$, for all $k \in \mathbb{N}^{\geq k_{\min}}$ and for all $j \in [p]$. As a consequence, POP (2.3) has CTP.

The proof of Theorem 4.2 being very similar to that of Theorem 3.5 by considering each clique of variables, is omitted.

Again by considering each clique of variables, the following result can be obtained from Theorem 4.2 in the same way Corollary 3.6 was obtained.

Corollary 4.3. *If (4.24) holds then Slater's condition for SDP (4.20) holds for all $k \in \mathbb{N}^{\geq k_{\min}}$.*

We are now in position to provide a general method to solve POPs with CS which have CTP.

Consider POP (2.3) with CS described in Section 4.1. Assume that POP (2.3) has CTP and let $k \in \mathbb{N}^{\geq k_{\min}}$ be fixed. For every $j \in [p]$, we denote by $\mathcal{S}_{j,k}$ the set of real symmetric matrices of size $s(k, n_j) + \sum_{i \in J_j} s(k - \lceil g_i \rceil, n_j)$ in a block diagonal form: $\mathbf{X} = \text{diag}(\mathbf{X}_0, (\mathbf{X}_i)_{i \in J_j})$ such that \mathbf{X}_0 is a block of size $s(k, n_j)$ and \mathbf{X}_i is a block of size $s(k - \lceil g_i \rceil, n_j)$ for $i \in J_j$.

Letting

$$\mathbf{X}_j = \mathbf{P}_k^{(j)} \mathbf{D}_k(\mathbf{y}, I_j) \mathbf{P}_k^{(j)}, \quad j \in [p], \quad (4.25)$$

SDP (4.22) can be rewritten as:

$$\tau_k^{\text{CS}} = \inf_{\mathbf{X}_j \in \mathcal{S}_{j,k}^+} \left\{ \sum_{j \in [p]} \langle \mathbf{C}_{j,k}, \mathbf{X}_j \rangle : \sum_{j \in [p]} \mathcal{A}_{j,k} \mathbf{X}_j = \mathbf{b}_k, j \in [p] \right\}, \quad (4.26)$$

where for every $j \in [p]$, $\mathcal{A}_{j,k} : \mathcal{S}_{j,k} \rightarrow \mathbb{R}^{\zeta_k}$ is a linear operator of the form $\mathcal{A}_{j,k} \mathbf{X} = (\langle \mathbf{A}_{j,k,1}, \mathbf{X} \rangle, \dots, \langle \mathbf{A}_{j,k,\zeta_k}, \mathbf{X} \rangle)$ with $\mathbf{A}_{j,k,i} \in \mathcal{S}_{j,k}$, $i \in [\zeta_k]$, $\mathbf{C}_{j,k} \in \mathcal{S}_{j,k}$, $j \in [p]$ and $\mathbf{b}_k \in \mathbb{R}^{\zeta_k}$. See Appendix A.4.2 for the conversion of SDP (4.22) to the form (4.26).

The dual of SDP (4.26) reads as:

$$\rho_k^{\text{CS}} = \sup_{\mathbf{y} \in \mathbb{R}^{\zeta}} \left\{ \mathbf{b}_k^\top \mathbf{y} : \mathcal{A}_{j,k}^\top \mathbf{y} - \mathbf{C}_{j,k} \in \mathcal{S}_{j,k}^+, j \in [p] \right\}, \quad (4.27)$$

where $\mathcal{A}_{j,k}^\top : \mathbb{R}^{\zeta} \rightarrow \mathcal{S}_{j,k}$ is the adjoint operator of $\mathcal{A}_{j,k}$, i.e., $\mathcal{A}_{j,k}^\top \mathbf{z} = \sum_{i \in [\zeta]} z_i \mathbf{A}_{j,k,i}$, $j \in [p]$. By Definition 4.1, it holds that for every $k \in \mathbb{N}^{\geq k_{\min}}$,

$$\left. \forall \mathbf{X}_j \in \mathcal{S}_{j,k}, j \in [p] \right\} \Rightarrow \text{trace}(\mathbf{X}_j) = a_k^{(j)}, j \in [p]. \quad (4.28)$$

After replacing $(\mathcal{A}_{j,k}, \mathbf{A}_{j,k,i}, \mathbf{b}_k, \mathbf{C}_{j,k}, \mathcal{S}_{j,k}, \zeta_k, \tau_k^{\text{CS}}, a_k^{(j)})$ by $(\mathcal{A}_j, \mathbf{A}_{i,j}, \mathbf{b}, \mathbf{C}_j, \mathcal{S}_j, \zeta, \tau, a_j)$, SDP (4.26) then becomes SDP (1.39); see Appendix A.2.2 with $\omega_j = m_j + 1$ and $s^{\max} = \max_{j \in [p]} s(k, n_j)$.

If there is a ball constraint on each clique of variables then by Corollary 4.3, strong duality holds for the pair (4.26)-(4.27), for every $k \in \mathbb{N}^{\geq k_{\min}}$.

The two algorithms (CGAL and SM) based on first-order methods are then leveraged to solve the primal-dual (4.26)-(4.27); see Appendix A.2.2 and Appendix A.3.2.

4.3 Verifying CTP for POPs with CS via LP

As in the dense case, we can verify CTP for a POP with CS via a series of LPs.

For every $k \in \mathbb{N}^{\geq k_{\min}}$ and for every $j \in [p]$, let $\hat{\mathcal{S}}_{k,j}$ be the set of real diagonal matrices of size $s(k, n_j)$ and consider the following LP:

$$\inf_{\xi, \mathbf{G}_i, \mathbf{u}_i} \left\{ \xi \mid \begin{array}{l} \mathbf{G}_0 - \mathbf{I}_0 \in \hat{\mathcal{S}}_{k,j}^+, \mathbf{G}_i - \mathbf{I}_i \in \hat{\mathcal{S}}_{k-\lceil g_i \rceil, j}^+, i \in J_j, \\ \xi = (\mathbf{v}_k^{I_j})^\top \mathbf{G}_0 \mathbf{v}_k^{I_j} + \sum_{i \in J_j} g_i (\mathbf{v}_{k-\lceil g_i \rceil}^{I_j})^\top \mathbf{G}_i \mathbf{v}_{k-\lceil g_i \rceil}^{I_j} \\ \quad + \sum_{i \in W_j} h_i (\mathbf{v}_{2(k-\lceil h_i \rceil)}^{I_j})^\top \mathbf{u}_i \end{array} \right\}, \quad (4.29)$$

where \mathbf{I}_i is the identity matrix, for every $i \in \{0\} \cup J_j$.

Lemma 4.4. *Let POP (2.3) with CS be described in Section 4.1. If LP (4.29) has a feasible solution $(\xi_k^{(j)}, \mathbf{G}_{i,k}^{(j)}, \mathbf{u}_{i,k}^{(j)})$, for every $k \in \mathbb{N}^{\geq k_{\min}}$ and for every $j \in [p]$, then POP (2.3) has CTP with $\mathbf{P}_k^{(j)} = \text{diag}(\mathbf{G}_{0,k}^{1/2}, (\mathbf{G}_{i,k}^{1/2})_{i \in J_i})$ and $a_k^{(j)} = \xi_k^{(j)}$, for $k \in \mathbb{N}^{\geq k_{\min}}$ and for $j \in [p]$.*

The proof of Lemma 4.4 is similar to that of Lemma 3.8.

For instance, for POPs with ball or annulus constraints on subsets of each clique of variables, CTP can be verified by LP.

Proposition 4.5. *Let POP (2.3) with CS be described in Section 4.1. Let $(T_i)_{i \in [r] \cup ([m] \setminus [2r])}$ be as in Assumption 3.10 and further assume that for every $j \in [p]$, $(\cup_{q \in J_j \cap [r]} T_q) \cup (\cup_{q \in J_j \setminus [2r]} T_q) = I_j$. Then LP (4.29) has a feasible solution for every $k \in \mathbb{N}^{\geq k_{\min}}$, and therefore POP (2.3) has CTP.*

The proof of Proposition 4.5 is postponed to Appendix A.9.

4.4 Main algorithm

Algorithm 2 below solves POP (2.3) with CS and whose CTP can be verified by LP.

Algorithm 2 SpecialPOP-CTP-CS

Input: POP (2.3) with CS and a relaxation order $k \in \mathbb{N}^{\geq k_{\min}}$

Output: The optimal value τ_k^{CS} of SDP (4.26)

- 1: **for** $j \in [p]$ **do**
 - 2: Solve LP (4.29) to obtain an optimal solution $(\xi_k^{(j)}, \mathbf{G}_{i,k}^{(j)}, \mathbf{u}_{j,k}^{(j)})$;
 - 3: Let $a_k^{(j)} = \xi_k^{(j)}$ and $\mathbf{P}_k^{(j)} = \text{diag}((\mathbf{G}_{0,k}^{(j)})^{1/2}, \dots, (\mathbf{G}_{m,k}^{(j)})^{1/2})$;
 - 4: Compute the optimal value τ_k^{CS} of SDP (4.26) by running an algorithm based on first-order methods and which exploits CTP.
-

In Step 4 of Algorithm 2 the two algorithms CGAL (Algorithm 4 in Appendix A.2.2 or SM (Algorithm 6 in Appendix A.3.2) are good candidates.

5 Numerical experiments

In this section we report results of numerical experiments obtained by solving the second-order Moment-SOS relaxation of various randomly generated instances of QCQPs with CTP. The experiments are performed in Julia 1.3.1 with the following software packages:

- **SumOfSquares** [43] is a modeling library for solving the Moment-SOS relaxations of dense POPs, based on JuMP (with Mosek 9.1 used as SDP solver).

Table 3: The notation

n	the number of variables of a POP
m	the number of inequality constraints of a POP
l	the number of equality constraints of a POP
u^{\max}	the largest size of variable cliques of a sparse POP
p	the number of variable cliques of a sparse POP
k	the relaxation order of the Moment-SOS hierarchy
t	the sparse order of the sparsity adapted Moment-SOS hierarchy (for TS and CS-TS)
ω	the number of psd blocks in an SDP
s^{\max}	the largest size of psd blocks in an SDP
ζ	the number of affine equality constraints in an SDP
a^{\max}	the largest constant trace
Mosek	the SDP relaxation modeled by <code>SumOfSquares</code> (for dense POPs) or <code>TSSOS</code> (for sparse POPs) and solved by Mosek 9.1
CGAL	the SDP relaxation modeled by our CTP-exploiting method and solved by the CGAL algorithm
LMBM	the SDP relaxation modeled by our CTP-exploiting method and solved by the SM algorithm with the LMBM solver
val	the optimal value of the SDP relaxation
gap	the relative optimality gap w.r.t. the value returned by Mosek, i.e., $\text{gap} = \text{val} - \text{val}(\text{Mosek}) / \text{val}(\text{Mosek}) $
time	the running time in seconds (including modeling and solving time)
–	the calculation runs out of space

- `TSSOS` [39–41] is a modeling library for solving Moment-SOS relaxations of sparse POPs based on JuMP (with Mosek 9.1 used as SDP solver).
- `LMBM` solves unconstrained non-smooth optimization with the limited-memory bundle method by Haarala et al. [11, 12] and calls Karmita’s Fortran implementation of the LMBM algorithm [20].
- `Arpack` [27] is used to compute the smallest eigenvalues and the corresponding eigenvectors of real symmetric matrices of (potentially) large size, which is based on the implicitly restarted Arnoldi method.

The implementation of algorithms 1 and 2 is available online via the link:

<https://github.com/maihoangnh/ctpPOP>.

We use a desktop computer with an Intel(R) Core(TM) i7-8665U CPU @ 1.9GHz \times 8 and 31.2 GB of RAM. The notation for the numerical results is given in Table 3.

For the examples tested in this paper, the modeling time of `SumOfSquares`, `TSSOS` and `ctpPOP` is typically negligible compared to the solving time of the packages Mosek, CGAL, and LMBM. Hence the total running time mainly depends on the solvers and we compare their performances below. As mentioned in the introduction, the current framework differs from our previous work [28], where we exploited CTP for equality constrained POPs on a sphere, which could be solved by `LMBM` efficiently. The reason is that the SDP relaxations of such equality constrained POPs involve a single psd matrix. For the benchmarks of this section, we consider POPs involving ball/annulus constraints, so the resulting relaxations include several psd matrices. Our numerical experiments confirm that for such SDPs, `LMBM` returns inaccurate values while `CGAL`

Table 4: Numerical results for minimizing a dense quadratic polynomial on a unit ball

- POP size: $m = 1, l = 0$; Relaxation order: $k = 2$; SDP size: $\omega = 2, a^{\max} = 3$.

POP size	SDP size		Mosek		CGAL		LMBM	
	s^{\max}	ζ	val	time	val	time	val	time
10	66	1277	-2.2181	0.3	-2.2170	0.2	-2.2187	0.3
20	231	16402	-3.7973	4	-3.7947	0.6	-3.7096	7
30	496	77377	-3.6876	3474	-3.6858	104	-3.8530	59
40	861	236202	—	—	-4.1718	33	-4.7730	179
50	1326	564877	—	—	-6.3107	1007	-7.3874	139
60	1891	1155402	—	—	-6.5326	1085	-7.4733	674
70	2556	2119777	—	—	-7.3379	1262	-9.5223	1486
80	3321	3590002	—	—	-7.9559	4988	-10.0260	1241
90	4186	5718077	—	—	-7.3425	5187	-9.4477	5313
100	5151	8676002	—	—	-7.7374	22451	-10.684	5355

(without sketching) performs better for this type of SDP in terms of accuracy and efficiency.

5.1 Randomly generated dense QCQPs with a ball constraint

Test problems: We construct randomly generated dense QCQPs with a ball constraint as follows:

1. Generate a dense quadratic polynomial objective function f with random coefficients following the uniform probability distribution on $(-1, 1)$.
2. Let $m = 1$ and $g_1 := 1 - \|\mathbf{x}\|_2^2$;
3. Take a random point \mathbf{a} in $S(g)$ w.r.t. the uniform distribution;
4. For every $j \in [l]$, generate a dense quadratic polynomial h_j by
 - (i) for each $\alpha \in \mathbb{N}_2^g \setminus \{\mathbf{0}\}$, taking a random coefficient $h_{j,\alpha}$ for h_j in $(-1, 1)$ w.r.t. the uniform distribution;
 - (ii) setting $h_{j,\mathbf{0}} := -\sum_{\alpha \in \mathbb{N}_2^g \setminus \{\mathbf{0}\}} h_{j,\alpha} \mathbf{a}^\alpha$.

Then \mathbf{a} is a feasible solution of POP (2.3).

The numerical results are displayed in Table 4 and 5.

Discussion: As one can see from Table 4 and 5, CGAL is typically the fastest solver and returns an optimal value of gap within 1% w.r.t. the one returned by Mosek when $n \leq 30$. Mosek runs out of memory when $n \geq 40$ while CGAL works well up to $n = 100$. We should point out that LMBM is less accurate or even fails to converge to the optimal value when $n \geq 20$. The reason might be that LMBM only solves the dual problem and hence loses information of the primal problem.

Table 5: Numerical results for randomly generated dense QCQPs with a ball constraint

- POP size: $m = 1$, $l = \lceil n/4 \rceil$; Relaxation order: $k = 2$; SDP size: $\omega = 2$, $a^{\max} = 3$.

POP size		SDP size		Mosek		CGAL		LMBM	
n	l	s^{\max}	ζ	val	time	val	time	val	time
10	3	66	1475	-2.0686	1.7	-2.0674	0.8	-2.0874	0.3
20	5	231	17557	-3.0103	61	-3.0075	7	-3.0750	18
30	8	496	81345	-3.3293	4573	-3.3249	80	-3.6863	123
40	10	861	244812	—	—	-4.6977	194	-5.3488	488
50	13	1326	582115	—	—	-4.2394	951	-6.1325	837
60	15	1891	1183767	—	—	-5.7793	1387	-7.5718	3781
70	18	2556	2165785	—	—	-6.1278	4335	-8.1181	15854

Table 6: Numerical results for minimizing a dense quadratic polynomial on an annulus

- POP size: $m = 2$, $l = 0$; Relaxation order: $k = 2$; SDP size: $\omega = 3$, $a^{\max} = 4$.

POP size		SDP size		Mosek		CGAL		LMBM	
n	s^{\max}	ζ	val	time	val	time	val	time	
10	66	1343	-3.0295	0.5	-3.0278	1	-3.0311	0.8	
20	231	16633	-3.6468	69	-3.6458	5	-3.7814	16	
30	496	77873	-3.9108	2546	-3.9079	9	-3.8941	51	
40	861	237063	—	—	-4.7469	28	-6.9780	119	
50	1326	566203	—	—	-6.4170	112	-11.1028	258	
60	1891	1157293	—	—	-5.5841	226	-9.2142	473	
70	2556	2122333	—	—	-7.9325	730	-12.7862	1669	
80	3321	3593323	—	—	-7.6164	1355	-10.068	317	
90	4186	5722263	—	—	-8.1900	3563	-12.439	8751	

5.2 Randomly generated dense QCQPs with annulus constraints

Test problems: We construct randomly generated dense QCQPs as in Section 5.1, where the ball constraint is now replaced by annulus constraints. Namely, in Step 2 we take $m = 2$, $g_1 := \|\mathbf{x}\|_2^2 - 1/2$ and $g_2 := 1 - \|\mathbf{x}\|_2^2$. The numerical results are displayed in Table 6 and 7.

Discussion: Same remarks as in Section 5.1.

5.3 Randomly generated dense QCQPs with box constraints

Test problems: We construct randomly generated dense QCQPs as in Section 5.1, where the ball constraint is now replaced by box constraints. Namely, in Step 2 we take $m = n$, $g_j := -x_j^2 + 1/n$, $j \in [n]$.

The numerical results are displayed in Table 8 and 9.

Table 7: Numerical results for randomly generated dense QCQPs with annulus constraints

- POP size: $m = 2$, $l = \lceil n/4 \rceil$; Relaxation order: $k = 2$; SDP size: $\omega = 3$, $a^{\max} = 4$.

POP size		SDP size		Mosek		CGAL		LMBM	
n	l	s^{\max}	ζ	val	time	val	time	val	time
10	3	66	1541	-2.7950	0.5	-2.7934	2	-2.7829	7
20	5	231	17788	-3.5048	95	-3.5027	10	-4.4491	46
30	8	496	81841	-3.3964	4237	-3.3937	45	-4.9592	111
40	10	861	245673	—	—	-4.6573	140	-6.7683	648
50	13	1326	583441	—	—	-3.8236	437	-6.9930	519
60	15	1891	1185658	—	—	-4.5246	1076	-7.5845	2917
70	18	2556	2168341	—	—	-6.2924	4783	-9.6145	2644

Table 8: Numerical results for minimizing a dense quadratic polynomial on a box

- POP size: $m = n$, $l = 0$; Relaxation order: $k = 2$; SDP size: $\omega = n + 1$, $a^{\max} = 3$.

POP size		SDP size		Mosek		CGAL		LMBM	
n		s^{\max}	ζ	val	time	val	time	val	time
10		66	1871	-2.7197	0.5	-2.7189	1	-2.7327	0.7
20		231	20791	-3.3560	98	-3.3501	57	-4.2987	18
30		496	91761	-4.6372	5150	-4.6242	285	-5.8805	156
40		861	269781	—	—	-4.5788	409	-6.5857	188
50		1326	629851	—	—	-4.2313	2083	-6.6163	323
60		1891	1266971	—	—	-4.0135	5525	-6.5792	814
70		2556	2296141	—	—	-5.4019	15172	-8.7669	1434

Table 9: Numerical results for randomly generated dense QCQPs with box constraints

- POP size: $m = n$, $l = \lceil n/7 \rceil$; Relaxation order: $k = 2$; SDP size: $\omega = n + 1$, $a^{\max} = 3$.

POP size		SDP size		Mosek		CGAL		LMBM	
n	l	s^{\max}	ζ	val	time	val	time	val	time
10	2	66	2003	-1.8320	0.6	-1.8321	3	-1.9692	4
20	3	231	21484	-3.1797	175	-3.1781	106	-4.0216	29
30	5	496	94241	-2.2949	6850	-2.2982	528	-3.9900	152
40	6	861	274947	—	—	-3.8651	933	-6.1379	298
50	8	1326	640459	—	—	-3.6267	6159	-6.3651	1494

Table 10: Numerical results for minimizing a dense quadratic polynomials on a simplex

- POP size: $m = n + 2$, $l = 0$; Relaxation order: $k = 2$; SDP size: $\omega = n + 3$, $a^{\max} = 5$.

POP size		SDP size		Mosek		CGAL		LMBM	
n		s^{\max}	ζ	val	time	val	time	val	time
10		66	2003	-1.9954	0.3	-1.9950	7	-2.2800	27
20		231	21253	-1.5078	58	-1.5055	116	-2.7237	32
30		496	92753	-2.0537	2804	-2.0480	377	-3.3114	917
40		861	271503	–	–	-2.3034	950	-4.0971	577
50		1326	632503	–	–	-1.8366	9539	-4.0541	13700

Table 11: Numerical results for randomly generated dense QCQPs with simplex constraints

- POP size: $m = n + 2$, $l = \lceil n/7 \rceil$; Relaxation order: $k = 2$; SDP size: $\omega = n + 3$, $a^{\max} = 5$.

POP size		SDP size		Mosek		CGAL		LMBM	
n	l	s^{\max}	ζ	val	time	val	time	val	time
10	2	66	2135	-1.0605	0.4	-1.0606	176	-2.2338	2
20	3	231	21946	-1.6629	72	-1.6628	512	-3.3538	93
30	5	496	95233	-1.0091	6206	-1.0249	1089	-2.9425	100
40	6	861	276669	–	–	-0.3256	2314	-2.9564	4431
50	8	1326	643111	–	–	-1.4200	10035	-5.4284	1310

Discussion: We observe similar behaviors of the solvers as in Section 5.1. The important point to note here is that solving a QCQP with box constraints is less efficient than solving the same one with ball constraints. This is because the efficiency of CGAL depends on the number of psd blocks involved in SDP. For instance, when $n = 50$, CGAL takes around 1000 seconds to solve the second-order moment relaxation of a QCQP with a ball constraint while it takes around 2100 seconds to solve this relaxation for a QCQP with box constraints.

5.4 Randomly generated dense QCQPs with simplex constraints

Test problems: We construct randomly generated dense QCQPs as in Section 5.1, where the ball constraint is now replaced by simplex constraints. Namely, in Step 2 we take g such that (3.17) holds with $L = R = 1$. The numerical results are displayed in Table 10 and 11.

Discussion: Again we observe a behavior of the solvers similar to that in Section 5.1. One can also see that solving a QCQP with simplex constraints by CGAL is significantly slower than solving the same one with box constraints. For instance, when $n = 50$, CGAL takes 2100 seconds to solve the second-order moment relaxation for a QCQP with box constraints while it takes 9500 seconds with simplex constraints.

Table 12: Numerical results for minimizing a random quadratic polynomial with TS on the unit ball

- POP size: $m = 1, l = 0$; Relaxation order: $k = 2$; Sparse order: $t = 1$; SDP size: $\omega = 4, a^{\max} = 3$.

POP size	SDP size		Mosek		CGAL		LMBM	
	s^{\max}	ζ	val	time	val	time	val	time
10	56	937	-1.5681	4	-1.5527	0.7	-1.5711	0.07
20	211	13722	-2.4275	36	-2.3996	1	-2.7301	0.6
30	466	68357	-3.0748	1930	-3.0577	8	-3.5188	8
40	821	214842	–	–	-3.6999	20	-4.9033	40
50	1276	523177	–	–	-4.1603	128	-5.3416	59
60	1831	1083362	–	–	-4.1914	655	-5.6983	303
70	2486	2005397	–	–	-4.9578	1461	-7.1968	1040
80	3241	3419282	–	–	-5.6452	7253	-7.9133	5759

5.5 Randomly generated QCQPs with TS and ball constraints

Test problems: We construct randomly generated QCQPs with TS and a ball constraint as follows:

1. Generate a quadratic polynomial objective function f such that for $\alpha \in \mathbb{N}_2^n$ with $|\alpha| \neq 2, f_\alpha = 0$ and for $\alpha \in \mathbb{N}_2^n$ with $|\alpha| = 2$, the coefficient f_α is randomly generated in $(-1, 1)$ w.r.t. the uniform distribution;
2. Take $m = 1$ and $g_1 := 1 - \|\mathbf{x}\|_2^2$;
3. Take a random point \mathbf{a} in $S(g)$ w.r.t. the uniform distribution;
4. For every $j \in [l]$, generate a quadratic polynomial h_j by
 - (i) setting $h_{j,\alpha} = 0$ for each $\alpha \in \mathbb{N}_2^n \setminus \{\mathbf{0}\}$ with $|\alpha| \neq 2$;
 - (ii) for each $\alpha \in \mathbb{N}_2^n \setminus \{\mathbf{0}\}$ with $|\alpha| = 2$, taking a random coefficient $h_{j,\alpha}$ for h_j in $(-1, 1)$ w.r.t. the uniform distribution;
 - (iii) setting $h_{j,\mathbf{0}} := -\sum_{\alpha \in \mathbb{N}_2^n \setminus \{\mathbf{0}\}} h_{j,\alpha} \mathbf{a}^\alpha$.

Then \mathbf{a} is a feasible solution of POP (2.3).

The numerical results are displayed in Table 12 and 13.

Discussion: The behavior of solvers is similar to that in the dense case.

5.6 Randomly generated QCQPs with TS and box constraints

Test problems: We construct randomly generated QCQPs with TS as in Section 5.5, where the ball constraint is now replaced by box constraints. The numerical results are displayed in Table 14 and 15.

Discussion: Again the behavior of solvers is similar to that in the dense case.

Table 13: Numerical results for randomly generated QCQPs with TS and a ball constraint

- POP size: $m = 1, l = \lceil n/4 \rceil$; Relaxation order: $k = 2$; Sparse order: $t = 1$; SDP size: $\omega = 4, a^{\max} = 3$.

POP size		SDP size		Mosek		CGAL		LMBM	
n	l	s^{\max}	ζ	val	time	val	time	val	time
10	3	56	1105	-0.60612	0.7	-0.60550	2	-0.60611	0.8
20	5	211	14777	-2.3115	47	-2.3097	17	-2.3952	3
30	8	466	72085	-2.8344	3102	-2.8321	112	-3.7588	128
40	10	821	223052	—	—	-3.4081	476	-4.4239	673
50	13	1276	539765	—	—	-3.3552	1845	-5.2568	729
60	15	1831	1110827	—	—	-3.5620	2992	-5.9898	1702

Table 14: Numerical results for minimizing a random quadratic polynomial with TS on a box

- POP size: $m = n, l = 0$; Relaxation order: $k = 2$; Sparse order: $t = 1$; SDP size: $a^{\max} = 3$.

POP size		SDP size			Mosek		CGAL		LMBM	
n	ω	s^{\max}	ζ	val	time	val	time	val	time	
10	22	56	1441	-1.0539	3	-1.0519	14	-1.11671	1	
20	42	211	17731	-1.3925	93	-1.3802	161	-2.2978	2	
30	62	466	81871	-2.2301	4392	-2.2128	567	-2.4544	533	
40	82	821	246861	—	—	-2.5209	1602	-4.6159	1036	
50	102	1276	585701	—	—	-3.0282	2583	-4.9146	376	
60	122	1831	1191391	—	—	-3.0470	10858	-5.7882	353	

Table 15: Numerical results for randomly generated QCQPs with TS and box constraints

- POP size: $m = n, l = \lceil n/7 \rceil$; Relaxation order: $k = 2$; Sparse order: $t = 1$;
SDP size: $\omega = n + 1, a^{\max} = 3$.

POP size		SDP size			Mosek		CGAL		LMBM	
n	l	ω	s^{\max}	ζ	val	time	val	time	val	time
10	2	22	56	1553	-0.77189	0.2	-0.77214	9	-0.78092	1
20	3	42	211	18364	-1.7962	71	-1.8009	150	-2.7771	4
30	5	62	466	84201	-1.8529	5814	-1.8625	650	-3.5891	268
40	6	82	821	251787	—	—	-2.1930	2994	-4.5890	317
50	8	102	1276	595909	—	—	-2.4655	8397	-5.1811	883

Table 16: Numerical results for minimizing a random quadratic polynomial with CS and ball constraints on each clique of variables

- POP size: $n = 1000$, $m = p$, $l = 0$, $u^{\max} = u + 1$; Relaxation order: $k = 2$; SDP size: $\omega = 2p$, $a^{\max} = 3$.

POP size		SDP size			Mosek		CGAL		LMBM	
u	p	ω	s^{\max}	ζ	val	time	val	time	val	time
11	91	182	91	222712	-240.54	124	-240.37	98	-508.35	15
16	63	126	171	550692	-205.45	1389	-205.19	280	-429.93	83
21	48	96	276	1107682	—	—	-175.60	321	-365.91	269
26	39	78	406	1955879	—	—	-165.65	559	-338.00	225
31	33	66	561	3167072	—	—	-149.10	973	-305.33	5280
36	28	56	741	4758727	—	—	-140.21	1315	-285.69	737
41	25	50	946	6839993	—	—	-126.55	1926	-265.28	622

5.7 Randomly generated QCQPs with CS and ball constraints on each clique of variables

Test problems: We construct randomly generated QCQPs with CS and ball constraints on each clique of variables as follows:

1. Take a positive integer u , $p := \lfloor n/u \rfloor + 1$ and let

$$I_j = \begin{cases} [u], & \text{if } j = 1, \\ \{u(j-1), \dots, uj\}, & \text{if } j \in \{2, \dots, p-1\}, \\ \{u(p-1), \dots, n\}, & \text{if } j = p; \end{cases} \quad (5.30)$$

2. Generate a quadratic polynomial objective function $f = \sum_{j \in [p]} f_j$ such that for each $j \in [p]$, $f_j \in \mathbb{R}[\mathbf{x}(I_j)]_2$, and the coefficient $f_{j,\alpha}$, $\alpha \in \mathbb{N}_2^{I_j}$ of f_j is randomly generated in $(-1, 1)$ w.r.t. the uniform distribution;
3. Take $m = p$ and $g_j := -\|\mathbf{x}(I_j)\|_2^2 + 1$, $j \in [m]$;
4. Take a random point \mathbf{a} in $S(g)$ w.r.t. the uniform distribution;
5. Let $r := \lfloor l/p \rfloor$ and

$$W_j := \begin{cases} \{(j-1)r+1, \dots, jr\}, & \text{if } j \in [p-1], \\ \{(p-1)r+1, \dots, l\}, & \text{if } j = p. \end{cases} \quad (5.31)$$

For every $j \in [p]$ and every $i \in W_j$, generate a quadratic polynomial $h_i \in \mathbb{R}[\mathbf{x}(I_j)]_2$ by

- (a) for each $\alpha \in \mathbb{N}_2^{I_j} \setminus \{\mathbf{0}\}$, taking a random coefficient $h_{i,\alpha}$ of h_i in $(-1, 1)$ w.r.t. the uniform distribution;
- (b) setting $h_{i,\mathbf{0}} := -\sum_{\alpha \in \mathbb{N}_2^{I_j} \setminus \{\mathbf{0}\}} h_{j,\alpha} \mathbf{a}^\alpha$.

Then \mathbf{a} is a feasible solution of POP (2.3).

The numerical results are displayed in Table 16 and 17.

Table 17: Numerical results for randomly generated QCQPs with CS and ball constraints on each clique of variables

- POP size: $n = 1000$, $m = p$, $l = 143$, $u^{\max} = u + 1$; Relaxation order: $k = 2$;
SDP size: $\omega = 2p$, $a^{\max} = 3$.

POP size		SDP size			Mosek		CGAL		LMBM	
u	p	ω	s^{\max}	ζ	val	time	val	time	val	time
11	91	182	91	235023	-224.15	163	-224.09	204	-500.34	13
16	63	126	171	572905	-192.45	1830	-192.30	335	-420.87	50
21	48	96	276	1139460	–	–	-162.79	537	-363.28	103
26	39	78	406	2005124	–	–	-148.77	1014	-336.42	263
31	33	66	561	3239573	–	–	-142.38	2115	-313.80	3679
36	28	56	741	4862292	–	–	-124.97	5304	-263.77	6598

Table 18: Numerical results for minimizing a random quadratic polynomial with CS and box constraints on each clique of variables

- POP size: $n = m = 1000$, $l = 0$, $u^{\max} = u + 1$; Relaxation order: $k = 2$;
Constant trace: $a^{\max} \in [3, 4]$.

POP size		SDP size			Mosek		CGAL		LMBM	
u	p	ω	s^{\max}	ζ	val	time	val	time	val	time
11	91	1181	91	313361	-204.89	443	-204.69	753	-555.51	223
16	63	1125	171	720323	-163.11	3082	-162.88	3059	-438.22	119
21	48	1095	276	1380918	–	–	-147.92	5655	-387.42	2213
26	39	1077	406	2357161	–	–	-131.00	8889	-340.04	5346

Discussion: The number of variables is fixed as $n = 1000$. We increase the clique size u so that the number of variable cliques p decreases accordingly. Again results in Table 16 and 17 show that CGAL is the fastest solver and returns an optimal value of gap within 1% w.r.t. the one returned by Mosek (for $u \leq 16$). Moreover Mosek runs out of memory when $u \geq 21$, and LMBM fails to converge to the optimal value.

5.8 Randomly generated QCQPs with CS and box constraints on each clique of variables

Test problems: We construct randomly generated QCQPs with CS as in Section 5.7, where ball constraints are now replaced by box constraints. Namely, in Step 3 we take $m = n$, $g_j := -x_j^2 + 1/u$, $j \in [n]$.

The numerical results are displayed in Table 18 and 19.

Discussion: The number of variables is fixed as $n = 1000$. We increase the clique size u so that the number of variable cliques p decreases accordingly. From results in Table 16 and 17, one observes that when the largest size of variable cliques is relatively small (say $u \leq 11$), Mosek is the fastest solver. However when the largest size of variable cliques is relatively large (say $u \leq 21$), Mosek runs out of memory while CGAL still works well.

Table 19: Numerical results for QCQPs with CS and box constraints on each clique of variables

- POP size: $n = m = 1000$, $l = 143$, $u^{\max} = u + 1$; Relaxation order: $k = 2$;
Constant trace: $a^{\max} \in [3, 4]$.

POP size		SDP size			Mosek		CGAL		LMBM	
u	p	ω	s^{\max}	ζ	val	time	val	time	val	time
11	91	1181	91	325672	-187.01	402	-186.98	1915	-570.68	184
16	63	1125	171	742536	-142.16	4323	-142.27	4126	-442.51	57
21	48	1095	276	1412696	–	–	-131.14	5334	-382.89	618
26	39	1077	406	2406406	–	–	-113.44	8037	-336.11	901

5.9 Randomly generated QCQPs with CS-TS and ball constraints on each clique of variables

Test problems: We construct randomly generated QCQPs with CS-TS and ball constraints on each clique of variables as follows:

1. Take a positive integer u , $p := \lfloor n/u \rfloor + 1$ and let $(I_j)_{j \in [p]}$ be defined as in (5.30).
2. Generate a quadratic polynomial objective function $f = \sum_{j \in [p]} f_j$ such that for each $j \in [p]$, $f_j \in \mathbb{R}[\mathbf{x}(I_j)]_2$ and the nonzero coefficient $f_{j,\alpha}$ with $\alpha \in \mathbb{N}_2^{I_j}$ and $|\alpha| = 2$ is randomly generated in $(-1, 1)$ w.r.t. the uniform distribution;
3. Take $m = p$ and $g_j := -\|\mathbf{x}(I_j)\|_2^2 + 1$, $j \in [m]$;
4. Take a random point \mathbf{a} in $S(g)$ w.r.t. the uniform distribution;
5. Let $r := \lfloor l/p \rfloor$ and $(W_j)_{j \in [p]}$ be as in (5.31). For every $j \in [p]$ and every $i \in W_j$, generate a quadratic polynomial $h_i \in \mathbb{R}[\mathbf{x}(I_j)]_2$ by
 - (a) for each $\alpha \in \mathbb{N}_2^{I_j} \setminus \{\mathbf{0}\}$ with $|\alpha| \neq 2$, taking $h_{i,\alpha} = 0$;
 - (b) for each $\alpha \in \mathbb{N}_2^{I_j}$ with $|\alpha| = 2$, taking a random coefficient $h_{i,\alpha}$ of h_i in $(-1, 1)$ w.r.t. the uniform distribution;
 - (c) setting $h_{i,\mathbf{0}} := -\sum_{\alpha \in \mathbb{N}_2^{I_j} \setminus \{\mathbf{0}\}} h_{j,\alpha} \mathbf{a}^\alpha$.

Then \mathbf{a} is a feasible solution of POP (2.3).

The numerical results are displayed in Table 20 and 21.

Discussion: The behavior of solvers is similar to that in Section 5.8. Here, we also emphasize that our framework is less efficient than interior-point methods for most benchmarks presented in [42]. The two underlying reasons are that (1) the block size of the resulting SDP relaxations is small, in which case Mosek performs more efficiently, e.g., for the benchmarks from [42, Section 5.2], and (2) it is harder to find the constant trace, e.g., for the benchmarks from [42, Section 5.4]. Thus our proposed method complements that in [42] when the block size of the SDP relaxations is large and/or when CTP can be efficiently verified.

5.10 Randomly generated QCQPs with CS-TS and box constraints on each clique of variables

Test problems: We construct randomly generated QCQPs with CS-TS as in Section 5.9, where ball constraints are now replaced by box constraints. Namely, in Step

Table 20: Numerical results for minimizing a random quadratic polynomial with CS-TS and ball constraints on each clique of variables

- POP size: $n = 1000$, $m = p$, $l = 0$, $u^{\max} = u + 1$; Relaxation order: $k = 2$;
Sparse order: $t = 1$; SDP size: $a^{\max} = 3$.

POP size		SDP size			Mosek		CGAL		LMBM	
u	p	ω	s^{\max}	ζ	val	time	val	time	val	time
11	91	364	79	169654	-160.05	163	-160.01	498	-489.87	98
16	63	252	154	448354	-135.78	1422	-135.74	768	-413.24	186
21	48	192	254	939619	—	—	-117.17	1605	-351.65	299
26	39	156	379	1705763	—	—	-106.26	3150	-318.15	347

Table 21: Numerical results for QCQPs with CS-TS and ball constraints on each clique of variables

- POP size: $n = 1000$, $m = p$, $l = 143$, $u^{\max} = u + 1$; Relaxation order: $k = 2$;
Sparse order: $t = 1$; SDP size: $a^{\max} = 3$.

POP size		SDP size			Mosek		CGAL		LMBM	
u	p	ω	s^{\max}	ζ	val	time	val	time	val	time
11	91	364	79	180303	-155.91	158	-155.87	604	-500.47	83
16	63	252	154	468290	127.42	1707	-127.36	1053	-412.42	236
21	48	192	254	939619	—	—	-114.85	2877	-363.23	128
26	39	156	379	1751556	—	—	-102.30	6878	-329.16	749

Table 22: Numerical results for minimizing a random quadratic polynomial with CS-TS and box constraints on each clique of variables

- POP size: $n = m = 1000$, $l = 0$, $u^{\max} = u + 1$; Relaxation order: $k = 2$; Sparse order: $t = 1$; Constant trace: $a^{\max} \in [3, 4]$.

POP size		SDP size			Mosek		CGAL		LMBM	
u	p	ω	s^{\max}	ζ	val	time	val	time	val	time
11	91	2362	79	248335	-126.15	151	-126.04	1982	-541.78	140
16	63	2250	154	601081	-100.75	2225	-100.64	7323	-429.78	88
21	48	2190	254	1191001	–	–	-87.804	10734	-363.88	157
26	39	2154	379	2080265	–	–	-81.908	20294	-338.00	1129

Table 23: Numerical results for QCQPs with CS-TS and box constraints on each clique of variables

- POP size: $n = m = 1000$, $l = 143$, $u^{\max} = u + 1$; Relaxation order: $k = 2$; Sparse order: $t = 1$; Constant trace: $a^{\max} \in [3, 4]$.

POP size		SDP size			Mosek		CGAL		LMBM	
u	p	ω	s^{\max}	ζ	val	time	val	time	val	time
11	91	2362	79	258984	-114.53	325	-114.27	482	-529.32	226
16	63	2250	154	621017	-96.199	4450	-96.079	1245	-433.34	519
21	48	2190	254	1220027	–	–	-83.013	8204	-372.97	554
26	39	2154	379	2126058	–	–	-74.532	27600	-258.90	764

3 we take $m = n$, $g_j := -x_j^2 + 1/u$, $j \in [n]$. The numerical results are displayed in Table 22 and 23.

Discussion: The behavior of solvers is similar to that in Section 5.8.

6 Conclusion

In this paper, we have proposed a general framework for exploiting the constant trace property (CTP) in solving large-scale SDPs, typically SDP-relaxations from the Moment-SOS hierarchy applied to POPs. Extensive numerical experiments strongly suggest that with this CTP formulation, the CGAL solver based on first-order methods is more efficient and more scalable than Mosek (based on IPM) without exploiting CTP, especially when the block size is large. In addition, the optimal value returned by CGAL is typically within 1% w.r.t. the one returned by Mosek.

We have also integrated sparsity-exploiting techniques into the CTP framework in order to handle larger size POPs. For SDP-relaxations of large-scale POPs with a term- and/or correlative-sparsity pattern, and in applications for which only a medium accuracy of optimal solutions is enough, we believe that our framework should be very useful.

As a topic of further investigation, we would like to improve the LP-based formulation for verifying CTP, for instance by relying on more general second-order cone

programming. We also would like to generalize the CTP-exploiting framework to non-commutative POPs [5, 21, 37] which have attracted a lot of attention in the quantum information community. Another line of research would be to investigate whether CTP could be efficiently exploited by interior-point solvers.

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A Appendix

A.1 Sparse POPs

For matrices \mathbf{A} and \mathbf{B} of same sizes, the Hadamard product of \mathbf{A} and \mathbf{B} , denoted by $\mathbf{A} \circ \mathbf{B}$, is the matrix with entries $[\mathbf{A} \circ \mathbf{B}]_{i,j} = A_{i,j}B_{i,j}$.

A.1.1 Term sparsity (TS)

Fix a relaxation order $k \in \mathbb{N}^{\geq k_{\min}}$ and a sparse order $t \in \mathbb{N} \setminus \{0\}$. We compute as in [39, Section 5] the following block diagonal (up to permutation) $(0, 1)$ -binary matrices: $\mathbf{G}_{k,t}^{(0)}$ of size $s(k)$; $\mathbf{G}_{k,t}^{(i)}$ of size $s(k - \lceil g_i \rceil)$, $i \in [m]$; $\mathbf{H}_{k,t}^{(i)}$ of size $s(k - \lceil h_i \rceil)$, $i \in [l]$. Then we consider the following sparse moment relaxation of POP (2.3):

$$\tau_{k,t}^{\text{ts}} := \inf_{\mathbf{y} \in \mathbb{R}^{s(2k)}} \left\{ L_{\mathbf{y}}(f) \mid \begin{array}{l} \mathbf{G}_{k,t}^{(0)} \circ \mathbf{M}_k(\mathbf{y}) \succeq 0, \mathbf{y}_0 = 1, \\ \mathbf{G}_{k,t}^{(i)} \circ \mathbf{M}_{k-\lceil g_i \rceil}(g_i \mathbf{y}) \succeq 0, i \in [m], \\ \mathbf{H}_{k,t}^{(i)} \circ \mathbf{M}_{k-\lceil h_i \rceil}(h_i \mathbf{y}) = 0, i \in [l] \end{array} \right\}. \quad (1.32)$$

One has $\tau_{k,t-1}^{\text{ts}} \leq \tau_{k,t}^{\text{ts}} \leq \tau_k \leq f^*$, for all (k, t) . Moreover, we have the following theorem.

Theorem A.1. (Wang et al. [39, Theorem 5.1]) *For each $k \in \mathbb{N}^{\geq k_{\min}}$, the sequence $(\tau_{k,t}^{\text{ts}})_{t \in \mathbb{N} \setminus \{0\}}$ converges to τ_k (the optimal value of SDP (2.7)) in finitely many steps.*

The dual of (1.32) reads as:

$$\rho_{k,t}^{\text{ts}} = \sup_{\xi, \mathbf{Q}_i, \mathbf{U}_i} \left\{ \xi \mid \begin{array}{l} \bar{\mathbf{Q}}_i = \mathbf{G}_{k,t}^{(i)} \circ \mathbf{Q}_i \succeq 0, i \in \{0\} \cup [m], \\ \bar{\mathbf{U}}_i = \mathbf{H}_{k,t}^{(i)} \circ \mathbf{U}_i, i \in [l], \\ f - \xi = \mathbf{v}_k^\top \bar{\mathbf{Q}}_0 \mathbf{v}_k + \sum_{i \in [m]} g_i \mathbf{v}_{k-\lceil g_i \rceil}^\top \bar{\mathbf{Q}}_i \mathbf{v}_{k-\lceil g_i \rceil} \\ \quad + \sum_{i \in [l]} h_i \mathbf{v}_{k-\lceil h_i \rceil}^\top \bar{\mathbf{U}}_i \mathbf{v}_{k-\lceil h_i \rceil} \end{array} \right\}. \quad (1.33)$$

A.1.2 Correlative-Term sparsity (CS-TS)

The basic idea of correlative-term sparsity is to exploit term sparsity for each clique. The clique structure of the initial set of variables is derived from correlative sparsity (Section 4.1).

Fix a relaxation order $k \in \mathbb{N}^{\geq k_{\min}}$. For every sparse order $t \in \mathbb{N} \setminus \{0\}$ and for every $j \in [p]$, we compute the following block diagonal (up to permutation) $(0, 1)$ -binary matrices (see [41]): $\mathbf{G}_{k,t,j}^{(0)}$ of size $s(n_j, k)$; $\mathbf{G}_{k,t,j}^{(i)}$ of size $s(n_j, k - \lceil g_i \rceil)$, $i \in J_j$; $\mathbf{H}_{k,t,j}^{(i)}$ of size $s(n_j, k - \lceil h_i \rceil)$, $i \in W_j$. Then let us consider the following CS-TS moment relaxation:

$$\tau_{k,t}^{\text{cs-ts}} := \inf_{\mathbf{y} \in \mathbb{R}^{s(2k)}} \left\{ L_{\mathbf{y}}(f) \left| \begin{array}{l} \mathbf{G}_{k,t,j}^{(0)} \circ \mathbf{M}_k(\mathbf{y}, I_j) \succeq 0, j \in [p], \mathbf{y}_0 = 1, \\ \mathbf{G}_{k,t,j}^{(i)} \circ \mathbf{M}_{k-\lceil g_i \rceil}(g_i \mathbf{y}, I_j) \succeq 0, i \in J_j, j \in [p], \\ \mathbf{H}_{k,t,j}^{(i)} \circ \mathbf{M}_{k-\lceil h_i \rceil}(h_i \mathbf{y}, I_j) = 0, i \in W_j, j \in [p] \end{array} \right. \right\}. \quad (1.34)$$

One has $\tau_{k,t-1}^{\text{cs-ts}} \leq \tau_{k,t}^{\text{cs-ts}} \leq \tau_k^{\text{cs}} \leq \tau_k \leq f^*$, for all (k, t) . Moreover, we have the following theorem.

Theorem A.2. (Wang et al. [41]) For each $k \in \mathbb{N}^{\geq k_{\min}}$, the sequence $(\tau_{k,t}^{\text{cs-ts}})_{t \in \mathbb{N} \setminus \{0\}}$ converges to τ_k^{cs} (the optimal value of SDP (4.21)) in finitely many steps.

The dual of (1.34) reads as:

$$\rho_{k,t}^{\text{cs-ts}} = \sup_{\xi, \bar{\mathbf{Q}}_i^{(j)}, \bar{\mathbf{U}}_i^{(j)}} \left\{ \xi \left| \begin{array}{l} \bar{\mathbf{Q}}_i^{(j)} = \mathbf{G}_{k,t,j}^{(i)} \circ \mathbf{Q}_i^{(j)} \succeq 0, i \in \{0\} \cup J_j, j \in [p], \\ \bar{\mathbf{U}}_i^{(j)} = \mathbf{H}_{k,t,j}^{(i)} \circ \mathbf{U}_i^{(j)}, i \in W_j, j \in [p], \\ f - \xi = \sum_{j \in [p]} \left((\mathbf{v}_k^{I_j})^\top \bar{\mathbf{Q}}_0^{(j)} \mathbf{v}_k^{I_j} \right. \right. \\ \quad \left. \left. + \sum_{i \in J_j} g_i (\mathbf{v}_{k-\lceil g_i \rceil}^{I_j})^\top \bar{\mathbf{Q}}_i^{(j)} \mathbf{v}_{k-\lceil g_i \rceil}^{I_j} \right. \right. \\ \quad \left. \left. + \sum_{i \in W_j} h_i (\mathbf{v}_{k-\lceil h_i \rceil}^{I_j})^\top \bar{\mathbf{U}}_i^{(j)} \mathbf{v}_{k-\lceil h_i \rceil}^{I_j} \right) \right\}. \quad (1.35)$$

A.2 Conditional gradient-based augmented Lagrangian (CGAL)

A.2.1 SDP with CTP

Let $s, l, s^{(j)} \in \mathbb{N}^{\geq 1}$, $j \in [\omega]$, be fixed such that $s = \sum_{j=1}^{\omega} s^{(j)}$. Let \mathcal{S} be the set of real symmetric matrices of size s in a block diagonal form: $\mathbf{X} = \text{diag}(\mathbf{X}_1, \dots, \mathbf{X}_\omega)$, such that \mathbf{X}_j is a block of size $s^{(j)}$, $j \in [\omega]$. Let $s^{\max} := \max_{j \in [\omega]} s^{(j)}$. Let \mathcal{S}^+ be the set of all $\mathbf{X} \in \mathcal{S}$ such that $\mathbf{X} \succeq 0$, i.e., \mathbf{X} has only nonnegative eigenvalues. Then \mathcal{S} is a Hilbert space with scalar product $\langle \mathbf{A}, \mathbf{B} \rangle = \text{trace}(\mathbf{B}^\top \mathbf{A})$ and \mathcal{S}^+ is a self-dual cone.

Let us consider the following SDP:

$$\tau = \inf_{\mathbf{X} \in \mathcal{S}^+} \{ \langle \mathbf{C}, \mathbf{X} \rangle : \mathcal{A}\mathbf{X} = \mathbf{b} \}, \quad (1.36)$$

where $\mathcal{A} : \mathcal{S} \rightarrow \mathbb{R}^\zeta$ is a linear operator of the form $\mathcal{A}\mathbf{X} = [\langle \mathbf{A}_1, \mathbf{X} \rangle, \dots, \langle \mathbf{A}_\zeta, \mathbf{X} \rangle]$, with $\mathbf{A}_i \in \mathcal{S}$, $i \in [\zeta]$, $\mathbf{C} \in \mathcal{S}$ is the cost matrix and $\mathbf{b} \in \mathbb{R}^\zeta$ is a vector.

The dual of SDP (1.36) reads as:

$$\rho = \sup_{\mathbf{y} \in \mathbb{R}^\zeta} \{ \mathbf{b}^\top \mathbf{y} : \mathcal{A}^\top \mathbf{y} - \mathbf{C} \in \mathcal{S}^+ \}, \quad (1.37)$$

where $\mathcal{A}^\top : \mathbb{R}^\zeta \rightarrow \mathcal{S}$ is the adjoint operator of \mathcal{A} , i.e., $\mathcal{A}^\top \mathbf{y} = \sum_{i \in [\zeta]} y_i \mathbf{A}_i$.

The following assumption will be used later on.

Assumption A.3. Consider the following conditions:

1. Strong duality of primal-dual (1.36)-(1.37) holds, i.e., $\rho = \tau$ and $\rho \in \mathbb{R}$.
2. Constant trace property (CTP): $\exists a > 0 : \forall \mathbf{X} \in \mathcal{S}, \mathcal{A}\mathbf{X} = \mathbf{b} \Rightarrow \text{trace}(\mathbf{X}) = a$.

For $\mathbf{X} \in \mathcal{S}$, the Frobenius norm of \mathbf{X} is defined by $\|\mathbf{X}\|_F := \sqrt{\langle \mathbf{X}, \mathbf{X} \rangle}$. We denote by $\|\mathcal{A}\|$ the operator norm of \mathcal{A} , i.e., $\|\mathcal{A}\| := \max_{\mathbf{X} \in \mathcal{S}} \|\mathcal{A}\mathbf{X}\|_2 / \|\mathbf{X}\|_F$. The smallest eigenvalue of a real symmetric matrix \mathbf{D} is denoted by $\lambda_{\min}(\mathbf{D})$.

Algorithm 3 CGAL-SDP-CTP

Input: SDP (1.36) such that Assumption A.3 holds; Parameter $K > 0$.

Output: $(\mathbf{X}_t)_{t \in \mathbb{N}}$.

- 1: Set $\mathbf{X}_0 := \mathbf{0}_S$ and $\mathbf{y}_0 := \mathbf{0}_{\mathbb{R}^c}$.
 - 2: **for** $t \in \mathbb{N}$ **do**
 - 3: Set $\beta_t := \sqrt{t+1}$ and $\eta_t := 2/(t+1)$;
 - 4: Take an eigenvector \mathbf{u}_t corresponding to $\lambda_{\min}(\mathbf{C} + \mathcal{A}^\top(\mathbf{y}_{t-1} + \eta_t(\mathcal{A}\mathbf{X}_{t-1} - \mathbf{b})))$;
 - 5: Set $\mathbf{X}_t := (1 - \eta_t)\mathbf{X}_{t-1} + \eta_t a \mathbf{u}_t \mathbf{u}_t^\top$;
 - 6: Select γ_t as the largest $\gamma \in [0, 1]$ such that:
 - 7: $\gamma \|\mathcal{A}\mathbf{X}_t - \mathbf{b}\|_2^2 \leq \beta_t \eta_t^2 a^2 \|\mathcal{A}\|^2$ and $\|\mathbf{y}_{t-1} + \gamma(\mathcal{A}\mathbf{X}_t - \mathbf{b})\|_2 \leq K$;
 - 8: Set $\mathbf{y}_t = \mathbf{y}_{t-1} + \gamma_t(\mathcal{A}\mathbf{X}_t - \mathbf{b})$.
-

Algorithm. In [45], Yurtsever et al. state Algorithm 3 (see below) to solve SDP (1.36) with CTP. This procedure is based on the augmented Lagrangian paradigm combined together with the conditional gradient method.

The convergence of the sequence $(\mathbf{X}_t)_{t \in \mathbb{N}}$ in Algorithm 3 to the set of optimal solutions of SDP (1.36) is guaranteed as follows:

Theorem A.4. [45, Fact 3.1] Consider SDP (1.36) such that Assumption A.3 holds. Let $(\mathbf{X}_t)_{t \in \mathbb{N}}$ be in the output of Algorithm 3. Then $\mathbf{X}_t \succeq 0$, for all $t \in \mathbb{N}$ and $\|\mathcal{A}\mathbf{X}_t - \mathbf{b}\|_2 \rightarrow 0$, $|\langle \mathbf{C}, \mathbf{X}_t \rangle - \tau| \rightarrow 0$ as $t \rightarrow \infty$, with the rate of order $\mathcal{O}(\sqrt{t})$.

Remark A.5. In order to achieve the best convergence rate for Algorithm 3, we scale the problem's input as follows: $\|\mathbf{C}\|_F = \|\mathcal{A}\| = a = 1$ and $\|\mathbf{A}_1\|_F = \dots = \|\mathbf{A}_\zeta\|_F$.

Remark A.6. Given $\varepsilon > 0$, the for loop in Algorithm 3 terminates when:

$$\frac{|\langle \mathbf{C}, \mathbf{X}_{t-1} \rangle - (a\lambda_{\min}(\mathbf{C} + \mathcal{A}^\top(\mathbf{y}_{t-1} + \eta_t(\mathcal{A}\mathbf{X}_{t-1} - \mathbf{b}))) - \mathbf{b}^\top \mathbf{y}_{t-1})|}{1 + \max\{|\langle \mathbf{C}, \mathbf{X}_{t-1} \rangle|, |a\lambda_{\min}(\mathbf{C} + \mathcal{A}^\top(\mathbf{y}_{t-1} + \eta_t(\mathcal{A}\mathbf{X}_{t-1} - \mathbf{b}))) - \mathbf{b}^\top \mathbf{y}_{t-1}|\}} \leq \varepsilon \quad (1.38)$$

and $\|\mathcal{A}\mathbf{X}_{t-1} - \mathbf{b}\|_2 / \max\{1, \|b\|_2\} \leq \varepsilon$. In our experiments, we choose $\varepsilon = 10^{-3}$. Note that the left hand side in (1.38) is the relative gap between the primal and dual approximate values obtained at each iteration.

Remark A.7. To save memory at each iteration, we can run Algorithm 3 with an implicit \mathbf{X}_t by setting $\mathbf{w}_t := \mathcal{A}\mathbf{X}_t - \mathbf{b}$. In this case, Step 5 becomes $\mathbf{w}_t := (1 - \eta_t)\mathbf{w}_{t-1} + \eta_t[\mathcal{A}(a\mathbf{u}_t\mathbf{u}_t^\top) - b]$. Thus we only obtain an approximate dual solution \mathbf{y}_t of SDP (1.36) when Algorithm 3 terminates. To recover an approximate primal solution \mathbf{X} of SDP (1.36), we rely on a process similar to steps 2 and 3 of Algorithm 5, which will be presented later on.

In Appendix A.2.2, we provide an analogous method to solve an SDP with CTP on each subset of blocks.

A.2.2 SDP with CTP on each subset of blocks

Let $p \in \mathbb{N}^{\geq 1}$, $s_j, \omega_j \in \mathbb{N}$, $j \in [p]$, and $s^{(i,j)} \in \mathbb{N}^{\geq 1}$, $i \in [\omega_p]$, $j \in [p]$, be fixed such that $s_j = \sum_{i \in [\omega_j]} s^{(i,j)}$, $j \in [p]$. For every $j \in [p]$, let \mathcal{S}_j be the set of real symmetric matrices of size s_j in a block diagonal form: $\mathbf{X}_j = \text{diag}(\mathbf{X}_{1,j}, \dots, \mathbf{X}_{\omega_j,j})$, such that $\mathbf{X}_{i,j}$ is a block of size $s^{(i,j)}$, $i \in [\omega_j]$. Let $s^{\max} := \max_{i \in [\omega_p], j \in [p]} s^{(i,j)}$. For every $j \in [p]$, let \mathcal{S}_j^+ be the set of all $\mathbf{X}_j \in \mathcal{S}_j$ such that $\mathbf{X}_j \succeq 0$. Then for every $j \in [p]$, \mathcal{S}_j is a Hilbert space with scalar product $\langle \mathbf{A}, \mathbf{B} \rangle = \text{trace}(\mathbf{B}^\top \mathbf{A})$ and \mathcal{S}_j^+ is a self-dual cone.

Let us consider the following SDP:

$$\tau = \inf_{\mathbf{X}_j \in \mathcal{S}_j^+} \left\{ \sum_{j \in [p]} \langle \mathbf{C}_j, \mathbf{X}_j \rangle : \sum_{j \in [p]} \mathcal{A}_j \mathbf{X}_j = \mathbf{b} \right\}, \quad (1.39)$$

where $\mathcal{A}_j : \mathcal{S}_j \rightarrow \mathbb{R}^\zeta$ is a linear operator of the form $\mathcal{A}_j \mathbf{X} = [\langle \mathbf{A}_{1,j}, \mathbf{X} \rangle, \dots, \langle \mathbf{A}_{\zeta,j}, \mathbf{X} \rangle]$, with $\mathbf{A}_{i,j} \in \mathcal{S}_j$, $i \in [\zeta]$, $\mathbf{C}_j \in \mathcal{S}_j$, $j \in [p]$, and $\mathbf{b} \in \mathbb{R}^\zeta$.

The dual of SDP (1.39) reads as:

$$\rho = \sup_{\mathbf{y} \in \mathbb{R}^\zeta} \left\{ \mathbf{b}^\top \mathbf{y} : \mathcal{A}_j^\top \mathbf{y} - \mathbf{C}_j \in \mathcal{S}_j^+, j \in [p] \right\}, \quad (1.40)$$

where $\mathcal{A}_j^\top : \mathbb{R}^\zeta \rightarrow \mathcal{S}_j$ is the adjoint operator of \mathcal{A}_j , i.e., $\mathcal{A}_j^\top \mathbf{z} = \sum_{i \in [\zeta]} z_i \mathbf{A}_{i,j}$, $j \in [p]$.

The following assumption will be used later on:

Assumption A.8. Consider the following conditions:

1. Strong duality of primal-dual (1.39)-(1.40) holds, i.e., $\rho = \tau$ and $\rho \in \mathbb{R}$.
2. Constant trace property (CTP): there exist $a_j > 0$ and $j \in [p]$, such that

$$\left. \begin{array}{l} \forall \mathbf{X}_j \in \mathcal{S}_j, j \in [p], \\ \sum_{j \in [p]} \mathcal{A}_j \mathbf{X}_j = \mathbf{b} \end{array} \right\} \Rightarrow \text{trace}(\mathbf{X}_j) = a_j, j \in [p]. \quad (1.41)$$

Recall that $\lambda_{\min}(\mathbf{D})$ stands for the smallest eigenvalue of a real symmetric matrix \mathbf{D} . We denote by $\prod_{j \in [p]} \mathcal{S}_j$ the set of all $\mathbf{X} = \text{diag}(\mathbf{X}_j)_{j \in [p]}$ such that $\mathbf{X}_j \in \mathcal{S}_j$, for $j \in [p]$. Let $\mathbf{C} := \text{diag}(\mathbf{C}_j)_{j \in [p]}$ and let $\mathcal{A} : \prod_{j \in [p]} \mathcal{S}_j \rightarrow \mathbb{R}^\zeta$ be a linear operator of the form: $\mathcal{A} \mathbf{X} = \sum_{j \in [p]} \mathcal{A}_j \mathbf{X}_j$, for all $\mathbf{X} = \text{diag}(\mathbf{X}_j)_{j \in [p]} \in \prod_{j \in [p]} \mathcal{S}_j$. Then for every $\mathbf{X} = \text{diag}(\mathbf{X}_j)_{j \in [p]} \in \prod_{j \in [p]} \mathcal{S}_j$, we have $\langle \mathbf{C}, \mathbf{X} \rangle = \sum_{j \in [p]} \langle \mathbf{C}_j, \mathbf{X}_j \rangle$ and $\mathcal{A} \mathbf{X} = \left[\langle \mathbf{A}^{(1)}, \mathbf{X} \rangle, \dots, \langle \mathbf{A}^{(\zeta)}, \mathbf{X} \rangle \right]$, where $\mathbf{A}^{(i)} := \text{diag}((\mathbf{A}_{i,j})_{j \in [p]})$, for $i \in [\zeta]$.

SDP (1.39) can be rewritten as $\tau = \inf_{\mathbf{X} \in \prod_{j \in [p]} \mathcal{S}_j^+} \{ \langle \mathbf{C}, \mathbf{X} \rangle : \mathcal{A} \mathbf{X} = \mathbf{b} \}$.

The dual operator $\mathcal{A}^\top : \mathbb{R}^\zeta \rightarrow \prod_{j \in [p]} \mathcal{S}_j$ of \mathcal{A} reads $\mathcal{A}^\top \mathbf{z} = \text{diag}((\mathcal{A}_j^\top \mathbf{z})_{j \in [p]})$. Note $\Delta_j := \{ \mathbf{X}_j \in \mathcal{S}_j^+ : \text{trace}(\mathbf{X}_j) = a_j \}$, for $j \in [p]$.

Algorithm. In order to solve SDP (1.39) with CTP on each subset of blocks, we use [44, Algorithm 1] due to Yurtsever et al. to describe Algorithm 4 with the following setting: $\mathcal{X} \leftarrow \Delta := \prod_{j \in [p]} \Delta_j$, $\mathcal{K} \leftarrow \{ \mathbf{b} \}$, $p \leftarrow \zeta$, $Ax \leftarrow \mathcal{A} \mathbf{X}$, $f(x) \leftarrow \langle \mathbf{C}, \mathbf{X} \rangle$, $\lambda_0 \leftarrow 1$, $\lambda_k \leftarrow \beta_k$, $\sigma_k \leftarrow \gamma_k$. $Dy_{k+1} \leftarrow K$, $L_f \leftarrow 0$, $\bar{r}_{k+1} \leftarrow \mathbf{b}$, $D_{\mathcal{X}}^2 \leftarrow 2 \sum_{j \in [p]} a_j^2$, $v_k \leftarrow \mathbf{C} + \mathcal{A}^\top \mathbf{z}_k$, $\arg \min_{x \in \mathcal{X}} \langle v_k, x \rangle \leftarrow \arg \min_{\mathbf{X} \in \Delta} \langle \mathbf{C} + \mathcal{A}^\top \mathbf{z}_k, \mathbf{X} \rangle$.

With fixed \mathbf{z}_k , we have:

$$\begin{aligned} \min_{\mathbf{X} \in \Delta} \langle \mathbf{C} + \mathcal{A}^\top \mathbf{z}_k, \mathbf{X} \rangle &= \min_{\text{diag}((\mathbf{X}_j)_{j \in [p]}) \in \prod_{j \in [p]} \Delta_j} \sum_{j \in [p]} \langle \mathbf{C}_j + \mathcal{A}_j^\top \mathbf{z}_k, \mathbf{X}_j \rangle \\ &= \sum_{j \in [p]} \min_{\mathbf{X}_j \in \Delta_j} \langle \mathbf{C}_j + \mathcal{A}_j^\top \mathbf{z}_k, \mathbf{X}_j \rangle = \sum_{j \in [p]} a_j \lambda_{\min}(\mathbf{C}_j + \mathcal{A}_j^\top \mathbf{z}_k). \end{aligned}$$

Let $\mathbf{u}_k^{(j)}$ be a uniform eigenvector corresponding to $\lambda_{\min}(\mathbf{C}_j + \mathcal{A}_j^\top \mathbf{z}_k)$, for $j \in [p]$. Then one has $\text{diag}((a_j \mathbf{u}_k^{(j)} (\mathbf{u}_k^{(j)})^\top)_{j \in [p]}) \in \arg \min_{\mathbf{X} \in \Delta} \langle \mathbf{C} + \mathcal{A}^\top \mathbf{z}_k, \mathbf{X} \rangle$. Thus we can set $s_k \leftarrow \text{diag}((a_j \mathbf{u}_k^{(j)} (\mathbf{u}_k^{(j)})^\top)_{j \in [p]})$ in [44, Algorithm 1].

Relying on [44, Theorem 3.1], we guarantee the convergence of the sequence $((\mathbf{X}_j^{(t)})_{j \in [p]})_{t \in \mathbb{N}}$ in Algorithm 4 to the set of optimal solutions of SDP (1.39) in the following theorem:

Algorithm 4 CGAL-SDP-CTP-Blocks

Input: SDP (1.39) such that Assumption A.8 holds; Parameter $K > 0$.

Output: $((\mathbf{X}_j^{(t)})_{j \in [p]})_{t \in \mathbb{N}}$.

- 1: Set $(\mathbf{X}_j^{(0)})_{j \in [p]} := (\mathbf{0}_S)_{j \in [p]}$ and $\mathbf{y}_0 := \mathbf{0}_{\mathbb{R}^{\zeta}}$.
 - 2: **for** $t \in \mathbb{N}$ **do**
 - 3: Set $\beta_t := \sqrt{t+1}$ and $\eta_t := 2/(t+1)$;
 - 4: Set $\mathbf{z}_t := \mathbf{y}_{t-1} + \eta_t(\sum_{j \in [p]} \mathcal{A}_j \mathbf{X}_j^{(t-1)} - \mathbf{b})$;
 - 5: **for** $j \in [p]$ **do**
 - 6: Take a uniform eigenvector $\mathbf{u}_t^{(j)}$ corresponding to $\lambda_{\min}(\mathbf{C}_j + \mathcal{A}_j^\top \mathbf{z}_t)$;
 - 7: Set $\mathbf{X}_j^{(t)} := (1 - \eta_t)\mathbf{X}_j^{(t-1)} + \eta_t a_j \mathbf{u}_t^{(j)} (\mathbf{u}_t^{(j)})^\top$;
 - 8: Select γ_t as the largest $\gamma \in [0, 1]$ such that:
 - 9: $\gamma \|\sum_{j \in [p]} \mathcal{A}_j \mathbf{X}_j^{(t)} - \mathbf{b}\|_2^2 \leq \beta_t \eta_t^2 (\sum_{j \in [p]} a_j^2) \|\mathcal{A}\|^2$ and $\|\mathbf{y}_{t-1} + \gamma (\sum_{j \in [p]} \mathcal{A}_j \mathbf{X}_j^{(t)} - \mathbf{b})\|_2 \leq K$;
 - 10: Set $\mathbf{y}_t = \mathbf{y}_{t-1} + \gamma_t (\sum_{j \in [p]} \mathcal{A}_j \mathbf{X}_j^{(t)} - \mathbf{b})$.
-

Theorem A.9. Consider SDP (1.39) such that Assumption A.8 holds. Let $((\mathbf{X}_j^{(t)})_{j \in [p]})_{t \in \mathbb{N}}$ be the output of Algorithm 4. Then $\mathbf{X}_j^{(t)} \succeq 0$, for all $j \in [p]$ and for all $t \in \mathbb{N}$ and $\|\sum_{j \in [p]} \mathcal{A}_j \mathbf{X}_j^{(t)} - \mathbf{b}\|_2 \rightarrow 0$ and $|\sum_{j \in [p]} \langle \mathbf{C}_j, \mathbf{X}_j^{(t)} \rangle - \tau| \rightarrow 0$ as $t \rightarrow \infty$ with the rate $\mathcal{O}(\sqrt{t})$.

Remark A.10. Before running Algorithm 4, we scale the problem's input as follows: $\|\mathbf{C}\|_F = \|\mathcal{A}\| = a_1 = \dots = a_p = 1$ and $\|\mathbf{A}^{(1)}\|_F = \dots = \|\mathbf{A}^{(\zeta)}\|_F$.

Remark A.11. Given $\varepsilon > 0$, the for loop in Algorithm 4 terminates when:

$$\frac{|\sum_{j \in [p]} \langle \mathbf{C}_j, \mathbf{X}_j^{(t-1)} \rangle - \sum_{j \in [p]} (a_j \lambda_{\min}(\mathbf{C}_j + \mathcal{A}_j^\top \mathbf{z}_t) - \mathbf{b}^\top \mathbf{y}_{t-1})|}{1 + \max\{|\sum_{j \in [p]} \langle \mathbf{C}_j, \mathbf{X}_j^{(t-1)} \rangle|, |\sum_{j \in [p]} (a_j \lambda_{\min}(\mathbf{C}_j + \mathcal{A}_j^\top \mathbf{z}_t) - \mathbf{b}^\top \mathbf{y}_{t-1})|\}} \leq \varepsilon$$

and $\|\sum_{j \in [p]} \mathcal{A}_j \mathbf{X}_j^{(t-1)} - \mathbf{b}\|_2 / \max\{1, \|\mathbf{b}\|_2\} \leq \varepsilon$. In our experiments, we choose $\varepsilon = 10^{-2}$.

Remark A.12. To save memory at each iteration, we can run Algorithm 4 with implicit $\mathbf{X}_j^{(t)}$, $j \in [p]$, by setting $\mathbf{w}_t := \sum_{j \in [p]} \mathcal{A}_j \mathbf{X}_j^{(t)} - \mathbf{b}$. In this case, Step 7 becomes $\mathbf{w}_t := (1 - \eta_t)\mathbf{w}_{t-1} + \eta_t [\sum_{j \in [p]} \mathcal{A}_j (a_j \mathbf{u}_t^{(j)} (\mathbf{u}_t^{(j)})^\top) - \mathbf{b}]$. Thus we only obtain an approximate dual solution \mathbf{y}_t of SDP (1.39) when Algorithm 4 terminates. To recover an approximate primal solution $(\mathbf{X}_j)_{j \in [p]}$ of SDP (1.39), we do a process similar to steps 2 and 3 of Algorithm 6 which will be presented later on.

A.3 Spectral method (SM)

A.3.1 SDP with CTP

Consider SDP with CTP described in Appendix A.2. The following assumption will be used later on:

Assumption A.13. Dual attainability: SDP (1.37) has an optimal solution.

Lemma A.14. Let Assumption A.3 hold and let $\varphi: \mathbb{R}^\zeta \rightarrow \mathbb{R}$ be a function defined by: $\mathbf{y} \mapsto \varphi(\mathbf{y}) := a \lambda_{\min}(\mathbf{C} - \mathcal{A}^\top \mathbf{y}) + \mathbf{b}^\top \mathbf{y}$. Then:

$$\tau = \sup_{\mathbf{y} \in \mathbb{R}^\zeta} \varphi(\mathbf{y}). \quad (1.42)$$

Moreover, if Assumption A.13 holds, then problem (1.42) has an optimal solution.

Notice that φ in Lemma A.14 is concave and continuous but not differentiable in general. The subdifferential of φ at \mathbf{y} reads: $\partial\varphi(\mathbf{y}) = \{\mathbf{b} - a\mathcal{A}\mathbf{U} : \mathbf{U} \in \text{conv}(\Gamma(\mathbf{C} - \mathcal{A}^\top \mathbf{y}))\}$, where for each $\mathbf{A} \in \mathcal{S}$, $\Gamma(\mathbf{A}) := \{\mathbf{u}\mathbf{u}^\top : \mathbf{A}\mathbf{u} = \lambda_{\min}(\mathbf{A})\mathbf{u}, \|\mathbf{u}\|_2 = 1\}$.

Given $r \in \mathbb{N}^{\geq 1}$ and $\mathbf{u}_j \in \mathbb{R}^s$, $j \in [r]$, consider the following convex quadratic optimization problem (QP):

$$\begin{aligned} \min_{\xi \in \mathbb{R}^r} \quad & \frac{1}{2} \left\| \mathbf{b} - a\mathcal{A} \left(\sum_{j \in [r]} \xi_j \mathbf{u}_j \mathbf{u}_j^\top \right) \right\|_2^2 \\ \text{s.t.} \quad & \sum_{j \in [r]} \xi_j = 1; \xi_j \geq 0, j \in [r]. \end{aligned} \quad (1.43)$$

Next, we describe Algorithm 5 to solve SDP (1.36), which is based on nonsmooth first-order optimization methods (e.g., LMBM [12, Algorithm 1]).

Algorithm 5 Spectral-SDP-CTP

Input: SDP (1.36) with unknown optimal value and optimal solution;
method (T) for solving convex nonsmooth unconstrained optimization problems (NSOP).

Output: the optimal value ρ and the optimal solution \mathbf{X}^* of SDP (1.36).

- 1: Compute the optimal value τ and an optimal solution $\bar{\mathbf{y}}$ of the NSOP (1.42) by using method (T);
 - 2: Compute $\lambda_{\min}(\mathbf{C} - \mathcal{A}^\top \bar{\mathbf{y}})$ and its corresponding uniform eigenvectors $\mathbf{u}_1, \dots, \mathbf{u}_r$;
 - 3: Compute an optimal solution $(\bar{\xi}_1, \dots, \bar{\xi}_r)$ of QP (1.43) and set $\mathbf{X}^* = a \sum_{j=1}^r \bar{\xi}_j \mathbf{u}_j \mathbf{u}_j^\top$.
-

Corollary A.15. *Let Assumption A.3 hold. Assume that the method (T) is globally convergent for NSOP (1.42) (e.g., (T) is LMBM). Then output τ of Algorithm 5 is well-defined. Moreover, if Assumption A.13 holds, the vector $\bar{\mathbf{y}}$ mentioned at Step 1 of Algorithm 5 exists and thus the output \mathbf{X}^* of Algorithm 5 is well-defined.*

A.3.2 SDP with CTP on each subset of blocks

Consider SDP with CTP on each subset of blocks described in Appendix A.2.2.

The following assumption will be used later on:

Assumption A.16. *Dual attainability: SDP (1.40) has an optimal solution.*

Lemma A.17. *Let Assumption A.8 hold and let $\psi : \mathbb{R}^\zeta \rightarrow \mathbb{R}$ be a function defined by: $\mathbf{y} \mapsto \psi(\mathbf{y}) := \mathbf{b}^\top \mathbf{y} + \sum_{j \in [p]} a_j \lambda_{\min}(\mathbf{C}_j - \mathcal{A}_j^\top \mathbf{y})$. Then:*

$$\tau = \sup_{\mathbf{y} \in \mathbb{R}^\zeta} \psi(\mathbf{y}). \quad (1.44)$$

Moreover, if of Assumption A.16 holds, then problem (1.44) has an optimal solution.

Proof. From (1.39) and Condition 4 of Assumption A.8,

$$\tau = \inf_{\mathbf{X}_j \in \mathcal{S}_j^+} \left\{ \sum_{j \in [p]} \langle \mathbf{C}_j, \mathbf{X}_j \rangle \mid \begin{array}{l} \sum_{j \in [p]} \mathcal{A}_j \mathbf{X}_j = \mathbf{b}, \\ \langle \mathbf{I}_j, \mathbf{X}_j \rangle = a_j, j \in [p] \end{array} \right\}, \quad (1.45)$$

where $\mathbf{I}_j \in \mathcal{S}_j$ is the identity matrix, for $j \in [p]$. Note that $\langle \mathbf{I}_j, \mathbf{X}_j \rangle = \text{trace}(\mathbf{X}_j)$, for $\mathbf{X}_j \in \mathcal{S}_j$, $j \in [p]$. The dual of this SDP reads as:

$$\rho = \sup_{(\xi, \mathbf{y}) \in \mathbb{R}^{p+\zeta}} \left\{ \sum_{j \in [p]} a_j \xi_j + \mathbf{b}^\top \mathbf{y} : \mathbf{C}_j - \mathcal{A}_j^\top \mathbf{y} - \xi_j \mathbf{I}_j \in \mathcal{S}_j^+, j \in [p] \right\}. \quad (1.46)$$

It implies that $\rho = \sup_{\xi, \mathbf{y}} \{ \sum_{j \in [p]} a_j \xi_j + \mathbf{b}^\top \mathbf{y} : \xi_j \leq \lambda_{\min}(\mathbf{C}_j - \mathcal{A}_j^\top \mathbf{y}), j \in [p] \}$. From this, the result follows since $\rho = \tau$. \square

Proposition A.18. *The function ψ in Lemma A.17 has the following properties:*

1. ψ is concave and continuous but not differentiable in general.
2. The subdifferential of ψ at \mathbf{y} satisfies: $\partial\psi(\mathbf{y}) = \mathbf{b} + \sum_{j \in [p]} a_j \partial\psi_j(\mathbf{y})$, where for every $j \in [p]$, $\psi_j : \mathbb{R}^\zeta \rightarrow \mathbb{R}$ is a function defined by $\psi_j(\mathbf{y}) = \lambda_{\min}(\mathbf{C}_j - \mathcal{A}_j^\top \mathbf{y})$ and $\partial\psi_j(\mathbf{y}) = \{-\mathcal{A}_j \mathbf{U} : \mathbf{U} \in \text{conv}(\Gamma(\mathbf{C}_j - \mathcal{A}_j^\top \mathbf{y}))\}$.

Proof. It is not hard to prove the first statement. Indeed, ψ is a positive combination of $\mathbf{z} \mapsto \mathbf{b}^\top \mathbf{z}$, ψ_j , $j \in [p]$, which are convex, continuous functions. The second statement follows by applying the subdifferential sum rule and notice that the domains of $\mathbf{z} \mapsto \mathbf{b}^\top \mathbf{z}$, ψ_j , $j \in [p]$, are both \mathbb{R}^n . \square

Lemma A.19. *If $\bar{\mathbf{z}}$ is an optimal solution of NSOP (1.44), then:*

1. For each $j \in [p]$, there exists $(\mathbf{X}_j^*) \in a_j \text{conv}(\Gamma(\mathbf{C}_j - \mathcal{A}_j^\top \bar{\mathbf{z}}))$, such that $\sum_{j \in [p]} \mathcal{A}_j \mathbf{X}_j^* = \mathbf{b}$.
2. For $j \in [p]$, $\mathbf{X}_j^* = a_j \sum_{i \in [r_j]} \bar{\xi}_{i,j} \mathbf{u}_{i,j} \mathbf{u}_{i,j}^\top$, where $(\mathbf{u}_{i,j})_{i \in [r_j]}$ are all uniform eigenvectors corresponding to $\lambda_{\min}(\mathbf{C}_j - \mathcal{A}_j^\top \bar{\mathbf{z}})$ and $((\bar{\xi}_{i,j})_{i \in [r_j]})_{j \in [p]}$ is an optimal solution of the convex quadratic problem:

$$\begin{aligned} \min_{\xi_{i,j}} \quad & \frac{1}{2} \left\| \mathbf{b} - \sum_{j \in [p]} a_j \mathcal{A}_j \left(\sum_{i \in [r_j]} \xi_{i,j} \mathbf{u}_{i,j} \mathbf{u}_{i,j}^\top \right) \right\|_2^2 \\ \text{s.t.} \quad & \sum_{i \in [r_j]} \xi_{i,j} = 1; \xi_{i,j} \geq 0, i \in [r_j], j \in [p]. \end{aligned} \quad (1.47)$$

3. $(\mathbf{X}_j^*)_{j \in [p]}$ is an optimal solution of SDP (1.39).

Proof. By [2, Theorem 4.2], $\mathbf{0} \in \partial\psi(\bar{\mathbf{z}})$. Combining this with Proposition A.18.2, the first statement follows, which in turn implies the second statement. We next prove the third statement. For $j \in [p]$, one has $\mathbf{X}_j^* \succeq \mathbf{0}$ since $\mathbf{X}_j^* = a_j \sum_{i \in [r_j]} \bar{\xi}_{i,j} \mathbf{u}_{i,j} \mathbf{u}_{i,j}^\top$ with $\bar{\xi}_{i,j} \geq 0$, $i \in [r_j]$. From this and since $\sum_{j \in [p]} \mathcal{A}_j \mathbf{X}_j^* = \mathbf{b}$, $(\mathbf{X}_j^*)_{j \in [p]}$ is a feasible solution of SDP (1.39). Moreover,

$$\begin{aligned} \sum_{j \in [p]} \langle \mathbf{C}_j, \mathbf{X}_j^* \rangle &= \sum_{j \in [p]} \langle \mathbf{C}_j - \mathcal{A}_j^\top \bar{\mathbf{z}}, \mathbf{X}_j^* \rangle + \sum_{j \in [p]} \langle \mathcal{A}_j^\top \bar{\mathbf{z}}, \mathbf{X}_j^* \rangle \\ &= \sum_{j \in [p]} a_j \sum_{i \in [r_j]} \bar{\xi}_{i,j} \langle \mathbf{C}_j - \mathcal{A}_j^\top \bar{\mathbf{z}}, \mathbf{u}_{i,j} \mathbf{u}_{i,j}^\top \rangle + \sum_{j \in [p]} \bar{\mathbf{z}}^\top (\mathcal{A}_j \mathbf{X}_j^*) \\ &= \sum_{j \in [p]} a_j \sum_{i \in [r_j]} \bar{\xi}_{i,j} \mathbf{u}_{i,j}^\top (\mathbf{C}_j - \mathcal{A}_j^\top \bar{\mathbf{z}}) \mathbf{u}_{i,j} + \bar{\mathbf{z}}^\top \mathbf{b} \\ &= \sum_{j \in [p]} a_j \lambda_{\min}(\mathbf{C}_j - \mathcal{A}_j^\top \bar{\mathbf{z}}) \sum_{i \in [r_j]} \bar{\xi}_{i,j} \|\mathbf{u}_{i,j}\|_2^2 + \bar{\mathbf{z}}^\top \mathbf{b} \\ &= \sum_{j \in [p]} a_j \lambda_{\min}(\mathbf{C}_j - \mathcal{A}_j^\top \bar{\mathbf{z}}) \sum_{i \in [r_j]} \bar{\xi}_{i,j} + \bar{\mathbf{z}}^\top \mathbf{b} \\ &= \sum_{j \in [p]} a_j \lambda_{\min}(\mathbf{C}_j - \mathcal{A}_j^\top \bar{\mathbf{z}}) + \bar{\mathbf{z}}^\top \mathbf{b} = \psi(\bar{\mathbf{z}}) = \tau. \end{aligned}$$

Thus, $\sum_{j \in [p]} \langle \mathbf{C}_j, \mathbf{X}_j^* \rangle = \tau$, yielding the third statement. \square

Next, we describe Algorithm 6 to solve SDP (1.39), which is based on nonsmooth first-order optimization methods (e.g., LMBM [12, Algorithm 1]).

The fact that Algorithm 6 is well-defined under certain conditions is a corollary of lemmas A.17, A.19 and [28, Lemma A.2].

Corollary A.20. *Let Assumption A.8 hold. Assume that the method (T) is globally convergent for NSOP (1.44) (e.g., (T) is LMBM). Then output τ of Algorithm 6 is well-defined. Moreover, if Assumption A.16 holds, the vector $\bar{\mathbf{y}}$ involved at Step 1 of Algorithm 6 exists and thus the output $(\mathbf{X}_j^*)_{j \in [p]}$ of Algorithm 6 is well-defined.*

Algorithm 6 Spectral-SDP-CTP-Blocks

Input: SDP (1.39) with unknown optimal value and optimal solution;
method (T) for solving NSOP.

Output: the optimal value ρ and the optimal solution $(\mathbf{X}_j^*)_{j \in [p]}$ of SDP (1.39).

- 1: Compute the optimal value τ and an optimal solution $\bar{\mathbf{y}}$ of the NSOP (1.44) by using method (T);
 - 2: For every $j \in [p]$, compute $\lambda_{\min}(\mathbf{C}_j - \mathcal{A}_j^\top \bar{\mathbf{y}})$ and its corresponding uniform eigenvectors $\mathbf{u}_{i,j}$, $i \in [r_j]$;
 - 3: Compute an optimal solution $((\bar{\xi}_{i,j})_{i \in [r_j]})_{j \in [p]}$ of QP (1.47) and set $\mathbf{X}_j^* = a_j \sum_{i \in [r_j]} \bar{\xi}_{i,j} \mathbf{u}_{i,j} \mathbf{u}_{i,j}^\top$, $j \in [p]$.
-

A.4 Converting the moment relaxation to the standard SDP

A.4.1 The dense case

Let $k \in \mathbb{N}^{\geq k_{\min}}$ be fixed. We will present a way to transform SDP (3.8) to the form (3.10). By adding slack variables $\mathbf{y}^{(i)} \in \mathbb{R}^{s(2(k-[g_i]))}$, $i \in [m]$, SDP (3.8) is equivalent to

$$\tau_k := \inf_{\mathbf{y}, \mathbf{y}^{(i)}} \left\{ L_{\mathbf{y}}(f) \left| \begin{array}{l} \mathbf{W}_k(\mathbf{y}, \mathbf{y}^{(1)}, \dots, \mathbf{y}^{(m)}) \in \mathcal{S}_k^+, \\ \mathbf{M}_{k-[g_i]}(\mathbf{y}^{(i)}) = \mathbf{M}_{k-[g_i]}(g_i \mathbf{y}), i \in [m], \\ \mathbf{M}_{k-[h_j]}(h_j \mathbf{y}) = \mathbf{0}, j \in [l] \end{array} \right. \right\}, \quad (1.48)$$

where $\mathbf{W}_k(\mathbf{y}, \mathbf{y}^{(1)}, \dots, \mathbf{y}^{(m)}) := \text{diag}(\mathbf{M}_k(\mathbf{y}), \mathbf{M}_{k-[g_1]}(\mathbf{y}^{(1)}), \dots, \mathbf{M}_{k-[g_m]}(\mathbf{y}^{(m)}))$.

Let $\mathcal{V} = \{\mathbf{M}_k(\mathbf{z}) : \mathbf{z} \in \mathbb{R}^{s(2k)}\}$ and $\mathcal{V}_i = \{\mathbf{M}_{k-[g_i]}(\mathbf{z}) : \mathbf{z} \in \mathbb{R}^{s(2(k-[g_i]))}\}$, $i \in [m]$. Then \mathcal{V} and \mathcal{V}_i , $i \in [m]$, are the linear subspaces of the spaces of real symmetric matrices of size $s(k)$ and $s(k - [g_i])$, $i \in [m]$, respectively.

Denote by \mathcal{V}^\perp , \mathcal{V}_i^\perp , $i \in [m]$, the orthogonal complements of \mathcal{V} , \mathcal{V}_i , $i \in [m]$, respectively. In [28, Appendix A.2], we show how to take a basis $\{\hat{\mathbf{A}}_j\}_{j \in [r]}$ of \mathcal{V}^\perp . Similarly we can take a basis $\{\hat{\mathbf{A}}_j^{(i)}\}_{j \in [r_i]}$ of \mathcal{V}_i^\perp , $i \in [m]$. Here $r = \dim(\mathcal{V}^\perp)$ and $r_i = \dim(\mathcal{V}_i^\perp)$, $i \in [m]$.

Notice that if \mathbf{X}_0 is a real symmetric matrix of size $s(k)$, then $\mathbf{X}_0 = \mathbf{M}_k(\mathbf{y})$ for some $\mathbf{y} \in \mathbb{R}^{s(2k)}$ if and only if $\langle \hat{\mathbf{A}}_j, \mathbf{X}_0 \rangle = 0$, $j \in [r]$. It implies that if $\mathbf{X} = \text{diag}(\mathbf{X}_0, \dots, \mathbf{X}_m) \in \mathcal{S}_k$, then there exist \mathbf{y} and $\mathbf{y}^{(i)}$, $i \in [m]$, such that $\mathbf{X} = \mathbf{W}_k(\mathbf{y}, \mathbf{y}^{(1)}, \dots, \mathbf{y}^{(m)}) \Leftrightarrow \langle \bar{\mathbf{A}}, \mathbf{X} \rangle = 0$, $\bar{\mathbf{A}} \in \mathcal{B}_1$, where \mathcal{B}_1 involves matrices $\bar{\mathbf{A}}$ defined as:

- $\bar{\mathbf{A}} = \text{diag}(\hat{\mathbf{A}}_j, \mathbf{0}, \dots, \mathbf{0})$ for some $j \in [r]$;
- $\bar{\mathbf{A}} = \text{diag}(\mathbf{0}, \hat{\mathbf{A}}_j^{(1)}, \dots, \mathbf{0})$ for some $j \in [r_1]$;
- ...
- $\bar{\mathbf{A}} = \text{diag}(\mathbf{0}, \mathbf{0}, \dots, \hat{\mathbf{A}}_j^{(m)})$ for some $j \in [r_m]$.

Notice that

$$|\mathcal{B}_1| = r + \sum_{i \in [m]} r_i = \frac{s(k)(s(k)+1)}{2} - s(2k) + \sum_{i \in [m]} \left(\frac{s(k-[g_i])(s(k-[g_i])+1)}{2} - s(2(k-[g_i])) \right). \quad (1.49)$$

The constraints $\mathbf{M}_{k-[g_i]}(\mathbf{y}^{(i)}) = \mathbf{M}_{k-[g_i]}(g_i \mathbf{y})$, $i \in [m]$, of SDP (1.48) are equivalent to $\mathbf{y}_\alpha^{(i)} = \sum_{\gamma \in \mathbb{N}_{2[g_i]}^n} g_i \mathbf{y}_{\alpha+\gamma}$, $\alpha \in \mathbb{N}_{2(k-[g_i])}^n$, $i \in [m]$. They can be written as

$\langle \bar{\mathbf{A}}, \mathbf{W}_k(\mathbf{y}, \mathbf{y}^{(1)}, \dots, \mathbf{y}^{(m)}) \rangle = 0$, for $\bar{\mathbf{A}} \in \mathcal{B}_2$, where \mathcal{B}_2 involves matrices $\bar{\mathbf{A}}$ defined by $\bar{\mathbf{A}} = \text{diag}(\tilde{\mathbf{A}}, \mathbf{0}, \dots, \mathbf{0}, \tilde{\mathbf{A}}^{(i)}, \mathbf{0}, \dots, \mathbf{0})$, with $\tilde{\mathbf{A}} = (\tilde{A}_{\mu, \nu})_{\mu, \nu \in \mathbb{N}_k^n}$ being defined as follows:

$$\tilde{A}_{\mu, \nu} = \begin{cases} g_{i, \gamma} & \text{if } \mu = \nu, \mu + \nu = \alpha + \gamma, \\ \frac{1}{2}g_{i, \gamma} & \text{if } \mu \neq \nu, (\mu, \nu) \in \{(\mu_1, \nu_1), (\nu_1, \mu_1)\} \\ & \text{with } (\mu_1, \nu_1) = \text{minimal}(\{(\bar{\mu}, \bar{\nu}) \in (\mathbb{N}_k^n)^2 : \bar{\mu} + \bar{\nu} = \alpha + \gamma\}), \\ 0 & \text{otherwise,} \end{cases} \quad (1.50)$$

and $\tilde{\mathbf{A}}^{(i)} = (\tilde{A}_{\mu, \nu}^{(i)})_{\mu, \nu \in \mathbb{N}_{k - \lceil g_i \rceil}^n}$ being defined as follows:

$$\tilde{A}_{\mu, \nu}^{(i)} = \begin{cases} -1 & \text{if } \mu = \nu, \mu + \nu = \alpha, \\ -\frac{1}{2} & \text{if } \mu \neq \nu, (\mu, \nu) \in \{(\mu_1, \nu_1), (\nu_1, \mu_1)\} \\ & \text{with } (\mu_1, \nu_1) = \text{minimal}(\{(\bar{\mu}, \bar{\nu}) \in (\mathbb{N}_k^n)^2 : \bar{\mu} + \bar{\nu} = \alpha\}), \\ 0 & \text{otherwise,} \end{cases} \quad (1.51)$$

for some $\alpha \in \mathbb{N}_{2(k - \lceil g_i \rceil)}^n$ and $i \in [m]$. Notice that $|\mathcal{B}_2| = \sum_{i \in [m]} 2(k - \lceil g_i \rceil)$. Here $\text{minimal}(T)$ is the minimal element of T , for every $T \subseteq \mathbb{N}^{2n}$ with respect to the graded lexicographic order.

The constraints $\mathbf{M}_{k - \lceil h_j \rceil}(h_j \mathbf{y}) = 0$, $j \in [l]$, can be simplified as $\sum_{\gamma \in \mathbb{N}_{2 \lceil h_j \rceil}^n} h_{j, \gamma} y_{\alpha + \gamma} = 0$, $\alpha \in \mathbb{N}_{2(k - \lceil h_j \rceil)}^n$, $j \in [l]$. They are equivalent to the following trace equality constraints: $\langle \bar{\mathbf{A}}, \mathbf{W}_k(\mathbf{y}, \mathbf{y}^{(1)}, \dots, \mathbf{y}^{(m)}) \rangle = 0$, $\bar{\mathbf{A}} \in \mathcal{B}_3$, where \mathcal{B}_3 involves matrices $\bar{\mathbf{A}} = \text{diag}(\tilde{\mathbf{A}}, \mathbf{0}, \dots, \mathbf{0})$, with $\tilde{\mathbf{A}} = (\tilde{A}_{\mu, \nu})_{\mu, \nu \in \mathbb{N}_k^n}$ being defined as follows:

$$\tilde{A}_{\mu, \nu} = \begin{cases} h_{j, \gamma} & \text{if } \mu = \nu, \mu + \nu = \alpha + \gamma, \\ \frac{1}{2}h_{j, \gamma} & \text{if } \mu \neq \nu, (\mu, \nu) \in \{(\mu_1, \nu_1), (\nu_1, \mu_1)\} \\ & \text{with } (\mu_1, \nu_1) = \text{minimal}(\{(\bar{\mu}, \bar{\nu}) \in (\mathbb{N}_k^n)^2 : \bar{\mu} + \bar{\nu} = \alpha + \gamma\}), \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $|\mathcal{B}_3| = \sum_{j \in [l]} 2(k - \lceil h_j \rceil)$.

Let $\cup_{j \in [3]} \mathcal{B}_j = (\bar{\mathbf{A}}_i)_{i \in [\zeta_k - 1]}$, where

$$\zeta_k = 1 + \sum_{j \in [3]} |\mathcal{B}_j| = 1 + \frac{s(k)(s(k) + 1)}{2} - s(2k) + \sum_{i \in [m]} \frac{s(k - \lceil g_i \rceil)(s(k - \lceil g_i \rceil) + 1)}{2} + \sum_{j \in [l]} s(2(k - \lceil h_j \rceil)).$$

The final constraint $\mathbf{y}_0 = 1$ can be rewritten as $\langle \bar{\mathbf{A}}_{\zeta_k}, \mathbf{W}_k(\mathbf{y}, \mathbf{y}^{(1)}, \dots, \mathbf{y}^{(m)}) \rangle = 1$ with $\bar{\mathbf{A}}_{\zeta_k} \in \mathcal{S}_k$ having zero entries except the top left one $[\bar{\mathbf{A}}_{\zeta_k}]_{\mathbf{0}, \mathbf{0}} = 1$. Thus we select real vector \mathbf{b}_k of length t_k such that all entries of \mathbf{b}_k are zeros except the final one $b_{\zeta_k} = 1$.

The function $L_{\mathbf{y}}(f) = \sum_{\gamma} f_{\gamma} y_{\gamma}$ is equal to $\langle \bar{\mathbf{C}}, \mathbf{W}_k(\mathbf{y}, \mathbf{y}^{(1)}, \dots, \mathbf{y}^{(m)}) \rangle$ with $\bar{\mathbf{C}} := \text{diag}(\tilde{\mathbf{C}}, \mathbf{0}, \dots, \mathbf{0})$, where $\tilde{\mathbf{C}} = (\tilde{C}_{\mu, \nu})_{\mu, \nu \in \mathbb{N}_k^n}$ is defined by:

$$\tilde{C}_{\mu, \nu} = \begin{cases} f_{\gamma} & \text{if } \mu = \nu, \mu + \nu = \gamma, \\ \frac{1}{2}f_{\gamma} & \text{if } \mu \neq \nu, (\mu, \nu) \in \{(\mu_1, \nu_1), (\nu_1, \mu_1)\} \\ & \text{with } (\mu_1, \nu_1) = \text{minimal}(\{(\bar{\mu}, \bar{\nu}) \in (\mathbb{N}_k^n)^2 : \bar{\mu} + \bar{\nu} = \gamma\}), \\ 0 & \text{otherwise.} \end{cases}$$

By noting $\bar{\mathbf{X}} = \mathbf{W}_k(\mathbf{y}, \mathbf{y}^{(1)}, \dots, \mathbf{y}^{(m)})$, SDP (1.48) has the standard form

$$\tau_k = \inf_{\bar{\mathbf{X}} \in \mathcal{S}_k^+} \{ \langle \bar{\mathbf{C}}, \bar{\mathbf{X}} \rangle : \bar{\mathbf{A}} \bar{\mathbf{X}} = \mathbf{b}_k \}, \quad (1.52)$$

where $\bar{\mathcal{A}} : \mathcal{S}_k \rightarrow \mathbb{R}^{\zeta_k}$ is a linear operator of the form $\bar{\mathcal{A}}\mathbf{X} = [\langle \bar{\mathbf{A}}_1, \mathbf{X} \rangle, \dots, \langle \bar{\mathbf{A}}_{\zeta_k}, \mathbf{X} \rangle]$. Since $\langle \mathbf{U}, \mathbf{V} \rangle = \langle \mathbf{P}_k^{-1} \mathbf{U} \mathbf{P}_k^{-1}, \mathbf{P}_k \mathbf{V} \mathbf{P}_k \rangle$, for all $\mathbf{U}, \mathbf{V} \in \mathcal{S}_k$, by noting $\mathbf{X} = \mathbf{P}_k \tilde{\mathbf{X}} \mathbf{P}_k$, SDP (1.52) can be written as (3.10) with $\mathbf{A}_{k,i} = \mathbf{P}_k^{-1} \bar{\mathbf{A}}_i \mathbf{P}_k^{-1}$, $i \in [\zeta_k]$, and $\mathbf{C}_k = \mathbf{P}_k^{-1} \bar{\mathbf{C}} \mathbf{P}_k^{-1}$.

A.4.2 The sparse case

Let $k \in \mathbb{N}^{\geq k_{\min}}$ be fixed. We will present a way to transform SDP (4.22) to the form (4.26). Doing a similar process as in Appendix A.4.1 on every clique, by noting (4.25), for every $j \in [p]$, the constraints

$$\begin{cases} \mathbf{D}_k(\mathbf{y}, I_j) \succeq 0, \mathbf{y}_0 = 1, \\ \mathbf{M}_{k-\lceil h_i \rceil}(h_i \mathbf{y}, I_j) = 0, i \in W_j, \end{cases} \quad (1.53)$$

become $\hat{\mathcal{A}}_j \mathbf{X}_j = \hat{\mathbf{b}}_j$ for some linear operator $\hat{\mathcal{A}}_j : \mathcal{S}_{j,k} \rightarrow \mathbb{R}^{\zeta_j}$ and vector $\hat{\mathbf{b}}_j \in \mathbb{R}^{\zeta_j}$. Moreover, $L_{\mathbf{y}}(f_j) = \langle \mathbf{C}_j, \mathbf{X}_j \rangle$ for some matrix $\mathbf{C}_j \in \mathcal{S}_{j,k}$ since $f_j \in \mathbb{R}[x(I_j)]$, for every $j \in [p]$. Then from (4.25), the objective function of SDP (4.22) is $L_{\mathbf{y}}(f) = \sum_{j \in [p]} \langle \mathbf{C}_j, \mathbf{X}_j \rangle$.

Next we describe the constraints depending on common moments on cliques. For every $\alpha \in \cup_{j \in [p]} \mathbb{N}_k^{I_j}$, note $T(\alpha) := \{j \in [p] : \alpha \in \mathbb{N}_k^{I_j}\}$. In other words, $T(\alpha)$ indices the cliques sharing the same moment y_α . For $\alpha \in \cup_{j \in [p]} \mathbb{N}_k^{I_j}$ such that $|T(\alpha)| \geq 2$, for every $j \in T(\alpha)$, let $\hat{\mathbf{A}}_j^{(\alpha)} \in \mathcal{S}_{j,k}$ be such that $\langle \hat{\mathbf{A}}_j^{(\alpha)}, \mathbf{X}_j \rangle = y_\alpha$. It implies the constraints $\langle \hat{\mathbf{A}}_{j_0}^{(\alpha)}, \mathbf{X}_{j_0} \rangle - \langle \hat{\mathbf{A}}_i^{(\alpha)}, \mathbf{X}_i \rangle = 0$, $i \in T(\alpha) \setminus \{j_0\}$, for every $\alpha \in \cup_{j \in [p]} \mathbb{N}_k^{I_j}$ such that $|T(\alpha)| \geq 2$, for some $j_0 \in T(\alpha)$. We denote by $\tilde{\mathcal{A}}\mathbf{X} = \mathbf{0}_{\mathbb{R}^{\zeta}}$ all these constraints with $\mathbf{X} = \text{diag}(\mathbf{X}_j)$.

Set $\zeta := \sum_{j \in [p]} \zeta_j + \tilde{\zeta}$ and $\mathbf{b} = [(\hat{\mathbf{b}}_j)_{j \in [p]}, \mathbf{0}_{\mathbb{R}^{\zeta}}] \in \mathbb{R}^{\zeta}$. Define the linear operator $\mathcal{A} : \prod_{j \in [p]} \mathcal{S}_{j,k} \rightarrow \mathbb{R}^{\zeta}$ such that $\mathcal{A}\mathbf{X} = [(\hat{\mathcal{A}}_j \mathbf{X}_j)_{j \in [p]}, \tilde{\mathcal{A}}\mathbf{X}]$, for all $\mathbf{X} = \text{diag}(\mathbf{X}_j)_{j \in [p]} \in \prod_{j \in [p]} \mathcal{S}_j$. From (4.25), the affine constraints of SDP (4.22) are now equivalent to $\mathcal{A}\mathbf{X} = \mathbf{b}$.

Let $\mathbf{A}^{(i)} := \text{diag}((\mathbf{A}_{i,j})_{j \in [p]}) \in \prod_{j \in [p]} \mathcal{S}_j$, $i \in [\zeta]$, be such that

$$\mathcal{A}\mathbf{X} = [\langle \mathbf{A}^{(1)}, \mathbf{X} \rangle, \dots, \langle \mathbf{A}^{(\zeta)}, \mathbf{X} \rangle],$$

for all $\mathbf{X} = \text{diag}(\mathbf{X}_j)_{j \in [p]} \in \prod_{j \in [p]} \mathcal{S}_j$. For every $j \in [p]$, define $\mathcal{A}_j : \mathcal{S}_j \rightarrow \mathbb{R}^{\zeta}$ as a linear operator of the form $\mathcal{A}_j \mathbf{X} := [\langle \mathbf{A}_{1,j}, \mathbf{X} \rangle, \dots, \langle \mathbf{A}_{\zeta,j}, \mathbf{X} \rangle]$. Then $\mathcal{A}\mathbf{X} = \sum_{j \in [p]} \mathcal{A}_j \mathbf{X}_j$, for all $\mathbf{X} = \text{diag}(\mathbf{X}_j)_{j \in [p]} \in \prod_{j \in [p]} \mathcal{S}_j$. Hence we obtain the data $(\mathbf{C}_{j,k}, \mathcal{A}_{j,k}, \mathbf{b}_k, \zeta_k) = (\mathbf{C}_j, \mathcal{A}_j, \mathbf{b}, \zeta)$ of the standard form (4.26) by plugging k .

A.5 Proof of Theorem 3.3

Proof. 1. Let $k \in \mathbb{N}^{\geq k_{\min}}$ and assume that $\mathbb{R}^{>0} \subseteq Q_k^\circ(g) + I_k(h)$. Then there exists $a_k > 0$ such that

$$a_k = \mathbf{v}_k^\top \mathbf{G}_0 \mathbf{v}_k + \sum_{i \in [m]} g_i \mathbf{v}_{k-\lceil g_i \rceil}^\top \mathbf{G}_i \mathbf{v}_{k-\lceil g_i \rceil} + \sum_{j \in [l]} h_j \mathbf{v}_{2(k-\lceil h_j \rceil)}^\top \mathbf{u}_j, \quad (1.54)$$

for some $\mathbf{G}_i \succ 0$, $i \in \{0\} \cup [m]$ and real vector \mathbf{u}_j , $j \in [l]$. We denote by $\mathbf{G}_i^{1/2}$ the square root of \mathbf{G}_i , $i \in \{0\} \cup [m]$. Then $\mathbf{G}_i^{1/2}$ is well-defined and $\mathbf{G}_i^{1/2} \succ 0$. Set $\mathbf{P}_k = \text{diag}(\mathbf{G}_0^{1/2}, \dots, \mathbf{G}_m^{1/2})$. Let $\mathbf{y} \in \mathbb{R}^{s(2k)}$ such that $\mathbf{M}_k(h_j \mathbf{y}) = 0$, $j \in [l]$, and $\mathbf{y}_0 = 1$. Then

$$L_{\mathbf{y}} \left(\sum_{j \in [l]} h_j \mathbf{v}_{2(k-\lceil h_j \rceil)}^\top \mathbf{u}_j \right) = \sum_{j \in [l]} \sum_{\alpha \in \mathbb{N}_{2(k-\lceil h_j \rceil)}^n} u_{j,\alpha} L_{\mathbf{y}}(h_j \mathbf{x}^\alpha) = 0. \quad (1.55)$$

From this and (1.54),

$$\begin{aligned}
a_k &= L_{\mathbf{y}}(\mathbf{v}_k^\top \mathbf{G}_0 \mathbf{v}_k + \sum_{i \in [m]} g_i \mathbf{v}_{k-\lceil g_i \rceil}^\top \mathbf{G}_i \mathbf{v}_{k-\lceil g_i \rceil}) \\
&= \text{trace}(\mathbf{M}_k(\mathbf{y}) \mathbf{G}_0) + \sum_{i \in [m]} \text{trace}(\mathbf{M}_{k-1}(g_i \mathbf{y}) \mathbf{G}_i) \\
&= \text{trace}(\mathbf{G}_0^{1/2} \mathbf{M}_k(\mathbf{y}) \mathbf{G}_0^{1/2}) + \sum_{i \in [m]} \text{trace}(\mathbf{G}_i^{1/2} \mathbf{M}_{k-1}(g_i \mathbf{y}) \mathbf{G}_i^{1/2}) \\
&= \text{trace}(\mathbf{P}_k \mathbf{D}_k(\mathbf{y}) \mathbf{P}_k),
\end{aligned}$$

yielding the first statement.

2. The “if” part comes from the first statement. Let us prove the “only if” part. Assume that POP (2.3) has CTP (Definition 3.1). Let $\mathbf{a} \in S(g)$, $\mathbf{y} = (y_\alpha)_{\alpha \in \mathbb{N}^n}$ be the moment sequence of the Dirac measure $\delta_{\mathbf{a}}$. Let $k \in \mathbb{N}^{\geq k_{\min}}$ be fixed. Since $\mathbf{P}_k \in \mathcal{S}_k$, $\mathbf{P}_k = \text{diag}(\mathbf{W}_0, \dots, \mathbf{W}_m)$. Then $\mathbf{W}_i^2 \succ 0$, $i \in \{0\} \cup [m]$ since $\mathbf{P}_k \succ 0$. Let us define the polynomial $w := \mathbf{v}_k^\top \mathbf{W}_0^2 \mathbf{v}_k + \sum_{i \in [m]} g_i \mathbf{v}_{k-\lceil g_i \rceil}^\top \mathbf{W}_i^2 \mathbf{v}_{k-\lceil g_i \rceil}$. By assumption,

$$\begin{aligned}
a_k &= \text{trace}(\mathbf{P}_k \mathbf{D}_k(\mathbf{y}) \mathbf{P}_k) \\
&= \text{trace}(\mathbf{W}_0 \mathbf{M}_k(\mathbf{y}) \mathbf{W}_0) + \sum_{i \in [m]} \text{trace}(\mathbf{W}_i \mathbf{M}_{k-1}(g_i \mathbf{y}) \mathbf{W}_i) \\
&= \text{trace}(\mathbf{M}_k(\mathbf{y}) \mathbf{W}_0^2) + \sum_{i \in [m]} \text{trace}(\mathbf{M}_{k-1}(g_i \mathbf{y}) \mathbf{W}_i^2) \\
&= L_{\mathbf{y}}(\mathbf{v}_k^\top \mathbf{W}_0^2 \mathbf{v}_k + \sum_{i \in [m]} g_i \mathbf{v}_{k-\lceil g_i \rceil}^\top \mathbf{W}_i^2 \mathbf{v}_{k-\lceil g_i \rceil}) = \int_{\mathbb{R}^n} w \delta_{\mathbf{a}} = w(\mathbf{a}),
\end{aligned}$$

It implies that $w - a_k$ vanishes on $S(g)$. Since $S(g)$ has nonempty interior, $w = a_k$, yielding the second statement. \square

A.6 Proof of Proposition 3.12

Proof. Let Assumption 3.10 hold. It is sufficient to show that (3.16) has a feasible solution for every $k \in \mathbb{N}^{\geq k_{\min}}$.

Let $\mathbf{u} = (u_j)_{j \in [n]} \subseteq \mathbb{N}^{\leq m}$ be defined by

$$u_j := |\{i \in [r] : j \in T_i\}| + |\{i \in [m] \setminus [2r] : j \in T_i\}|, \quad \forall j \in [n]. \quad (1.56)$$

Since $(\cup_{i \in [r]} T_i) \cup (\cup_{i \in [m] \setminus [2r]} T_i) = [n]$, one has $u_j \in \mathbb{N}^{\geq 1}$, $j \in [n]$. Moreover,

$$\|\mathbf{u} \circ \mathbf{x}\|_2^2 = \sum_{i \in [r]} \|\mathbf{x}(T_i)\|_2^2 + \sum_{i \in [m] \setminus [2r]} \|\mathbf{x}(T_i)\|_2^2. \quad (1.57)$$

With $R := \sum_{i \in [r]} (\underline{R}_i + \overline{R}_i) + \sum_{i \in [m] \setminus [2r]} \overline{R}_i$, by replacing \mathbf{x} by $\mathbf{u} \circ \mathbf{x}$ in Lemma 3.4, one obtains that for all $k \in \mathbb{N}^{\geq k_{\min}}$,

$$(R+1)^k = (1 + \|\mathbf{u} \circ \mathbf{x}\|_2^2)^k + \Lambda_{k-1} \sum_{i \in [m]} \delta_i g_i, \quad (1.58)$$

where $\Lambda_{k-1} := \sum_{j=0}^{k-1} (R+1)^j (1 + \|\mathbf{u} \circ \mathbf{x}\|_2^2)^{k-j-1}$ and

$$\delta_i := \frac{\underline{R}_i}{\overline{R}_i - \underline{R}_i}, \quad \delta_{i+r} := \frac{\overline{R}_i}{\overline{R}_i - \underline{R}_i}, \quad i \in [r], \quad \text{and } \delta_q = 1, \quad q \in [m] \setminus [2r]. \quad (1.59)$$

It is due to the fact that

$$R - \|\mathbf{u} \circ \mathbf{x}\|_2^2 = \sum_{i \in [r]} (\underline{R}_i + \overline{R}_i - \|\mathbf{x}(T_i)\|_2^2) + \sum_{i \in [m] \setminus [2r]} (\overline{R}_i - \|\mathbf{x}(T_i)\|_2^2), \quad (1.60)$$

and $\underline{R}_i + \overline{R}_i - \|\mathbf{x}(T_i)\|_2^2 = \delta_i g_i + \delta_{i+r} g_{i+r}$, for all $i \in [r]$. For each $k \in \mathbb{N}^{\geq k_{\min}}$, let $(\theta_{k,\alpha})_{\alpha \in \mathbb{N}_k^n} \subseteq \mathbb{R}^{>0}$ and $(\eta_{k-1,\alpha})_{\alpha \in \mathbb{N}_{k-1}^n} \subseteq \mathbb{R}^{>0}$ be such that

$$(1 + \|\mathbf{u} \circ \mathbf{x}\|_2^2)^k = \sum_{\alpha \in \mathbb{N}_k^n} \theta_{k,\alpha} \mathbf{x}^{2\alpha} \quad \text{and} \quad \Lambda_{k-1} = \sum_{\alpha \in \mathbb{N}_{k-1}^n} \eta_{k-1,\alpha} \mathbf{x}^{2\alpha},$$

and define the diagonal matrices

$$\mathbf{G}_k^{(0)} := \text{diag}((\theta_{k,\alpha})_{\alpha \in \mathbb{N}_k^n}) \quad \text{and} \quad \mathbf{G}_{k-1}^{(i)} := \text{diag}((\delta_i \eta_{k-1,\alpha})_{\alpha \in \mathbb{N}_{k-1}^n}), \quad i \in [m]. \quad (1.61)$$

Then (1.58) yields that for every $k \in \mathbb{N}^{\geq k_{\min}}$:

$$(R+1)^k = \mathbf{v}_k^\top \mathbf{G}_k^{(0)} \mathbf{v}_k + \sum_{i \in [m]} g_i \mathbf{v}_{k-1}^\top \mathbf{G}_{k-1}^{(i)} \mathbf{v}_{k-1}.$$

Hence $((R+1)^k, \mathbf{G}_k^{(i)}, \mathbf{0})$ is a feasible solution of (3.16), for every $k \in \mathbb{N}^{\geq k_{\min}}$. \square

A.7 Proof of Proposition 3.14

Proof. Let Assumption 3.13 hold with $u := \lceil g_i \rceil$, $i \in [n+1]$. For every $k \in \mathbb{N}^{\geq k_{\min}}$, letting $\Lambda_{k-1} := \sum_{j=0}^{k-1} (R+1)^j (1 + \|\mathbf{x}\|_2^2)^{k-j-1}$ and $\Theta_t := (1 + \|\mathbf{x}\|_2^2)^t$, for $t \in \mathbb{N}$, Lemma 3.4 yields: $(R+1)^k = \Theta_k + g_m \Lambda_{k-1}$. It implies that for every $k \in \mathbb{N}^{\geq k_{\min}}$,

$$(R+1)^k = (\Theta_k - \frac{L}{L+1} \Theta_{k-u}) + \frac{1}{L+1} \Theta_{k-u} \sum_{i \in [m-1]} g_i + g_m \Lambda_{k-1}. \quad (1.62)$$

It is due to the fact that $\sum_{i \in [m-1]} g_i = L$. For each $k \in \mathbb{N}^{\geq k_{\min}}$, let us consider the following sequences:

- $(\nu_{k,\alpha})_{\alpha \in \mathbb{N}_k^n} \subseteq \mathbb{R}^{>0}$ such that $\Theta_k - \frac{L}{L+1} \Theta_{k-u} = \sum_{\alpha \in \mathbb{N}_k^n} \nu_{k,\alpha} \mathbf{x}^{2\alpha}$;
- $(\theta_{k-u,\alpha})_{\alpha \in \mathbb{N}_{k-u}^n} \subseteq \mathbb{R}^{>0}$ such that $\frac{1}{L+1} \Theta_{k-u} = \sum_{\alpha \in \mathbb{N}_{k-u}^n} \theta_{k-u,\alpha} \mathbf{x}^{2\alpha}$;
- $(\eta_{k-1,\alpha})_{\alpha \in \mathbb{N}_{k-1}^n} \subseteq \mathbb{R}^{>0}$ such that $\Lambda_{k-1} = \sum_{\alpha \in \mathbb{N}_{k-1}^n} \eta_{k-1,\alpha} \mathbf{x}^{2\alpha}$.

For each $k \in \mathbb{N}^{\geq k_{\min}}$, define the diagonal matrices: $\mathbf{G}_k^{(0)} := \text{diag}((\nu_{k,\alpha})_{\alpha \in \mathbb{N}_k^n})$,

$$\mathbf{G}_{k-u}^{(1)} := \text{diag}((\theta_{k-u,\alpha})_{\alpha \in \mathbb{N}_{k-u}^n}), \quad \text{and} \quad \mathbf{G}_{k-1}^{(2)} := \text{diag}((\eta_{k-1,\alpha})_{\alpha \in \mathbb{N}_{k-1}^n}).$$

Then (1.62) yields that for every $k \in \mathbb{N}^{\geq k_{\min}}$,

$$(R+1)^k = \mathbf{v}_k^\top \mathbf{G}_k^{(0)} \mathbf{v}_k + \mathbf{v}_{k-u}^\top \mathbf{G}_{k-u}^{(1)} \mathbf{v}_{k-u} + \sum_{i \in [m-1]} g_i + \mathbf{v}_{k-1}^\top \mathbf{G}_{k-1}^{(2)} \mathbf{v}_{k-1} g_m. \quad (1.63)$$

Hence $((R+1)^k, \mathbf{G}_k^{(i)}, \mathbf{0})$ is a feasible solution of (3.16), for every $k \in \mathbb{N}^{\geq k_{\min}}$. By using Lemma 3.8, the conclusion follows. \square

A.8 Proof of Corollary 3.17

Proof. Let $\tilde{g} := \{\tilde{g}_i\}_{i \in [m+2]}$. Then $\{\tilde{g}_i\}_{i \in [m]}$ have the equivalent degree, i.e., there exists $u \in \mathbb{N}$ such that $\lceil \tilde{g}_i \rceil = u$, for all $i \in [m]$. Thus Assumption 3.13 holds for $g \leftarrow \tilde{g}$, $m \leftarrow m+2$. By Proposition 3.14, (3.16) has a feasible solution with $g \leftarrow \tilde{g}$ for every order $k \in \mathbb{N}^{\geq k_{\min}}$. It implies that for every $k \in \mathbb{N}^{\geq k_{\min}}$, there exist $\mathbf{u}_k^{(j)} \in \mathbb{R}^{s(2(k-\lceil h_j \rceil))}$, $j \in [l]$, and

$$(\eta_{k,\alpha})_{\alpha \in \mathbb{N}_k^n} \subseteq \mathbb{R}^{>0}, \quad (\eta_{k-u,\alpha})_{\alpha \in \mathbb{N}_{k-u}^n} \subseteq \mathbb{R}^{>0}, \quad i \in [m+1], \quad (\eta_{k-1,\alpha}^{(m+2)})_{\alpha \in \mathbb{N}_{k-1}^n} \subseteq \mathbb{R}^{>0}$$

such that

$$1 = \mathbf{v}_k^\top \text{diag}((\eta_{k,\alpha}^{(0)})_{\alpha \in \mathbb{N}_k^n}) \mathbf{v}_k + \sum_{i \in [m+1]} \tilde{g}_i \mathbf{v}_{k-u}^\top \text{diag}((\eta_{k-u,\alpha}^{(i)})_{\alpha \in \mathbb{N}_{k-u}^n}) \mathbf{v}_{k-u} + \tilde{g}_{m+2} \mathbf{v}_{k-1}^\top \text{diag}((\eta_{k-1,\alpha}^{(m+2)})_{\alpha \in \mathbb{N}_{k-1}^n}) \mathbf{v}_{k-1} + \sum_{j \in [l]} h_j \mathbf{v}_{2(k-\lceil h_j \rceil)}^\top \mathbf{u}_k^{(j)}.$$

Let $k \in \mathbb{N}^{\geq k_{\min}}$ be fixed. We define the following polynomials:

- $\sigma_0 := \mathbf{v}_k^\top \text{diag}((\eta_{k,\alpha}^{(0)})_{\alpha \in \mathbb{N}_k^n}) \mathbf{v}_k = \sum_{\alpha \in \mathbb{N}_k^n} \eta_{k,\alpha}^{(0)} \mathbf{x}^{2\alpha}$,
- $\sigma_i := \mathbf{v}_{k-u}^\top \text{diag}((\eta_{k-u,\alpha}^{(i)})_{\alpha \in \mathbb{N}_{k-u}^n}) \mathbf{v}_{k-u} = \sum_{\alpha \in \mathbb{N}_{k-u}^n} \eta_{k-u,\alpha}^{(i)} \mathbf{x}^{2\alpha}$, $i \in [m+1]$,
- $\sigma_{m+2} := \mathbf{v}_{k-1}^\top \text{diag}((\eta_{k-1,\alpha}^{(m+2)})_{\alpha \in \mathbb{N}_{k-1}^n}) \mathbf{v}_{k-1} = \sum_{\alpha \in \mathbb{N}_{k-1}^n} \eta_{k-1,\alpha}^{(m+2)} \mathbf{x}^{2\alpha}$,
- $\psi_j := \mathbf{v}_{2(k-\lceil h_j \rceil)}^\top \mathbf{u}_k^{(j)}$, $j \in [l]$.

From these and since $\tilde{g}_i := g_i(1 + \|\mathbf{x}\|_2^2)^{u-\lceil g_i \rceil}$, for $i \in [m]$, one has

$$1 = \sigma_0 + \sum_{i \in [m]} \sigma_i \tilde{g}_i + \sum_{j \in [l]} \psi_j h_j = \sigma_0 + \sum_{i \in [m]} \sigma_i (1 + \|\mathbf{x}\|_2^2)^{u-\lceil g_i \rceil} g_i + \tilde{g}_{m+1} \sigma_{m+1} + \tilde{g}_{m+2} \sigma_{m+2} + \sum_{j \in [l]} \psi_j h_j. \quad (1.64)$$

Then there exist $(\theta_{k-\lceil g_i \rceil, \alpha}^{(i)})_{\alpha \in \mathbb{N}_{k-\lceil g_i \rceil}^n} \subseteq \mathbb{R}^{>0}$, $i \in [m]$, such that

$$\sigma_i (1 + \|\mathbf{x}\|_2^2)^{u-\lceil g_i \rceil} = \sum_{\alpha \in \mathbb{N}_{k-\lceil g_i \rceil}^n} \theta_{k-\lceil g_i \rceil, \alpha}^{(i)} \mathbf{x}^{2\alpha}, \quad i \in [m]. \quad (1.65)$$

Thus (1.64) becomes

$$\begin{aligned} 1 = & \mathbf{v}_k^\top \text{diag}((\eta_{k,\alpha}^{(0)})_{\alpha \in \mathbb{N}_k^n}) \mathbf{v}_k + \sum_{i \in [m]} g_i \mathbf{v}_{k-\lceil g_i \rceil}^\top \text{diag}((\theta_{k-\lceil g_i \rceil, \alpha}^{(i)})_{\alpha \in \mathbb{N}_{k-\lceil g_i \rceil}^n}) \mathbf{v}_{k-\lceil g_i \rceil} \\ & + \tilde{g}_{m+1} \mathbf{v}_{k-u}^\top \text{diag}((\eta_{k-u,\alpha}^{(m+1)})_{\alpha \in \mathbb{N}_{k-u}^n}) \mathbf{v}_{k-u} \\ & + \tilde{g}_{m+2} \mathbf{v}_{k-1}^\top \text{diag}((\eta_{k-1,\alpha}^{(m+2)})_{\alpha \in \mathbb{N}_{k-1}^n}) \mathbf{v}_{k-1} + \sum_{j \in [l]} h_j \mathbf{v}_{2(k-\lceil h_j \rceil)}^\top \mathbf{u}_k^{(j)} \\ & \in Q_k^\circ(g \cup \{\tilde{g}_{m+1}, \tilde{g}_{m+2}\}) + I_k(h), \end{aligned} \quad (1.66)$$

since

- $\text{diag}((\eta_{k,\alpha}^{(0)})_{\alpha \in \mathbb{N}_k^n}) \succ 0$, $\text{diag}((\theta_{k-\lceil g_i \rceil, \alpha}^{(i)})_{\alpha \in \mathbb{N}_{k-\lceil g_i \rceil}^n}) \succ 0$, $i \in [m]$,
- $\text{diag}((\eta_{k-u,\alpha}^{(m+1)})_{\alpha \in \mathbb{N}_{k-u}^n}) \succ 0$, and $\text{diag}((\eta_{k-1,\alpha}^{(m+2)})_{\alpha \in \mathbb{N}_{k-1}^n}) \succ 0$.

It yields that (3.16) has a feasible solution with $g \leftarrow g \cup \{\tilde{g}_{m+1}, \tilde{g}_{m+2}\}$, for every order $k \in \mathbb{N}^{\geq k_{\min}}$. \square

A.9 Proof of Proposition 4.5

Proof. To prove that POP (2.3) has CTP on each clique of variables, it is sufficient to show that (4.29) has a feasible solution, for every $k \in \mathbb{N}^{\geq k_{\min}}$ and for every $j \in [p]$ due to Lemma 4.4.

For every $j \in [p]$, let $\mathbf{u}^{(j)} = (u_i^{(j)})_{i \in I_j} \subseteq \mathbb{N}^{\leq |J_j|}$ be defined by

$$u_i^{(j)} = |\{q \in J_j \cap [r] : i \in T_q\}| + |\{q \in J_j \setminus [2r] : i \in T_q\}|, \quad i \in I_j. \quad (1.67)$$

For every $j \in [p]$, one has $u_i^{(j)} \in \mathbb{N}^{\geq 1}$, $i \in I_j$, according to $(\cup_{q \in J_j \cap [r]} T_q) \cup (\cup_{q \in J_j \setminus [2r]} T_q) = I_j$. Moreover,

$$\|\mathbf{u}^{(j)} \circ \mathbf{x}(I_j)\|_2^2 = \sum_{i \in J_j \cap [r]} \|\mathbf{x}(T_i)\|_2^2 + \sum_{i \in J_j \setminus [2r]} \|\mathbf{x}(T_i)\|_2^2, \quad \forall j \in [p]. \quad (1.68)$$

For every $j \in [p]$, with $R^{(j)} := \sum_{i \in J_j \cap [r]} (\underline{R}_i + \overline{R}_i) + \sum_{i \in J_j \setminus [2r]} \overline{R}_i$, by replacing \mathbf{x} (resp. R) by $\mathbf{u}^{(j)} \circ \mathbf{x}(I_j)$ (resp. $R^{(j)}$) in Lemma 3.4, we obtain

$$(R^{(j)} + 1)^k = (1 + \|\mathbf{u}^{(j)} \circ \mathbf{x}(I_j)\|_2^2)^k + \Lambda_{k-1}^{(j)} \sum_{i \in J_j} \delta_i g_i, \quad \forall j \in [p], \forall k \in \mathbb{N}^{\geq k_{\min}}, \quad (1.69)$$

where $\Lambda_{k-1}^{(j)} := \sum_{r=0}^{k-1} (R^{(j)} + 1)^r (1 + \|\mathbf{u}^{(j)} \circ \mathbf{x}(I_j)\|_2^2)^{k-r-1}$ and

$$\delta_i := \frac{\underline{R}_i}{\bar{R}_i - \underline{R}_i}, \delta_{i+r} := \frac{\bar{R}_i}{\bar{R}_i - \underline{R}_i}, i \in J_j \cap [r] \text{ and } \delta_q = 1, q \in J_j \setminus [2r]. \quad (1.70)$$

It is due to the fact that

$$R^{(j)} - \|\mathbf{u}^{(j)} \circ \mathbf{x}(I_j)\|_2 = \sum_{i \in J_j \cap [r]} (\underline{R}_i + \bar{R}_i - \|\mathbf{x}(T_i)\|_2^2) + \sum_{i \in J_j \setminus [2r]} (\bar{R}_i - \|\mathbf{x}(T_i)\|_2^2), \quad (1.71)$$

and $\underline{R}_i + \bar{R}_i - \|\mathbf{x}(T_i)\|_2^2 = \delta_i g_i + \delta_{i+r} g_{i+r}$, $i \in J_j \cap [r]$. For every $j \in [p]$, for each $k \in \mathbb{N}^{\geq k_{\min}}$, let $(\theta_{k,\alpha}^{(j)})_{\alpha \in \mathbb{N}_k^{I_j}} \subseteq \mathbb{R}^{>0}$ and $(\eta_{k-1,\alpha}^{(j)})_{\alpha \in \mathbb{N}_{k-1}^{I_j}} \subseteq \mathbb{R}^{>0}$ be such that

$$(1 + \|\mathbf{u}^{(j)} \circ \mathbf{x}(I_j)\|_2^2)^k = \sum_{\alpha \in \mathbb{N}_k^{I_j}} \theta_{k,\alpha}^{(j)} \mathbf{x}^{2\alpha} \quad \text{and} \quad \Lambda_{k-1}^{(j)} = \sum_{\alpha \in \mathbb{N}_{k-1}^{I_j}} \eta_{k-1,\alpha}^{(j)} \mathbf{x}^{2\alpha},$$

and define the diagonal matrices:

$$\mathbf{G}_k^{(j,0)} := \text{diag}((\theta_{k,\alpha}^{(j)})_{\alpha \in \mathbb{N}_k^{I_j}}) \quad \text{and} \quad \mathbf{G}_{k-1}^{(j,i)} := \text{diag}((\delta_i \eta_{k-1,\alpha}^{(j)})_{\alpha \in \mathbb{N}_{k-1}^{I_j}}), \quad i \in J_j. \quad (1.72)$$

For every $j \in [p]$, (1.69) yields that for every $k \in \mathbb{N}^{\geq k_{\min}}$,

$$(R^{(j)} + 1)^k = (\mathbf{v}_k^{I_j})^\top \mathbf{G}_k^{(j,0)} \mathbf{v}_k^{I_j} + \sum_{i \in J_j} g_i (\mathbf{v}_{k-1}^{I_j})^\top \mathbf{G}_{k-1}^{(j,i)} \mathbf{v}_{k-1}^{I_j}. \quad (1.73)$$

Hence $((R^{(j)} + 1)^k, \mathbf{G}_k^{(j,i)}, \mathbf{0})$ is a feasible solution of (4.29), for every $k \in \mathbb{N}^{\geq k_{\min}}$ and for every $j \in [p]$. \square

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