Bayesian Contextual Choices under Imperfect Perception of Attributes

Junnan He

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Sciences Po, Department of Economics, CNRS, Paris, France

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Abstract

The classical rational choice theory proposes that preferences are context-independent, e.g. independent of irrelevant alternatives. Empirical choice data, however, display several contextual choice effects that seem inconsistent with rational preferences. We study a choice model with a fixed underlying utility function and explain contextual choices with a novel information friction: the agent's perception of the options is affected by an attribute-specific noise. Under this friction, the agent learns useful information when she sees more options. Therefore, the agent chooses contextually, exhibiting intransitivity, joint-separate evaluation reversal, attraction effect, compromise effect, similarity effect, and phantom decoy effect. Nonetheless, because the noise is attribute-specific and common across alternatives, the agent's choice is perfectly rational whenever an option clearly dominates others.

JEL codes: D81, D83.

1 Introduction

Classically, rationality is defined by consistency axioms. Under consistency, rational preferences are transitive and independent of contexts, representable by utility functions. However, empirical research has long found violations of these consistency axioms. For example, intransitivity was spotted as early as Tversky (1969), and some recent evidence is surveyed in Rieskamp et al. (2006). Empirical studies on other aspects of contextual dependence include Huber et al. (1982), Pratkanis and Farquhar (1992) and Hsee (1996). Here, by contextual effects we mean the following type of observations. In different decision problems involving objects $\mathbf{x}$ and $\mathbf{y}$, their observed choice probabilities or the reported evaluations differ in a way that implies some objects are evaluated differently. For instance, in Huber et al. (1982) and Pratkanis and Farquhar (1992), experimenters offer the subjects two choice problems. One involves only two options $\mathbf{x}$ and $\mathbf{y}$ and the other includes a third choice $\mathbf{z}$. They find that the inclusion of $\mathbf{z}$ can reverse the relative choice frequency between $\mathbf{x}$ and

*Department of Economics, Sciences Po, 28 Rue des Saints-Pères, 75007 Paris

y, even though **z** itself is rarely chosen (attraction effect) or listed as unavailable (phantom decoy effect). Another example is the joint-separate valuation reversal (henceforth j-s reversal) in Hsee (1996). When the willingness to pay for **x** or **y** is elicited separately, **x** can be valued higher than **y**, but when elicited together, **x** becomes inferior to **y**. Intransitive choices can also be interpreted as a type of contextual effects.

Within the literature, some contextual effects, such as the similarity effect (Tversky and Russo, 1969), attraction effect (Huber et al. , 1982) and compromise effect (Simonson, 1989), can be rationalized as the maximization of a context-independent preference under informational constraints (see e.g. Hausman and Wise (1978), Wernerfelt (1995), Kamenica (2008), Guo (2016), Natenzon (2019)). However, others such as phantom decoy effect (Pratkanis and Farquhar, 1992), j-s reversal (Hsee, 1996) and stochastic intransitivity (Tversky, 1969) have not yet been rationalized.[1]

This paper proposes a model to systematically rationalize and predict the aforementioned empirical findings. In our model, a decision maker maximizes a context-independent preference under the constraint of a novel informational friction. We show that the decision maker can exhibit the stochastic intransitivity, the j-s reversal and the compromise effect when there are trade-offs between attributes as empirically observed. We define the *decoy choice pattern*, a comparative static that captures the attraction effect, the phantom decoy effect and the compromise effect, and show in Theorem 4.1 that for the relevant choice problems, the model predicts the decoy choice pattern.[2] In comparison to models that require different parameters to explain different phenomena, our model is parsimonious in the sense that several contextual effects can be explained within one simple parametric setting. Despite the explanation power, our model still possesses desirable regularities. For instance, when there is a dominating alternative, context effects disappear as choices maximize the context-independent preference with overwhelming probability (see e.g. Theorem 4.2). This identifies a subclass of choice problems where the classical rationality in choice is retained.

[1] The "phantom decoy effect" refers to the observation that an unavailable third option, dominating the target but not the competitor, *increases* the attractiveness of the target. See evidence in Pratkanis and Farquhar (1992) and later in Highhouse (1996), Pettibone and Wedell (2000), Pettibone and Wedell (2007) and Hedgcock et al. (2009) etc.

[2] Incidentally, Highhouse (1996) has similarly argued that the attraction effect and the phantom decoy effect are likely caused by the same mechanism.

Our novel information friction is that the perception of attributes is noisy. Each option $\mathbf{x}$ has precise attribute levels $\mathbf{x}^*$ over which the agent's utility function is defined. However the agent cannot observe these precise attributes, but a noisy signal $X|\mathbf{x}^*$. Conditional on the true attributes, the noisy signals across different alternatives are correlated. Therefore, although the distribution of $X|\mathbf{x}^*$ is fixed, the agent makes different inferences about $\mathbf{x}^*$ when she is presented with other different alternatives. For example, in the choice problem $\{\mathbf{x}, \mathbf{y}\}$, the agent observes the signals X, Y but not the actual attribute levels $\mathbf{x}^*$ and $\mathbf{y}^*$. She forms a posterior belief, say about $\mathbf{x}^*$, conditional on the signals X, Y. When she faces the choice problem $\{\mathbf{x}, \mathbf{z}\}$, the posterior belief about $\mathbf{x}^*$ is conditioned on X, Z. These two posterior beliefs about $\mathbf{x}^*$ are generally different, and so are the posterior expected utilities of $\mathbf{x}$ in $\{\mathbf{x}, \mathbf{y}\}$ and in $\{\mathbf{x}, \mathbf{z}\}$. Then intuitively, even if the agent is (stochastically) indifferent both in the choice problem $\{\mathbf{x}, \mathbf{y}\}$ and in the problem $\{\mathbf{x}, \mathbf{z}\}$, she would *not* be indifferent in $\{\mathbf{y}, \mathbf{z}\}$. In other words, the (stochastic) indifference curves can cross, giving rise to intransitivity.

From now on, we assume a type of noise termed *imperfect perception of attributes*. This noise is specific to each attribute, but not specific to each alternative. In other words, for each attribute, the noise is common across all items. It perturbs the perceived attribute levels of the items while keeping their relative differences unchanged. Under this noisy signal, if the agent over-perceives an attribute in an object, she over-perceives the same attribute in other objects. This noise in the attributes is qualitatively different from a noise in utilities. As shown in Proposition 3.3, our model does not satisfy monotonicity, and hence *cannot* be interpreted as any random utility model.

The assumption of imperfect perception is intuitively sensible because such correlated signals arise easily in perception tasks. Imagine when choosing apartments, a decision maker prefers rooms with abundant natural light. She visits two apartments on the same day, and sees that apartment $\mathbf{x}$ is brighter than $\mathbf{y}$. Although she does not know how bright the apartments typically are (she does not observe $\mathbf{x}^*, \mathbf{y}^*$), she learns the noisy signals from her visits. Each signal may be inaccurate, but the difference between signals can clearly indicates which room is typically brighter. After all, she is seeing both apartments at roughly the same time, under the same weather. There is a natural common component in the noise of the signals. The same intuition holds in perceiving other attributes such as the noisiness of the neighborhood, the length of commuting time etc.

This type of uncertainty in perception can also arise from misinterpreting attributes measured and displayed in scientific units, such as megabytes of memory space, lumens of light, and decibels of sound etc. As is similarly argued by Ariely et al. (2003) and Kamenica (2008), such units can be hard to interpret precisely. When choosing a light bulb for decoration, the brightness level is important for utility evaluation. However, it is difficult to read of the brightness in lumens and evaluate directly. When interpreting these measurements, people are uncertain about how one lumen of light relates to perceived brightness levels, and may interpret from experience etc. This interpretation process is noisy, and (under-) over-interpreting a unit can lead to (under-) over-perceived attribute levels for *all* the alternatives. Therefore, the uncertainty in the understanding of the units can cause misinterpretation, resulting in imperfect perception of the attributes.

In general, imperfect perception arises whenever the decision making agent *believes* that there is a common component in the uncertainty in attribute perception. We elaborate further in Section 2.

For a Bayesian agent, this imperfect perception causes a *contrast effect* in the perception of each attribute. The contrast effect is a well-known psychological phenomenon that refers to the strengthening or weakening of the perception of any attribute when the object is contrasted with surrounding objects of different levels in the same attribute.[3] To illustrate with the apartment example, suppose that the decision maker on the same day also visited another apartment $\mathbf{z}$ that is much brighter than both $\mathbf{x}$ and $\mathbf{y}$. The Bayesian decision maker infers that it is unlikely for any apartment to be so bright on every day, implying an upward bias in the common component of all the signals. Hence after visiting the apartment $\mathbf{z}$, she revises downwards the perceived brightness of $\mathbf{x}$ and $\mathbf{y}$. The judgments about other attributes of the apartments can also be affected similarly. For example, an apartment can be perceived as quieter in the presence of a really noisy one.

When the decision maker's preference is determined by a single attribute monotonically, this contrast effect is inconsequential: she always chooses to maximize (or minimize) that

[3] See e.g. Schwarz and Bless (1992) and Plous (1993) page 38 - 41.

attribute in the model.[4] However, if her preference involves at least two attributes, a different set of competing options can simultaneously affect the perception of two attributes differently (increase one and decrease the other). Hence the same two options can have different posterior expected utilities when contrasted with different sets of alternatives.

The compromise effect is one such example. Suppose in choosing apartments, the decision maker faces a trade-off between natural lighting and quietness. She prefers better lighting as well as a quieter living place. As before, she observes correlated signals X and Y in both attributes of the two apartments $\{\mathbf{x}, \mathbf{y}\}$. Suppose $\mathbf{x}$ has good natural lighting but is subject to noises from the street, whereas $\mathbf{y}$ has a gloomy interior but is very quiet. Suppose the decision maker is inclined to choose $\mathbf{y}$ between the two. Now introduce a third option $\mathbf{z}$ that is even brighter than $\mathbf{x}$ but is also much noisier. As explained previously, conditional on X, Y *and* Z, both the (posterior) perceived brightness levels for $\mathbf{x}$ and $\mathbf{y}$ are lower than those conditional on only X and Y (the contrast effect in perception of light). And similarly, with the additional signal Z both the (posterior) perceived quietness for $\mathbf{x}$ and $\mathbf{y}$ increase. Now, reducing the perceived brightness of $\mathbf{x}$ and $\mathbf{y}$ affects both apartments negatively, but more so for $\mathbf{y}$ because of diminishing marginal utility in lighting. And increasing the perceived quietness of $\mathbf{x}$ and $\mathbf{y}$ affects both apartments positively, but more so for $\mathbf{x}$, due to the diminishing marginal utility in quietness. Consequently, $\mathbf{x}$ has a higher expected utility level relative to $\mathbf{y}$ after $\mathbf{z}$ is introduced.

Besides the assumption of imperfect perception, Bayesian rationality is also an important component of our model. If there is no updating at all, presenting the alternative $\mathbf{z}$ will not affect the preference between $\mathbf{x}$ and $\mathbf{y}$. We use Bayesian updating because it is the rational benchmark in modeling information and learning. Despite the reliance of our model on Bayesian rationality, we do not claim that in reality people perform sophisticated Bayesian updating and calculate posterior expectations. Instead, we interpret the model as an as-if representation of the decision process. Nonetheless, the analysis of this as-if channel does parallel some intuitive explanations of contextual choices as illustrated above.

We present the general set-up in the Section 2. In Section 3, we apply a parametric special case of the model to explain intransitive choices, j-s reversal, and compromise effect in

[4] I.e, if the agent only cares about lighting, she always chooses the brightest apartment with certainty.

detail. The analysis of the general model is presented in Section 4, where we study the decoy choice pattern, the choice for dominating options and show how the assumption of imperfect perception can be weakened. Section 5 contains some further discussion. Additional proofs are in the appendix.

1.1 Related Literature

This paper contributes to the literature on rationalizing contextual choices by proposing a new and parsimonious informational channel that complements existing explanations.

Our model differs from the class of reference-dependent models where utilities are directly assumed to depend on menus. See e.g. Simonson (1989), Tversky and Simonson (1993), Koszegi and Rabin (2006), Bordalo et al. (2013), Ok et al. (2015) and Tserenjigmid (2016) etc. In a reference-dependent model, the preference is relative to the reference point which is typically a function of the choice set. Therefore, the "utility" or "welfare" of options are not comparable across choice sets. Hence they do not speak up to evaluation tasks such as j-s reversal – any option in a singleton budget set would have the same "utility". In contrast, our model contains a rational behavioral benchmark $u(.)$ that can be used to assess welfare across different choice problems, so it is relevant for such evaluation tasks. Also different from reference-dependent models, the fixed $u(.)$ allows us to identify potential mistakes (i.e. failures to maximize $u(.)$) or inefficiencies caused by various frictions.

Our model also differs from the general random utility framework of Block and Marschak (1960) and Falmagne (1978), which includes Thurstone (1927), Luce (1959), Tversky (1972), Hausman and Wise (1978) and more recently, Gul et al. (2014). As detailed in Section 3.3, because random utility models are monotonic, they cannot explain the increase in the absolute choice probability in compromise effect. Since our model can explain this phenomenon (see Proposition 3.3), it is not a random utility mode.

There are also other models that violate monotonicity in the literature, including some of the reference dependent models discussed above. Rieskamp et al. (2006) surveyed many well-known models and classified them based on five consistency principles, including monotonicity and stochastic transitivity. According to Rieskamp et al. (2006), all these models, including Mixed Logit Models, Decision Field Theory (Busemeyer and Townsend, 1993; Roe, Busemeyer and Townsend, 2001), Componential Context Theory (Tversky and Simonson, 1993) etc., satisfy at least one of the two properties (See Table 5 in Rieskamp et al. (2006)), and are therefore different from our model.

There are papers in the literature rationalizing different contextual choices through informational channels. Wernerfelt (1995) and Kamenica (2008) both study consumer-retailer games where the set of alternatives conveys information in equilibrium. In contrast, our model focuses on a single agent decision environment when market interaction is not of major concern. Guo (2016) and Natenzon (2019) both study informational channels in a single agent decision environment, and their information structure differ from ours. Guo (2016) assumes that the available information in different contexts is identical. However, because the incentive to acquire information changes with contexts, the agent eventually uses different (acquired) information in decision making. Different from Guo (2016), we do not study a model of information acquisition. Instead, we show how learning under a family of exogenous information structure can predict several contextual effects. Therefore, our mechanism focuses on scenarios where selectively acquiring information is not the main driver. Also different from our model, Natenzon (2019) studies a transitive choice model where the agent learns about the mean utilities from a noisy signal. In his paper, the covariance structure of the noise is used as free parameters to explain data. In contrast, the uncertainty in our model lies in the more primitive attribute space. As a result, we explain different contextual choices, such as intransitivity, j-s reversal and phantom decoy effects despite our correlation across objects is held fixed.

Most of the models in the literature do not explicitly model attributes, including also the drift diffusion models in neuroscience (see e.g. Ratcliff (1978), Busemeyer and Townsend (1993), Usher and McClelland (2004), Woodford (2014), and Fehr and Rangel (2011)), and extensions or variations of Luce's logit model (see e.g. Masatlioglu et al. (2012)). In contrast, because we take attributes as model primitives, we have the advantage to make strong natural predictions for clearly dominating alternatives (see Theorem 4.2).

2 The Model, its Assumptions and Motivations

Most empirical research on contextual choices focus on options with *two or more* attributes. Therefore, we take the primitives of our model to be the attributes of each object. In particular, we use $\mathbb{R}^n$ for $n \geq 2$ to represent the attribute space. The attributes of each item $\mathbf{x}$ are represented as a vector $\mathbf{x}^* := (x_1^*, \ldots, x_n^*)$ in the space, with each coordinate given by the corresponding attribute level. In many of the experiments, contextual choices can occur when there are only two different attributes. Therefore we restrict our discussion to $\mathbb{R}^2$ in this paper for mathematical simplicity.[5]

The vector of attributes $\mathbf{x}^*$ is not directly observed by the agent. The agent observes a noisy signal about the attributes and tries to maximize her payoff given the signals. In accordance with the classical theory, the agent is assumed to be rational in two senses. Firstly, she has a context-independent preference over the attribute space that can be represented by a vNM utility function. Following standard assumptions from consumer theory, we assume that all attributes are both goods, so that utility is insatiable along all axes. At the same time, the utility has diminishing returns and there is weak complementarity between attributes. We call a preference *standard* if it displays these properties.

Assumption 2.1 (Standard Preference) *The decision maker's preference over distributions on $\mathbb{R}^2$ can be represented by a vNM utility function $u : \mathbb{R}^2 \to \mathbb{R}$ that is differentiable, increasing (i.e. $u_1 > 0, u_2 > 0$), and exhibits decreasing marginal sensitivity (i.e. $u_{11} < 0, u_{22} < 0$) and weak complementarity (i.e. $u_{12} \geq 0$). Any utility function representing a standard preference is called a standard utility function.*

Secondly, the agent is Bayesian rational with a prior belief over $\mathbb{R}^2$. The prior distribution represents the agent's anticipation about the attribute levels before she observes any options. We endow the agent with a normal prior distribution. Without loss of generality, we translate and scale the attribute space such that the prior mean is at the origin and the prior variance is $\Omega := \begin{bmatrix} 1 & r \\ r & 1 \end{bmatrix}$ for some $r \in (-1, 1)$.[6] By Bayesian rationality, we mean the

[5] The mechanisms for the main theorems can be extended to higher attribute dimensions.

[6] Such a correlation can arise when, for example, the two attributes are price and quality. One can interpret $r < 0$ as the agent having a prior belief that a good price is associated with low quality.

convention that the agent has a fixed vNM utility $u(.)$ which she aims to maximize under imperfect information.[7]

Assumption 2.2 (Normal-Bayesian Rationality) *The decision maker is Bayesian with a normal prior $\mathcal{N}(0, \Omega)$ and maximizes posterior expected utility.*

Next, we assume a novel type of noise in the perception of attributes. The noise is specific only to the attributes, and hence is common across alternatives. Let capital letters (i.e. $X = (X_1, X_2)$) denote the noisy signal of an object's attributes. For instance, in the choice set $\{\mathbf{x}, \mathbf{y}\}$, the attribute levels $\mathbf{x}^*$ and $\mathbf{y}^*$ are signaled by $X = \mathbf{x}^* + \epsilon$ and $Y = \mathbf{y}^* + \epsilon$ where ϵ has the same realization for all objects. Hence the agent perceives better the relative differences in attributes between the items, i.e. $\mathbf{x}^* - \mathbf{y}^* = X - Y$, than the the absolute locations $\mathbf{x}^*$ and $\mathbf{y}^*$ in the attribute space. This common noise across alternatives is assumed for mathematical simplicity and can be relaxed. Section 4.4 shows that the noise does not have to be identical across alternatives. Instead, the noise for each attribute needs to be positively correlated across alternatives.

Assumption 2.3 (Imperfect Perception) *For any n alternatives $\{\mathbf{x}^1, \dots, \mathbf{x}^n\}$ each with attributes $\mathbf{x}^{1*}, \dots, \mathbf{x}^{n*} \in \mathbb{R}^2$, the agent receives signals $X^1, \dots, X^n$ where $X^i - \mathbf{x}^{i*} = \epsilon$ for all i. The noise term $\epsilon \sim \mathcal{N}(0, T^{-1})$ is normal with variance matrix*

$$T^{-1} = \begin{bmatrix} 1/t_1^2 & R/(t_1 t_2) \\ R/(t_1 t_2) & 1/t_2^2 \end{bmatrix} \text{ for some } \frac{1}{t_1^2} + \frac{1}{t_2^2} > 0, \text{ and some } R \in (-1, 1).$$

Several motivating arguments can be made for this assumption. First, as discussed in the apartment hunting example, our direct perception for many attributes, such as light, sound and time, can be affected by imperfect perception. Furthermore, perception through seemingly noiseless numerical and textual information can still be susceptible to the imperfect perception. Experiments show that *even* when the attributes are measured in technical units and described precisely, the participants are *not* able to effectively perceive these numerical information perfectly (Green and Srinivasan, 1978; Ariely et al., 2003). For instance, Ariely et al. (2003) find that reading the measured volume of noises in scientific units does

[7] See e.g. Savage (1954).

not provide any more information about the volume than simply hearing the noises. And the information provided visually is as efficient, if not more efficient than that provided in numbers and words (Green and Srinivasan, 1978). One explanation for these findings is that the decision maker is subjectively uncertain in interpretating scientific units. As Kamenica (2008) argues, in general *"interpreting technical units of quality can be difficult."* For instance, a person who is used to seeing temperature in Celsius finds it hard to interpret Fahrenheit. In fact, even in Celsius, the same person's perception of the numeric temperature is not perfect. Due to such difficulty, precise measurements serve only as noisy indicators of the attribute levels. When the decision maker is uncertain in interpreting the technical units, her interpretation of the measured attributes may be smaller or larger than objective, resulting in under- or over-perceived attribute levels in all options.

To further illustrate this point, in our apartment-choice example, suppose the decision maker is also concerned with the safety of the neighborhoods. She can obtain a signal of this attribute by consulting the last-year crime statistics published by the same authority.[8] Even though for each neighborhood, this attribute is measured in simple units as "number of crimes per year per ten thousand people", it is still a noisy signal from the perspective of the decision maker, because for instance, it is not clear how strict the definition of crime is in this context. The signal can be an exaggeration (understatement) for all neighborhoods if the local authority applies a broader (narrower) definition of crime than she thinks. Consequently, the uncertainty in interpretation can result in imperfect perception of the safty of the neighborhoods.

The above explanation provides one reason for a decision maker to believe there is a common component in her noisy perception across options. There can be other reasons that we cannot enumerate due to limited scope. However, according to our model, contextual choices can occur *as long as the decision maker believes* there can be a common noise in her perception of the attributes. Regardless the justification for her belief in this common noise, when she forms the posterior taking into consideration a potential common component in the noise, contextual effects can occur from posterior utility maximization.

Second, instead of providing an arguement that details how the imperfect perception

[8]E.g. local police department and city websites, or the *Uniform Crime Reports* by FBI in US.

occurs, one can also argue that the assumption is instrumental. In other words, this assumption is made because in experiments, respondents' perceptions are inconsistent in a way as if they are affected by the imperfect perception. Indeed, some experiments in Ariely et al. (2003) show that people's perception can be affected by a "random anchor", causing "coherent arbitrariness". In evaluation tasks, participants usually evaluate the absolute value of an attribute level arbitrarily, but the difference in valuation across alternatives is coherent with the difference in their attribute levels. This finding is robust to whether the attributes are displayed in technical units or not. As put by Ariely et al. (2003),

> *"[W]e show that consumers' absolute valuation of experience goods is surprisingly arbitrary, even under "full information" conditions. However, we also show that consumers' relative valuations of different amounts of the good appear orderly ..."*

For a fixed preference, these observations can be interpreted as follow. When an attribute of an option is perceived higher (and hence of higher utility), such attribute for other options are also perceived higher (so are also of higher utility). Meanwhile, the difference in the perceived levels (and also in the utility values) is consistent with the difference in the actual attribute levels among the options. Hence the difference between options is perceived coherently, but the absolute value of the attribute is perceived arbitrarily. In light of these findings, the imperfect perception assumption can be interpreted as a random anchoring affecting the perception of attributes. Hence the agent either over-perceives or under-perceives each attribute across all alternatives.

In terms of the distributional aspect, the usual adoption of normality leads to prior-signal conjugacy. We allow the standard deviations in attributes to differ as long as one of them is strictly positive (i.e. $\frac{1}{t_1^2} + \frac{1}{t_2^2} > 0$) and the other can be zero (e.g. $t_1 = \infty$). Our assumption also allows the noise across attributes to have a non-zero correlation R.[9]

We now summarize some additional notations in the paper. Bold letters (e.g.$\mathbf{x}, \mathbf{y}, \mathbf{z}$) denote different alternatives. Letters with an asterisk (e.g. $\mathbf{x}^*, \mathbf{y}^*, \mathbf{z}^*$) denote the true attribute levels of an object in $\mathbb{R}^2$. We denote more than three alternatives with superscripts.

[9]Such a correlation can arise when attributes are closely related, such as the sugar content and calories in a soft drink, one might expect a correlation in the noise across these attributes.

Capital letters denote the initial noisy signals. Calligraphic letters (i.e. $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$) denote the agent's posterior beliefs about the true attributes. Subscripts distinguish the respective attribute-dimensions for a given vector. We use $C(\mathbf{x}^l, \{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^i, (\mathbf{x}^{i+1}, \ldots, \mathbf{x}^{i+j})\})$ to denote the choice probability of $\mathbf{x}^l$ from the set $\{\mathbf{x}^1 \ldots \mathbf{x}^{i+j}\}$ in which $\{\mathbf{x}^{i+1}, \ldots, \mathbf{x}^{i+j}\}$ are unavailable. A $C(.,.)$ that assigns a probability for any $\mathbf{x}$ in every nonempty finite set of alternatives S, with any $S' \subsetneq S$ specifying the unavailable objects, is called the *choice behavior* of an agent. The choice behavior satisfies

$$\sum_{k=1}^{i} C(\mathbf{x}^k, \{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^i, (\mathbf{x}^{i+1}, \ldots, \mathbf{x}^{i+j})\}) = 1.$$

3 A Parametric Special Case

In this section, we show the existence of stochastic intransitivity, the j-s reversal and the compromise effect with the following parametric setting. The preference is described by a regular exponential utility $u : \mathbb{R}^2 \to \mathbb{R}$

$$u(x_1, x_2) = -e^{-3x_1} - e^{-3x_2}.$$

The noise structure is simple and one dimensional. The first attribute is perfectly perceived, and noise exists only in the perception of the second. Therefore, the noise has no variance in the first attribute,

$$\epsilon \sim \mathcal{N}\left(0, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right).$$

Finally, the agent's prior is taken to be the standard bivariate normal centered at the origin.

3.1 Violation of Weak Stochastic Transitivity

Weak stochastic transitivity refers to the postulate that if $C(\mathbf{x}, \{\mathbf{x}, \mathbf{y}\}) \geq 0.5$ and $C(\mathbf{y}, \{\mathbf{y}, \mathbf{z}\}) \geq 0.5$, then $C(\mathbf{x}, \{\mathbf{x}, \mathbf{z}\}) \geq 0.5$. Early evidence of its violations can be found in Tversky (1969), and more recently Rieskamp et al. (2006). They suggest that weak transitivity can be violated when there is no clear domination among $\mathbf{x}, \mathbf{y}, \mathbf{z}$, which agrees with our model as shown below. In this subsection, a decision maker is said to *display intransitivity* if there are

$\mathbf{x}, \mathbf{y}, \mathbf{z}$ such that the choice behavior C satisfies $C(\mathbf{x}, \{\mathbf{x}, \mathbf{y}\}) > 0.5$, $C(\mathbf{y}, \{\mathbf{y}, \mathbf{z}\}) > 0.5$, and $C(\mathbf{z}, \{\mathbf{x}, \mathbf{z}\}) > 0.5$. In short, intransitivity in our model results from crossing of stochastic indifference curves.

Due to the randomness ϵ in the information, the choice between any two objects $\mathbf{x}$ and $\mathbf{y}$ depends on their fixed attribute levels $\mathbf{x}^*, \mathbf{y}^*$ and the realization of ϵ. Hence given the attribute levels, we can determine the probability of choice, $C(\mathbf{x}, \{\mathbf{x}, \mathbf{y}\})$, from the distribution of ϵ. We say $\mathbf{x}$ is *stochastically indifferent* to $\mathbf{y}$ (writes $\mathbf{x} \sim \mathbf{y}$) if

$$C(\mathbf{x}, \{\mathbf{x}, \mathbf{y}\}) = 0.5.$$

Similarly, the *stochastic indifference curve* of $\mathbf{x}$ is the set of alternatives that are stochastically indifferent to $\mathbf{x}$. On the space of attributes, this set of alternatives corresponds to the following set of attributes $\{\mathbf{y}^* \in \mathbb{R}^2 | \mathbf{x} \sim \mathbf{y}\}$.

Consider two alternatives $\mathbf{x}, \mathbf{y}$ such that $x_1^* > y_1^*$ and $y_2^* > x_2^*$. When is $\mathbf{x}$ chosen over $\mathbf{y}$? Since the agent is Bayesian rational, she chooses $\mathbf{x}$ whenever the posterior expected utility of $\mathbf{x}$ is greater than that of $\mathbf{y}$. Under the notation, the posterior beliefs about $\mathbf{x}^*$ and $\mathbf{y}^*$ are respectively the random variables $\mathcal{X}|X, Y$ and $\mathcal{Y}|X, Y$. So $\mathbf{x}$ is chosen over $\mathbf{y}$ if and only if

$$\mathbb{E}[u(\mathcal{X})|X, Y] > \mathbb{E}[u(\mathcal{Y})|X, Y].$$

We obtain the posterior belief from Bayesian updating, using the fact that $X - \mathbf{x}^* = Y - \mathbf{y}^*$,

$$\mathcal{X}_1|X, Y = x_1^*, \text{ and } \mathcal{X}_2|X, Y \sim \mathcal{N}\left(\frac{1}{3}(2X_2 - Y_2), \frac{1}{3}\right).$$ [10]

The belief about the first attribute, $\mathcal{X}_1|X, Y$, is equal to the true attribute level x_1^* since it is noiseless. The belief about the second attribute exhibits the contrast effect. If $\mathbf{y}$ is very good in the second attribute (i.e. if Y_2 is very large), then in contrast, $\mathbf{x}$ is perceived to be poorer in the second attribute (i.e. then $\frac{1}{3}(2X_2 - Y_2)$ is very small). Substituting the belief into the expected utility formula gives that $\mathbf{x}$ is chosen over $\mathbf{y}$ if and only if

$$\mathbb{E}[u(\mathcal{X})|X, Y] = -e^{-3x_1^*} - e^{-(2X_2 - Y_2) + 3/2} > -e^{-3y_1^*} - e^{-(2Y_2 - X_2) + 3/2} = \mathbb{E}[u(\mathcal{Y})|X, Y].$$ [11]

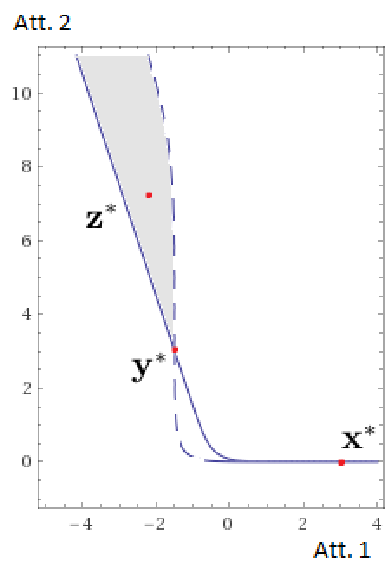
To get the choice probability, substitute in $X - \mathbf{x}^* = Y - \mathbf{y}^* = \epsilon$ to get an equivalent inequality

$$-\frac{3}{2} + \ln\left(\frac{e^{-3y_1^*} - e^{-3x_1^*}}{e^{y_2^* - 2x_2^*} - e^{x_2^* - 2y_2^*}}\right) > -\epsilon_2.$$

Since the $\epsilon_2 \sim \mathcal{N}(0,1)$, the choice probability can be expressed using the normal c.d.f Φ,

$$C(\mathbf{x}, \{\mathbf{x}, \mathbf{y}\}) = \Phi\left(-\frac{3}{2} + \ln\left(\frac{e^{-3y_1^*} - e^{-3x_1^*}}{e^{y_2^*-2x_2^*} - e^{x_2^*-2y_2^*}}\right)\right).$$

For interpretation, first recall that $x_1^* > y_1^*$ and $y_2^* > x_2^*$. Therefore, both $e^{-3y_1^*} - e^{-3x_1^*}$ and $e^{y_2^*-2x_2^*} - e^{x_2^*-2y_2^*}$ are positive. Moreover, since both Φ and ln are increasing functions, the choice probability is increasing in x_1^* and x_2^*, and decreasing in y_1^* and y_2^*. Intuitively, the agent is more likely to choose $\mathbf{x}$ if the true attribute levels of $\mathbf{x}$ improve, less so if the attributes of $\mathbf{y}$ become more desirable.[12]



Empirically, intransitivity can happen when the difference is easier to discriminate in attribute one, and harder in attribute two (Tversky, 1969; Leland, 1994). I.e. if the difference in attribute two is not large enough to be "consequential", individuals choose the option higher in attribute one. However, individuals choose the option higher in attribute two if its difference is large enough.
In our model, such observation can happen locally near $\mathbf{y}$ *as shown by the indifference curves of* $\mathbf{y}$ *(dashed) and of* $\mathbf{x}$ *(solid). For options slightly worse than* $\mathbf{y}$ *in attribute one, a large difference in attribute two is needed for them to compare favourably to* $\mathbf{y}$*. And for options (slightly) better than* $\mathbf{y}$ *in attribute one, also a sizable difference in attribute two of approximately* $y_2^* - x_2^*$ *is needed to compare unfavourably to* $\mathbf{y}$*. Now that* $z_2^* - x_2^* > y_2^* - x_2^*$ *is more than enough,* $\mathbf{x}$ *compares unfavourably to* $\mathbf{z}$ *even though* x_1^* *is mucb better than* z_1^**.*

Figure 1: **Crossing Stochastic Indifference Curves**

The indifference curve can be traced out using the definition $C(\mathbf{x}, \{\mathbf{x}, \mathbf{y}\}) = 0.5$. Because $\Phi(0) = 0.5$, we have $\mathbf{x} \sim \mathbf{y}$ if and only if

$$0 = -\frac{3}{2} + \ln\left(\frac{e^{-3y_1^*} - e^{-3x_1^*}}{e^{y_2^*-2x_2^*} - e^{x_2^*-2y_2^*}}\right).$$

Any $\mathbf{x}$ and $\mathbf{y}$ with attributes satisfying the above equation are stochastically indifferent.

Generically, if $\mathbf{x} \sim \mathbf{y}$, their indifference curves cross. For illustration, we let $\mathbf{x}^* = (3, 0)$ and $\mathbf{y}^* = (3 - \frac{1}{3}\ln(1 - e^{9/2} + e^{27/2}), 3)$ and check that $\mathbf{x} \sim \mathbf{y}$. As shown in Figure 1, the red dots are the corresponding true attribute levels, and the indifference curve of $\mathbf{x}$ is the

[12]We will show that if $x_1^* > y_1^*$ and $y_2^* > x_2^*$ does not hold, the dominating option will be chosen with probability 1 in the next section.

solid curve, whereas that of $\mathbf{y}$ is dashed. The two curves intersect at $\mathbf{x}^*$ and $\mathbf{y}^*$. The curves are indistinguishable for large values in the first attribute. Because the curves are distinct, intransitivity can occur when we consider any $\mathbf{z}$ with attributes in the shaded area. As in Figure 1, $\mathbf{z}^*$ is below the $\mathbf{y}$-curve and above the $\mathbf{x}$-curve. So $C(\mathbf{y}, \{\mathbf{y}, \mathbf{z}\}) > 0.5$ and $C(\mathbf{x}, \{\mathbf{x}, \mathbf{z}\}) < 0.5$. But as readily seen, slight improving $\mathbf{x}^*$ in either attribute will cause $C(\mathbf{x}, \{\mathbf{x}, \mathbf{y}\}) > 0.5$. Thereby strictly violating weak transitivity. The example is itself a proof of the following existence result.

Proposition 3.1 *Suppose there is imperfect perception in one of the attributes. There exists a normal-Bayesian rational agent with a standard preference who displays intransitivity.*

3.2 Joint-Separate Valuation Reversal

The effect refers to the reversal of evaluations for the alternatives in two contexts. In an experiment of Hsee (1996), the subjects (as company owners) were asked for their valuations in terms of willingness to pay to hire different job candidates as programmers. Candidate $\mathbf{x}$ has a college GPA of 4.9 out of 5 and has written 10 programs in the computer language KY. Candidate $\mathbf{y}$ has a GPA of 3.0 from the same school, and has written 70 similar programs in the same language. When the subjects were asked to evaluate $\mathbf{x}$ alone, the average valuation was about 32.7k dollars; when asked to evaluate $\mathbf{y}$ alone, the average valuation was less and about 26.8k. However, when the two candidates were presented together, the evaluations reversed. The average valuation for $\mathbf{x}$ in the presence of $\mathbf{y}$ became 31.2k, less than the new valuation for $\mathbf{y}$, 33.2k. With abuse of notation, we denote by $\$(\mathbf{x})$ and $\$(\mathbf{y})$ the valuation or the average willingness to pay for $\mathbf{x}$ and $\mathbf{y}$ in dollars, and denote by $\$(\mathbf{x}|\mathbf{x}, \mathbf{y})$ the average valuation for $\mathbf{x}$ in the presence of $\mathbf{y}$, and $\$(\mathbf{y}|\mathbf{x}, \mathbf{y})$ for $\mathbf{y}$ in the presence of $\mathbf{x}$. A decision maker is said to *display j-s reversal* if there exist $\mathbf{x}, \mathbf{y}$ such that both $\$(\mathbf{x}) > \$(\mathbf{y})$ and $\$(\mathbf{x}|\mathbf{x}, \mathbf{y}) < \$(\mathbf{y}|\mathbf{x}, \mathbf{y})$ holds.

In the experiment, the two attributes are the GPAs and the programming experience. While the GPA (scaled out of 5) is easy to interpret, the programming experience is hard. Although the programming experience is explicitly measured in numbers of programs written, it is not clear how advanced the computer language KY is, and how difficult it is to write programs in. The subjects as "company owners" may not be experts in programming, and are uncertain of their subjective interpretation. Hence it is reasonable to model programming experience with imperfect perception.

To show the existence of reversal in our model, we need to find a pair of $\mathbf{x}$ and $\mathbf{y}$ such that $x_1^* > y_1^*$ and $x_2^* < y_2^*$, and that $\$(\mathbf{x}) > \$(\mathbf{y})$ and $\$(\mathbf{x}|\mathbf{x},\mathbf{y}) < \$(\mathbf{y}|\mathbf{x},\mathbf{y})$ hold simultaneously. In this subsection, we use the *average posterior expected utility* as a proxy for average willingness to pay. That is, $\$(\mathbf{x})$ is understood as the average posterior expected utility of $\mathbf{x}$ in $\{\mathbf{x}\}$, $\$(\mathbf{y})$ that of $\mathbf{y}$ in $\{\mathbf{y}\}$. And $\$(\mathbf{x}|\mathbf{x},\mathbf{y})$, $\$(\mathbf{y}|\mathbf{x},\mathbf{y})$ that of $\mathbf{x}$, of $\mathbf{y}$ in $\{\mathbf{x},\mathbf{y}\}$.

When there is only one option, the posterior is based only on its own signal. For noiseless perception, $\mathcal{X}_1|X = x_1^*$. The noisy one has Bayesian posterior $\mathcal{X}_2|X \sim \mathcal{N}(\frac{1}{2}X_2, \frac{1}{2})$. Hence the average posterior expected utility is

$$\$(\mathbf{x}) := \mathbb{E}_X[\mathbb{E}_{\mathcal{X}_2}[-e^{-3x_1^*} - e^{-3\mathcal{X}_2}|X]] = -e^{-3x_1^*} - e^{-\frac{3}{2}x_2^*+\frac{27}{8}}.$$ [13]

When there are two options, from similar analysis in previous subsection we obtain

$$\$(\mathbf{x}|\mathbf{x},\mathbf{y}) := \mathbb{E}_{X,Y}[\mathbb{E}_{\mathcal{X}_2}[-e^{-3x_1^*} - e^{-3\mathcal{X}_2}|X,Y]] = -e^{-3x_1^*} - e^{-(2x_2^*-y_2^*)+2}.$$

Also, an analogous expression holds for $\$(\mathbf{y}|\mathbf{x},\mathbf{y})$. The two inequalities $\$(\mathbf{x}) > \$(\mathbf{y})$ and $\$(\mathbf{x}|\mathbf{x},\mathbf{y}) < \$(\mathbf{y}|\mathbf{x},\mathbf{y})$ are then

$$\begin{cases} -e^{-3x_1^*} - e^{-\frac{3}{2}x_2^*+\frac{27}{8}} > -e^{-3y_1^*} - e^{-\frac{3}{2}y_2^*+\frac{27}{8}} \\ -e^{-3x_1^*} - e^{-(2x_2^*-y_2^*)+2} < -e^{-3y_1^*} - e^{-(2y_2^*-x_2^*)+2}. \end{cases}$$

There are many pairs of alternatives that satisfy both inequalities. For illustration, let $\mathbf{x}^*$ be $(3, 0)$, and Figure 2 plots the shaded region where both inequalities are satisfied. The dashed curve is the boundary defined by the first inequality above, and the solid curve by the second. Any $\mathbf{y}$ with attributes $\mathbf{y}^*$ in the shaded region is an example of the desired reversal.

This mechanism that causes the reversal is intuitively shown in Figure 2. A $\mathbf{y}$ that is bad in the first attribute easily satisfies $\$(\mathbf{y}) < \$(\mathbf{x})$ in separate valuations. Because the utility function is concave and the perception is noisy, a strong y_2^* attribute cannot effectively increase the overall valuation. However, in joint valuation, there is a clear contrast in the second attributes for $x_2^* < y_2^*$. In comparison, $\mathcal{X}|X,Y$ is perceived as much worse off, and

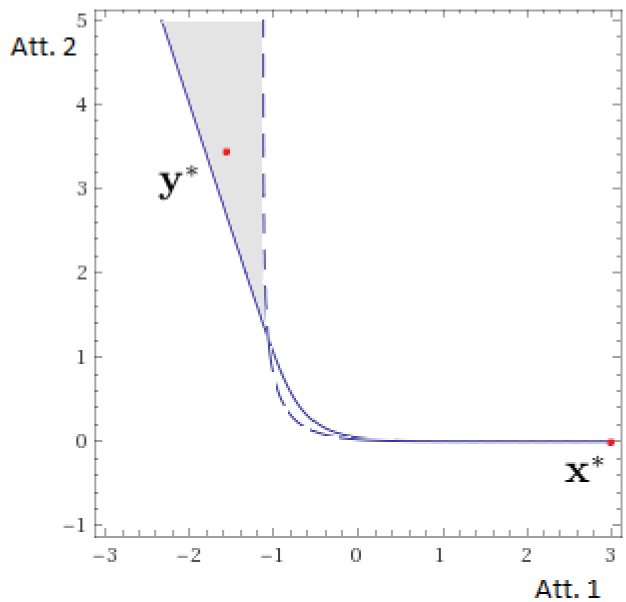


Figure 2: Joint-Separate Valuation Reversal
Empirically, j-s reversal can be observed if attribute one is easier to evaluate than attribute two (Hsee et al., 1999). In separate evaluation, this attribute one primarily determines the evaluation outcome. In a joint evaluation, comparison allows better evaluation of the attribute two, increasing its impact on the evaluation outcome, leading to the reversal.
In our model, this observation can be interpreted as the case where attribute one is perceived with less noise than attribute two. The figure indicates the equi-value curve of $\mathbf{x}$ *in separate evaluation (dashed), and that in joint evaluation (solid). When the difference* $x_1^* - y_1^*$ *is significant enough,* $\mathbf{x}$ *is easily better valued in separate evaluation, even for* $\mathbf{y}$ *with rather large* y_2^* *in the shaded area. However, in joint evaluation, attribute two is better valued and hence better substitutes for attribute one, as illustrated by the flatter solid curve. So a large enough* y_2^* *is enough to compensate for the difference in* $x_1^* - y_1^*$*, and overall* $\mathbf{y}$ *becomes better valued in joint evaluation (i.e. being above the solid cuve).*

$\mathcal{Y}|X, Y$ much better off, resulting in the reversal. The above analysis proves the following existence result.

Proposition 3.2 *Suppose there is imperfect perception in one of the attributes. There exists a normal-Bayesian rational agent with standard preference who displays j-s reversal.*

3.3 The Compromise Effect and Ternary Choices

The compromise effect involves choice problems of two and three options. As in Figure 3, suppose there is a binary choice problem with options $\mathbf{x}, \mathbf{y}$ where $\mathbf{x}$ is better than $\mathbf{y}$ in the first attribute but $\mathbf{y}$ is better in the second. The *compromise effect* (Simonson (1989)) refers to introducing a third $\mathbf{z}$ in or near the region C where $\mathbf{z}^*$ is extremely favorable in the first attribute but extremely unfavorable in the second one. Empirically, at the introduction of $\mathbf{z}$, subjects are generally led to choose the "compromising option" $\mathbf{x}$, increasing its choice frequency. Mathematically, let the initial choice set be $\{\mathbf{x}, \mathbf{y}\}$ and the extended choice set be $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ where $z_1^* > x_1^* > y_1^*$ and $y_2^* > x_2^* > z_2^*$. The compromise effect refers to $C(\mathbf{x}, \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}) > C(\mathbf{x}, \{\mathbf{x}, \mathbf{y}\})$ for all $\mathbf{z}$ "extreme enough".

Let Pr denote the probability measure for ϵ. We have seen previously that

$$C(\mathbf{x}, \{\mathbf{x}, \mathbf{y}\}) = \Pr\left(\mathbb{E}[u(\mathcal{X})|X, Y] > \mathbb{E}[u(\mathcal{Y})|X, Y]\right) = \Pr\left(\epsilon_2 > \frac{3}{2} - \ln\left(\frac{e^{-3y_1^*} - e^{-3x_1^*}}{e^{y_2^*-2x_2^*} - e^{x_2^*-2y_2^*}}\right)\right), \tag{3.1}$$

Similarly, we can also express the ternary probability as

$$\begin{aligned} &C(\mathbf{x}, \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}) \\ =& \Pr\left(\Big\{\mathbb{E}[u(\mathcal{X})|X, Y, Z] > \mathbb{E}[u(\mathcal{Y})|X, Y, Z]\Big\} \cap \Big\{\mathbb{E}[u(\mathcal{X})|X, Y, Z] > \mathbb{E}[u(\mathcal{Z})|X, Y, Z]\Big\}\right), \end{aligned}$$

where the first term in the intersection is the event that $\mathbf{x}$ is perceived better than $\mathbf{y}$,

$$\{\mathbb{E}[u(\mathcal{X})|X, Y, Z] > \mathbb{E}[u(\mathcal{Y})|X, Y, Z]\} = \{\epsilon_2 > \frac{3}{2} - \frac{4}{3}\ln\left(\frac{(e^{-3y_1^*} - e^{-3x_1^*})}{e^{-\frac{3}{4}(3x_2^*-y_2^*-z_2^*)} - e^{-\frac{3}{4}(3y_2^*-x_2^*-z_2^*)}}\right)\}, \tag{3.2}$$

and the second the event that $\mathbf{x}$ is perceived better than $\mathbf{z}$,

$$\{\mathbb{E}[u(\mathcal{X})|X, Y, Z] > \mathbb{E}[u(\mathcal{Z})|X, Y, Z]\} = \{\epsilon_2 < \frac{3}{2} - \frac{4}{3}\ln\left(\frac{e^{-3x_1^*} - e^{-3z_1^*}}{e^{-\frac{3}{4}(3z_2^*-x_2^*-y_2^*)} - e^{-\frac{3}{4}(3x_2^*-y_2^*-z_2^*)}}\right)\}. \tag{3.3}$$

In these two events, both fractions inside the logarithm are positive, because $z_1^* > x_1^* > y_1^*$ and $y_2^* > x_2^* > z_2^*$. It is clear that both sets are monotonic in the attributes of $\mathbf{x}$, the better the attributes for $\mathbf{x}$ are, the larger the event that $\mathbf{x}$ is the most preferred. Through a similar rationale, it is intuitive to see in Equation (3.3) that the event that $\mathbf{x}$ is preferred to $\mathbf{z}$ is monotonically decreasing in $\mathbf{z}$'s attributes.

More subtle is the influence of attributes of $\mathbf{z}$ on the preference between $\mathbf{x}$ and $\mathbf{y}$. From Equation (3.2), it is clear that the first attribute of z_1^* does not affect the preference between

$\mathbf{x}$ and $\mathbf{y}$, because the first attribute is noiseless for all. The second attribute is not. The (main component of the) perceived second attribute of $\mathbf{x}$ is $3x_2^* - y_2^* - z_2^*$.[14] Hence the term $-e^{-\frac{3}{4}(3x_2^*-y_2^*-z_2^*)}$ is (the main component of) the posterior utility of $\mathbf{x}$ from the second attribute. A weak attribute level of z_2^* contrasts with that of $\mathbf{x}$, increasing $\mathbf{x}$'s perceived level and its posterior utility level. Therefore, $\mathbf{x}$ appears more appealing in the context of an undesirable $\mathbf{z}$. Similarly, such an undesirable $\mathbf{z}$ also increases the posterior utility of $\mathbf{y}$. However, since $y_2^* > x_2^*$, $\mathbf{y}$ is more satiated than $\mathbf{x}$ in the second attribute. Hence the increase in perceived levels benefits $\mathbf{x}$ more. Mathematically, both the posterior utility of $\mathbf{y}$ and of $\mathbf{x}$ from the second attribute increase as z_2^* decreases, but their gap

$$
\begin{aligned}
e^{-\frac{3}{4}(3x_2^*-y_2^*-z_2^*)} - e^{-\frac{3}{4}(3y_2^*-x_2^*-z_2^*)} &= - e^{-\frac{3}{4}(3y_2^*-x_2^*-z_2^*)} - \left(-e^{-\frac{3}{4}(3x_2^*-y_2^*-z_2^*)}\right) \\
&= \left(-e^{-\frac{3}{4}(3y_2^*-x_2^*)} - (-e^{-\frac{3}{4}(3x_2^*-y_2^*)})\right) \exp(\frac{3}{4}z_2^*)
\end{aligned}
$$

decreases. Therefore, from Equation (3.2), a low z_2^* benefits $\mathbf{x}$ more, causing $\mathbf{x}$ to be preferred to $\mathbf{y}$.

To show that the compromise effect occurs, we take the limit that $z_1^* \to x_1^*$ from the right and see from Equation (3.3) that $\mathbf{x}$ is perceived better than $\mathbf{z}$ with probability approaching 1. I.e. $\Pr\left(\left\{\mathbb{E}[u(\mathcal{X})|X,Y,Z] > \mathbb{E}[u(\mathcal{Z})|X,Y,Z]\right\}\right) \to 1$ as $z_1^* \searrow x_1^*$. Moreover, for z_2^* small enough, the event in Equation (3.2) becomes a superset of the event in Equation (3.1). I.e. $\Pr\left(\left\{\mathbb{E}[u(\mathcal{X})|X,Y,Z] > \mathbb{E}[u(\mathcal{Y})|X,Y,Z]\right\}\right) > C(\mathbf{x},\{\mathbf{x},\mathbf{y}\})$ for z_2^* small enough. Therefore $C(\mathbf{x},\{\mathbf{x},\mathbf{y},\mathbf{z}\}) > C(\mathbf{x},\{\mathbf{x},\mathbf{y}\})$ for inferior enough $\mathbf{z}$. We have just proved the following result.

Proposition 3.3 (The Compromise Effect) *Assume the parametrization in this section. For any $\mathbf{x},\mathbf{y}$ with $x_1^* > y_1^*$ and $x_2^* < y_2^*$, there exists $\delta > 0$ and $D \in \mathbb{R}$ such that for all $\mathbf{z}$ with $z_1^* - x_1^* \in (0,\delta)$ and $z_2^* < D$, the inequality $C(\mathbf{x},\{\mathbf{x},\mathbf{y},\mathbf{z}\}) > C(\mathbf{x},\{\mathbf{x},\mathbf{y}\})$ holds.*

The result above points out an important distinction between our model and a large class of models that satisfy Monotonicity (also called Regularity). This includes the class of all random utility models (see e.g. Block and Marschak (1960) and Falmagne (1978) and

[14] Seen from the posterior belief being $\mathcal{X}_2 \sim \mathcal{N}(\frac{1}{4}(3X_2 - Y_2 - Z_2), \frac{1}{4})$.

section 5 of Rieskamp et al. (2006)). In the random utility framework, the utility of the options $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are random variables $U_\mathbf{x}, U_\mathbf{y}, U_\mathbf{z}$, i.e. measurable functions from a probability space to $\mathbb{R}$. The decision maker chooses $\mathbf{x}$ if and only if the event $\{U_\mathbf{x} > U_\mathbf{y} \text{ and } U_\mathbf{x} > U_\mathbf{z}\}$ is realized. A very general random utility model allows $U_\mathbf{x}, U_\mathbf{y}$, and $U_\mathbf{z}$ to be correlated in arbitrary ways. Nonetheless for a random utility model, it always holds that

$$\{U_\mathbf{x} > U_\mathbf{y}\} \subseteq \{U_\mathbf{x} > U_\mathbf{y} \text{ and } U_\mathbf{x} > U_\mathbf{z}\}, \text{ and hence } C(\mathbf{x}, \{\mathbf{x}, \mathbf{y}\}) < C(\mathbf{x}, \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}).$$

According to Proposition 3.3, our model directly violate this property, and hence it cannot be reinterpreted as any random utility model.

Through a similar mechanism, our model also captures two other effects in Figure 3. The *phantom decoy* effect (Pratkanis and Farquhar, 1992) occurs in the situation when $\mathbf{z}$ is positioned near the area P. Usually, the phantom alternative is better than $\mathbf{x}$ in the first attribute and no worse than $\mathbf{x}$ in the second. Also, it is worse than $\mathbf{y}$ in the second attribute. In experimental settings, the subjects are told that such a $\mathbf{z}$ is unavailable to choose and hence the subject has to choose from $\{\mathbf{x}, \mathbf{y}\}$. Empirically, the phantom decoy increases the frequency of choosing $\mathbf{x}$.[15] The *attraction effect* (Huber et al. , 1982) corresponds to introducing a third option $\mathbf{z}$ in or near the region A in Figure 3. In general, $\mathbf{z}$ needs to be inferior to $\mathbf{x}$ in the second attribute, and no better in the first. In addition, $\mathbf{z}$ needs to be better than $\mathbf{y}$ in the first attribute. Empirically, such a third option itself is hardly chosen, but increases the choice frequency of $\mathbf{x}$. Both findings can violate Monotonicity.

Because our model predicts these two effects through a similar channel, it suggests that there can be some commonality among the effects, as argued by Highhouse (1996). Here, we omit their formal proofs to avoid repetition. Nonetheless, a proof of the attraction effect (phantom decoy effect) parallels the following intuition. Suppose there is imperfect perception in the second (first) attribute. Again, let $\mathbf{x}$ be inferior in the second attribute and $\mathbf{y}$ inferior in the first. Now introduce the third object $\mathbf{z}$ near A (P) that is extremely bad in the second attribute (good in the first attribute). In comparison, $\mathbf{z}$ causes both $\mathbf{x}$ and $\mathbf{y}$ to be perceived better in the second attribute (worse in the first attribute) than before. However, because $\mathbf{y}$ was already good enough in the second (barely acceptable in the first)

[15] See e.g. Pratkanis and Farquhar (1992), Highhouse (1996), Pettibone and Wedell (2000), Pettibone and Wedell (2007) and Hedgcock et al. (2009) etc.

attribute in $\{\mathbf{x}, \mathbf{y}\}$, overall $\mathbf{x}$ turns out relatively more favorable (less repulsive) than $\mathbf{y}$.

3.4 Remarks on the Parametric Model

First, the above examples illustrate how intransitivity, j-s reversal and the compromise effect can be explained by the parametric model where only attribute one is precisely perceived. Further derivation reveals that, under the same model, these effects can occur under a range of budget sets that are expected from the behavioral literature.

Second, to explaining the empirical observations, a more general parametric model can be used to estimate preferences and predict choice probabilities. For example, let the utility function be additively exponential $u(x) := u(x_1, x_2) = -e^{\gamma x_1} - e^{\rho x_2}$ where $\gamma, \rho < 0$ are preference parameters. And the noise be $\epsilon \sim N\left(0, \begin{bmatrix} 1/t_1^2 & 0 \\ 0 & 1/t_2^2 \end{bmatrix}\right)$ with parameters $t_1, t_2 \in (0, \infty]$, one of them potentially infinite. Under this parametrization, the parameters can be estimated easily from choice data. For example, in our parametrization, the choice probability for any binary problem is given analytically in Lemma 3.1.

Lemma 3.1 *For any $\mathbf{x}, \mathbf{y}$ where $x_1^* > y_1^*$ and $y_2^* > x_2^*$, the parametric model in the subsection gives $C(\mathbf{x}, \{\mathbf{x}, \mathbf{y}\}) = \Phi\left(\theta(\gamma, \rho, \mathbf{x}^*, \mathbf{y}^*, t)\right)$, where Φ is the standard normal c.d.f. function and $\theta(\gamma, \rho, \mathbf{x}^*, \mathbf{y}^*, t)$ is defined as*

$$\theta := \frac{1}{\sqrt{\left(\frac{\rho\sqrt{t_2}}{2+t_2^2}\right)^2 + \left(\frac{\gamma\sqrt{t_1}}{2+t_1^2}\right)^2}} \left[\frac{\gamma^2}{2(2+t_1^2)} - \frac{\rho^2}{2(2+t_2^2)} + \ln\left(\frac{\exp\left(\gamma\frac{(t_1^2+1)y_1^*-x_1^*}{2+t_1^2}\right) - \exp\left(\gamma\frac{(t_1^2+1)x_1^*-y_1^*}{2+t_1^2}\right)}{\exp\left(\rho\frac{(t_2^2+1)x_2^*-y_2^*}{2+t_2^2}\right) - \exp\left(\rho\frac{(t_2^2+1)y_2^*-x_2^*}{2+t_2^2}\right)}\right)\right].$$

When an attribute becomes noiseless (i.e. $t_1 \to \infty$), the above Lemma reduces to Equation 3.1. As seen previously, an $\mathbf{x}$ with better attributes results in a higher θ and higher $C(\mathbf{x}, \{\mathbf{x}, \mathbf{y}\})$, and the reverse holds for $\mathbf{y}$. Moreover, because $x_1^* > y_1^*$, and γ is the preference parameter in the first attribute, a larger γ^2 implies that the first attribute is more decisive, and hence more likely to choose $\mathbf{x}$.

As the Lemma specifies choice probabilities in terms of parameters, it can be used to estimate exponential utility functions when there are observations for different menus. When the parameters are estimated, the model can be used to predict choice probabilities in new menus. Here, we adopt an implicit assumption similar to that in Koszegi and Szeidl (2013).

To maintain empirical identifiability and avoid excessive degrees of freedom, the definitions and measurements of the attributes must be determined *before* fitting the model to data. They should not be free parameters but part of the data that the model seeks to explain.[16]

Although the expression in the Lemma can be useful for experimenters, the agent in the model does not evaluate this complicated algebra before making the choice. She simply chooses the choice item that maximizes her expected utility while being unaware of the choice probabilities her actions generate.

4 The General Results

Last section shows that one simple parametric setting can explain and predict several contextual effects. These results are *not* outcomes of parametric flexibility. On the contrary, the next subsection shows the model is robust in the sense that even without the parametric restrictions, contextual effects will always occur to some "rightly designed" choice problems. In contrast, many other models are not robust in this sense, that even for the right choice problems, there are parameters for which context effects cannot occur.

While the "rightly designed" choice problems have been a focus in experimental research, contextual effects are rarely observed in many other choice problems. There, the observed choices are usually more "rational". Subsection 4.2 shows this is in accordance of our model. The agent's choice conforms to the classical rational choice theory in a class of choice problems where contextual effects are not empirically observed. We also discuss the type of sensible regularity akin to rationality that our model always satisfy.

4.1 The Decoy Choice Pattern

In this subsection, we first define the term "decoy choice pattern" as an abstraction of the attraction, compromise, and phantom decoy effect. We then show that the general model predicts the decoy choice pattern under the general class of preferences and prior-signal dis-

[16] While it is easier to satisfy this procedure in marketing experiments where the attributes of each object are specified by the experimenter, it is sometimes difficult to include other relevant attributes in real life decision-making processes. For example, when shopping (online or in person), individuals may base their decisions on attributes that are not listed on the product descriptions. For instance, decisions may be made based on the retailer's customer service, which is usually not listed in the product labels. Hence it is difficult to account for these influences.

tributions as described in Section 2. Start with a binary choice problem where $\mathbf{x}$ is better than $\mathbf{y}$ in the first attribute but $\mathbf{y}$ is better in the second, as shown in Figure 3. As empirically observed, a third object $\mathbf{z}$ in the lower right corner of Figure 3 generally increases the choice probability of $\mathbf{x}$. Due to symmetry, it is also true empirically that if instead of $\mathbf{z}$, a third object $\mathbf{w}$ lies in the upper left corner of the same figure will increase the choice probability of $\mathbf{y}$ (e.g. a compromise effect where $\mathbf{y}$ is the compromising option). These empirical effects share a common feature that $\mathbf{z}$ or $\mathbf{w}$ is either unavailable (as a phantom decoy) or rarely chosen (as in compromise effect or attraction effect). Therefore, one can reasonably infer that both the attraction effect and the compromise effect will remain qualitatively unchanged when the third option is unavailable. To summarize these observations, there exists some $\mathbf{w}$ and $\mathbf{z}$ where the difference $\mathbf{z}^* - \mathbf{w}^*$ points towards the lower-right half plane, such that the *unavailable third option* $\mathbf{w}$ increases the choice probability of $\mathbf{y}$ whereas the *unavailable third option* $\mathbf{z}$ increases the choice probability of $\mathbf{x}$. We call this comparative statics the *decoy choice pattern.*

Definition 4.1 *The choice behavior is said to display the* decoy choice pattern *if there exists a vector* $\Delta \in \mathbb{R}^2$ *with* $\Delta_1 > \Delta_2$, *such that for any* $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}$ *with attributes in* $\mathbb{R}^2$ *satisfying* $x_1^* > y_1^*$, $x_2^* < y_2^*$ *and* $\mathbf{z}^* = \mathbf{w}^* + \Delta$, *the inequality* $C(\mathbf{x}, \{\mathbf{x}, \mathbf{y}, (\mathbf{z})\}) > C(\mathbf{x}, \{\mathbf{x}, \mathbf{y}, (\mathbf{w})\})$ *holds.*

Our model predicts the decoy choice pattern, which can be interpreted as an empirically testable implication in two ways. First, when there are at least two attributes under consideration, there exists menus for which the agent does not satisfy the Luce's IIA. Second, the agent violates Luce's IIA in a specific way. I.e. making $\mathbf{x}$ the compromising option in the experiments *does not reduce* the choice probability of $\mathbf{x}$.

Theorem 4.1 *Any normal-Bayesian rational agent with standard preference and imperfect perception displays the decoy-choice pattern.*

Observe that Theorem 4.1 is a sufficiency result. Intuitively, it states that if $\mathbf{z}^*$ is to the right or to the bottom of $\mathbf{w}^*$, such a $\mathbf{z}$ affects the choice probability of $\mathbf{x}$ positively as opposed to $\mathbf{w}$. Another interesting implication of the theorem is that the attraction effect

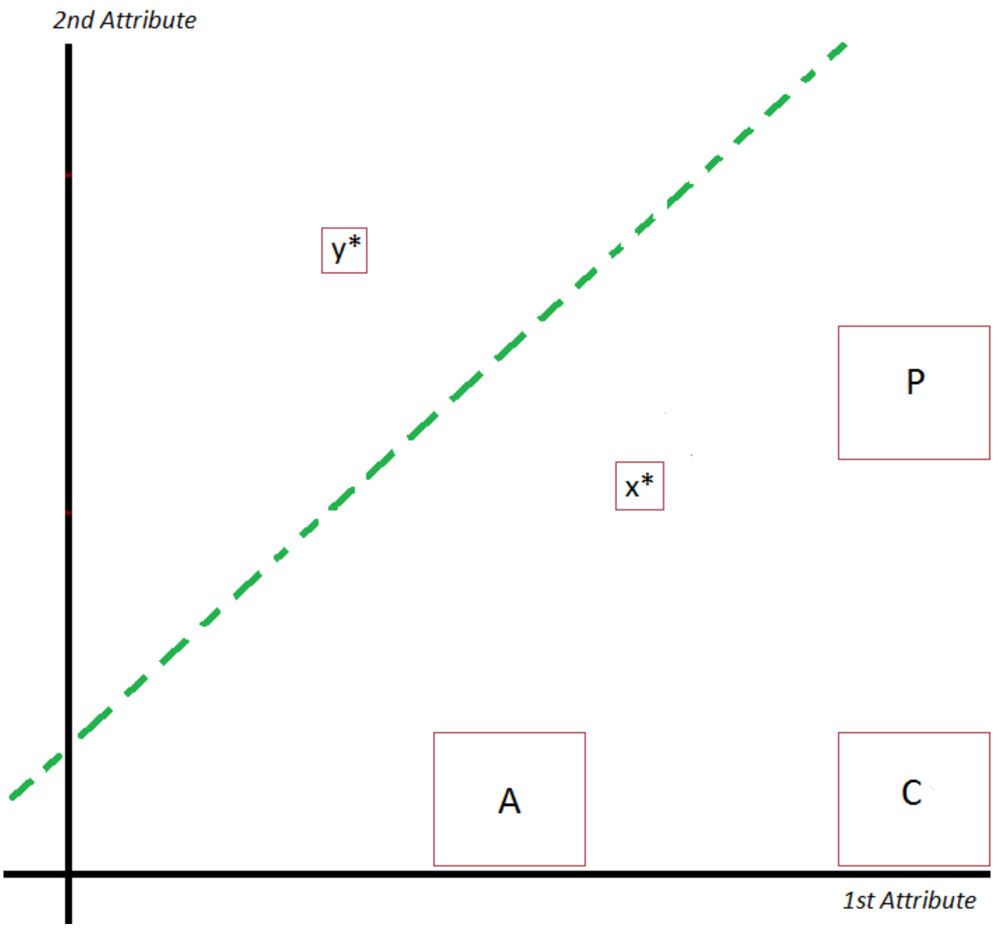


Figure 3: Areas for the phantom decoy effect (P), the compromise effect (C) and the attraction effect (A)

and the compromise effects should still exist even when **z** is unavailable. Since **z** is rarely chosen in experiments, such a prediction is reasonable to expect, but is special to our model. Other choice models usually do not consider unavailable options.

4.2 Choice under Dominance

We have seen previously that when there is a trade-off between the alternatives, i.e. some alternatives are better in the first attribute while others are better in the second, contextual choices can arise in the model. A natural question is what would the model predict when such a trade-off is absent. Intuitively, if we are given two alternatives **x** and **z** where $\mathbf{z}^* > \mathbf{x}^*$, a rational agent should always choose **z** due to the monotonicity of the utility function.[17]

[17] The vector inequality $\mathbf{z}^* > \mathbf{x}^*$ means $z_1^* \geq x_1^*$ and $z_2^* \geq x_2^*$ with at least one inequality being strict.

The prediction of our model fits this intuition. Since the error ϵ in perception is the same for each of $\mathbf{x}$ and $\mathbf{z}$, the perturbed signal $X = \epsilon + \mathbf{x}^*$ and $Z = \epsilon + \mathbf{z}^*$ preserves the inequality: $Z > X$. A Bayesian rational agent can hence correctly infer the inequality and choose optimally.

Theorem 4.2 *For any $\{\mathbf{x}, \mathbf{z}\}$ with $\mathbf{x}^*, \mathbf{z}^* \in \mathbb{R}^2$, a normal-Bayesian rational agent with standard preference and imperfect perception chooses $\mathbf{z}$ with probability 1 if $\mathbf{z}^* > \mathbf{x}^*$.*

It is clear that the above theorem also predicts the following intuitive choice effect described and observed in Tversky (1972) and Tversky and Russo (1969). Consider an individual that is choosing between a trip to Paris ($\mathbf{x}$) and a trip to Rome ($\mathbf{y}$). If she is interested to see both places and doesn't have a strong preference for one over the other, her choice probability for Paris ($\mathbf{x}$) would be roughly 1/2. Now if we offer the individual a new choice problem with two alternatives, a trip to Paris ($\mathbf{x}$) or a trip to Paris plus a $1 bonus ($\mathbf{z}$), he would probably not hesitate to choose the option with the extra dollar. In other words, choosing $\mathbf{z}$ over $\mathbf{x}$ is of probability 1. However, if we offer him a third choice problem that consists of a trip to Paris plus $1 and a trip to Rome, it is intuitive that the choice probability should still be roughly 1/2.

Another implication of the above theorem is that transitivity holds with overwhelming probability for choice objects among which each either dominates or is dominated by another. Therefore, violation of weak stochastic transitivity can happen only when the options do not dominate each other. The proof of the following result is immediate.

Corollary 4.1 *Suppose $\mathbf{x}, \mathbf{y}, \mathbf{z}$ have attributes $\mathbf{x}^* > \mathbf{y}^* > \mathbf{z}^*$, then $1 = C(\mathbf{x}, \{\mathbf{x}, \mathbf{y}\}) = C(\mathbf{y}, \{\mathbf{y}, \mathbf{z}\}) = C(\mathbf{x}, \{\mathbf{x}, \mathbf{z}\}) > 1/2$.*

Theorem 4.2 can also be generalized to the following statement. When $S = \{\mathbf{x}^1, \ldots, \mathbf{x}^n\}$ is the choice set involving multiple options, if $\mathbf{x}^i$ is dominated in the set S, then $C(\mathbf{x}^i, S) = 0$. In other words, objects are chosen with positive probability only when they are on the "attribute possibility frontier". This is a desirable regularity condition that the our model satisfies, and it rules out many other types of irregular choice behaviors.[18]

[18] It is also one of the distinctions between our model and Natenzon (2019). In his model, a dominated object $\mathbf{x}$ with $\mathbf{x}^* < \mathbf{y}^*$ can still be chosen with probability significantly greater than 0.

A tempting conjecture is that a similar result holds for j-s reversal. But it is not true in general. It is true that Corollary 2 follows immediately from the proof of Theorem 4.2.

Corollary 4.2 *For any* $\{\mathbf{x}, \mathbf{z}\}$ *with* $\mathbf{x}^* < \mathbf{z}^* \in \mathbb{R}^2$*, it holds that* $\$(\mathbf{x}|\mathbf{x}, \mathbf{z}) < \$(\mathbf{z}|\mathbf{x}, \mathbf{z})$.

However, it does not follow that $\$(\mathbf{x}) < \$(\mathbf{z})$ if $\mathbf{x}^* < \mathbf{z}^*$ when the correlations r and R are not restricted. Consider the apartment-choice problem and the two attributes are convenience and safety. The decision maker values both attributes, and her prior believes that the two attributes are negatively correlated: on average, a convenient location is usually less safe, and a safe location is farther away and hence less convenient. Let $\mathbf{x}$ and $\mathbf{z}$ be two apartments that are exactly of the same safety level, but $\mathbf{z}$ is more convenient, i.e., $\mathbf{x}^* < \mathbf{z}^*$. It clearly holds that $\$(\mathbf{x}|\mathbf{x}, \mathbf{z}) < \$(\mathbf{z}|\mathbf{x}, \mathbf{z})$ in a joint valuation. But the decision maker cannot see this comparison in the separate valuations. When she only sees the very convenient $\mathbf{z}$, her prior makes her believe that $\mathbf{z}$ is likely unsafe. If she values safety much more than convenience, her valuation for $\$(\mathbf{z})$ can be low. On the other hand, if she sees only $\mathbf{x}$, since $\mathbf{x}$ is not so convenient, her posterior assumes $\mathbf{x}$ is safe. As a result, she may value $\$(\mathbf{x})$ highly. In this case, $\$(\mathbf{x}) > \$(\mathbf{z})$ is still allowed by our model even though $\mathbf{x}^* < \mathbf{z}^*$. As a numerical example, let the prior be $\mathcal{N}\left(\mathbf{0}, \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}\right)$ and the noise be $\epsilon \sim \mathcal{N}(0, I_2)$. A signal $X = (0, 0)$ would result in the posterior $\mathcal{X}|X \sim \mathcal{N}\left(\mathbf{0}, \frac{1}{15}\begin{bmatrix} 7 & -2 \\ -2 & 7 \end{bmatrix}^{-1}\right)$. A dominating signal $Z = (1, 0)$ would result in the posterior $\mathcal{Z}|Z \sim \mathcal{N}\left(\begin{bmatrix} 7/15 \\ -2/15 \end{bmatrix}, \frac{1}{15}\begin{bmatrix} 7 & -2 \\ -2 & 7 \end{bmatrix}^{-1}\right)$. If the utility function values the second attribute much more than the first, then $\mathbb{E}[u(\mathcal{Z})|Z] < \mathbb{E}[u(\mathcal{X})|X]$. This observation is a novel prediction of our model, that J-S reversal is possible even though one option dominates the other in the attribute space.

4.3 Limiting Noise Structure

As shown in Proposition 3.3, our model does not satisfy Monotonicity, a fundamental property of all random utility models. Despite this difference, one interesting question may be whether such non-Monotonic predictions disappear in some limiting parameters of our model. For example, if the noise in the signal goes to zero, does our model converge to some well-known models? We discuss below that as the noise term becomes small, our model approximates the well-known conditional probit model of Hausman and Wise (1978). Because Hausman and Wise (1978)'s model is a random utility model, it satisfies Monotonicity. We also remark that because the conditional probit model can explain the similarity effect, a corollary of this subsection is that our model can also explain the similarity effect.

Again, restrict our discussion to the exponential utility functions so that $u(x_1, x_2) = -e^{\gamma x_1} - e^{\rho x_2}$. Given a finite choice set $S = \{\mathbf{x}^1, \ldots \mathbf{x}^n\}$, the posterior belief of the ith alternative under imperfect perception is

$$\mathcal{X}^i | X^1, \ldots X^n \sim \mathcal{N}\left((T + n\Omega^{-1})^{-1} \left(TX^i + n\Omega^{-1} X^i - \sum_{j=1}^{n} \Omega^{-1} X^j \right), (T + n\Omega^{-1})^{-1} \right).$$

When the noise variance converges to zero, i.e. $T^{-1} \to \mathbf{0}$, the posterior belief $\mathcal{X}^i | X^1, \ldots X^n$ is approximately $\mathcal{N}(X^i, T^{-1}) = \mathcal{N}(\mathbf{x}^{i*} + \epsilon, T^{-1})$. When the utility function is smooth enough near $\mathbf{x}^{i*}$, we approximate the expected utility using the utility of the expected attributes

$$\mathbb{E}[u(\mathcal{X}) | X^1, \ldots X^n] \approx u(\mathbf{x}^{i*} + \epsilon)$$

which is already a random utility model. Under exponential utility, this approximates the Hausman and Wise (1978),

$$u(\mathbf{x}^{i*} + \epsilon) = -e^{\gamma(x_1^{i*} + \epsilon_1)} - e^{\rho(x_2^{i*} + \epsilon_2)} \approx u_1(x_1^{i*}) + u_2(x_2^{i*}) + \beta_1 u_1(x_1^{i*}) + \beta_2 u_2(x_2^{i*}),$$

where we have used the first order approximation at $\mathbf{x}^{i*}$ with the notation that $u_1(x_1) = -e^{\gamma x_1}, u_2(x_2) = -e^{\rho x_2}$ and $\beta_1 = \gamma \epsilon_1, \beta_2 = \rho \epsilon_2$. It is clear that the form of the approximation coincide with equation (3.6) in Hausman and Wise (1978).

4.4 Relaxing the Perfect Correlation

Imperfect perception assumes that signals across options bear the same noise. In other words, the noise is perfectly correlated across the alternatives. The strength of this assumption simplifies the notation and derivation. However, it is not necessary and can be weakened. For example, when the noise is positively correlated across options, the quantitative properties of our model still holds approximately. In this case, from the decision maker's perspective, it means that she does not have to believe that the noise is the identical across options. It suffices for the her to believe only a component of the noise is common, so that the noise is not too different across alternatives.

For instance, when there are alternatives $\{\mathbf{x}, \mathbf{y}\}$ with signals X and Y, let the noise be $\epsilon_x = X - \mathbf{x}^*$ and $\epsilon_y = Y - \mathbf{y}^*$. Let us use μ to denote the posterior belief of $(\mathcal{X}, \mathcal{Y})|X, Y$ when the noise is perfectly correlated, i.e. $\epsilon_x = \epsilon_y$. And similarly, use μ_a to denote the posterior belief when perfect correlation *does not* hold. Suppose under the belief μ, the agent chooses $\mathbf{x}$, i.e., $\mathbb{E}_\mu[u(\mathcal{X})] > \mathbb{E}_\mu[u(\mathcal{Y})]$. Notice that when μ_a is close enough to μ, it also holds that $\mathbb{E}_{\mu_a}[u(\mathcal{X})] > \mathbb{E}_{\mu_a}[u(\mathcal{Y})]$ and so $\mathbf{x}$ is also chosen under the belief μ_a. Therefore, due to this continuity, one can locally relax the assumption and allow $\epsilon_x \neq \epsilon_y$, and at the same time the resulting μ_a would be closed enough to μ for the model predictions to be quantitatively similar.

Formally, the following convergence result shows when correlation across alternatives is high enough, the posterior μ_a is close enough to μ. For commonly used utility functions, such closeness is sufficient to maintain the choice decisions in a given problem.

Proposition 4.1 *Let the prior for each option be $\mathcal{N}(0, \Omega)$. For any n options with realized signals $X^1, \dots, X^n$, let the noise for each signal be ϵ^i. Suppose $(\epsilon^1, \dots, \epsilon^n)' \sim \mathcal{N}(0, \Sigma_a)$ for some positive definite Σ_a, and the resulting posterior belief be μ_a. Denote by Σ the $2n \times 2n$ matrix*

$$\Sigma = \begin{bmatrix} T^{-1} & T^{-1} & \dots & T^{-1} \\ \vdots & \vdots & \dots & \vdots \\ T^{-1} & T^{-1} & \dots & T^{-1} \end{bmatrix},$$

and by μ the posterior belief when $\epsilon^i = \epsilon^j$ almost surely. Then $\mu_a(\mathcal{X}^i)$ for each i is normally distributed and converges to $\mu(\mathcal{X}^i)$ weakly as $\Sigma_a \to \Sigma$.

5 Discussion and Conclusion

We present a choice model with underlying rational preferences. Through noisy attribute perception, the model generically predicts the compromise effect, the attraction effect and the phantom decoy effect, and the existence of choice cycles and j-s reversal.

Despite the context-dependent predictions, in our model, the noise of an alternative has the same exogeneously fixed distribution in all contexts. Hence the noise is independent of contexts in the same way as a random utility model is. In a random utility model, the distribution of a random utility (correlated with others or not) is not context-dependent, and the agent observes only the utilities of the options available. Analogously in our model, the agent observes only the signals of alternatives on the menu. However, due to the endogenously updated Bayesian posterior belief, the posterior utility maximizing choices in our model can explain contextual effects.

In additional to our simple mechanism, it is likely that other mechanisms are also at play in reality. Imagine a choice problem with many options, An agent is asked to rank the options, or is asked to choose one from each pair. Our model implies that these two tasks yield the same ordering since the posterior utilities are the same. This is a property of our working assumption that the signal precision is fixed regardless of availability and contexts. So the agent always learns all the available information in a choice problem. Therefore, our model is not intended to explain empirical phenomena such as the choice overload (Iyengar and Lepper, 2000), for which the main driver is likely the limited information capacity.[19] For these choice effects, a more suitable model would likely cover endogenous attention and information acquisition. See Guo (2016) for one such model in explaining choice overload. On the other hand, when the agent's finite power to process information is not of first-order importance, our model captures the systematic mechanism of contextual choices through imperfect attribute perception.

Similarly, another related interesting mechanism is limited memory. For example, after the agent sees $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$, the current model predicts the same choice behavior for the choice set $\{\mathbf{x}, \mathbf{y}, (\mathbf{z})\}$ where $\mathbf{z}$ is shown but not available, and the choice set $\{\mathbf{x}, \mathbf{y}\}$ where $\mathbf{z}$ is later removed. This does not explain the findings in Sivakumar and Cherian (1995) that the choice probability of $\mathbf{x}$ (the target) is significantly reduced following the removal of $\mathbf{z}$, although it does not fully recover to the level at which $\mathbf{z}$ was never shown. A possible future extension to capture such empirical findings is a model where the agent partially forgets what she has learned when stimuli are removed.

[19] I thank the associate editor and the referees for pointing this out.

6 APPENDICES

6.1 Proof of Lemma

Proof. Proof of Lemma 3.1 We calculate directly the expected utility

$$
\begin{aligned}
\mathbb{E}[u(\mathcal{X})|X,Y] =& \mathbb{E}\left[-e^{\gamma\mathcal{X}_1} - e^{\rho\mathcal{X}_2}|X,Y\right] \\
=& -\exp\left(\gamma\frac{(t_1^2+1)X_1 - Y_1}{2+t_1^2} + \gamma^2\frac{1}{2(2+t_1^2)}\right) - \exp\left(\rho\frac{(t_2^2+1)X_2 - Y_2}{2+t_2^2} + \rho^2\frac{1}{2(2+t_2^2)}\right) \\
=& -\exp\left(\gamma\frac{(t_1^2+1)x_1^* - y_1^* + t_1^2\epsilon_1}{2+t_1^2} + \gamma^2\frac{1}{2(2+t_1^2)}\right) \\
& -\exp\left(\rho\frac{(t_2^2+1)x_2^* - y_2^* + t_2^2\epsilon_2}{2+t_2^2} + \rho^2\frac{1}{2(2+t_2^2)}\right)
\end{aligned}
$$

where the second equality is due to the normally distributed exponents. The third equality is due to the identities $\mathbf{x}^* + \epsilon = X$, $\mathbf{y}^* + \epsilon = Y$. Similarly,

$$
\begin{aligned}
\mathbb{E}[u(\mathcal{Y})|X,Y] =& -\exp\left(\gamma\frac{(t_1^2+1)y_1^* - x_1^* + t_1^2\epsilon_1}{2+t_1^2} + \gamma^2\frac{1}{2(2+t_1^2)}\right) \\
& -\exp\left(\rho\frac{(t_2^2+1)y_2^* - x_2^* + t_2^2\epsilon_2}{2+t_2^2} + \rho^2\frac{1}{2(2+t_2^2)}\right)
\end{aligned}
$$

Hence given $\mathbf{x}^*, \mathbf{y}^*$ and ϵ, the agent would choose $\mathbf{x}$ over $\mathbf{y}$ iff $\mathbb{E}[u(\mathcal{X})|X,Y] > \mathbb{E}[u(\mathcal{Y})|X,Y]$. Suppose $x_1^* > y_1^*$ and $y_2^* > x_2^*$, then we see that $\mathbf{x}$ is chosen over $\mathbf{y}$ iff

$$
\exp\left(\frac{\gamma^2}{2(2+t_1^2)} - \frac{\rho^2}{2(2+t_2^2)}\right)\frac{\exp\left(\gamma\frac{(t_1^2+1)y_1^*-x_1^*}{2+t_1^2}\right) - \exp\left(\gamma\frac{(t_1^2+1)x_1^*-y_1^*}{2+t_1^2}\right)}{\exp\left(\rho\frac{(t_2^2+1)x_2^*-y_2^*}{2+t_2^2}\right) - \exp\left(\rho\frac{(t_2^2+1)y_2^*-x_2^*}{2+t_2^2}\right)} \geq \exp\left(\frac{\rho t_2^2\epsilon_2}{2+t_2^2} - \frac{\gamma t_1^2\epsilon_1}{2+t_1^2}\right). \tag{$\dagger$}
$$

Since $x_1^* > y_1^*$ and $y_2^* > x_2^*$, we can take natural-log on both hand sides of (†) to obtain the following equivalent condition

$$
\frac{\gamma^2}{2(2+t_1^2)} - \frac{\rho^2}{2(2+t_2^2)} + \ln\left(\frac{\exp\left(\gamma\frac{(t_1^2+1)y_1^*-x_1^*}{2+t_1^2}\right) - \exp\left(\gamma\frac{(t_1^2+1)x_1^*-y_1^*}{2+t_1^2}\right)}{\exp\left(\rho\frac{(t_2^2+1)x_2^*-y_2^*}{2+t_2^2}\right) - \exp\left(\rho\frac{(t_2^2+1)y_2^*-x_2^*}{2+t_2^2}\right)}\right) \geq \frac{\rho t_2^2\epsilon_2}{2+t_2^2} - \frac{\gamma t_1^2\epsilon_1}{2+t_1^2}.
$$

Notice that RHS follows a normal distribution $\mathcal{N}\left(0, \left(\frac{\rho}{2+t_2^2}\right)^2 t_2 + \left(\frac{\gamma}{2+t_1^2}\right)^2 t_1\right)$. We can standardize both hand side by multiplying $1/\sqrt{\left(\frac{\rho\sqrt{t_2}}{2+t_2^2}\right)^2 + \left(\frac{\gamma\sqrt{t_1}}{2+t_1^2}\right)^2}$. Hence $\mathbf{x}^*$ is chosen over

$\mathbf{y}^*$ iff some standard normal random variable Z is below the threshold θ defined below:

$$\theta(\gamma, \rho, \mathbf{x}^*, \mathbf{y}^*, t) := \frac{1}{\sqrt{\left(\frac{\rho\sqrt{t_2}}{2+t_2^2}\right)^2 + \left(\frac{\gamma\sqrt{t_1}}{2+t_1^2}\right)^2}} \times$$
$$\left[\frac{\gamma^2}{2(2+t_1^2)} - \frac{\rho^2}{2(2+t_2^2)} + \ln\left(\frac{\exp\left(\gamma\frac{(t_1^2+1)y_1^*-x_1^*}{2+t_1^2}\right) - \exp\left(\gamma\frac{(t_1^2+1)x_1^*-y_1^*}{2+t_1^2}\right)}{\exp\left(\rho\frac{(t_2^2+1)x_2^*-y_2^*}{2+t_2^2}\right) - \exp\left(\rho\frac{(t_2^2+1)y_2^*-x_2^*}{2+t_2^2}\right)}\right)\right].$$

■

6.2 Proof of Theorem 4.1

Proof. Proof of Theorem 4.1 It suffices to show that under our assumptions, for every realization of ϵ the following inequality holds

$$\mathbb{E}[u(\mathcal{X})|X,Y,Z] - \mathbb{E}[u(\mathcal{Y})|X,Y,Z] > \mathbb{E}[u(\mathcal{X})|X,Y,W] - \mathbb{E}[u(\mathcal{Y})|X,Y,W].$$

Conditional on X, Y, W, the posterior for $\mathcal{X}$ is

$$\begin{aligned}
&\Pr(\mathcal{X}|X,Y,W)\\
\propto &\exp\left(-\frac{\mathcal{X}'\Omega^{-1}\mathcal{X}}{2}\right)\exp\left(-\frac{\mathcal{Y}'\Omega^{-1}\mathcal{Y}}{2}\right)\\
&\exp\left(-\frac{\mathcal{W}'\Omega^{-1}\mathcal{W}}{2}\right)\exp\left(-\frac{(X-\mathcal{X})'T(X-\mathcal{X})}{2}\right)\times 1_{\{X-\mathcal{X}=Y-\mathcal{Y}=W-\mathcal{W}\}}\\
\propto &\exp\left(-\frac{1}{2}\left[\mathcal{X}'\left(3\Omega^{-1}+T\right)\mathcal{X} - 2\left(TX-\Omega^{-1}(Y+W-2X)\right)'\mathcal{X}\right]\right)\\
\propto &\exp\left(-\frac{1}{2}\left(\mathcal{X}-\left(3\Omega^{-1}+T\right)^{-1}\left(TX-\Omega^{-1}(Y+W-2X)\right)\right)'\right.\\
&\left.\left(3\Omega^{-1}+T\right)\left(\mathcal{X}-\left(3\Omega^{-1}+T\right)^{-1}\left(TX-\Omega^{-1}(Y+W-2X)\right)\right)\right)
\end{aligned}$$

So we denote the above posterior distribution of $\mathcal{X}|X,Y,W$ by $\mathcal{N}\left(\mu(\mathbf{x}^*;\mathbf{y}^*,\mathbf{w}^*,\epsilon),\hat{\Omega}\right)$, where

$$\begin{aligned}
\mu(\mathbf{x}^*;\mathbf{y}^*,\mathbf{w}^*,\epsilon) &:= \left(3\Omega^{-1}+T\right)^{-1}\left(T\mathbf{x}^*+T\epsilon-\Omega^{-1}(\mathbf{y}^*+\mathbf{w}^*-2\mathbf{x}^*)\right)\\
&= \left(3\Omega^{-1}+T\right)^{-1}\left(TX-\Omega^{-1}(Y+W-2X)\right),\\
\text{and}\quad \hat{\Omega} &:= \left(3\Omega^{-1}+T\right)^{-1}.
\end{aligned}$$

Denote the density of $\mathcal{X}|X,Y,W \sim \mathcal{N}(\mu,\hat{\Omega})$ by $\phi(\mathcal{X}-\mu,\hat{\Omega})$. The posterior expected utility is therefore

$$\begin{aligned}\mathbb{E}[u(\mathcal{X})|X,Y,W] &= \int_{\mathbb{R}^2} u(\mathcal{X}) \times \phi\left(\mathcal{X}-\mu(\mathbf{x}^*;\mathbf{y}^*,\mathbf{w}^*,\epsilon),\hat{\Omega}\right) \mathrm{d}\mathcal{X} \\ &= \int_{\mathbb{R}^2} u(\mathbf{s}+\mu(\mathbf{x}^*;\mathbf{y}^*,\mathbf{w}^*,\epsilon)) \times \phi\left(\mathbf{s},\hat{\Omega}\right) \mathrm{d}\mathbf{s}.\end{aligned}$$

Similarly,

$$\mathcal{Y}|X,Y,W \sim \mathcal{N}\left(\mu(\mathbf{y}^*;\mathbf{x}^*,\mathbf{w}^*,\epsilon),\hat{\Omega}\right).$$

Because

$$\begin{aligned}\mu(\mathbf{x}^*;\mathbf{y}^*,\mathbf{w}^*,\epsilon) :=& \hat{\Omega}\left(T\mathbf{x}^* + T\epsilon - \Omega^{-1}(\mathbf{y}^*+\mathbf{w}^*-2\mathbf{x}^*)\right) \\ =& \mu(\mathbf{y}^*;\mathbf{x}^*,\mathbf{w}^*,\epsilon) - (\mathbf{y}^*-\mathbf{x}^*),\end{aligned}$$

we have

$$\mathbb{E}[u(\mathcal{Y})|X,Y,W] = \int_{\mathbb{R}^2} u(\mathbf{s}+(\mathbf{y}^*-\mathbf{x}^*)+\mu(\mathbf{x}^*;\mathbf{y}^*,\mathbf{w}^*,\epsilon)) \times \phi\left(\mathbf{s},\hat{\Omega}\right) \mathrm{d}\mathbf{s}.$$

Recall that $\mu(\mathbf{x}^*;\mathbf{y}^*,\mathbf{w}^*,\epsilon) = \hat{\Omega}T\mathbf{x}^* + \hat{\Omega}T\epsilon - \hat{\Omega}\Omega^{-1}\mathbf{y}^* - \hat{\Omega}\Omega^{-1}\mathbf{w}^* + 2\hat{\Omega}\Omega^{-1}\mathbf{x}^*$. Substitute in $\mathbf{z}^* := \mathbf{w}^* + \Delta$ for $\mathbf{w}^*$ we have

$$\begin{aligned}&\mathbb{E}[u(\mathcal{X})|X,Y,Z] - \mathbb{E}[u(\mathcal{Y})|X,Y,Z] \\ =& \int_{\mathbb{R}^2} u(\mathbf{s}+\mu(\mathbf{x}^*;\mathbf{y}^*,\mathbf{z}^*,\epsilon)) \times \phi\left(\mathbf{s},\hat{\Omega}\right) \mathrm{d}\mathbf{s} - \int_{\mathbb{R}^2} u(\mathbf{s}+(\mathbf{y}^*-\mathbf{x}^*)+\mu(\mathbf{x}^*;\mathbf{y}^*,\mathbf{z}^*,\epsilon)) \times \phi\left(\mathbf{s},\hat{\Omega}\right) \mathrm{d}\mathbf{s} \\ =& \int_{\mathbb{R}^2} \left[u\left(\mathbf{s}+\mu(\mathbf{x}^*;\mathbf{y}^*,\mathbf{w}^*,\epsilon)-\hat{\Omega}\Omega^{-1}\Delta\right) - u\left(\mathbf{s}+(\mathbf{y}^*-\mathbf{x}^*)+\mu(\mathbf{x}^*;\mathbf{y}^*,\mathbf{w}^*,\epsilon)-\hat{\Omega}\Omega^{-1}\Delta\right)\right] \phi\left(\mathbf{s},\hat{\Omega}\right) \mathrm{d}\mathbf{s}.\end{aligned}$$

Since u is standard, and $y_1^* < x_1^*$, and $y_2^* > x_2^*$, if $-\hat{\Omega}\Omega^{-1}\Delta \in (-\infty,0)\times(0,\infty)$, i.e. the second quadrent, then

$$\begin{aligned}&u\left(\mathbf{s}+\mu(\mathbf{x}^*;\mathbf{y}^*,\mathbf{w}^*,\epsilon)-\hat{\Omega}\Omega^{-1}\Delta\right) - u\left(\mathbf{s}+(\mathbf{y}^*-\mathbf{x}^*)+\mu(\mathbf{x}^*;\mathbf{y}^*,\mathbf{w}^*,\epsilon)-\hat{\Omega}\Omega^{-1}\Delta\right) \\ >& u\left(\mathbf{s}+\mu(\mathbf{x}^*;\mathbf{y}^*,\mathbf{w}^*,\epsilon)\right) - u\left(\mathbf{s}+(\mathbf{y}^*-\mathbf{x}^*)+\mu(\mathbf{x}^*;\mathbf{y}^*,\mathbf{w}^*,\epsilon)\right)\end{aligned}$$

for all $\mathbf{s}$ and ϵ. When we integrate out $\mathbf{s}$, we have $\mathbb{E}[u(\mathcal{X})|X,Y,Z] - \mathbb{E}[u(\mathcal{Y})|X,Y,Z] > \mathbb{E}[u(\mathcal{X})|X,Y,W] - \mathbb{E}[u(\mathcal{X})|X,Y,W]$ for every realization of ϵ.

Therefore, one sufficient condition is that $-\hat{\Omega}\Omega^{-1}\Delta \in (-\infty, 0) \times (0, \infty)$. If this condition holds, we have $-\hat{\Omega}\Omega^{-1}\Delta = \mathbf{w}$ for some $w_1 < 0$, and $w_2 > 0$. In order to show the decoy choice patter, we just need to show there exists Δ with $\Delta_1 > \Delta_2$ such that this condition holds.

Recall that we had normalized Ω so that for some $r \in (-1, 1)$,

$$\Omega = \begin{bmatrix} 1 & r \\ r & 1 \end{bmatrix}$$

and the noise has variance

$$T^{-1} = \begin{bmatrix} 1/t_1^2 & R/(t_1 t_2) \\ R/(t_1 t_2) & 1/t_2^2 \end{bmatrix}.$$

We can calculate

$$\Omega^{-1} = \begin{bmatrix} 1/(1-r^2) & -r/(1-r^2) \\ -r/(1-r^2) & 1/(1-r^2) \end{bmatrix} \text{ and } T = \begin{bmatrix} t_1 & 0 \\ 0 & t_2 \end{bmatrix} \begin{bmatrix} 1/(1-R^2) & -R/(1-R^2) \\ -R/(1-R^2) & 1/(1-R^2) \end{bmatrix} \begin{bmatrix} t_1 & 0 \\ 0 & t_2 \end{bmatrix};$$

It follows that

$$\begin{aligned}
\Delta =& -\Omega\hat{\Omega}^{-1}\mathbf{w} = -\Omega(3\Omega^{-1} + T)\mathbf{w} = -(3I + \Omega T)\mathbf{w} \\
=& -\left(\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 1 & r \\ r & 1 \end{bmatrix} \begin{bmatrix} t_1 & 0 \\ 0 & t_2 \end{bmatrix} \begin{bmatrix} 1/(1-R^2) & -R/(1-R^2) \\ -R/(1-R^2) & 1/(1-R^2) \end{bmatrix} \begin{bmatrix} t_1 & 0 \\ 0 & t_2 \end{bmatrix} \right) \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \\
=& -\begin{bmatrix} 3 + \frac{t_1^2 - t_1 t_2 r R}{1-R^2} & \frac{t_2^2 r - t_1 t_2 R}{1-R^2} \\ \frac{t_1^2 r - t_1 t_2 R}{1-R^2} & 3 + \frac{t_2^2 - t_1 t_2 r R}{1-R^2} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}
\end{aligned}$$

Since $w_1 < 0$, and $w_2 > 0$, the sufficient condition to holds when Δ is some positive linear combinations of the two vectors

$$\{ \begin{bmatrix} 3(1-R^2) + t_1^2 - t_1 t_2 r R \\ t_1^2 r - t_1 t_2 R \end{bmatrix}, - \begin{bmatrix} t_2^2 r - t_1 t_2 R \\ 3(1-R^2) + t_2^2 - t_1 t_2 r R \end{bmatrix} \}.$$

And the decoy choice pattern holds when there exists such a Δ with $\Delta_1 > \Delta_2$. In other words, the decoy choice pattern holds if

$$\begin{cases} 3(1-R^2)+t_1^2-t_1t_2rR > t_1^2r - t_1t_2R \\ \text{or} \\ -(t_2^2r-t_1t_2R) > -(3(1-R^2)+t_2^2-t_1t_2rR), \end{cases} \Leftrightarrow \begin{cases} 3(1-R^2) > (r-1)(t_1^2+t_1t_2R) \\ \text{or} \\ 3(1-R^2) > (r-1)(t_2^2+t_1t_2R). \end{cases}$$

Because $r, R \in (-1,1)$ and $t_1, t_2 > 0$, it is impossible for both $t_1 + t_2R < 0$ and $t_2 + t_1R < 0$ to hold simultaneously. Therefore the decoy choice pattern holds. ■

6.3 Proof of Theorem 4.2

Proof. Proof of Theorem 4.2 As before, we start with the Bayesian posterior

$$\begin{aligned} \Pr(\mathcal{X}|X,Z) &\propto \exp\left(-\frac{\mathcal{X}'\Omega^{-1}\mathcal{X}}{2}\right)\exp\left(-\frac{\mathcal{Z}'\Omega^{-1}\mathcal{Z}}{2}\right)\exp\left(-\frac{(X-\mathcal{X})'T(X-\mathcal{X})}{2}\right)\times 1_{\{X-\mathcal{X}=Z-\mathcal{Z}\}} \\ &= \exp\left(-\frac{1}{2}\left[\mathcal{X}'\left(2\Omega^{-1}+T\right)\mathcal{X} - 2\left(TX-\Omega^{-1}(Z-X)\right)'\mathcal{X}\ldots\right]\right) \\ &\propto \exp\left(-\frac{1}{2}\left(\mathcal{X}-\left(2\Omega^{-1}+T\right)^{-1}\left(TX-\Omega^{-1}(Z-X)\right)\right)'\left(2\Omega^{-1}+T\right)(\mathcal{X}-\ldots)\right) \end{aligned}$$

Therefore, the posterior inference for $\mathbf{x}^*$ is

$$\begin{aligned} \mathcal{X}|X,Z \sim &\mathcal{N}\left(\left(2\Omega^{-1}+T\right)^{-1}\left(TX-\Omega^{-1}(Z-X)\right), \left(2\Omega^{-1}+T\right)^{-1}\right) \\ =&\mathcal{N}\left(\left(2\Omega^{-1}+T\right)^{-1}\left(T\mathbf{x}^*+T\epsilon-\Omega^{-1}(\mathbf{z}^*-\mathbf{x}^*)\right), \left(2\Omega^{-1}+T\right)^{-1}\right) \\ :=&\mathcal{N}\left(\mu(\mathbf{x}^*;\mathbf{z}^*,\epsilon),\hat{\Omega}\right) \end{aligned}$$

Similarly, $\mathcal{Z}|X,Z \sim \mathcal{N}\left(\mu(\mathbf{z}^*;\mathbf{x}^*,\epsilon),\hat{\Omega}\right)$. Observe that they have the same variance, and that

$$\begin{aligned} &\mu(\mathbf{z}^*;\mathbf{x}^*,\epsilon) - \mu(\mathbf{x}^*;\mathbf{z}^*,\epsilon) \\ =&\left(2\Omega^{-1}+T\right)^{-1}\left(T\mathbf{z}^*+T\epsilon-\Omega^{-1}(\mathbf{x}^*-\mathbf{z}^*)\right) - \left(2\Omega^{-1}+T\right)^{-1}\left(T\mathbf{x}^*+T\epsilon-\Omega^{-1}(\mathbf{z}^*-\mathbf{x}^*)\right) \\ =&\mathbf{z}^*-\mathbf{x}^* > 0. \end{aligned}$$

Therefore the posterior inference distribution for $\mathbf{z}^*$ is that for $\mathbf{x}^*$ translated by the vector $\mathbf{z}^* - \mathbf{x}^* > 0$. Since standard preference is increasing in both attributes, we have for every $\epsilon \in \mathbb{R}^2$

$$\mathbb{E}[u(\mathcal{X})|X,Z] < \mathbb{E}[u(\mathcal{Z})|X,Z].$$

Hence the rational agent chooses $\mathbf{z}$ over $\mathbf{x}$ with probability 1. ■

6.4 Proof of Proposition 4.1

Proof. Proof of Proposition 4.1

Denote $\tilde{X} = (X^1, \ldots, X^n)$, and $\tilde{\mathcal{X}} = (\mathcal{X}^1, \ldots, \mathcal{X}^n)$. Let's use $\phi_k(., A)$ to denote the density of the k dimensional normal distribution $\mathcal{N}(0, A)$. Under Σ_a, the posterior distribution for X^1 is

$$\Pr(\mathcal{X}^1 | X^1, \ldots, X^n) = \frac{\int \prod_{j=1}^n \phi_2(\mathcal{X}^j, \Omega) \times \phi_{2n}\left((\tilde{X} - \tilde{\mathcal{X}}), \Sigma_a\right) \mathrm{d}\mathcal{X}^2 \times \cdots \times \mathrm{d}\mathcal{X}^n}{\int \prod_{j=1}^n \phi_2(\mathcal{X}^j, \Omega) \times \phi_{2n}\left((\tilde{X} - \tilde{\mathcal{X}}), \Sigma_a\right) \mathrm{d}\tilde{\mathcal{X}}}$$

It is easy to see this is a normal distribution, establishing the claim of normality of the posterior. Let v be any bounded continuous function. Then

$$\mathbb{E}_{\mu_a}[v(\mathcal{X}^1)] = \frac{\int v(\mathcal{X}^1) \prod_{j=1}^n \phi_2(\mathcal{X}^j, \Omega) \times \phi_{2n}\left((\tilde{X} - \tilde{\mathcal{X}}), \Sigma_a\right) \mathrm{d}\tilde{\mathcal{X}}}{\int \prod_{j=1}^n \phi_2(\mathcal{X}^j, \Omega) \times \phi_{2n}\left((\tilde{X} - \tilde{\mathcal{X}}), \Sigma_a\right) \mathrm{d}\tilde{\mathcal{X}}}$$

As $\Sigma_a \to \Sigma$, the measure $\phi_{2n}\left((\tilde{\mathcal{X}} - \tilde{X}), \Sigma_a\right) \mathrm{d}\tilde{\mathcal{X}}$ converges weakly to

$$\phi_2\left((\mathcal{X}^1 - X^1), T^{-1}\right) \times \prod_{i=2}^n 1_{\mathcal{X}^i - X^i = \mathcal{X}^1 - X^1} \mathrm{d}\tilde{\mathcal{X}},$$

the measure where $\epsilon^i = \epsilon^j$ for all i, j. Therefore, by weak convergence we have

$$\begin{aligned}\mathbb{E}_{\mu_a}[v(\mathcal{X}^1)] \to & \frac{\int v(\mathcal{X}^1) \prod_{j=1}^n \phi_2(\mathcal{X}^j, \Omega) \times \phi_2\left((\mathcal{X}^1 - X^1), T^{-1}\right) \times \prod_{i=2}^n 1_{\mathcal{X}^i - X^i = \mathcal{X}^1 - X^1} \mathrm{d}\tilde{\mathcal{X}}}{\int \prod_{j=1}^n \phi_2(\mathcal{X}^j, \Omega) \times \phi_2\left((\mathcal{X}^1 - X^1), T^{-1}\right) \times \prod_{i=2}^n 1_{\mathcal{X}^i - X^i = \mathcal{X}^1 - X^1} \mathrm{d}\tilde{\mathcal{X}}} \\ = & \mathbb{E}_{\mu}[v(\mathcal{X}^1)]\end{aligned}$$

as $\Sigma_a \to \Sigma$. This completes the proof. ■

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