Supermodularity in Unweighted Graph Optimization II: Matroidal Term Rank Augmentation

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Abstract

Ryser's max term rank formula with graph theoretic terminology is equivalent to a characterization of degree sequences of simple bipartite graphs with matching number at least ℓ. In a previous paper [1] by the authors, a generalization was developed for the case when the degrees are constrained by upper and lower bounds. Here two other extensions of Ryser's theorem are discussed. The first one is a matroidal model, while the second one settles the augmentation version. In fact, the two directions shall be integrated into one single framework.

1 Introduction

Ryser [16] derived a formula for the maximum term rank of a $(0,1)$-matrix with specified row- and column-sums. In graph theoretic terms, his theorem is equivalent to a characterization for the existence of a degree-specified simple bipartite graph (bigraph for short) with matching number at least ℓ. Several natural extensions, like the min-cost and the subgraph version, turned out to be **NP**-hard, but in a previous paper [1], we could extend Ryser's theorem to the degree-constrained case when, instead of exact degree-specifications, lower and upper bounds are imposed on the degrees of the bigraph. An even more general problem was also solved when, in addition, lower and upper bounds were imposed on the number of edges. The main tool in [1] for proving these extensions was a general framework for covering an intersecting supermodular function by degree-constrained simple bipartite graphs.

In the present paper we consider two other extensions of Ryser's theorem: the augmentation and the matroidal version. In the first one, a given initial bigraph is to be augmented to get a simple degree-specified bigraph with matching number at least ℓ. In original matrix terms, this means that some of the entries of the $(0,1)$-matrix are specified to be 1. The solvability of this version is in sharp contrast with the **NP**-completeness of another variation when some entries of the matrix are specified to be 0. (This follows from the **NP**-completeness of the problem that seeks to decide whether an initial bigraph G_0 has a perfectly matchable degree-specified subgraph, see [11], [13], [14].)

In the matroidal extension of Ryser's theorem, there is a matroid on S and there is a matroid on T, and the goal is to find a degree-specified simple bigraph including a matching that covers bases in both matroids. These results will be consequences of a general framework including both the augmentation and the matroidal cases.

The starting point in deriving the main result is the supermodular arc-covering theorem by Frank and Jordán [9] (Theorem 1 below). Since [9] describes a polynomial algorithm, relying on the ellipsoid method, to compute the optima in question, our matroidal term rank augmentation problem also admits a polynomial algorithm. One of the most important applications in [9] is the directed node-connectivity augmentation problem. Végh and Benczúr [17] developed for this special case a pretty intricate but purely combinatorial algorithm (not relying on the ellipsoid method). Although not mentioned explicitly in [17], their algorithm can probably be extended to work on the supermodular arc covering theorem when the function in question is ST-crossing supermodular, but the details have not been worked out. (In the special case of node-connectivity augmentation, this general oracle was realized via network flow computations.)

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Therefore the algorithm of Végh and Benczúr seems to be adaptable to the term rank problem, too. In a forthcoming paper [3], we shall develop a much simpler algorithm along with a natural unification of the matroidal augmentation and the degree-constrained term rank problems.

1.1 Notions and notation

We use the notation of [1]. Here we briefly repeat the most important notions. For a family $\mathcal{T}$ of sets, let $\cup\mathcal{T}$ denote the union of the members of $\mathcal{T}$. For a subpartition $\mathcal{T} = \{T_1, \dots, T_q\}$, we always assume that its members T_i are non-empty but $\mathcal{T}$ is allowed to be empty (that is, $q = 0$).

An arc st **enters** or **covers** a set X if $s \notin X$, $t \in X$. A digraph **covers** X if it contains an arc covering X. Let S and T be two non-empty subsets of a ground-set V. By an ST**-arc** we mean an arc st with $s \in S$ and $t \in T$. Two sets X and Y are ST**-independent** if $X \cap Y \cap T = \emptyset$ or $S - (X \cup Y) = \emptyset$, that is, no ST-arc enters both sets. Two subsets X and Y are **comparable** if $X \subseteq Y$ or $Y \subseteq X$. Two non-comparable sets X and Y are T**-intersecting** if $X \cap Y \cap T \neq \emptyset$ and ST**-crossing** if $X \cap Y \cap T \neq \emptyset$ and $S - (X \cup Y) \neq \emptyset$. A set-function p is called **positively** T**-intersecting (**ST**-crossing) supermodular** if the supermodular inequality

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y)$$

holds for T-intersecting (resp. ST-crossing) subsets X and Y for which $p(X) > 0$ and $p(Y) > 0$. The function is **fully supermodular** if the supermodular inequality holds for every pair X and Y of subsets.

For a function $m : V \to \mathbf{R}$, the set-function $\widetilde{m}$ is defined by $\widetilde{m}(X) = \sum[m(v) : v \in X]$. A set-function p can analogously be extended to families $\mathcal{F}$ of sets by $\widetilde{p}(\mathcal{F}) = \sum[p(X) : X \in \mathcal{F}]$.

The following min-max theorem of Frank and Jordán [9] will be a basic tool in the proof of the main theorem.

Theorem 1 (Supermodular arc-covering, set-function version). *A positively ST-crossing supermodular set-function p for which $p(V') \leq 0$ holds when no ST-arc enters V' can be covered by γ (possibly parallel) ST-arcs if and only if $\widetilde{p}(\mathcal{I}) \leq \gamma$ holds for every ST-independent family $\mathcal{I}$ of subsets of V. There is an algorithm, which is polynomial in $|S| + |T|$ and in the maximum value of $p(X)$, to compute the minimum number of ST-arcs to cover p and an ST-independent family $\mathcal{I}$ of subsets maximizing $\widetilde{p}(\mathcal{I})$.*

Henceforth we assume that S and T are two disjoint non-empty sets and $V := S \cup T$. Let $G^* = (S, T; E^*)$ denote the complete bipartite graph on bipartition (S, T). Let $D^* = (S, T; A^*)$ be the digraph arising from G^* by orienting each of its edges from S to T, that is, A^* consists of all ST-arcs. More generally, for a bigraph $H = (S, T; F)$, let $\overrightarrow{H} = (S, T; \overrightarrow{F})$ denote the digraph arising from H by orienting each of its edges from S toward T.

Throughout we are given a simple bigraph $H_0 = (S, T; F_0)$ serving as an initial bigraph to be augmented. For $E_0 := E^* - F_0$, the bigraph $G_0 = (S, T; E_0)$ is called the **bipartite complement** of H_0, that is, F_0 and E_0 partition E^*. Note that a bigraph $G = (S, T; E)$ is a subgraph of G_0 precisely if the augmented bigraph $G^+ = (S, T; F_0 + E)$ is simple. For $X \subseteq S$ and $Y \subseteq T$, let $d_{G_0}(X, Y)$ denote the number of edges of G_0 connecting X and Y.

2 Matroidal covering and augmentation

Let p_T be a positively intersecting supermodular set-function on T. In [1], we studied the problem of finding a simple degree-specified bigraph $G = (S, T; E)$ covering p_T in the sense that

$$|\Gamma_G(Y)| \geq p_T(Y) \text{ for every subset } Y \subseteq T$$

where $\Gamma_G(Y)$ denotes the set of neighbours of Y. Here we consider a framework which is more general in two directions. First, for a given initial simple bigraph $H_0 = (S, T; F_0)$, we want to find a degree-specified bigraph G in such a way that $G^+ := G + H_0$ is simple and covers p_T. This kind of problems is often referred to as augmentation problems to be distinguished from the synthesis problems where F_0 is empty. If $p_T \equiv 0$, the augmentation problem is equivalent to finding a degree-specified subgraph of the bipartite complement of H_0.

Second, we extend the notion of covering to matroidal covering in the following sense. Let $M_S = (S, r_S)$ be a matroid on S with rank function r_S. A bigraph G is said to **M_S-cover** p_T if

$$r_S(\Gamma_G(Y)) \geq p_T(Y) \text{ for every subset } Y \subseteq T. \tag{1}$$

Clearly, when M_S is the free matroid, we are back at the original notion of covering by a bigraph.

2.1 Degree-specified matroidal augmentation

Let $m_V = (m_S, m_T)$ be a degree-specification. A bigraph $G = (S, T; E)$ is said to **fit** m_V if $d_G(v) = m_V(v)$ for every $v \in S \cup T$. Our main goal is to describe a characterization for the existence of a bigraph G fitting m_V so that $G + H_0$ is simple and M_S-covers p_T. The more general problem, when there are upper and lower bounds on V, will be discussed in [3]. This degree-constrained version was solved in [1] in the special case when H_0 has no edges and M_S is the ℓ-uniform matroid on S.

Our main result is as follows.

Theorem 2. *We are given a simple bigraph $H_0 = (S, T; F_0)$, a matroid $M_S = (S, r_S)$, a positively intersecting supermodular set-function p_T on T, and a degree-specification $m_V = (m_S, m_T)$ on $V := S \cup T$ for which $\widetilde{m}_S(S) = \widetilde{m}_T(T) = \gamma$. There is a bigraph $G = (S, T; E)$ fitting m_V for which $G^+ = G + H_0$ is simple and M_S-covers p_T if and only if*

$$\widetilde{m}_S(X) + \widetilde{m}_T(Y) - d_{G_0}(X, Y) + \textstyle\sum_{i=1}^{q}[p_T(T_i) - r_S(X \cup \Gamma_{H_0}(T_i))] \leq \gamma$$
$$\text{whenever } Y \subseteq T,\ X \subseteq S, \text{ and } \mathcal{T} = \{T_1, \dots, T_q\} \text{ is a subpartition of } T - Y, \tag{2}$$

where G_0 is the bipartite complement of H_0.

Proof. Proof. Necessity. Suppose that there is a requested bigraph $G = (S, T; E)$ and let $G^+ = (S, T; E \cup F_0)$. Note that the simplicity of G^+ is equivalent to the requirement that G is a subgraph of G_0. Let $X \subseteq S$ and $Y \subseteq T$ be subsets and let $\{T_1, \dots, T_q\}$ be a subpartition of $T - Y$. Let $W_i := \Gamma_G(T_i) - [X \cup \Gamma_{H_0}(T_i)] = \Gamma_{G^+}(T_i) - [X \cup \Gamma_{H_0}(T_i)]$. Then we have

$$p_T(T_i) \leq r_S(\Gamma_{G^+}(T_i)) \leq r_S(\Gamma_{H_0}(T_i) \cup X) + r_S(W_i) \leq$$

$$r_S(\Gamma_{H_0}(T_i) \cup X) + |W_i| \leq r_S(\Gamma_{H_0}(T_i) \cup X) + d_G(T_i, W_i)$$

from which $d_G(T_i, W_i) \geq p_T(T_i) - r_S(X \cup \Gamma_{H_0}(T_i))$. Therefore G has at least $\sum_{i=1}^{q}[p_T(T_i) - r_S(X \cup \Gamma_{H_0}(T_i))]$ edges connecting $T - Y$ and $S - X$, and G has at least $\widetilde{m}_S(X) + \widetilde{m}_T(Y) - d_{G_0}(X, Y)$ edges with end-nodes in X or in Y, from which the inequality in (2) follows.

Sufficiency. Let $\mathcal{H}_0 := \{V' \subseteq V : \text{no arc of } \overrightarrow{H_0} \text{ enters } V'\}$. Then $\mathcal{H}_0$ is closed under taking union and intersection. In the following definition of set-function p_0, we have $X \subseteq S$, $Y \subseteq T$, and $y \in T$.

$$p_0(V') := \begin{cases} \max\{p_T(y) - r_S(X),\ m_T(y) - |X| + d_{H_0}(y)\} & \text{if } V' = X + y \in \mathcal{H}_0, \\ p_T(Y) - r_S(X) & \text{if } V' = X \cup Y \in \mathcal{H}_0,\ |Y| \geq 2 \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

The definition of p_0 implies that $p_0(V')$ can be positive only if $V' \in \mathcal{H}_0$.

Lemma 3. *The set-function p_0 is positively T-intersecting supermodular.*

Proof. Proof. Let X_1, X_2 be subsets of S and let Y_1, Y_2 be subsets of T for which $Y_1 \cap Y_2 \neq \emptyset$. Suppose that $p_0(V_i) > 0$ for $V_i = X_i \cup Y_i$ $(i = 1, 2)$. Then each of the sets V_1, V_2, $V_1 \cap V_2$, and $V_1 \cup V_2$ belongs to $\mathcal{H}_0$. We distinguish three cases.

Case 1 $p_0(V_i) = p_T(Y_i) - r_S(X_i)$ for $i = 1, 2$. Then

$$p_0(V_1) + p_0(V_2) = [p_T(Y_1) - r_S(X_1)] + [p_T(Y_2) - r_S(X_2)] \leq$$

$$p_T(Y_1 \cap Y_2) - r_S(X_1 \cap X_2) + p_T(Y_1 \cup Y_2) - r_S(X_1 \cup X_2) \leq p_0(V_1 \cap V_2) + p_0(V_1 \cup V_2).$$

Case 2 $p_0(V_i) > p_T(Y_i) - r_S(X_i)$ for $i = 1, 2$. Then $Y_1 = Y_2 = \{y\}$ for some $y \in T$, and $p_0(V_i) = m_T(y) - |X_i| + d_{H_0}(y)$. We have

$$p_0(V_1) + p_0(V_2) = m_T(y) - |X_1| + d_{H_0}(y) + m_T(y) - |X_2| + d_{H_0}(y) =$$

$$m_T(y) - |X_1 \cap X_2| + d_{H_0}(y) + m_T(y) - |X_1 \cup X_2| + d_{H_0}(y) \le p_0(V_1 \cap V_2) + p_0(V_1 \cup V_2).$$

Case 3 $p_0(V_1) = p_T(Y_1) - r_S(X_1)$ and $p_0(V_2) > p_T(Y_2) - r_S(X_2)$. (The situation is analogous when the indices $i = 1, 2$ are interchanged.) Then $Y_2 = \{y\}$ for some $y \in T$ and $y \in Y_1$. Since

$$r_S(X_1 \cup X_2) - r_S(X_1) \le |(X_1 \cup X_2) - X_1| = |X_2| - |X_1 \cap X_2|,$$

we have $-r_S(X_1) - |X_2| \le -r_S(X_1 \cup X_2) - |X_1 \cap X_2|$ and hence

$$p_0(V_1) + p_0(V_2) = [p_T(Y_1) - r_S(X_1)] + [m_T(y) - |X_2| + d_{H_0}(y)] =$$

$$[p_T(Y_1 \cup Y_2) - r_S(X_1)] + [m_T(y) - |X_2| + d_{H_0}(y)] \le$$

$$p_T(Y_1 \cup Y_2) - r_S(X_1 \cup X_2) + m_T(y) - |X_1 \cap X_2| + d_{H_0}(y) \le p_0(V_1 \cup V_2) + p_0(V_1 \cap V_2),$$

as required. • □

Claim 4. *$m_S(s) \le d_{G_0}(s)$ for each $s \in S$.*

Proof. Proof. By applying (2) to $Y = T$, $X = \{s\}$, and $\mathcal{T} = \emptyset$, the claim follows. • □

For $s \in S$, let $V_s := \{v \in V - s : \ sv \notin F_0\}$. Note that $V_s \in \mathcal{H}_0$ for $s \in S$. Let a set-function p_1 on V be defined as follows.

$$p_1(V') := \begin{cases} m_S(s) & \text{if } V' = V_s \text{ for some } s \in S \\ p_0(V') & \text{otherwise.} \end{cases} \tag{4}$$

The definition of p_1 implies that $p_1(V')$ can be positive only if $V' \in \mathcal{H}_0$.

Claim 5. *$p_1(V_s) \ge p_0(V_s)$ holds for every $s \in S$.*

Proof. Proof. Consider first the case when $V_s \cap T = \{y\}$ for some $y \in T$. By applying (2) to $X = S - s$, $Y = \{y\}$, and $\mathcal{T} = \emptyset$, we get

$$m_T(y) - |S - s| + d_{H_0}(y) = m_T(y) - d_{G_0}(S - s, y) \le m_S(s).$$

By applying (2) to $X = S - s$, $Y = \emptyset$, and $\mathcal{T} = \{y\}$, we get $p_T(y) - r_S(S - s) \le m_S(s)$, from which

$$m_S(s) \ge \max\{p_T(y) - r_S(S - s),\ m_T(y) - |S - s| + d_{H_0}(y)\} = p_0(V_s).$$

Second, assume that $|V_s \cap T| \ge 2$. By applying (2) to $X = S - s$, $Y = \emptyset$, and $\mathcal{T} = \{V_s \cap T\}$, we get

$$p_0(V_s) = p_T(V_s \cap T) - r_S(S - s) \le m_S(s). \qquad \bullet$$

□

Claim 6. *The set-function p_1 is positively ST-crossing supermodular.*

Proof. Proof. It follows from Claim 5 that p_1 arises from p_0 by increasing its values on sets V_s ($s \in S$). Let $V' \subset V$ be a set which is ST-crossing with V_s (in particular, V' and V_s are not comparable). Then $S \not\subseteq V_s \cup V'$ and hence $V' \cap S \subseteq V_s \cap S$. Therefore $V' \cap T \not\subseteq V_s \cap T$, that is, there is an element $t \in (V' - V_s) \cap T$. Since st is an arc of $\overrightarrow{H_0}$ entering V', we conclude that $p_1(V') = 0$, implying that p_1 is indeed positively ST-crossing supermodular. • □

Let ν denote the maximum total p_1-value of ST-independent sets.

Lemma 7. *$\nu = \gamma$.*

Proof. Proof. Since the family $\mathcal{L} = \{V_s : s \in S\}$ is ST-independent, $\nu \geq \widetilde{p}_1(\mathcal{L}) = \widetilde{m}_S(S) = \gamma$. Suppose indirectly that $\nu > \gamma$ and let $\mathcal{I}$ be an ST-independent family for which $\widetilde{p}_1(\mathcal{I}) = \nu$. We can assume that $|\mathcal{I}|$ is minimal in which case $p_1(V') > 0$ for each $V' \in \mathcal{I}$.

Claim 8. *There are no two T-intersecting members V_1 and V_2 of $\mathcal{I}$ for which $p_1(V_i) = p_0(V_i)$ $(i = 1, 2)$.*

Proof. Proof. Suppose indirectly the existence of such T-intersecting members V_1 and V_2 of $\mathcal{I}$. Since $\mathcal{I}$ is ST-independent, we must have $S \subseteq V_1 \cup V_2$ and hence $p_0(V_1 \cup V_2) = 0$. Since p_0 is positively T-intersecting supermodular,

$$p_1(V_1) + p_1(V_2) = p_0(V_1) + p_0(V_2) \leq$$
$$p_0(V_1 \cap V_2) + p_0(V_1 \cup V_2) = p_0(V_1 \cap V_2) \leq p_1(V_1 \cap V_2).$$

Now $\mathcal{I}' = \mathcal{I} - \{V_1, V_2\} + \{V_1 \cap V_2\}$ is also ST-independent and $\widetilde{p}_1(\mathcal{I}') \geq \widetilde{p}_1(\mathcal{I})$, but we must have equality by the optimality of $\mathcal{I}$, contradicting the minimality of $|\mathcal{I}|$. • □

We say that a member $V' \in \mathcal{I}$ is of Type I if $V' = X_t + t$ for some $t \in T$ and $X_t \subseteq S$ and

$$p_1(X_t + t) = p_0(X_t + t) = m_T(t) - |X_t| + d_{H_0}(t) > p_T(t) - r_S(X_t).$$

Let $\mathcal{I}_1$ $(\subseteq \mathcal{I})$ denote the family of sets of Type I. Claim 8 implies that if $X_1 + t_1 \in \mathcal{I}_1$ and $X_2 + t_2 \in \mathcal{I}_1$ for which $X_1 + t_1 \neq X_2 + t_2$ $(X_i \subseteq S,\ t_i \in T)$, then $t_1 \neq t_2$. Let

$$Y := \{t \in T : \text{there is a member } X_t + t \in \mathcal{I}_1\}.$$

Note that $|Y| = |\mathcal{I}_1|$.

We say that a member $V' \in \mathcal{I}$ is of Type II if

$$p_1(V') = p_0(V') = p_T(V' \cap T) - r_S(V' \cap S).$$

Let $\mathcal{I}_2 = \{V_1, V_2, \cdots, V_q\}$ $(\subseteq \mathcal{I})$ denote the family of sets of Type II. Let

$$\mathcal{T} := \{T_1, \ldots, T_q\} \text{ where } T_i := V_i \cap T \text{ for } i = 1, \ldots, q.$$

Since $p_1(V_i) > 0$, the members of $\mathcal{T}$ are non-empty. Furthermore, Claim 8 implies that $\mathcal{T}$ is a subpartition of $T - Y$.

Let $\mathcal{I}_3 := \mathcal{I} - (\mathcal{I}_1 \cup \mathcal{I}_2)$. The members of $\mathcal{I}_3$ are called of Type III. Then each member V' of $\mathcal{I}_3$ is of form $V' = V_s$ for some $s \in S$ such that $m_S(s) = p_1(V') > p_0(V')$. Let

$$X := \{s \in S : V_s \in \mathcal{I}_3\}.$$

It follows from the definitions that $\mathcal{I}_1, \mathcal{I}_2$, and $\mathcal{I}_3$ form a partition of $\mathcal{I}$.

Claim 9. *Let $t \in Y$ and $X_t + t \in \mathcal{I}_1$. Then $X \subseteq X_t$.*

Proof. Proof. Suppose indirectly that there is an element $s \in X - X_t$. By the ST-independence of the sets $X_t + t$ and V_s, the element t cannot be in V_s. Therefore the arc st belongs to $\overrightarrow{F_0}$. Since st enters $X_t + t$, we have $p_1(X_t + t) = 0$, a contradiction. • □

Claim 10. $\sum[|X_t| - d_{H_0}(t) : t \in Y] \geq d_{G_0}(X, Y)$.

Proof. Proof. What we prove is that $|X_t| - d_{H_0}(t) \geq d_{G_0}(X, t)$ for $t \in Y$ and $X_t + t \in \mathcal{I}_1$. Since no arc of $\overrightarrow{H_0}$ enters $X_t + t$ and since $X \subseteq X_t$ by Claim 9, we have

$$|X_t| - d_{H_0}(t) = |X_t| - d_{H_0}(X_t, t) = d_{G_0}(X_t, t) \geq d_{G_0}(X, t),$$

as required. • □

Claim 11. *$X \cup \Gamma_{H_0}(T_i) \subseteq V_i \cap S$ holds for each $i = 1, \ldots, q$.*

Proof. Proof. As $V_i \in \mathcal{H}_0$, we have $\Gamma_{H_0}(T_i) \subseteq V_i \cap S$. If, indirectly, there is an $s \in X - V_i$, then the ST-independence of V_s and V_i implies that $V_s \cap V_i \cap T = \emptyset$. In this case, an element $t \in V_i \cap T$ cannot be in V_s implying that $st \in \overrightarrow{F_0}$. But then $p_1(V_i) = 0$, contradicting the property $p_1(V') > 0$ for each $V' \in \mathcal{I}$. • □

Recall that $\mathcal{T}$ is a subpartition of $T - Y$. This and the last two claims imply

$$\gamma < \nu = \widetilde{p}_1(\mathcal{I}) = \widetilde{p}_1(\mathcal{I}_1) + \widetilde{p}_1(\mathcal{I}_2) + \widetilde{p}_1(\mathcal{I}_3) =$$

$$\sum [m_T(t) - |X_t| + d_{H_0}(t) : X_t + t \in \mathcal{I}_1] + \sum_{i=1}^{q} [p_T(T_i) - r_S(V_i \cap S)] + \sum [m_S(s) : V_s \in \mathcal{I}_3] \leq$$

$$\sum [m_T(t) : X_t + t \in \mathcal{I}_1] - d_{G_0}(X, Y) + \sum_{i=1}^{q} [p_T(T_i) - r_S(X \cup \Gamma_{H_0}(T_i))] + \widetilde{m}_S(X) =$$

$$\widetilde{m}_T(Y) - d_{G_0}(X, Y) + \sum_{i=1}^{q} [p_T(T_i) - r_S(X \cup \Gamma_{H_0}(T_i))] + \widetilde{m}_S(X),$$

in a contradiction with (2), completing the proof of the lemma. • • □

Claim 12. *If $p_1(V')$ is positive, then $\overrightarrow{G_0}$ covers V'.*

Proof. Proof. As already observed after (4), $V' \in \mathcal{H}_0$. Assume to the contrary that $\overrightarrow{G_0}$ does not cover V'. As G_0 denotes the bipartite complement of H_0, this is only possible if $V' \cap T = \emptyset$ or $S \subseteq V'$.

If $V' = V_s$ for some $s \in S$, then $s \notin V'$, hence $V' \cap T = \emptyset$. This means that $st \in F_0$ for each $t \in T$. By applying (2) to $X = \{s\}$, $Y = T$ and $\mathcal{T} = \emptyset$, we get $p_1(V') = m_S(s) \leq 0$, a contradiction.

Therefore, we must have $p_1(V') = p_0(V')$. As p_0 was defined to be 0 for sets not intersecting T, we can assume that $S \subseteq V'$ holds. If $p_0(V') = p_T(V' \cap T) - r_S(V' \cap S)$, then (2), when applied to $Y = \emptyset$, $X = S$ and $\mathcal{T} = \{V' \cap T\}$, gives $p_0(V') = p_T(V' \cap T) - r_S(S) \leq 0$, a contradiction. Therefore $p_0(V')$ is defined by the first line of (3). Hence $V' = S + y$ for some $y \in T$ and $p_0(V') = m_T(y) - |S| + d_{H_0}(y)$. Now (2), when applied to $Y = \{y\}$, $X = S$ and $\mathcal{T} = \emptyset$, gives $p_0(V') = m_T(y) - |S| + d_{H_0}(y) = m_T(y) - d_{G_0}(S, y) \leq 0$, thus leading to a contradiction again. • □

By Claim 12, Theorem 1 can be applied to p_1. This means that there is a digraph $D = (V, A)$ on V with $\nu = \gamma$ ST-arcs that covers p_1, that is, $\varrho_D(V') \geq p_1(V')$ for every subset $V' \subseteq V$. Let $G = (S, T; E)$ denote the undirected bipartite graph underlying D.

Claim 13. *$d_G(s) = m_S(s)$ for every $s \in S$ and $d_G(t) = m_T(t)$ for every $t \in T$.*

Proof. Proof. Since $d_G(s) = \delta_D(s) \geq \varrho_D(V_s) \geq p_1(V_s) = m_S(s)$ for every $s \in S$, we have $\widetilde{m}_S(S) = |E| = \sum [d_G(s) : s \in S] \geq \widetilde{m}_S(S)$, from which $d_G(s) = m_S(s)$ follows for every $s \in S$.

Since $d_G(t) = \varrho_D(t) \geq \varrho_D(\Gamma_{H_0}(t) + t) \geq p_0(\Gamma_{H_0}(t) + t) \geq m_T(t)$ for every $t \in T$, we have $\widetilde{m}_T(T) = |E| = \sum [d_G(t) : t \in T] \geq \widetilde{m}_T(T)$, from which $d_G(t) = m_T(t)$ follows for every $t \in T$. • □

Claim 14. *The bigraph $G^+ = (S, T; E + F_0)$ is simple.*

Proof. Proof. The minimality of D implies that each arc of D enters a subset V' with $p_1(V') > 0$. Since $p_1(V')$ can be positive only if no arc of $\overrightarrow{H_0}$ enters V', we can conclude that no edge of G is parallel to an edge of H_0.

Suppose indirectly that there are two parallel edges e and e' of G connecting s and t for some $s \in S$ and $t \in T$. Let $X := \Gamma_{H_0}(t)$. Then $p_1(X + t) \geq m_T(t) = \varrho_D(t)$. For $V' = X + s + t$, we have $\varrho_D(t) - 2 \geq \varrho_D(V') \geq p_1(V') \geq p_1(X + t) - 1 \geq m_T(t) - 1 = \varrho_D(t) - 1$, a contradiction. • □

Claim 15. *$r_S(\Gamma_{G^+}(Y)) \geq p_T(Y)$ for every subset $Y \subseteq T$.*

Proof. Proof. Let $X := \Gamma_{G^+}(Y)$ and $V' := Y \cup X$. Then $0 = \varrho_D(V') \geq p_1(V') \geq p_T(Y) - r_S(X)$, from which the claim follows. • □

We conclude that G meets all the requirements of the theorem, and the proof is complete. • • • □

2.2 Variations

2.2.1 Degree-specification only on S

With the proof technique used above, one can derive the following variation where the degrees are specified only for the nodes in S. Namely, the definition of p_0 in (3) should be modified as follows.

$$p_0(V') := \begin{cases} p_T(Y) - r_S(X) & \text{if } V' = X \cup Y \in \mathcal{H}_0,\ X \subseteq S,\ Y \subseteq T \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Theorem 16. *We are given a simple bigraph $H_0 = (S, T; F_0)$, a matroid $M_S = (S, r_S)$, a positively intersecting supermodular function p_T on T, and a degree-specification m_S on S for which $\widetilde{m}_S(S) = \gamma$. There is a bigraph $G = (S, T; E)$ fitting m_S for which $G^+ = G + H_0$ is simple and M_S-covers p_T if and only if*

$$m_S(s) + d_{H_0}(s) \leq |T| \text{ for every } s \in S \tag{6}$$

and

$$\widetilde{m}_S(X) + \textstyle\sum_{i=1}^{q} [p_T(T_i) - r_S(X \cup \Gamma_{H_0}(T_i))] \leq \gamma$$
$$\text{whenever } X \subseteq S \text{ and } \mathcal{T} = \{T_1, \dots, T_q\} \text{ a subpartition of } T. \tag{7}$$

One reason why we do not go into the details is that the proof is quite similar to (and, in fact, slightly simpler than) the proof of Theorem 2. Another reason is that, in a forthcoming work [3], we solve a common generalization of Theorems 2 and 16 where, instead of degree-specifications, there are both upper and lower bounds for the degrees of all nodes in $S \cup T$.

2.2.2 Fully supermodular p_T

In the special case when $p_T \equiv 0$, it suffices to require the inequality in (2) only for the empty $\mathcal{T}$, in which case Theorem 2 reduces to the following classic result (which actually holds for non-simple bigraphs, too).

Theorem 17 (Ore [12])**.** *A simple bigraph $G_0 = (S, T; E_0)$ has a subgraph fitting a degree-specification (m_S, m_T) with $\widetilde{m}_S(S) = \widetilde{m}_T(T) = \gamma$ if and only if*

$$\widetilde{m}_S(X) + \widetilde{m}_T(Y) - d_{G_0}(X, Y) \leq \gamma \text{ whenever } X \subseteq S,\ Y \subseteq T. \tag{8}$$

The content of the next result is that the condition in Theorem 2 can also be simplified when p_T is fully supermodular.

Theorem 18. *We are given a simple bigraph $H_0 = (S, T; F_0)$, a matroid $M_S = (S, r_S)$, a fully supermodular function p_T on T, and a degree-specification $m_V = (m_S, m_T)$ for which $\widetilde{m}_S(S) = \widetilde{m}_T(T) = \gamma$. There is a bigraph $G = (S, T; E)$ fitting m_V for which $G^+ = G + H_0$ is simple and M_S-covers p_T if and only if (8) holds and*

$$\widetilde{m}_S(X) + \widetilde{m}_T(Y) - d_{G_0}(X, Y) + p_T(T_0) - r_S(X \cup \Gamma_{H_0}(T_0)) \leq \gamma$$
$$\text{whenever } Y \subseteq T,\ X \subseteq S,\ T_0 \subseteq T - Y, \tag{9}$$

where G_0 is the bipartite complement of H_0.

Proof. Proof. Conditions (8) and (9) correspond to the special cases of Condition (2) when $|\mathcal{T}| = 0$ and $|\mathcal{T}| = 1$, respectively. Therefore their necessity was proved earlier. To see sufficiency, by Theorem 2 it suffices to show that (2) holds in general. Suppose, indirectly, that there are X, Y, and $\mathcal{T}$ violating (2). Assume that $|\mathcal{T}|$ is minimal. Then (8) and (9) imply that $|\mathcal{T}| \geq 2$. Let T_1, T_2 be two members of $\mathcal{T}$. Since

$$p_T(T_1 \cup T_2) - r_S(X \cup \Gamma_{H_0}(T_1 \cup T_2)) \geq p_T(T_1) + p_T(T_2) - r_S(X \cup \Gamma_{H_0}(T_1)) - r_S(X \cup \Gamma_{H_0}(T_2)),$$

the unchanged sets X, Y and the partition $\mathcal{T}'$ obtained from $\mathcal{T}$ by replacing T_1 and T_2 with the single set $T_1 \cup T_2$ also violate (2), contradicting the minimal choice of $\mathcal{T}$. • □

It is worth formulating Theorem 18 in the special case when H_0 has no edges.

Corollary 19. *We are given a matroid $M_S = (S, r_S)$, a fully supermodular function p_T on T, and a degree-specification $m_V = (m_S, m_T)$ for which $\widetilde{m}_S(S) = \widetilde{m}_T(T) = \gamma$. There is a simple bigraph $G = (S, T; E)$ fitting m_V and M_S-covering p_T if and only if*

$$\widetilde{m}_S(X) + \widetilde{m}_T(Y) - |X||Y| \leq \gamma \quad \textit{whenever } X \subseteq S,\ Y \subseteq T \tag{10}$$

and

$$\widetilde{m}_S(X) + \widetilde{m}_T(Y) - |X||Y| + p_T(T_0) - r_S(X) \leq \gamma \textit{ whenever } Y \subseteq T,\ X \subseteq S,\ T_0 \subseteq T - Y. \tag{11}$$

If, in addition, p_T is monotone non-decreasing, then T_0 in (11) can be chosen to be $T_0 = T - Y$, that is,

$$\widetilde{m}_S(X) + \widetilde{m}_T(Y) - |X||Y| + p_T(T - Y) - r_S(X) \leq \gamma \textit{ whenever } X \subseteq S,\ Y \subseteq T. \tag{12}$$

Proof. Proof. The first part is a special case of Theorem 18. When p_T, in addition, is monotone non-decreasing in the second part, we can choose T_0 in (11) as large as possible, that is, $T_0 = T - Y$. • □

3 Matroidal max term rank

Let $\mathcal{G}(m_S, m_T)$ denote the set of simple bigraphs $G = (S, T; E)$ fitting a degree-specification (m_S, m_T) with $\widetilde{m}_S(S) = \widetilde{m}_T(T) = \gamma$. It follows from Theorem 17 that $\mathcal{G}(m_S, m_T)$ is non-empty if and only if (10) holds. In [1] (Theorem 26), we reformulated Ryser's classic max term rank formula in graph theoretic language.

Theorem 20 (Ryser)**.** *Let $\ell \leq |T|$ be an integer. Suppose that $\mathcal{G}(m_S, m_T)$ is non-empty. Then $\mathcal{G}(m_S, m_T)$ has a member G with matching number $\nu(G) \geq \ell$ if and only if*

$$\widetilde{m}_S(X) + \widetilde{m}_T(Y) - |X||Y| + (\ell - |X| - |Y|) \leq \gamma \quad \textit{whenever } X \subseteq S,\ Y \subseteq T. \tag{13}$$

Moreover, (13) holds if the inequality in it is required only when X consists of the i largest values of m_S and Y consists of the j largest values of m_T $(i = 0, 1, \ldots, |S|,\ j = 0, 1, \ldots, |T|)$.

We keep using graph terminology, but the original expression (max term rank) of Ryser is retained. Our present goal is to extend Ryser's theorem in two directions. In the augmentation version an initial bigraph is to be augmented while in the matroidal form the matching is expected to cover a basis of a matroid M_S on S and a basis of matroid M_T on T. Actually, we shall integrate the two generalizations into one single framework.

In what follows, $M_S = (S, r_S)$ and $M_T = (T, r_T)$ will be matroids of rank ℓ. In [1], the complementary set-function p of a set-function b was defined by $p(Y) := b(S) - b(S - Y)$. Clearly, b is submodular if and only if p is supermodular, and p is monotone non-decreasing if and only if b is monotone non-decreasing. The complementary function p_T of the rank function r_T of M_T is called the **co-rank function** of M_T. It can easily be shown that $p_T(Y) = \min\{|Y \cap B| : B \text{ a basis of } M_T\}$.

The following extension of Edmonds' matroid intersection theorem [6] will be used. For notational convenience, the bipartite graph in the theorem is denoted by G^+.

Theorem 21 (Brualdi, [4])**.** *Let $G^+ = (S, T; E^+)$ be a bigraph with a matroid $M_S = (S, r_S)$ on S and with a matroid $M_T = (T, r_T)$ on T for which $r_S(S) = r_T(T) = \ell$. There is a matching of G^+ covering bases of M_S and M_T if and only if*

$$\begin{gathered} r_S(X') + r_T(Y') \geq \ell \\ \textit{whenever } X' \cup Y' \textit{ hits every edge of } G^+ \quad (X' \subseteq S,\ Y' \subseteq T). \end{gathered} \tag{14}$$

We need the following equivalent version of Theorem 21.

Lemma 22. *Let $G^+ = (S, T; E^+)$ be a bigraph. Let M_S be a matroid on S with rank function r_S and M_T a matroid on T with co-rank function p_T for which $r_S(S) = p_T(T) = \ell$. There is a matching of G^+ covering bases of M_S and M_T if and only if*

$$r_S(\Gamma_{G^+}(Y)) \geq p_T(Y) \textit{ for every } Y \subseteq T. \tag{15}$$

Proof. Proof. The necessity is straightforward. The sufficiency follows from Theorem 21 once we show that (14) holds. Since $X' \cup Y'$ hits every edge of G^+, for $Y := T - Y'$ we have $\Gamma_{G^+}(Y) \subseteq X'$. Therefore (15) implies that $r_S(X') \geq r_S(\Gamma_{G^+}(Y)) \geq p_T(Y) = r_T(T) - r_T(Y') = \ell - r_T(Y')$ and hence (14) indeed holds.

Theorem 23. *We are given a simple bigraph $H_0 = (S, T; F_0)$, a matroid $M_S = (S, r_S)$ and a matroid $M_T = (T, r_T)$ with $r_S(S) = r_T(T) = \ell$, and a degree-specification $m_V = (m_S, m_T)$ on $V := S \cup T$ for which $\widetilde{m}_S(S) = \widetilde{m}_T(T) = \gamma$. There is a bigraph $G = (S, T; E)$ fitting m_V for which $G^+ = G + H_0$ is simple and includes a matching covering a basis of M_S and a basis of M_T if and only if (8) holds and*

$$\widetilde{m}_S(X) + \widetilde{m}_T(Y) - d_{G_0}(X, Y) + \ell - r_S(X') - r_T(Y') \leq \gamma$$
$$\text{whenever } X \subseteq X' \subseteq S,\ Y \subseteq Y' \subseteq T, \text{ and } X' \cup Y' \text{ hits all the edges of } H_0, \tag{16}$$

where G_0 is the bipartite complement of H_0.

Proof. Proof. Necessity. Suppose that the requested bigraph G and its ℓ-element matching M exist. The number of edges of G with at least one end-node in $X \cup Y$ is at least $\widetilde{m}_S(X) + \widetilde{m}_T(Y) - d_{G_0}(X, Y)$. The number of edges in M with at least one end-node in $X' \cup Y'$ is at most $r_S(X') + r_T(Y')$. Therefore M has at least $\ell - r_S(X') - r_T(Y')$ elements connecting $S - X'$ and $T - Y'$. But these elements must be in E since $X' \cup Y'$ hits all edges of H_0. Therefore the total number of edges of G is at least $\widetilde{m}_S(X) + \widetilde{m}_T(Y) - d_{G_0}(X, Y) + \ell - r_S(X') - r_T(Y')$, and (16) follows.

Sufficiency. Let p_T denote the co-rank function of M_T, that is, $p_T(Z) = \ell - r_T(T - Z)$ for $Z \subseteq T$. Note that p_T is fully supermodular.

Claim 24. *Condition (9) is satisfied.*

Proof. Proof. For the present p_T, Condition (9) requires

$$\widetilde{m}_S(X) + \widetilde{m}_T(Y) - d_{G_0}(X, Y) + \ell - r_T(T - T_0) - r_S(X \cup \Gamma_{H_0}(T_0)) \leq \gamma. \tag{17}$$

No sets $X \subseteq S$, $Y \subseteq T$, and $T_0 \subseteq T - Y$ can violate this inequality since then, by letting $Y' := T - T_0$ and $X' := X \cup \Gamma_{H_0}(T_0)$, the quadruple $(X,\ Y,\ X',\ Y')$ would violate (16). • □

By Theorem 18, there is a bigraph G fitting m_V for which $G^+ = G + H_0$ is simple and M_S-covers p_T. The latter property, by definition, means that (15) holds, and therefore Lemma 22 implies that G^+ has a requested matching.• • □

When $m_V \equiv 0$ and $\gamma = 0$, it suffices to require (16) only for $X = Y = \emptyset$ in which case it transforms to

$$\ell - r_S(X') - r_T(Y') \leq 0 \text{ whenever } X' \subseteq S,\ Y' \subseteq T, \text{ and } X' \cup Y' \text{ hits all the edges of } H_0, \tag{18}$$

which is the same as (14). In other words, Theorem 23 may be considered as a straight generalization of Brualdi's theorem.

The content of the next corollary is that in the special case of Theorem 8 when $F_0 = \emptyset$ it suffices to require (16) only in a simplified form.

Corollary 25. *Let S and T be two disjoint sets and (m_S, m_T) a degree-specification on $S \cup T$ for which $\widetilde{m}_S(S) = \widetilde{m}_T(T) = \gamma$ and $\mathcal{G}(m_S, m_T)$ is non-empty, that is, (10) holds. Let $M_S = (S, r_S)$ and $M_T = (T, r_T)$ be matroids for which $r_S(S) = r_T(T) = \ell$. There is a simple bigraph $G = (S, T; E)$ fitting (m_S, m_T) that includes a matching covering bases of M_S and M_T if and only if*

$$\widetilde{m}_S(X) + \widetilde{m}_T(Y) - |X||Y| + \ell - r_S(X) - r_T(Y) \leq \gamma \tag{19}$$

holds for every $X \subseteq S$ and $Y \subseteq T$.

Proof. Proof. Consider Theorem 23 in the special case when $F_0 = \emptyset$. Then the bipartite complement G_0 of H_0 is a complete bigraph and hence $d_{G_0}(X, Y) = |X||Y|$. Therefore Condition (10) requested in the corollary is the same as Condition (8) requested in Theorem 23. Furthermore (19) is the special case of (16) when $X' = X$ and $Y' = Y$, and hence (19) is necessary.

We claim, conversely, that (19) implies (16). Indeed, if $X \subseteq X'$ and $Y \subseteq Y'$ violate the inequality in (16), then the monotonicity of matroid rank functions imply that $\gamma < \widetilde{m}_S(X) + \widetilde{m}_T(Y) - |X||Y| + \ell - r_S(X') - r_T(Y') \leq \widetilde{m}_S(X) + \widetilde{m}_T(Y) - |X||Y| + \ell - r_S(X) - r_T(Y)$, contradicting (19). Therefore the requested bigraph exists by Theorem 23. • □

Note that the inequalities in Conditions (10) and (19) can be integrated into the following single form:

$$\widetilde{m}_S(X) + \widetilde{m}_T(Y) - |X||Y| + (\ell - r_S(X) - r_T(Y))^+ \leq \gamma. \tag{20}$$

By specializing Theorem 23 to the case when M_S and M_T are ℓ-uniform matroids on S and T, respectively, one obtains the following.

Corollary 26. *We are given a simple bigraph $H_0 = (S, T; F_0)$, an integer ℓ, and a degree-specification (m_S, m_T) for which $\widetilde{m}_S(S) = \widetilde{m}_T(T) = \gamma$. There is a bigraph $G = (S, T; E)$ fitting (m_S, m_T) for which $G^+ = G + H_0$ is simple and includes an ℓ-element matching if and only if* (8) *holds and*

$$\widetilde{m}_S(X) + \widetilde{m}_T(Y) - d_{G_0}(X, Y) + \ell - |X'| - |Y'| \leq \gamma$$
$$\text{whenever } X \subseteq X' \subseteq S,\ Y \subseteq Y' \subseteq T, \text{ and } X' \cup Y' \text{ hits all the edges of } H_0, \tag{21}$$

where G_0 is the bipartite complement of H_0.

Proof. Proof. Consider Theorem 23 in the special case when M_S and M_T are ℓ-uniform matroids on S and on T, respectively. Since matroid rank functions are subcardinal, (21) is implied by (16) and hence (21) is necessary.

We claim, conversely, that (21) implies (16), that is, $\alpha + \ell - r_S(X') - r_T(Y') \leq \gamma$ where $\alpha := \widetilde{m}_S(X) + \widetilde{m}_T(Y) - d_{G_0}(X, Y)$. Indeed, if $\max\{|X'|, |Y'|\} \leq \ell$, then $\alpha + \ell - r_S(X') - r_T(Y') = \alpha + \ell - \min\{\ell, |X'|\} - \min\{\ell, |Y'|\} = \alpha + \ell - |X'| - |Y'| \leq \gamma$, where the last inequality follows by (21). If $\max\{|X'|, |Y'|\} > \ell$, and, say, $|X'| > \ell$, then $\alpha + \ell - r_S(X') - r_T(Y') = \alpha + \ell - \min\{\ell, |X'|\} - \min\{\ell, |Y'|\} \leq \alpha + \ell - \ell - 0 = \alpha \leq \gamma$, where the last inequality follows by (8). Therefore the requested bigraph exists by Theorem 23. • □

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