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A CHARACTERIZATION OF COMBINATORIAL DEMAND

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Abstract

We prove that combinatorial demand functions are characterized by two properties: continuity and the law of demand.

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1 Introduction

We prove that combinatorial demand functions are characterized by two properties: continuity and the law of demand. Suppose given a finite collection of items. We are interested in the demand for packages, or bundles of items. For each vector of item prices, we are given a collection of demanded packages, and we want to know if there exists a valuation function for packages such that the demanded packages are optimal. Utility is quasilinear in money. So the valuation has to be such that, for each price vector, the demanded packages maximize the value of the packages when one subtracts the sum of the prices for the items in the package.

The two properties that characterize optimal combinatorial demand are upper hemicontinuity and the law of demand. The continuity property is technical, but familiar. The law of demand captures the economic nature of our problem. Demand for a single item must "slope down," meaning that higher prices correspond to smaller demands. For combinatorial demand, the law of demand says that the change in demanded items should have a negative value, when evaluated by the change in prices. The law of demand is an aggregate, or average, version of the downward sloping demand property, and it has a long history in economics (see for example Samuelson (1948)).

While very natural, our result appears to be new. A long literature investigates the combinatorial demands that satisfy specific behavioral properties, such as gross substitutes: Murota (2003) provides a description of the literature. Our result is more basic, in that we seek to understand optimal demand behavior alone, without additional behavioral properties. Brown and Calsamiglia (2007) investigate a similar question to ours in the context of bundles of infinitely divisible goods, but their result does not extend to combinatorial demand. Finally, we should mention the paper by Baldwin and Klemperer (2012) which introduces a new framework for the study of discrete demand, and investigates many of its properties.

Our main result follows along the lines of Rochet's approach to revealed preference theory (see Rochet (1987)). The property of cyclic monotonicity is crucial to obtain a rationalizing valuation. We use the results of Lavi et al. (2003) or Saks and Yu (2005) (in a version due to Ashlagi et al. (2010)) to establish that the law of demand is sufficient for cyclic monotonicity. The main issue in adapting these various results to our problem is that cyclic monotonicity is not enough to obtain a strict rationalization: the difficulty is that one may add optimal packages when constructing the rationalization from cyclic monotonicity. The crucial idea to overcome this difficulty is contained in Lemma 4 in the proof.

2 Results

2.1 Notation:

Let X be a finite set. Let S be the set of all nonempty subsets of 2^X (so the empty set is not in S, but $\{\varnothing\}$ is).

We identify a set $A \subseteq X$ with its indicator function $1_A \in \mathbf{R}^X$. The inner product of a vector $p \in \mathbf{R}^X$ and 1_A is denoted by $\langle p, A \rangle = \sum_{x \in A} p_x$.

2.2 Results

A *demand function* is a function $D : \mathbf{R}^X_{++} \to S$ with the property that there is $\bar{p} \in \mathbf{R}^X_{++}$ such that $D(p) = \{\varnothing\}$ for all $p \geq \bar{p}$.

The relevant properties for a demand function are three: A demand function D

- is *quasilinear rationalizable* if there exists $v : 2^X \to \mathbf{R}$ such that

$$D(p) = \operatorname{argmax}\{v(A) - \langle p, A \rangle : A \subseteq X\};$$

- satisfies the *law of demand* if for all $p, q \in \mathbf{R}^X_{++}$, and all $A \in D(p)$ and $B \in D(q)$,

$$\langle p - q, A - B \rangle \leq 0;$$

- is *upper hemicontinuous* if, for all $p \in \mathbf{R}^X_{++}$, there is a neighborhood V of p such that $D(q) \subseteq D(p)$ when $q \in V$.

Theorem 1. *A demand function is quasilinear rationalizable iff it is upper hemicontinuous and satisfies the law of demand.*

A stronger condition places more restrictions on the rationalization. We say a function $g : \mathbf{R}^X_+ \to \mathbf{R}$ is *monotone* if for all $x, y \in \mathbf{R}^X_+$, $x \leq y$ (coordinatewise) implies $g(x) \leq g(y)$. D is *monotone, concave, quasilinear rationalizable (MCQ-rationalizable)* if there exists a monotone, concave $g : \mathbf{R}^X_+ \to \mathbf{R}$ such that $v(A) = g(1_A)$, and

$$D(p) = \operatorname{argmax}\{v(A) - \langle p, A \rangle : A \subseteq X\}.$$

An easy corollary, demonstrated by our proof is the following:

Corollary 2. *If a demand function is quasilinear rationalizable, then it is MCQ-rationalizable.*

The corollary demonstrates that there is no additional empirical content delivered by the hypotheses of concavity and monotonicity.

3 Proof of Theorem 1

Lemma 3. *If D is quasilinear rationalizable then it is upper hemicontinuous and satisfies the law of demand*

Proof. Let v rationalize B. Let $u(p) = \max\{v(A) - \langle p, A \rangle : A \subseteq X\}$.

First we show that D is upper hemicontinuous. Since X is finite, there is $\varepsilon > 0$ such that $u(p) - (v(B') - \langle p, B' \rangle) > \varepsilon$ for all $B' \notin D(p)$. Let V be a ball with center p and radius small enough that for all $q \in V$, and all $B' \notin D(p)$, $u(q) - (v(B') - \langle q, B' \rangle) > \varepsilon$. Then $D(q) \subseteq D(p)$ for all $q \in V$.

Second we show the law of demand. Let $A \in D(p)$ and $A' \in D(p')$. Then $v(A) - \langle p, A \rangle \geq v(A') - \langle p, A' \rangle$ and $v(A') - \langle p', A' \rangle \geq v(A) - \langle p', A \rangle$. Adding these two inequalities and rearranging yields $\langle p - p', A - A' \rangle \leq 0$. $\square$

Lemma 3 establishes the necessity direction in the theorem. We now turn to sufficiency. The upper hemicontinuity of D implies the following property: A demand function D satisfies condition ♠ if for all p and $B \notin D(p)$ there is $A \in D(p)$ and p' such that $A \in D(p')$ and $\langle p', A - B\rangle > \langle p, A - B\rangle$.

Lemma 4. *If D is upper hemicontinuous, then it satisfies condition ♠.*

Proof. Let $p \in \mathbf{R}^X_{++}$ and $B \notin D(p)$. Let V be a neighborhood of p as in the definition of upper hemicontinuity. So $D(q) \subseteq D(p)$ for all $q \in V$.

Let $W = \cup_{A' \in D(p)}(A' \setminus B)$ and $E = \cup_{A' \in D(p)}(B \setminus A')$. Note that

$$(B \setminus A') \cup (A' \setminus B) \neq \varnothing \tag{1}$$

for each $A' \in D(p)$.

Let $\lambda, \lambda' > 0$. By definition of W, $\langle 1_W, B\rangle = 0$. So

$$\langle \lambda 1_W - \lambda' 1_E, A' - B\rangle = \lambda\langle 1_W, A'\rangle - \lambda'\langle 1_E, A'\rangle + \lambda'\langle 1_E, B\rangle.$$

Then for each $A' \in D(p)$, (1) implies that $\langle 1_W, A'\rangle \neq 0$ or $\langle 1_E, B\rangle \neq 0$, or both. Moreover, if $\langle 1_W, A'\rangle = 0$ then it must be true that $A' \subsetneq B$, which implies that

$$-\langle 1_E, A'\rangle + \langle 1_E, B\rangle = \langle 1_E, B - A'\rangle > 0. \tag{2}$$

Choose $\lambda, \lambda' > 0$ such that $\lambda\langle 1_W, A'\rangle - \lambda'\langle 1_E, A'\rangle + \lambda'\langle 1_E, B\rangle > 0$ for all $A' \in D(p)$. This is possible by equation (2), and for example by letting $\lambda/\lambda' > |X|$. Also choose λ, λ' such that $p' = p + (\lambda 1_W - \lambda' 1_E) \in V$.

Now, for any $A' \in D(p')$,

$$\langle p', A' - B\rangle - \langle p, A' - B\rangle = \langle(\lambda 1_W - \lambda' 1_E), A' - B\rangle > 0.$$

Moreover, $A' \in D(p)$, as $p' \in V$ and thus $D(p') \subseteq D(p)$. □

A demand function satisfies *cyclic monotonicity* if, for all n, and using summation mod n,

$$\sum_{i=1}^{n} \langle p_i, A_i - A_{i+1}\rangle \leq 0,$$

where $A_i \in D(p_i)$, for all sequences $\{p_i\}_{i=1}^n$.

The following argument is mostly standard, adapting the construction of Rockafellar (1966) and Rochet (1987). A potential novelty is the use of the upper hemicontinuity condition in guaranteeing strict inequalities when necessary.

Lemma 5. *If D satisfies cyclic monotonicity, and condition ♠, then it is quasilinear rationalizable.*

Proof. We have assumed that there is p^* for which $\{\varnothing\} = D(p^*)$. For any $A \subseteq X$, define:

$$v(A) = \inf \langle p_1, A - A_1 \rangle + \ldots + \langle p^*, A_k - \varnothing \rangle,$$

where the infimum is taken over all finite sequences $(p_i, A_i)_{i=1}^k$ for which $A_i \in D(p_i)$.

Observe that by cyclic monotonicity, $v(\varnothing) \in \mathbf{R}$; in fact $v(\varnothing) \geq 0$. By construction, v is nondecreasing, as it is the lower envelope of nondecreasing functions. Hence $v(A) \in \mathbf{R}$ for all A. Finally, observe that v is the lower envelope of restriction of affine functions on $\mathbf{R}^X$. Conclude that v is the restriction of a concave function on $\mathbf{R}^X$.

Finally, observe by construction that if $A \in D(p)$, then for any $B \subseteq X$,

$$v(B) \leq \langle p, B - A \rangle + v(A),$$

from which we obtain $v(A) - \langle p, A \rangle \geq v(B) - \langle p, B \rangle$.

Finally, to prove the lemma we need to show that if in addition $B \notin D(p)$ then $v(A) - \langle p, A \rangle > v(B) - \langle p, B \rangle$, or that $v(A) > \langle p, A - B \rangle + v(B)$. By condition ♠, there is $A' \in D(p)$ and p' such that $A' \in D(p')$ and $\langle p', A' - B \rangle > \langle p, A' - B \rangle$.

Suppose that $\{(A_i, p_i)\}$ is a sequence as in the definition of $v(A')$. Then

$$v(B) \leq \langle p', B - A' \rangle + \sum_{i=1}^n \langle p_i, A_i - A_{i+1} \rangle < \langle p, B - A' \rangle + \sum_{i=1}^n \langle p_i, A_i - A_{i+1} \rangle,$$

so $v(B) < \langle p, B - A' \rangle + v(A')$; and thus

$$v(A) - \langle p, A \rangle = v(A') - \langle p, A' \rangle > v(B) - \langle p', B \rangle.$$

$\square$

We finish the proof by using a recent result in the mechanism design literature, establishing conditions under which monotonicity (a condition that coincides with the law of demand) implies cyclic monotonicity: see Lavi et al. (2003) and Saks and Yu (2005).

Lemma 6. *A demand function satisfies cyclic monotonicity if it satisfies the law of demand.*

Proof. So let D satisfy the law of demand and suppose towards a contradiction that there is a sequence $(p_i, A_i)_{i=1}^n$, with $A_i \in D(p_i)$ and $\sum_{i=1}^n \langle p_i, A_i - A_{i+1}\rangle > 0$ (summation mod n), but no such sequence with $n \leq 2$. Choose such a sequence with minimal n, and observe that $n \geq 3$.

For any selection $f(p) \in D(p)$, if f is monotone then it is cyclically monotone, see *e.g.* Saks and Yu (2005) or Ashlagi et al. (2010), Theorem S.7 in the supplementary material.[1] Since D satisfies the law of demand, any selection f is monotone, and therefore cyclically monotone.

If $p_i \neq p_j$ for all $i, j = 1, \ldots, n$ with $i \neq j$, then we can choose a selection f of D with $f(p_i) = A_i$, violating cyclic monotonicity of f, and hence contradicting the fact that it is monotone.

We now claim that in fact it is the case that $p_i \neq p_j$ for all $i \neq j$.

Observe first that if $p_i = p_{i+1}$ for some i, then $\langle p_i A_i - A_{i+1}\rangle + \langle p_{i+1} A_{i+1} - A_{i+2}\rangle = \langle p_i A_i - A_{i+2}\rangle$, implying the existence of a shorter sequence, a contradiction.

Suppose then that $p_i = p_j$. By the preceding, we know that $j = i + 1$ is false, and $i = j + 1$ is false. Without loss, suppose that $i = 1$. Then $j \neq n$ and $j \neq 2$. Further, $\langle p_j, A_j - A_{j+1}\rangle = \langle p_j, A_j - A_1\rangle + \langle p_1, A_1 - A_{j+1}\rangle$, so

$$0 < \sum_{i=1}^n \langle p_i, A_i - A_{i+1}\rangle = \langle p_1, A_1 - A_2\rangle + \cdots + \langle p_j, A_j - A_1\rangle + \langle p_1, A_1 - A_{j+1}\rangle$$
$$+ \langle p_{j+1}, A_{j+1} - A_{j+2}\rangle + \cdots + \langle p_n, A_1 - A_n\rangle.$$

[1] Technically, the Ashlagi et al. (2010) result requires the output of f to be a probability measure. To modify the construction to fit our environment, simply let $y^* \notin X$, and consider the set $Y \subseteq \mathbf{R}^{X \cup \{y^*\}}$ given by $Y = \{(p, 0) : p \in \mathbf{R}^X_{++}\}$. Define the function $f^* : Y \to \Delta(X \cup \{y^*\})$ by $f^*(p,0)(x) = \frac{1_{x \in f(p)}}{|X|}$ and $f^*(p,0)(y^*) = 1 - \frac{|f(p)|}{|X|}$. Observe that $\langle (q,0), f^*(p,0)\rangle = \langle q, f(p)\rangle \frac{1}{|x|}$, and therefore monotonicity of f is equivalent to that of f^* and cyclic monotonicity of f is equivalent to that of f^*.

Consequently, either $\langle p_1, A_1 - A_2 \rangle + \cdots + \langle p_j, A_j - A_1 \rangle > 0$ or $\langle p_1, A_1 - A_{j+1} \rangle + \cdots + \langle p_{j+1}, A_{j+1} - A_{j+2} \rangle + \cdots + \langle p_n, A_1 - A_n \rangle > 0$. In either case, we have demonstrated the existence of a shorter cycle violating cyclic monotonicity, a contradiction.

$\square$

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