Macroscopic pricing schemes for the utilization of pool ride-hailing vehicles in bus lanes

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***Abstract*— With the increasing popularity of ride-hailing services, new modes of transportation are having a significant impact on the overall performance of transportation networks. As a result, there is a need to ensure that both the various transportation alternatives and the spatial network resources are used efficiently. In this work, we analyze a network configuration where part of the urban transportation network is devoted to dedicated bus lanes. Apart from buses, we let pool ride-hailing trips use the dedicated bus lanes which, contingent upon the demand for the remaining modes, may result in faster trips for users opting for the pooling alternative. Under an aggregated modelling framework, we characterize the spatial configuration and the multi-modal demand split for which this strategy achieves a system optimum. For these specific scenarios, we compute the equilibrium when ride-hailing users can choose between solo and pool services, and we provide a pricing scheme for mitigating the gap between total user delays of the system optimum and user equilibrium solutions, when needed.**

I. Introduction

The flexibility and convenience of ride-hailing services are key features of this on-demand transportation alternative. This is because they offer door-to-door rides, and they are characterized by low fares and short waiting times. However, soon after their launch, many problems have surfaced regarding their operation, safety, and impact on traffic. In particular, the high number of idle ride-hailing vehicles, while providing a good level of service, significantly deteriorates traffic conditions as shown in [1]. Ride-splitting, where passengers pool their rides with other users, can mitigate this negative impact because it allows ride-hailing drivers to travel for shorter distances while serving the same demand. Pooling passengers receive a fare discount to compensate for the longer travel time they incur but the pool engagement levels are still moderate.

To improve the overall understanding of the operation and impact of ride-splitting, many researchers have focused on modelling ride-hailing services to i) explore the pricing mechanism of these markets and ii) quantify their impact on traffic and other transportation modes. Through a comprehensive modelling of ride-hailing supply and demand in [2], the authors highlighted the importance of surge pricing in the event of a supply shortage. In [3], the authors computed the maximum achievable solo and pool demand that can be serviced by the network and proved that pooling is able to reduce the network travel time for pool riders and private vehicles concurrently utilizing the same network.

Since the dynamics in urban transportation networks are highly complex, it is necessary to formulate tractable models suitable for theoretical analysis. One such approach is the use of network-level Macroscopic Fundamental Diagrams (MFDs), which represent an aggregate relationship between traffic flow, density, and speed in a region [4]. Their potential to provide an accurate estimate of aggregate traffic measures is observed in multiple locations as shown in [5]. This paved the way for the use of MFD modelling for many different applications, such as multi-region perimeter control [6] which goal is to reduce traffic congestion in urban areas. Beyond macroscopic modelling of car traffic dynamics, MFDs are very versatile as they can also capture the interactions between vehicle and bus traffic, as in [7] where the authors used 3D-MFD to show that the marginal influence on traffic of buses is not equivalent to that of cars. This latter tool is used in [8] to suggest a multi-modal spatial allocation policy, where using MFD and 3D-MFD theory, the authors assessed the benefits of allowing pool ride-hailing vehicles in the bus network.

While this macroscopic approach to model the delays of different modes in a shared network is novel to our knowledge, different delay functions for different actors and/or objectives of the actors at the link level in transportation networks have been previously analyzed. The existence of a Wardrop equilibrium for multi-class transportation network was shown in [9] and [10]. These works present an analysis of a multi-modal setting where public transit vehicles interact with private vehicles on the same roads. In [11], [12], a situation where both autonomous and regular vehicles share the same route is studied, and the vehicle classes are assumed to have a different impact on the total and common delay for each link. Another type of routing game is one in which different classes of users may have different objectives, but their decisions cause delays to other classes through the use of common resources. In [13], the authors examine a situation in which a company tries to minimize the total travel time for its fleet while interacting with regular drivers who are trying to minimize their own travel time.

The contribution of this paper is twofold. First, we develop delay functions for multi-modal transportation networks, and use them to estimate aggregate modal- and network-dependent travel times. We also illustrate how these delay functions replicate the uncongested behavior of the MFD function. Second, we analyze the system optimum for the allocation strategy in which private vehicles and solo ride-

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hailing users utilize the vehicle network, while buses and ride-hailing vehicles with high occupancy use the bus network. We do so mainly to determine the network space configuration and multi-modal demand profiles for which such strategy is advantageous. Unlike in [8], we furthermore assess the user equilibrium, which we also refer to as Wardrop equilibrium, and advance a tolling scheme under settings where the user equilibrium and system optimum solutions do not coincide. This analysis is of particular interest because the user equilibrium we are analyzing only determines the number of solo and pool ride-hailing trips, while the system optimum considers the total multi-modal delays in both networks.

The remainder of the paper is outlined as follows. In the next Section II, we introduce the delay model for each mode of transportation and illustrate how this model relates to the theory of MFD. In the following Section, III, we analyze the properties of the system optimum and we define some conditions that guarantee the efficiency of the proposed strategy. In Section IV, we characterize the Wardrop equilibrium for the efficient set of network configurations, and we identify solutions for which the user equilibrium and system optimum coincide naturally. In Section V, we employ these results to set forward a tolling scheme for the cases where the allocation strategy is beneficial yet the Wardrop solution does not coincide with the system optimum. Numerical examples are presented in Section VI, and the paper concludes with some suggestions for future research.

II. Model

In this section, we first introduce the delay model for a multi-modal transportation network and then provide a link between this model and macroscopic traffic flow theory.

A. Macroscopic Multi-modal Delay Model

In this framework, we study a multi-modal network with private vehicles, ride-hailing, and buses, which demand we denote by $x^{pv} > 0$, $x^{rs} > 0$, and $x^b > 0$, respectively. In ride-hailing, users have the option to either ride alone or to pool, and we denote the demand for solo and pool trips by x^s and x^p, respectively. Naturally, $x^{rs} = x^s + x^p$. The entire network infrastructure is split into a vehicle network which we denote by $\mathcal{V}$, and a bus network which we denote by $\mathcal{B}$, according to a spatial division factor $\alpha \in (0,1)$ where α is the space allocated for the vehicle network and $\bar{\alpha} = 1-\alpha$ is the space allocated to the bus network. The private vehicles and solo ride-hailing users are required to utilize the vehicle network, and the bus and pool ride-hailing users utilize the bus network. Note that in this framework, we focus on pool rides with no more than two passengers. A schematic sketch of the problem setting is displayed in Figure 1. To further describe the proposed allocation scheme, we first define the baseline delay function that we adopt in this work.

Let $x \in \mathbb{R}_{\geq 0}$ be the network flow expressed in vehicles per hour, then the average travel time in the network t :

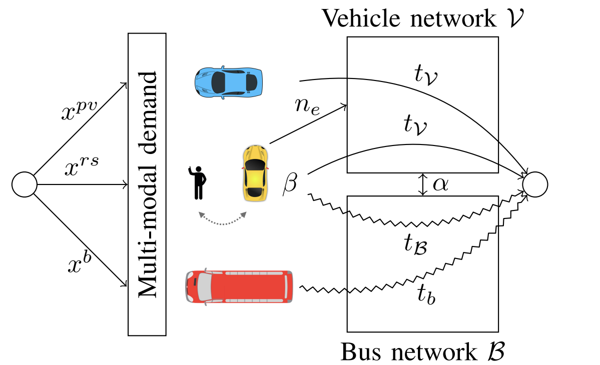

Fig. 1. Schematic sketch of the problem structure. Users travel by private vehicle, ride-hailing, or buses whose demand rates we denote as x^{pv}, x^{rs}, and x^b respectively. The network space is partitioned into two parts, with a portion $\alpha \in (0,1)$ assigned to the vehicle network $\mathcal{V}$, and the remaining portion assigned to the bus network $\mathcal{B}$. Private vehicles and solo trip users utilize the vehicle network $\mathcal{V}$ with an average trip time of $t_{\mathcal{V}}$ whereas buses and pool ride-hailing vehicles travel exclusively in the bus network $\mathcal{B}$ with a travel time of $t_{\mathcal{B}}$ and t_b respectively. The demand rates are exogenous but the split between solo and pool ride-hailing trips β is endogenous.

$\mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$, also called the delay function, is given by

$$t(x) = t_f\left(1 + a\left(\frac{x}{C}\right)^b\right), \tag{1}$$

where $t_f > 0$ is the free flow travel time, $C > 0$ is the network capacity, and $a > 0$ and $b > 0$ are the network-specific delay function parameters. The travel time in the vehicle network $\mathcal{V}$, which occupies a fraction α of the total network space, $t_{\mathcal{V}} : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times (0,1) \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ is given by

$$t_{\mathcal{V}}(x^s, x^{pv}, \alpha, n_e) = t_f\left(1 + a\left(\frac{x^{pv} + x^s}{\omega(n_e)\alpha C}\right)^b\right),$$

where $\omega : \mathbb{R}_{\geq 0} \to (0,1]$ is a continuously differentiable function that depends on the number of empty ride-hailing vehicles $n_e \geq 0$. This category of vehicles is usually roaming around in the network, waiting for a pick-up request. Assuming that empty vehicles are only allowed to travel in the vehicle network $\mathcal{V}$, the purpose of ω is to capture the capacity drop in the vehicle network $\mathcal{V}$ due to the existence of idling vehicles n_e in $\mathcal{V}$ such that $\frac{\mathrm{d}\omega}{\mathrm{d}n_e} \leq 0$ and $\omega(0) = 1$. The traffic flow in this network consists of the flow for private vehicles x^{pv}, and the flow for solo ride-hailing trips x^s.

Similarly, knowing that the bus network occupies a space $\bar{\alpha} = 1 - \alpha$ of the total network infrastructure, we compute the delay function for pool vehicles utilizing the bus network, $t_{\mathcal{B}} : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times (0,1) \to \mathbb{R}_{>0}$, using

$$t_{\mathcal{B}}(x^p, f_b, \alpha) = t_f\left(1 + a\left(\frac{\frac{x^p}{o^p} + f_b}{(1-\alpha)C}\right)^b\right)\Delta_p k\left(f_b\right), \tag{2}$$

where the constant $o^p > 1$ is the pool vehicle occupancy, $\Delta_p > 1$ is the normalized detour factor of passengers which reflects the extra distance travelled by passengers due to them sharing their rides with other passengers. The variable $f_b > 0$ is the average bus flow in the bus network, and

$k : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ is a continuously differentiable function that estimates the average influence of bus flow on travel time in the bus network such that $\frac{\mathrm{d}k}{\mathrm{d}f_b} > 0$ and $k(0) > 1$. Within our framework, the bus flow is assumed to be constant such that buses maintain the same frequency at stops. Moreover, we assume that bus flow is always less than the bus demand such that $f_b < x^b$. The flow in the bus network consists of the pool vehicles trips equal to $\frac{x^p}{o^p}$ and the average bus flow f_b.

The bus user delays, $t_b : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times (0,1) \to \mathbb{R}_{>0}$ are computed in a similar manner as in (2) as shown in

$$t_b(x^p, f_b, \alpha) = t_f \left(1 + a \left(\frac{\frac{x^p}{o^p} + f_b}{(1-\alpha)C}\right)^b\right) \Delta_b k\left(f_b\right) + \gamma\,,$$

except that the normalized detour ratio Δ_p is replaced by Δ_b such that $\Delta_b > 1$ to account for the extra time bus users must travel compared to the direct trip, and $\gamma > 0$ is a constant describing the extra time passengers incur due to the boarding and alighting of other bus users.

In this paper, we analyze the optimal split between solo and pool rides for a given ride-hailing demand. To do so, we introduce the split $\beta \in [0, 1]$ that determines the fraction of pooled rides, i.e., $x^p = \beta x^{rs}$ and $x^s = (1-\beta)x^{rs}$. The equilibrium of the system is achieved when β is determined in two different ways: first, through the case where a central authority dictates this split, i.e., the system optimum, or second through the case where ride-hailing users attempt to minimize their own travel time when choosing between a solo trip or a pooled trip, which is commonly referred to as the Wardrop or user equilibrium. If the two equilibria are different, the Wardrop equilibrium may perform worse overall than the system optimum, and incur a so-called Price of Anarchy (PoA). In Section IV, we categorize when this is not the case. Irrespectively, we note that in the particular case where the system optimum occurs for $\beta = 0$, the network configuration and the demand profile do not allow for the utilization of bus lanes by pool ride-hailing vehicles.

Since the main focus of this paper is on the influence of solo and pool rides, we will omit the dependence on these demands, i.e., with a slight abuse of notation, we let $t_{\mathcal{V}}(x^s) = t_{\mathcal{V}}(x^s, x^{pv}, \alpha, n_e), t_{\mathcal{B}}(x^p) = t_{\mathcal{B}}(x^p, f_b, \alpha)$ and $t_b(x^p) = t_b(x^p, f_b, \alpha)$.

Before proceeding to the equilibrium analysis, we will show how the proposed delay functions relate to macroscopic traffic flow theory for urban transportation networks.

B. Relationship to Macroscopic Traffic Flow Models

The proposed aggregate delay functions reflect modal-specific travel times at the network level. Under our static network equilibrium framework, these functions relate to the well-established theory of Macroscopic Fundamental Diagrams, which is often used to model urban traffic at the macroscopic level. The MFD provides a relationship between average accumulation, i.e, the total number of network vehicles, and average traffic flow in the network. It captures the relationship of the traffic dynamics on the aggregate level, and under specific settings, it can very accurately represent the network conditions. The MFD usually assumes that flow increases up to a certain level of accumulation, often referred to as the critical accumulation. Another common assumption is that the MFD is concave. The following theorem shows that the functional form of the delay function provided in (1) is able to capture the MFD relationship.

Theorem 1: When the average travel time in the network is given by (1), the flow-accumulation relation $x(n)$ is increasing with accumulation n and is concave with respect to this accumulation.

Proof By observing that the accumulation in the network for a given flow is given by $n(x) = xt(x)$, it follows that the accumulation is strictly increasing with the flow, since

$$\frac{\mathrm{d}n}{\mathrm{d}x} = x\frac{\mathrm{d}t}{\mathrm{d}x} + t(x) > 0\,.$$

Thus, the function $n(x)$ is invertible, and let $x(n)$ be its inverse. It then follows from the inverse function rule that $x(n)$ is strictly increasing in n. For the second part, we observe that $n(x)$ is convex with respect to x, since

$$\frac{\mathrm{d}^2 n}{\mathrm{d}x^2} = x\frac{\mathrm{d}^2 t}{\mathrm{d}x^2} + 2\frac{\mathrm{d}t}{\mathrm{d}x} = t_f \frac{ab(b+1)}{C}\left(\frac{x}{C}\right)^{b-1} > 0\,.$$

Then it follows from [14, 12.21] that $x(n)$ is concave. ∎

Remark 1: The MFD flow often decreases beyond a critical accumulation or density. However, in this paper we focus on the increasing regime, i.e., we assume that the average demand flow never exceeds the network capacity.

The following example illustrates how the delay functions correspond to an MFD for one set of parameters.

Example 1: In (1) let $a = 0.8$, $b = 6$, $C = 150000$ pax/hr, and $t_f = 0.1$ hr. We can express the flow as a function of accumulation for the entire network under consideration, but also for the two subnetworks, by setting the network fractional split α to 0.8, and multiplying the capacity by α and $\bar{\alpha}$ for the vehicle and bus networks, respectively. Figure 2 shows that the functions obtained reproduce what we observe in large-scale networks when traffic measures are aggregated. This suggests that the delay function we propose in (1) can potentially be used to capture the relationship between flow and accumulation at the aggregate traffic level.

Having established the macroscopic modelling framework for the proposed space allocation strategy, we proceed next with analyzing the system optimum properties of the multi-modal user delays.

III. Efficiency of the System Optimum

In the following section, we analyze the properties of the system optimum. The ultimate goal of this analysis is to determine the set $\{\alpha, x^{pv}, x^{rs}, f_b\}$ for which the system optimum yields a value of $\beta = 0$, i.e., for when our proposed strategy is not able to return lower delays compared to the scenario where ride-hailing vehicles are not allowed in bus lanes. This total multi-modal delays in the network under consideration, also referred to as Passenger Hour Travelled

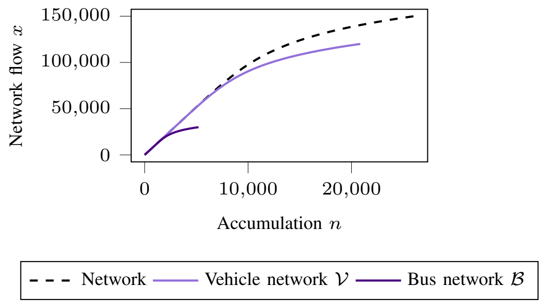

Fig. 2. The Macroscopic Fundamental Diagrams (MFDs) in Example 1.

(PHT) for all users of the network, is computed by taking into account the demand and delays of each network user. Therefore, for a given split β and a fixed α, the PHT is given by

$$\begin{aligned}\mathrm{PHT}(\beta) =& x^{pv} t_{\mathcal{V}}((1-\beta)x^{rs}) + (1-\beta)x^{rs} t_{\mathcal{V}}((1-\beta)x^{rs}) \\ &+ \beta x^{rs} t_{\mathcal{B}}(\beta x^{rs}) + x^{b} t_{b}(\beta x^{rs})\,.\end{aligned}$$

The aim of the system optimum is to find the split between solo and pool demand β^{SO} such that the total PHT of multi-modal commuters including ride-hailing users is minimized as follows

$$\beta^{\mathrm{SO}} \in \operatorname*{arg\,min}_{\beta\in[0,1]} \mathrm{PHT}(\beta)\,. \tag{3}$$

The following proposition guarantees that the optimization problem (3) has a unique solution.

Proposition 1: If $b > 1$, then the solution to the system optimum β^{SO} in (3) is unique.

The proof is given in Appendix A.

Knowing that the minimum is unique, we can characterize how it changes with α for fixed private vehicles, ride-hailing, and bus demands.

Proposition 2: If $b > 1$, the solution to the system optimum β^{SO} is a decreasing function of α.

The proof is given in Appendix B.

Next, we utilize the convexity of the $\mathrm{PHT}(\beta)$ function to identify some sufficient conditions for the system optimum

Proposition 3: Suppose that $b > 1$. For a fixed demand and a given α, the following two sufficient conditions hold:

1) If $\omega \frac{\alpha}{1-\alpha}\left(\frac{x^{rs}}{o^p} + f_b\right) > \left(\frac{b+1}{\frac{b}{o^p}+1}\right)^{\frac{1}{b}} x^{pv}$ then $\beta^{\mathrm{SO}} < 1$.
2) If $\omega \frac{\alpha}{1-\alpha} f_b \geq \left(\frac{b+1}{\frac{b}{o^p}+1}\right)^{\frac{1}{b}} (x^{pv} + x^{rs})$ then $\beta^{\mathrm{SO}} = 0$.

Moreover, there exists an $\underline{\alpha} > 0$ such that $\beta^{\mathrm{SO}} = 1$ for all $\alpha < \underline{\alpha}$.

The proof is given in Appendix C.

It can be concluded that for any space and demand configuration Ω such that $\Omega = \{\alpha, x^{pv}, x^{rs}, f_b \mid \frac{\alpha}{1-\alpha} f_b \geq \left(\frac{b+1}{\frac{b}{o^p}+1}\right)^{\frac{1}{b}} (x^{pv} + x^{rs})\}$, pool ride-hailing users should not be allowed in bus lanes under the system optimum solution. Nevertheless, in situations where the benefits of the pooling in bus lanes scenario is evident, i.e., when $\beta^{SO} = (0, 1]$, it is useful to look at the user's choice under this strategy.

IV. Efficiency of the User Equilibrium

We proceed in this section with the analysis of the properties of the user equilibrium, and we concertize some conditions for which this Wardrop equilibrium solution is efficient under the proposed allocation strategy by utilizing the Price of Anarchy formulation.

A. Properties of the User Equilibrium

Since the demands x^{pv} and f_b are strictly positive, and n_e and α are fixed. Then, it holds that

$$\begin{aligned}\frac{\partial t_{\mathcal{V}}}{\partial x^s} &= \frac{t_f ab}{\omega\alpha C}\left(\frac{x^{pv}+x^s}{\omega\alpha C}\right)^{b-1} > 0\,,\\ \frac{\partial t_{\mathcal{B}}}{\partial x^p} &= \frac{t_f ab}{o^p \bar{\alpha} C}\left(\frac{\frac{x^p}{o^p}+f_b}{\bar{\alpha}C}\right)^{b-1} \Delta_p k > 0\,.\end{aligned}$$

It follows from [15, Theorem 2.5] that the Wardrop equilibrium β^{UE} is unique and can be determined by

$$\beta^{\mathrm{UE}} \in \operatorname*{arg\,min}_{\beta\in[0,1]} \int_0^{(1-\beta)x^{rs}} t_{\mathcal{V}}(s)\mathrm{d}s + \int_0^{\beta x^{rs}} t_{\mathcal{B}}(s)\mathrm{d}s\,.$$

Moreover, since the immediate consequence is that the delay functions are strictly increasing, it holds that for any $\beta \in (0,1)$, it is a Wardrop equilibrium if and only if $t_{\mathcal{V}}((1-\beta)x^{rs}) = t_{\mathcal{B}}(\beta x^{rs})$.

For the remainder of this section, we will investigate how the Wardrop equilibrium depends on the space allocation α. We start with the case where ride-hailing users utilize both networks.

Proposition 4: The solution to the system optimum β^{UE} is a decreasing function of α.

The proof is given in Appendix D.

While the previous proposition makes sure that a user equilibrium always decreases, the following proposition identifies some sufficient conditions for the user equilibrium.

Proposition 5: For a fixed demand and a given α, the following two sufficient conditions hold:

1) If $(1-\alpha)x^{pv} < \omega\alpha\left(\frac{x^{rs}}{o^p} + f_b\right)$ then $\beta^{\mathrm{UE}} < 1$.
2) If $(1-\alpha)(x^{pv} + x^{rs}) < \omega\alpha f_b$ then $\beta^{\mathrm{UE}} = 0$.

Moreover, there exists an $\underline{\alpha} > 0$ such that $\beta^{\mathrm{UE}} = 1$ for all $\alpha < \underline{\alpha}$.

The proof is given in Appendix E.

Remark 2: Even if the average pool detour times captured by Δ_p and the slowing down of pool vehicles by buses captured by k are not fixed but rather demand-dependent, the conditions in Proposition 5 are still valid.

B. Price of Anarchy

To quantify the potential performance decrease a user optimal split inflicts on the system, we examine the Price of Anarchy (PoA), i.e., the ratio of Passengers Hours Travelled at user equilibrium to system optimum such that

$$\mathrm{PoA}(\beta^{\mathrm{UE}}, \beta^{\mathrm{SO}}) = \frac{\mathrm{PHT}(\beta^{\mathrm{UE}})}{\mathrm{PHT}(\beta^{\mathrm{SO}})}\,.$$

The following theorem summarizes some of the main findings and shows that, for certain values of α, the need for intervention is limited since the system optimum coincides with the choice of users.

Theorem 2: Suppose $b > 1$, then there exist $\underline{\alpha} > 0$ and $\bar{\alpha} < 1$, so that for all $\alpha < \underline{\alpha}$ and all $\alpha > \bar{\alpha}$ the Price of Anarchy is 1. Moreover, if $\beta^{\text{SO}} \in (0,1)$ and satisfies

$$(x^{pv} + (1-\beta^{\text{SO}})x^{rs})\frac{\partial t_{\mathcal{V}}}{\partial \beta} + \beta x^{rs}\frac{\partial t_{\mathcal{B}}}{\partial \beta} + x^{b}\frac{\partial t_{b}}{\partial \beta} = 0\,,$$

then the Price of Anarchy is also 1.

Proof The statement about $\underline{\alpha}$ is a direct consequence of the first statements in Proposition 5 and Proposition 3 where $\beta^{\text{UE}} = \beta^{\text{SO}} = 1$. This yields a PoA $= 1$.

For the statement about $\bar{\alpha}$, we observe that the condition for $\beta^{\text{UE}} = 0$ in Proposition 5 reads $\frac{\alpha}{1-\alpha} > \frac{x^{pv}+x^{rs}}{\omega f_b}$, and the condition for $\beta^{\text{SO}} = 0$ in Proposition 3 reads

$$\frac{\alpha}{1-\alpha} \geq \frac{1}{\omega f_b}\left(\frac{b+1}{\frac{b}{o^p}+1}\right)^{\frac{1}{b}}(x^{pv}+x^{rs})\,.$$

Hence, since $\frac{\alpha}{1-\alpha} \to +\infty$ when $\alpha \to 1^-$, it is possible to find a value $\bar{\alpha} < 1$ such that $\beta^{\text{UE}} = \beta^{\text{SO}} = 0$ for all $\alpha > \bar{\alpha}$. ■

In cases of efficient allocation strategies yet inefficient user equilibrium, i.e., when $\beta^{\text{SO}} \in (0,1]$ and PoA > 1, tolling is required to bridge the gap between the system optimum and the user equilibrium.

V. Tolling

In the following section, we provide a tolling scheme for when the multi-modal space allocation policy is efficient, i.e., $\beta^{\text{SO}} \in (0,1]$, and the user equilibrium solution does not leverage the full potential of this policy, i.e., $\beta^{\text{SO}} \neq \beta^{\text{UE}}$. More specifically, we provide an additive toll to the pool user delay function $t_{\mathcal{B}}$ that incentivizes or deters ride-hailing users from pooling in the bus network $\mathcal{B}$.

Proposition 6: If $\tau_p \in \mathbb{R}$ is the toll for utilizing the bus lanes by pool ride-hailing users, then by letting

$$\tau_p = -t_f ab\left(\frac{x^{pv}+(1-\beta^{\text{SO}})x^{rs}}{\omega\alpha C}\right)^{b} + \frac{t_f abk}{o^p\bar{\alpha}C}\left(\frac{\beta^{\text{SO}}\frac{x^{rs}}{o^p}+f_b}{\bar{\alpha}C}\right)^{b-1}(x^{rs}\Delta_p + x^b\Delta_b)\,, \quad (4)$$

the socially optimal solution is achieved.

The proof is given in Appendix F.

Remark 3: We note that the expression of tolling holds for $\beta^{\text{SO}} = 1$ and is intrinsically equal to 0 when $\beta^{\text{SO}} = \beta^{\text{UE}}$ and $\beta^{\text{SO}} \in (0,1)$. The latter point is straightforwardly observed from Theorem 2 which leads to $\tau_p = 0$. The former is the outcome of looking at the necessary conditions for when $\beta^{\text{SO}} = 1$. Therefore, when all ride-hailing users opt for pooling, we have that

$$x^{pv}\frac{\partial t_{\mathcal{V}}}{\partial \beta} + x^{rs}\frac{\partial t_{\mathcal{B}}}{\partial \beta} + x^{b}\frac{\partial t_{b}}{\partial \beta} \leq x^{rs}(t_{\mathcal{V}} - t_{\mathcal{B}})\,.$$

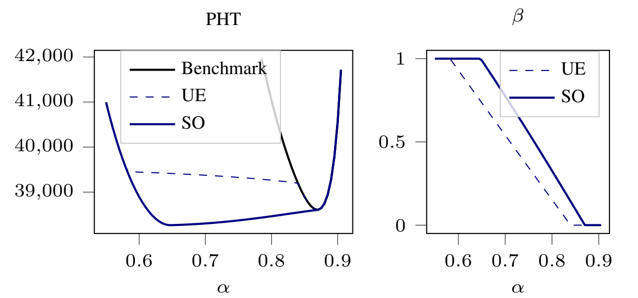

Fig. 3. Comparison of PHT and β for $x^{pv} = 80000$, $x^{rs} = 35000$, and $x^b = 100000$.

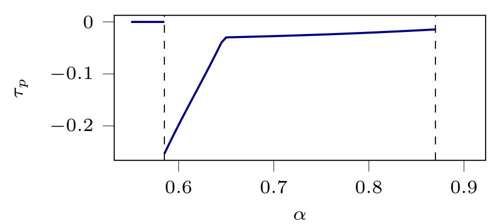

Fig. 4. Tolls for the different values of α

When $t_{\mathcal{V}} < t_{\mathcal{B}}$ for $\beta^{\text{SO}} = 1$, then we have that $\beta^{\text{UE}} = 0$ at system optimum. Introducing a toll τ_p is therefore necessary to yield $x^{rs}\tau_p - x^{rs}(t_{\mathcal{V}} - t_{\mathcal{B}}) \leq 0$, which implies that $\beta^{\text{UE}} = 1$. Nevertheless, when $t_{\mathcal{V}} \geq t_{\mathcal{B}}$, tolls are not needed. However, adding τ_p to the cost of pool ride-hailing users will not modify their choices. Therefore, the tolling function is still valid though unnecessary.

VI. Numerical Examples

Next, we illustrate the efficiency of the space allocation strategy with a numerical example and compare the optimal solo and pool ride-hailing split for the system optimum and the user equilibrium for different values of the network split α.

For this purpose, we use the delay function defined in Example 1. In terms of the constants, we set the pool and bus trip detour factors Δ_p to 1.2 and Δ_b to 1.4 respectively, the additional travel time of bus users due to stopping at stations γ to 0.05 hr, the average pool trip occupancy o^p to 1.6 pax/veh, and the bus flow f_b to 12000 bus/hr. Note that despite considering pool trips with no more than two passengers, the pool vehicle occupancy is less than 2 because users do not have exactly the same origins and destinations. Furthermore, we assume that the number of idle vehicles n_e is constant, so $\omega(n_e) = 0.97$ captures the capacity drop in network $\mathcal{V}$ due to n_e. To account for the potential of buses to slow down pool vehicles due to their frequent stops, we set the factor k to 1.15. The example under consideration assumes that the private vehicle x^{pv}, bus x^b, and ride-hailing x^{rs} demand are equal to 80000, 35000, and 100000 pax/hr respectively. We note that the result section only shows solutions for values of α where the capacity limits are not exceeded in both the vehicle and bus networks.

TABLE I

PHT FOR DIFFERENT α

	$\alpha = 0.869$			$\alpha = 0.647$		
	BM	UE	SO	BM	UE	SO
PHT pv	11053	11053	11018	-	11884	9937
PHT rs	4836	4836	4823	-	5199	5383
PHT b	22717	22717	22763	-	22331	22944
Total	38606	38606	38604	-	39414	38264

Figure 3 displays the results of the PHT under the benchmark (BM), the user equilibrium, and the system optimum scenarios for different network configurations α. Note that the benchmark scenario describes ordinary network settings where all ride-hailing users utilize the vehicle network $\mathcal{V}$, i.e., $\beta = 0$. It can be observed that for high values of α, the three different curves merge together, reflecting that when the network space allocated to buses is relatively compact, pooling must not be allowed in bus lanes to avoid penalizing bus users. This coincides with ride-hailing users' choice who find utilizing the bus network very costly compared to travelling solo in the vehicle network. On the contrary, for relatively low values of α, the PHT curves merge because the large network space allocated for buses implies that pooling in bus lanes is a convenient solution from both a system optimum and user equilibrium perspective. For intermediate values of α, we observe a gap between the PHT curves. This gap can be explained by looking at the values of β in Figure 3. Since a pool trip length is distance-wise longer, ride-hailing users prefer the solo over the pool option, which substantiates why $\beta^{\text{UE}} \leq \beta^{\text{SO}}$ for the different space configuration values α. Therefore, a tolling scheme is needed to mostly incentivize ride-hailing users to pool in bus lanes, i.e., the tolls are in fact discounts. The toll values are displayed in Figure 4 where it can be seen that for low values of α, no tolling is needed as the system optimum and user equilibrium solutions occur naturally for $\beta = 1$, while for high values of α, the space allocation strategy proposed is inefficient and should be therefore not activated.

To further understand the efficiency of our strategy, we compare in Table I the PHT values for private vehicles PHT pv, ride-hailing users PHT rs, and PHT b for $\alpha = 0.869$ and $\alpha = 0.647$. The choice of these values corresponds to the network configuration where the delays are minimal under the benchmark and the system optimum scenario. While the total demand under the benchmark scenario exceeds the network capacity for $\alpha = 0.647$, we observe an improvement in total delays under the SO settings compared to scenarios where $\alpha = 0.869$ where the benchmark, the user equilibrium, and the system optimum mostly yield a solution with no pooling in bus lanes.

VII. CONCLUSIONS

In this work, we propose a space allocation policy where pool ride-hailing users are allowed to travel in the bus network to compensate for the extra detour caused by pooling, while solo users perform their trips in the vehicle network concurrently with private vehicles. We use macroscopic-level delay functions to estimate aggregate modal and network-dependent travel times. We also show that this approach replicates well the static analysis with network level MFDs. We then assess and compare the system optimum and user equilibrium, narrowing down the user choice to solo and pool for the user equilibrium, while considering the compounded delays for all network users for the system optimum. Finally, we propose a pricing scheme to ensure the efficiency of the user equilibrium in instances where the space allocation scheme proposed reduces overall multi-modal user delays.

In the future, we plan to investigate a similar space allocation policy for pool trips with more than two passengers, where the pool vehicle occupancy and pool detour are demand-dependent. Moreover, we aim to capture in future work how the number of idle vehicles changes with the ride-hailing demand.

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Appendix

A. Proof of Proposition 1

Proof The second derivative of PHT with respect to β is

$$\frac{\partial^2 \text{PHT}}{\partial \beta^2} = x^{pv}\frac{\partial^2 t_{\mathcal{V}}}{\partial \beta^2} + (1-\beta)x^{rs}\frac{\partial^2 t_{\mathcal{V}}}{\partial \beta^2} - 2x^{rs}\frac{\partial t_{\mathcal{V}}}{\partial \beta} + \beta x^{rs}\frac{\partial^2 t_{\mathcal{B}}}{\partial \beta^2} + 2x^{rs}\frac{\partial t_{\mathcal{B}}}{\partial \beta} + x^{b}\frac{\partial^2 t_{b}}{\partial \beta^2},$$

where $\frac{\partial t_{\mathcal{V}}}{\partial \beta} = -\frac{t_f a b x^{rs}}{\omega\alpha C}\left(\frac{x^{pv}+(1-\beta)x^{rs}}{\omega\alpha C}\right)^{b-1} < 0$ and $\frac{\partial t_{\mathcal{B}}}{\partial \beta} = \frac{t_f a b x^{rs}}{o^p\bar{\alpha}C}\left(\frac{\beta\frac{x^{rs}}{o^p}+f_b}{\bar{\alpha}C}\right)^{b-1}\Delta_p k > 0$. The second derivative of the delays with respect to β for both $t_{\mathcal{V}}$ and $t_{\mathcal{B}}$ is $\frac{\partial^2 t_{\mathcal{V}}}{\partial \beta^2} = \frac{t_f a b(b-1)x^{rs2}}{(\omega\alpha C)^2}\left(\frac{x^{pv}+(1-\beta)x^{rs}}{\omega\alpha C}\right)^{b-2} > 0$ and $\frac{\partial^2 t_{\mathcal{B}}}{\partial \beta^2} = \frac{t_f a b(b-1)x^{rs2}}{(o^p\bar{\alpha}C)^2}\left(\frac{\beta\frac{x^{rs}}{o^p}+f_b}{\bar{\alpha}C}\right)^{b-2}\Delta_p k > 0$ respectively, since $b > 1$. The second derivative of t_b with respect to β is the same as $\frac{\partial^2 t_{\mathcal{B}}}{\partial \beta^2}$, except for Δ_p that is replaced by Δ_b. We conclude that $\frac{\partial^2 \text{PHT}}{\partial \beta^2} > 0$ indicating that the PHT is strictly convex with respect to β. ∎

B. Proof of Proposition 2

Proof Starting with the necessary condition for the system optimum when $\beta \in (0,1)$, $\frac{\partial \text{PHT}}{\partial \beta} = 0$, we get

$$(x^{pv} + (1-\beta^{\text{SO}})x^{rs})\frac{\partial t_{\mathcal{V}}}{\partial \beta} + \beta^{\text{SO}}x^{rs}\frac{\partial t_{\mathcal{B}}}{\partial \beta} + x^{b}\frac{\partial t_{b}}{\partial \beta} = x^{rs}(t_{\mathcal{V}} - t_{\mathcal{B}}), \quad (5)$$

and computing the implicit derivative $\frac{\partial \beta^{\text{SO}}}{\partial \alpha}$, we get that

$$\frac{\partial \beta^{\text{SO}}}{\partial \alpha} = -\frac{1}{x^{rs}E}\Bigg(\frac{A^b(b+1)}{\alpha} + \frac{B^b\Delta_p k}{\bar{\alpha}} + \left(\frac{kB^{b-1}}{o^p\bar{\alpha}^2 C} + \frac{k(b-1)B^{b-1}}{o^p\bar{\alpha}^2 C}\right)(x^b\Delta_b + \beta^{\text{SO}}x^{rs}\Delta_p)\Bigg),$$

with $A = \frac{x^{pv}+(1-\beta^{\text{SO}})x^{rs}}{\omega\alpha C}$, $B = \frac{\beta^{\text{SO}}\frac{x^{rs}}{o^p}+f_b}{\bar{\alpha}C}$, and

$$E = A^{b-1}(b+1)\omega\alpha C + \frac{B^{b-1}\Delta_p k}{o^p\bar{\alpha}C} + \frac{k\Delta_p B^{b-1}}{o^p\bar{\alpha}C} + \frac{k(b-1)B^{b-2}}{(o^p\bar{\alpha}C)^2}(x^b\Delta_b + \beta^{\text{SO}}x^{rs}\Delta_p).$$

Since $A > 0, B > 0$ and $E > 0$ due to the fact that $b > 1$, it follows that $\frac{\partial \beta^{\text{SO}}}{\partial \alpha} < 0$. Moreover, for $\beta = 1$, we observe that $\text{PHT}(1) \to +\infty$ when $\alpha \to 0^+$, which concludes the proof. ∎

C. Proof of Proposition 3

Proof We start by observing that

$$\frac{\partial \text{PHT}}{\partial \beta} = -t_f a b x^{rs}\left(\frac{x^{pv}+(1-\beta)x^{rs}}{\omega\alpha C}\right)^b - t_f x^{rs}\left(1 + a\left(\frac{x^{pv}+(1-\beta)x^{rs}}{\omega\alpha C}\right)^b\right) + t_f x^{rs}\left(1 + a\left(\frac{\beta\frac{x^{rs}}{o^p}+f_b}{\bar{\alpha}C}\right)^b\right)\Delta_p k + \frac{t_f a b x^{rs}}{o^p\bar{\alpha}C}\left(\frac{\beta\frac{x^{rs}}{o^p}+f_b}{\bar{\alpha}C}\right)^{b-1}(x^b\Delta_b + \beta x^{rs}\Delta_p). \quad (6)$$

Due to the convexity of $\text{PHT}(\beta)$, $\beta^{\text{SO}} = 1$ implies $\frac{\partial \text{PHT}}{\partial \beta}(1) \le 0$. Hence, by utilizing (6), we get that

$$-ab\left(\frac{x^{pv}}{\omega\alpha C}\right)^b + \frac{x^{rs}ab\Delta_p k}{o^p\bar{\alpha}C}\left(\frac{\frac{x^{rs}}{o^p}+f_b}{\bar{\alpha}C}\right)^{b-1} + \frac{x^b ab\Delta_b k}{o^p\bar{\alpha}C}\left(\frac{\frac{x^{rs}}{o^p}+f_b}{\bar{\alpha}C}\right)^{b-1} - a\left(\frac{x^{pv}}{\omega\alpha C}\right)^b + a\left(\frac{\frac{x^{rs}}{o^p}+f_b}{\bar{\alpha}C}\right)^b\Delta_b k \le 1 - \Delta_p k.$$

Since $\Delta_p k > 1$, we have that $1 - \Delta_p k < 0$, and therefore

$$\frac{b}{o^p\bar{\alpha}C}\left(\frac{\frac{x^{rs}}{o^p}+f_b}{\bar{\alpha}C}\right)^{b-1}(x^{rs}\Delta_p k + x^b\Delta_b k) + \left(\frac{\frac{x^{rs}}{o^p}+f_b}{\bar{\alpha}C}\right)^b \le (b+1)\left(\frac{x^{pv}}{\omega\alpha C}\right)^b.$$

Since $f_b < x^b$, $\Delta_p k > 1$, and $\Delta_b k > 1$, it follows that $\left(\frac{b}{o^p}+1\right)\left(\frac{\frac{x^{rs}}{o^p}+f_b}{\bar{\alpha}C}\right)^b \le (b+1)\left(\frac{x^{pv}}{\omega\alpha C}\right)^b$. Finally, since $b >$, we get that $\omega\frac{\alpha}{1-\alpha}\left(\frac{x^{rs}}{o^p}+f_b\right) \le \left(\frac{b+1}{\frac{b}{o^p}+1}\right)^{\frac{1}{b}}x^{pv}$. Therefore, we know that if $\omega\frac{\alpha}{1-\alpha}\left(\frac{x^{rs}}{o^p}+f_b\right) > \left(\frac{b+1}{\frac{b}{o^p}+1}\right)^{\frac{1}{b}}x^{pv}$, then the system optimum can not occur for $\beta = 1$.

For the second statement, starting with $\beta^{\text{SO}} > 0$ implies $\frac{\partial \text{PHT}}{\partial \beta}(0) < 0$, we get that

$$-ab\left(\frac{x^{pv}+x^{rs}}{\omega\alpha C}\right)^b + \frac{x^b ab\Delta_b k}{o^p\bar{\alpha}C}\left(\frac{f_b}{\bar{\alpha}C}\right)^{b-1} - a\left(\frac{x^{pv}+x^{rs}}{\omega\alpha C}\right)^b + a\left(\frac{f_b}{\bar{\alpha}C}\right)^b\Delta_p k < 1 - \Delta_p k.$$

Since $1 - \Delta_p k < 0$, it follows that

$$\frac{x^b b\Delta_b k}{o^p\bar{\alpha}C}\left(\frac{f_b}{\bar{\alpha}C}\right)^{b-1} + \left(\frac{f_b}{\bar{\alpha}C}\right)^b\Delta_p k < b\left(\frac{x^{pv}+x^{rs}}{\omega\alpha C}\right)^b + \left(\frac{x^{pv}+x^{rs}}{\omega\alpha C}\right)^b.$$

Since $f_b < x^b$ and $\Delta_p k > 1$, we get that

$$\left(\frac{b}{o^p}+1\right)\left(\frac{f_b}{\bar{\alpha}C}\right)^b < (b+1)\left(\frac{x^{pv}+x^{rs}}{\omega\alpha C}\right)^b .$$

Finally, by utilizing that $b > 1$, we obtain that

$$\omega\frac{\alpha}{1-\alpha}f_b < \left(\frac{b+1}{\frac{b}{o^p}+1}\right)^{\frac{1}{b}}(x^{pv}+x^{rs}) .$$

Therefore, we know that if $\omega\frac{\alpha}{1-\alpha}f_b \geq \left(\frac{b+1}{\frac{b}{o^p}+1}\right)^{\frac{1}{b}}(x^{pv}+x^{rs})$, then $\beta^{\text{SO}} = 0$.

To prove the last part, we observe that for a fixed set of demands and all $\beta \in [0,1]$, it holds that when $\alpha \to 0^+$ then $\frac{\partial \text{PHT}}{\partial \beta} \to -\infty$ and hence $\beta^{\text{SO}} = 1$. This combined with Proposition 2 proves the statement. ■

D. Proof of Proposition 4

Proof We start with the observation that if $\beta^{\text{UE}} \in (0,1)$, then it must hold that $t_{\mathcal{V}} = t_{\mathcal{B}}$. Therefore,

$$1+a\left(\frac{x^{pv}+(1-\beta^{\text{UE}})x^{rs}}{\omega\alpha C}\right)^b = \left(1+a\left(\frac{\beta^{\text{UE}}\frac{x^{rs}}{o^p}+f_b}{\bar{\alpha}C}\right)^b\right)\Delta_p k\,(f_b) .$$

By taking the implicit derivative of the above expression with respect to α, we obtain that

$$\frac{\partial\beta^{\text{UE}}}{\partial\alpha} = -\frac{\frac{1}{\alpha}\left(\frac{x^{pv}+(1-\beta^{\text{UE}})x^{rs}}{\omega\alpha C}\right)^b + \frac{1}{\bar{\alpha}}\Delta_p k\left(\frac{\beta^{\text{UE}}\frac{x^{rs}}{o^p}+f_b}{\bar{\alpha}C}\right)^b}{\frac{x^{rs}}{\omega\alpha C}\left(\frac{x^{pv}+(1-\beta^{\text{UE}})x^{rs}}{\omega\alpha C}\right)^{b-1} + \frac{x^{rs}}{o^p\bar{\alpha}C}\Delta_p k\left(\frac{\beta^{\text{UE}}\frac{x^{rs}}{o^p}+f_b}{\bar{\alpha}C}\right)^{b-1}} ,$$

which is always negative because all the variables in the expression are strictly positive. Moreover, for $\beta = 1$, we observe that $\text{PHT}(1) \to +\infty$ when $\alpha \to 0^+$, implying that the user equilibrium solution β^{UE} always decreases with α. ■

E. Proof of Proposition 5

Proof To prove the first part of the statement, we will show that $\beta^{\text{UE}} = 1$ implies that $(1-\alpha)x^{pv} \geq \omega\alpha\left(\frac{x^{rs}}{o^p}+f_b\right)$. In fact, if $\beta^{\text{UE}} = 1$, it must hold that the solo trip delays in the vehicle network $t_{\mathcal{V}}$ are greater than or equal to the pool trip delays in the bus network $t_{\mathcal{B}}$, and therefore

$$t_f\left(1+a\left(\frac{x^{pv}}{\omega\alpha C}\right)^b\right) \geq t_f\left(1+a\left(\frac{\frac{x^{rs}}{o^p}+f_b}{\bar{\alpha}C}\right)^b\right)\Delta_p k .$$

Since $t_f > 0$ and $\Delta_p k > 1$, we get that

$$\left(\frac{x^{pv}}{\omega\alpha C}\right)^b \geq \left(\frac{\frac{x^{rs}}{o^p}+f_b}{\bar{\alpha}C}\right)^b .$$

Moreover, since $b > 0$, it follows that $(1-\alpha)x^{pv} \geq \omega\alpha\left(\frac{x^{rs}}{o^p}+f_b\right)$. Hence, $(1-\alpha)x^{pv} < \omega\alpha\left(\frac{x^{rs}}{o^p}+f_b\right)$ implies $\beta^{\text{UE}} < 1$.

For the second part of the statement, starting from $(1-\alpha)(x^{pv}+x^{rs}) < \omega\alpha f_b$, we get that $\frac{x^{pv}+x^{rs}}{\omega\alpha C} < \frac{f_b}{\bar{\alpha}C}$. Since $a > 0$ and $b > 0$, it follows that

$$1+a\left(\frac{x^{pv}+x^{rs}}{\omega\alpha C}\right)^b < 1+a\left(\frac{f_b}{\bar{\alpha}C}\right)^b .$$

Given that $t_f > 0$ and $\Delta_p k > 1$, then

$$t_f\left(1+a\left(\frac{x^{pv}+x^{rs}}{\omega\alpha C}\right)^b\right) < t_f\left(1+a\left(\frac{f_b}{\bar{\alpha}C}\right)^b\right)\Delta_p k ,$$

which implies that $t_{\mathcal{V}}(x^{rs}) < t_{\mathcal{B}}(0)$.

To prove the last part of the statement, we first show the existence of a value of α such that $t_{\mathcal{V}}((1-\beta)x^{rs}) < t_{\mathcal{B}}(\beta x^{rs})$ for all $\beta \in [0,1]$. Since for a fixed set of demands and every choice of β, it holds that $t_{\mathcal{V}}((1-\beta)x^{rs}) \to +\infty$ when $\alpha \to 0^+$, such an α must exist. This together with Proposition 4 proves the last part. ■

F. Proof of Proposition 6

Proof A split between solo and pool β is an interior solution for the system optimum if and only if it satisfies

$$(x^{pv}+(1-\beta)x^{rs})\frac{\partial t_{\mathcal{V}}}{\partial\beta} + \beta x^{rs}\frac{\partial t_{\mathcal{B}}}{\partial\beta} + x^b\frac{\partial t_b}{\partial\beta} = x^{rs}(t_{\mathcal{V}} - t_{\mathcal{B}}) .$$

By adding a toll to the utilization of pool users in bus lanes, a user equilibrium split between solo and pool β^{UE} is given by

$$\beta^{\text{UE}} \in \underset{\beta\in[0,1]}{\arg\min} \int_0^{(1-\beta)x^{rs}} t_{\mathcal{V}}(s)\mathrm{d}s + \int_0^{\beta x^{rs}} t_{\mathcal{B}}(s) + \tau_p \mathrm{d}s .$$

Differentiating the two integrals, and using the uniqueness property of the solution, we get that a solution to the user equilibrium problem with tolling satisfies $x^{rs}\tau_p = x^{rs}(t_{\mathcal{V}} - t_{\mathcal{B}})$. Therefore, setting $\tau_p = (\frac{x^{pv}}{x^{rs}} + (1-\beta^{\text{SO}}))\frac{\partial t_{\mathcal{V}}}{\partial\beta} + \frac{\partial t_{\mathcal{B}}}{\partial\beta} + \frac{x^b}{x^{rs}}\frac{\partial t_b}{\partial\beta}$, we recover the necessary condition for the system optimum from (5). Rewriting the toll function and replacing the derivatives with their expressions, we get (4). ■