Research Article

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On the tangent model for the density of lines and a Monte Carlo method for computing hypersurface area

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Abstract: Methods to estimate surface areas of geometric objects in 3D are well known. A number of these methods are of Monte Carlo type, and some are based on the Cauchy–Crofton formula from integral geometry. Employing this formula requires the generation of sets of random lines that are uniformly distributed in 3D. One model to generate sets of random lines that are uniformly distributed in 3D is called the tangent model (see [4]). In this paper, we present an extension of this model to higher dimensions, and we examine its performance by estimating hypersurface areas of n-ellipsoids. Then we apply this method to estimate surface areas of hypersurfaces defined by Fermat-type varieties of even degree.

Keywords: Tangent model, Cauchy–Crofton formula, hypersurface area, Monte Carlo method

MSC 2010: 52A38, 28A75, 53A05

1 Introduction

The problem of computing surface areas of geometric objects is of interest in many areas of applications. Studying and computing such measures is a central topic in integral geometry and geometric probability. The classical integral expression for the surface area from analytic geometry is often difficult to reduce analytically for dimensions $n > 3$. These difficulties are also encountered, for instance, when employing quadrature methods. It is well known that Monte Carlo-type methods become the method of choice for estimating high dimensional integrals. The purpose of this article is to develop further a Monte Carlo-type method based on the Cauchy–Crofton formula (CCF) from integral geometry to compute hypersurface areas of compact convex bodies.

A interesting review of the early history of the CCF and its extensions is given in [3]. One can find the relevant theoretical treatment of integral geometry and geometric probability for example in [7, 9]. The CCF transforms the problem of finding the surface area into counting intersections of the surface with a set of uniformly distributed lines. The algorithm we employ is based on the CCF coupled with a comparison principle. More specifically, suppose we know the surface area S_1 of a reference bounding object Σ_1 containing in its interior the object with boundary Σ whose surface area S we wish to compute. Consider a random sample of a set of N lines from the set of lines that intersect Σ_1. Let k_1 and k be the total number of intersection points with Σ_1 and Σ, respectively. Then $S \approx \frac{k}{k_1} S_1$. The origins of this algorithm can be traced back to the work of W. M. Crofton (in 1867) and to E. Czuber (in 1884), who gave this algorithm for computing experimentally the perimeters of closed convex curves (see [3, p. 9]). It is advantageous to take Σ_1 as a circumscribing hypersphere

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(circumsphere). This gives this algorithm some generality in that it allows one to take advantage of the well-known uniform sampling from spheres and balls. This algorithm was employed by Li, Wang, Martin and Bowyer in [4] to compute surface areas for constructive solid geometry models in 3D ($n = 3$). In this study, we examine the implementation and application of this algorithm in dimensions $n \geq 3$. We note that this method may apply to compute surface areas of certain bodies which are not necessarily restricted to be convex – examples in 3D of this sort were examined in [4].

A key ingredient for applying this algorithm is to generate a set of uniformly distributed lines in Euclidean space $\mathbf{E}^n$. The theory for the density of lines in $\mathbf{E}^n$ is given in Santalo [7]. For $n = 3$, Li, Wang, Martin and Bowyer give in [4] two models for generating lines; the chord model and the tangent model. In Section 2, we visit briefly the problem of estimating surface areas in 3D. In Section 3, we develop the tangent model further to compute hypersurface areas for $n = 4$ (or 4D), and we give an extension that applies in dimensions $n \geq 5$. In Section 4, we apply the results developed in Section 3 to compute hypersurface areas of Fermatoids – a class of compact convex hypersurfaces that are defined by Fermat varieties of even degree.

Before we present our results, we recall briefly a result on the surface areas of n-dimensional ellipsoids developed in [11]. We employ this result to verify our algorithm and computations. The hypersurfaces of n-dimensional ellipsoids are defined by the equation

$$\sum_{i=1}^{n} \left(\frac{x_i}{a_i}\right)^2 = 1, \quad n > 2,$$

where a_i are constants. In [11], Tee gives a reduction of the hypersurface area integral to an abelian integral on $[0, 1]$, which is well suited for numerical evaluations. Let

$$\delta_i := 1 - \frac{a_n^2}{a_i^2}, \quad k_i(x) := 1 - \delta_i(1 - x^2)^2, \quad i = 1, 2, \dots, n-1, \qquad B_n := \frac{4a_1 \cdots a_{n-1}\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n+1}{2})}.$$

For $n \geq 3$, the surface area of an n-dimensional ellipsoid is given by ([11, equation (93)])

$$A = B_n \int_0^1 x^{n-2} \sqrt{\frac{(2 - x^2)^{n-3}}{k_1(x)k_2(x)\cdots k_{n-1}(x)}} \left(\frac{1-\delta_1}{k_1(x)} + \frac{1-\delta_2}{k_2(x)} \cdots + \frac{1-\delta_{n-1}}{k_{n-1}(x)}\right) dx. \tag{1}$$

In Table 2, we give, in the column labeled SA(1), estimates from (1) with the parameters a_i for various ellipsoids in dimensions 3, 4, 5 and 8.

2 3D models

In [4], two models for generating uniformly distributed lines in Euclidean three space $\mathbf{E}^3$ are described; namely, the chord model and the tangent model. Also, a quasi-Monte Carlo method for computing surface areas in 3D for constructive solid geometry models was developed. The prefix “quasi” was used to indicate the use of low-discrepancy sequences, which were employed to improve the efficiency and accuracy of the estimates. We employ the low-discrepancy Halton sequences in all of our computations. To set the stage for further development in higher dimensions, we briefly discuss in this section the implementation of these models and fill in some additional details. For illustration, we compute the surface areas of some 3-dimensional ellipsoids with known surface areas (obtained from equation (1)).

2.1 The Chord model

In this model, a random line is defined as a line passing through two independently uniformly distributed points on a sphere S_R^2 of radius R in $\mathbf{E}^3$. Each random chord within the sphere can be associated with a random line which can be considered as its carrier. By considering the chord length distribution, Solomon [9] showed

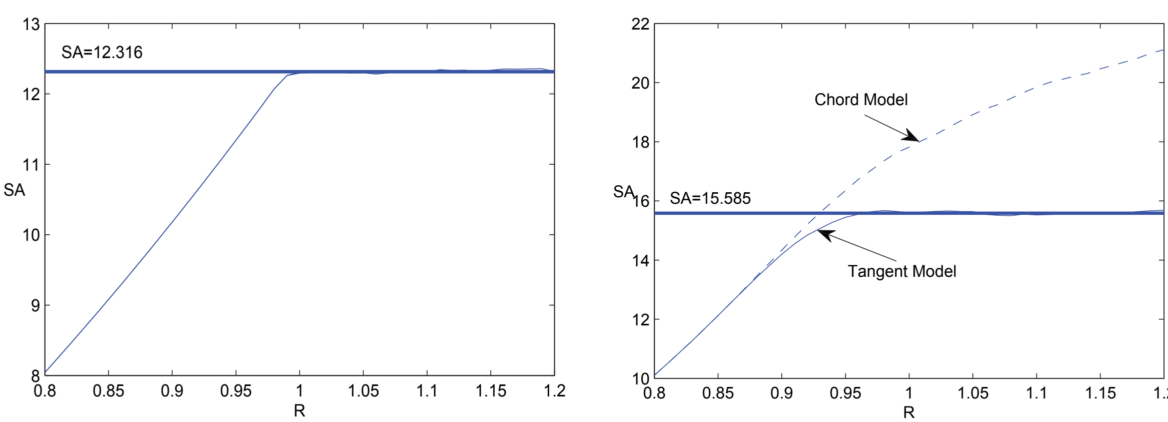
Dimension	Parameters	SA(1)	Chord model
3	$\frac{1}{2}, \frac{3}{4}, 1$	6.9715	6.9768
	$\frac{98}{100}, \frac{99}{100}, 1$	12.3160	12.2949
4	$\frac{1}{2}, \frac{4}{6}, \frac{5}{6}, 1$	7.9904	10.7618
	$\frac{94}{100}, \frac{96}{100}, \frac{98}{100}, 1$	18.0090	19.2201
	$\frac{17}{20}, \frac{18}{20}, \frac{19}{20}, 1$	15.5850	17.8186

Table 1. Surface area estimates from equation (1), and the chord model for 3D and 4D ellipsoids.

that the associated random lines have a uniform distribution. An alternative justification for this uniform distributivity is given in [4]. Now, two approaches to generate uniform distributions of points on S_R^2 were developed in [5, 6, 8, 10]. One method to obtain a uniform distribution of points within S_R^2 is to take

$$x = 2Ru^{\frac{1}{3}} w^{\frac{1}{2}} (1-w)^{\frac{1}{2}} \cos\theta, \quad y = 2Ru^{\frac{1}{3}} w^{\frac{1}{2}} (1-w)^{\frac{1}{2}} \sin\theta, \quad z = Ru^{\frac{1}{3}}(1-2w),$$

where u and w are uniform on $[0, 1]$, and θ is uniform on $[0, 2\pi]$. Setting $u = 1$, one obtains a set of uniformly distributed points on the surface of S_R^2. These formulas follow from results given by Tashiro in [10]. Tests of the CCF using this model on ellipsoids of various dimension are shown in Table 1. Moreover at this point, we also show in Table 1 results obtained for the chord model when extended and applied to compute surface areas of ellipsoids in 4D. This naive extension of the chord model from 3D to 4D does not work. Indeed, this method of generating chords fails in 2D as well, and perhaps the 3D case is an exception. In support of this empirical finding, and for comparison purposes with the tangent model (developed and discussed below), we examine the effect of scaling the radius of the circumsphere on the surface area estimates. We think of this invariance to perturbation of the radius of the circumsphere as a kind of a stability analysis. Figures 1a and 1b show the stability analysis in 3D and 4D, for both the chord model and the tangent model. It is clear that in 3D, both models are stable to scaling and give almost identical results and confirm further the findings in [4]. However, in the 4D case, we see that the chord model does not give good estimates and it is not stable to scaling of the circumsphere radius.



(a) Stability analysis for 3D model.

(b) Stability analysis for 4D model.

Figure 1

2.2 The tangent model

An alternative method for generating uniformly distributed lines in $\mathbf{E}^3$ intersecting the reference sphere was developed by Li, Wang, Martin and Bowyer in [4], and was named by them the *tangent model*. This model is based on the density of random straight lines in $\mathbf{E}^3$ obtained by Beckers and Smeulder in [1], where the analysis was based on heuristic invariance principles. Beckers and Smeulder employed a 4D space parameterized by (r, θ, ϕ, ψ), and showed that the density of lines in $\mathbf{E}^3$ is proportional to $r\sin\phi$. More specifically, each chord from a uniform density of chords on a reference sphere, S_R, centered at the origin, can be viewed as a tangent to a small sphere, S_r, centered at the origin with $r < R$ (see [4] for more details). The implementation of the tangent model in 3D has two steps: first generate a suitable random point P on S_r, then pick a random line (or chord) from the "flat" pencil of tangents on the plane tangent to S_r at P. To implement the tangent model, generate the points on the surface of a sphere of radius $r < R$ by

$$x = 2Ru^{\frac{1}{2}}w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}\cos\theta, \quad y = 2Ru^{\frac{1}{2}}w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}\sin\theta, \quad z = Ru^{\frac{1}{2}}(1-2w),$$

where u and w are uniform on $[0, 1]$, and θ is uniform on $[0, 2\pi]$. Clearly, $x^2 + y^2 + z^2 = R^2u$ – which indicates a compatibility with the form of the density $cr\sin\phi$. Recall that in the 2D case, a uniform distribution of chords on the disc of radius R is generated by using for each chord, a radius obtained from the uniform distribution on $[0, R]$, and an angle θ obtained from the uniform distribution on $[0, 2\pi]$, to define its midpoint. This is indeed the method used by Crofton for selecting random lines on the disc in 2D (see [3, p. 7]). Observe that the tangent model in 3D can be viewed as an extension of this 2D model; and as we will see below, this observation extends, at least empirically, into higher dimensions as well.

We now outline a procedure for finding the chords. Pick one point $P = (x_1, y_1, z_1)$ on S_r, and denote its position vector by $\mathbf{r}_1$. This vector is normal to the tangent plane at P. Take a point (x, y, z) in this plane, and form the vector $\mathbf{r}_2 = (x - x_1, y - y_1, z - z_1)$. As $\mathbf{r}_1$ and $\mathbf{r}_2$ are orthogonal, their dot product must be zero and so we obtain the equation of the tangent plane H: $x_1x + y_1y + z_1z = r^2$.

The second part of the tangent model entails picking a line form the pencil of tangent lines centered at P. To do this, pick a great circle and use its tangent as a reference direction – this tangent is a fixed element of the pencil of lines on H passing through P. The intercept of H with the z-axis is $(0, 0, \frac{r^2}{z_1})$, and so we have the vector along the reference line in the direction of $\mathbf{r}_3 := (x_1, y_1, z_1 - \frac{r^2}{z_1})$. This exists with the exception of a set of measure zero (with respect to surface area Lebesgue measure). Next, generate the angle ψ uniform on $[0, \pi]$. We now need to find the equation of the line passing through (x_1, y_1, z_1) making an angle ψ with $\mathbf{r}_3$. This will be the equation of the line that gives us the chord we are seeking. To do that, it suffices to take a point (X, Y, Z) on the tangent plane such that the vector $\mathbf{r}_4 := (X - x_1, Y - y_1, Z - z_1)$ has unit length. Now this ψ is the angle between $\mathbf{r}_3$ and $\mathbf{r}_4$. The angle between these two vectors is thus given by the dot product formula

$$\cos\psi = \frac{\mathbf{r}_3.\mathbf{r}_4}{|\mathbf{r}_3||\mathbf{r}_4|},$$

which reduces to

$$\cos\psi = r\frac{-Z + z_1}{\sqrt{r^2 - z_1^2}}.$$

Since we pick ψ, we can find Z. Therefore, we have the following two equations to determine X and Y:

$$(X - x_1)^2 + (Y - y_1)^2 = A, \quad x_1X + y_1Y = B,$$

where $A = 1 - (Z - z_1)^2$, and $B = r^2 - z_1Z$. So we end up with a quadratic in Y. Pick any of the roots to obtain

$$Y = \frac{1}{2(y_1^2 + x_1^2)}(2y_1B + 2Q), \quad X = \frac{-y_1(Y) + B}{x_1},$$

where

$$Q = \sqrt{-2y_1^2x_1^4 + 2y_1^2Bx_1^2 - y_1^4x_1^2 + y_1^2Ax_1^2 - x_1^6 - x_1^2B^2 + 2x_1^4B + x_1^4A}.$$

Then from the points (x_1, y_1, z_1) and (X, Y, Z) we can find the equation of the random line we are seeking.

Next we determine whether this line does or does not intersect the surface Σ by solving a polynomial using a global method that finds all the roots at once. If all the roots are complex, then the chord does not intersect Σ; otherwise, it does. The results from this method are shown in Table 2, in the column labeled SA(a).

3 4D and higher dimensions

In this section, we present the extension of the tangent model to 4D, and to higher dimensions. One may proceed along the same lines in 4D as in the 3D case. That is, first pick a point P on the sphere S_r^3; then pick two random coordinates for the point on the 3-sphere centered at P and solve for the other two coordinates. The results from this case are reported in the column labeled SA(a) of Table 2. This method is a bit awkward to extend as n increases.

A better alternative method is to first generate a point P on S_r^3 using

$$\begin{aligned} x_1 &= Ru^{\frac{1}{3}} w^{\frac{1}{2}} \sin\theta_1, \\ y_1 &= Ru^{\frac{1}{3}} w^{\frac{1}{2}} \cos\theta_1, \\ z_1 &= Ru^{\frac{1}{3}} (1-w)^{\frac{1}{2}} \sin\theta_2, \\ t_1 &= Ru^{\frac{1}{3}} (1-w)^{\frac{1}{2}} \cos\theta_2, \end{aligned}$$

where u and w are uniform on $[0, 1]$, and θ_1, θ_2 are uniform on $[0, 2\pi]$. Save for the factor $Ru^{\frac{1}{3}}$, these formulas give the coordinates of points uniform on the surface of S^3 given in [8]. Now the second step in the tangent model is to pick a line that is uniformly distributed on the hypersphere centered at P that is embedded within the hyperplane (a higher dimensional analogue of the unit disc centered at P we utilized in the 3D case). Therefore, all we need now is to find another suitable random point on this sphere to form the equation of a random line that we can use to find the intersections with Σ. The equation of the hyperplane H tangent to S_r^3 at the point $P = (x_1, y_1, z_1, t_1)$ is given by $x_1 x + y_1 y + z_1 z + t_1 t = r^2$.

Now we resort to a different approach for sampling the uniform distribution on spheres (see [6]). Pick $\mathbf{a} = (a_1, a_2, a_3, a_4)$ from the standard normal distribution $N(0, 1)$. Find the perpendicular line passing through $\mathbf{a}$ in the direction of the vector OP. The foot of this line on H, is the projection $\mathbf{b}$ of $\mathbf{a}$ onto H, and is given by

$$\begin{aligned} b_1 &= a_1 + x_1\Big(-\frac{x_1 a_1 + y_1 a_2 + z_1 a_3 + t_1 a_4 - r^2}{r^2}\Big), \\ b_2 &= a_2 + y_1\Big(-\frac{x_1 a_1 + y_1 a_2 + z_1 a_3 + t_1 a_4 - r^2}{r^2}\Big), \\ b_3 &= a_3 + z_1\Big(-\frac{x_1 a_1 + y_1 a_2 + z_1 a_3 + t_1 a_4 - r^2}{r^2}\Big), \\ b_4 &= a_4 + t_1\Big(-\frac{x_1 a_1 + y_1 a_2 + z_1 a_3 + t_1 a_4 - r^2}{r^2}\Big). \end{aligned}$$

Now let

$$D = \sqrt{(b_1 - x_1)^2 + (b_2 - y_1)^2 + (b_3 - z_1)^2 + (b_4 - t_1)^2}.$$

Next, compute $\mathbf{c} = (c_1, c_2, c_3, c_4)$, the normalization of $\mathbf{b}$, as

$$c_1 = \frac{b_1}{D}, \quad c_2 = \frac{b_2}{D}, \quad c_3 = \frac{b_3}{D}, \quad c_4 = \frac{b_4}{D}.$$

Now, it is easy to verify that $x_1 c_1 + y_1 c_2 + z_1 c_3 + t_1 c_4 = \frac{r^2}{D}$. Therefore, to bring $\mathbf{c}$ back onto the sphere within the hyperplane, project again, and so we get $\mathbf{d} = (d_1, d_2, d_3, d_4)$ given by

$$\begin{aligned} d_1 &= c_1 + x_1\Big(-\frac{x_1 c_1 + y_1 c_2 + z_1 c_3 + t_1 c_4 - r^2}{r^2}\Big), \\ d_2 &= c_2 + y_1\Big(-\frac{x_1 c_1 + y_1 c_2 + z_1 c_3 + t_1 c_4 - r^2}{r^2}\Big), \\ d_3 &= c_3 + z_1\Big(-\frac{x_1 c_1 + y_1 c_2 + z_1 c_3 + t_1 c_4 - r^2}{r^2}\Big), \\ d_4 &= c_4 + t_1\Big(-\frac{x_1 c_1 + y_1 c_2 + z_1 c_3 + t_1 c_4 - r^2}{r^2}\Big). \end{aligned}$$

It is easy to check that $\mathbf{d}$ is on the unit sphere centered at P within the hyperplane. Set $\mathbf{d} = (x_2, y_2, z_2, t_2)$. This is the second point we need to find the equation of the random line defining a chord, and then we proceed

Dimension	Parameters	SA(1)	SA(a)	SA(b)	SA(c)
3	$\frac{1}{2}, \frac{3}{4}, 1$	6.9715	7.0020	–	–
	$\frac{98}{100}, \frac{99}{100}, 1$	12.3160	12.3088	–	–
4	$\frac{1}{2}, \frac{4}{6}, \frac{5}{6}, 1$	7.9904	8.1819	8.0773	7.9904
	$\frac{94}{100}, \frac{96}{100}, \frac{98}{100}, 1$	18.0090	18.0298	18.0278	18.0239
	$\frac{17}{20}, \frac{18}{20}, \frac{19}{20}, 1$	15.5850	15.6137	15.6117	15.5841
5	$\frac{3}{5}, \frac{7}{10}, \frac{4}{5}, \frac{9}{10}, 1$	10.3980	–	10.4644	10.8250
	$\frac{92}{100}, \frac{94}{100}, \frac{96}{100}, \frac{98}{100}, 1$	22.3310	–	22.4001	22.4974
	$\frac{16}{20}, \frac{17}{20}, \frac{18}{20}, \frac{19}{20}, 1$	17.1460	–	17.0389	17.4600
8	$\frac{3}{10}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{7}{10}, \frac{4}{5}, \frac{9}{10}, 1$	1.1207	–	1.1140 (10^6), 1.0293 (10^4)	–
	$\frac{86}{100}, \frac{88}{100}, \frac{90}{100}, \frac{92}{100}, \frac{94}{100}, \frac{96}{100}, \frac{98}{100}, 1$	19.4140	–	19.4364	–
	$\frac{13}{20}, \frac{14}{20}, \frac{15}{20}, \frac{16}{20}, \frac{17}{20}, \frac{18}{20}, \frac{19}{20}, 1$	8.0283	–	8.0265	–

Table 2. Estimates of the surface areas using the tangent model, for various n-ellipsoids using a chord sample size equal to 10^4.

as we did in the 3D case to find and count intersections. The results from this method are shown in Table 2, in the column labeled SA(c).

A variant of this method that we have also tested differs only in the way we obtain the point (x_1, y_1, z_1, t_1). In this case, this point is obtained using normal variates as well. Create a point (X, Y, Z, T) from $N(0, 1)$. Then, to get it onto a sphere of radius r, we take

$$x_1 = r\frac{X}{\sqrt{X^2 + Y^2 + Z^2 + T^2}},$$

with similar expressions for y_1, z_1, t_1, where $r = w^{\frac{1}{3}}$ and w uniform on $[0, R]$. The results from this modification are shown in Table 2, under the column labeled SA(b). The advantage of this algorithm is that it can be extended to higher dimensions with a minimal effort. We present in Table 2 some results from this algorithm to estimate the hypersurface areas of ellipsoids in 5D and 8D. In instances where the ratio of the area of the circumsphere to the surface area of the n-ellipsoid is rather large, we note that the accuracy of the method is reduced, however, increasing the sample size helps. For example, the first result obtained for the 8D case in Table 2 is much more accurate with 10^6 samples than with 10^4 samples.

4 Surface areas of Fermatoids

In this section we present our estimates of hypersurface areas of Fermatoids. Methods to obtain such estimates in 2D were given in [2]. Consider the n-dimensional hyper-ellipsoid of degree $2m$, $m = 1, 2, \dots$, centered at the origin of a rectangular coordinate axes with semi-axes $a_1, \dots, a_n$ given by

$$\sum_{i=1}^{n} \frac{x_i^{2m}}{a_i^{2m}} = 1.$$

The area of this $(n-1)$-dimensional surface is given by

$$A = 2^n \int_{x_1=0}^{a_1} \int_{x_2=0}^{a_2 \sqrt[2m]{1-\frac{x_1^{2m}}{a_1^{2m}}}} \cdots \int_{x_{n-1}=0}^{a_{n-1} \sqrt[2m]{1-\frac{x_1^{2m}}{a_1^{2m}}-\cdots-\frac{x_{n-2}^{2m}}{a_{n-2}^{2m}}}} \sqrt{1 + \left(\frac{\partial x_n}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial x_n}{\partial x_{n-1}}\right)^2}\, dx_{n-1} \cdots dx_1.$$

For the n-ellipsoid ($m = 1$) case, as was mentioned in the introduction, Tee [11, Section 4] gave a reduction of this n-dimensional surface area integral to an abelian integral on $[0, 1]$. However, it appears that it is not an easy task to carry out a similar useful reduction for the hyper-ellipsoid case ($m > 1$), and so this remains an open question at this point. Now if $a_1 = \cdots = a_n = 1$, we obtain the hypersurfaces that we call Fermatoids, as

n	*m*	SA(a)	SA(b)	SA(c)
3	2	17.5627	–	–
	3	19.6357	–	–
	4	20.6846	–	–
4	2	37.7529	37.6077	37.6524
	3	45.6923	45.8344	45.9055
	4	50.1593	50.4222	50.2813
	7	56.0131	56.3181	56.3298
	10	58.6559	58.8483	58.1813
5	2	–	73.0219	–
	3	–	99.3930	–
	4	–	114.5534	–
	7	–	135.1825	–
	10	–	142.8273	–
8	2	–	363.3965	–
	3	–	743.1145	–
	4	–	1006.9137	–
	7	–	1433.0306	–
	10	–	1614.6608	–

Table 3. Estimates of the hypersurface areas of Fermatoids of various dimensions and degrees using the tangent model with a sample size equal to 10^4.

they are associated with the Fermat varieties $\sum_{i=1}^{n} x_i^{2m} = 1$, $m > 1$. These power equations can be viewed as equations of n-dimensional l^{2m} unit spheres. Despite the symmetrical properties of Fermatoids, numerical quadrature methods could become tedious for $n > 3$. As an alternative, we employ the Monte Carlo method developed above to compute estimates of hypersurface areas for a few Fermatoids. The results appear in Table 3 for Fermatoids of various dimensions and degrees.

5 Conclusion

In general, Monte Carlo type methods are the main tool for evaluating multidimensional integrals of high dimensions. In this article we presented a Monte Carlo type method based on the Cauchy–Crofton formula from integral geometry to compute hypersurface area integrals. For the practical implementation of this method, it is necessary to have a model to generate a set of uniformly distributed lines in Euclidean space $\mathbf{E}^n$. The chord model and the tangent model are two known models for generating such lines in $\mathbf{E}^3$ (see [4]). To our knowledge practical models for generating a uniform density of lines for dimensions $n > 3$ are not available. In this study we extended the tangent model for dimensions $n > 3$. We also found out that the chord model that works well for $n = 3$, and which is known to fail for $n = 2$, also fails for $n = 4$. To test the performance of this Monte Carlo method, we carried out experiments to compute the hypersurface areas of n-ellipsoids. For most tests, our results agree very well with estimates for these hypersurface areas computed by another method given in the literature (see [11]).

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