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# NUMERICAL METHOD FOR SINGULARLY PERTURBED PARABOLIC EQUATIONS IN UNBOUNDED DOMAINS IN THE CASE OF SOLUTIONS GROWING AT INFINITY

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**Abstract** — An initial-boundary value problem is considered in an unbounded domain on the $x$-axis for a singularly perturbed parabolic reaction-diffusion equation; the highest-order derivative of the equation is multiplied by the parameter $\varepsilon^2$, where $\varepsilon \in (0, 1]$. When $x$ tends to $\infty$, the right-hand side of the equation and the initial function increase unboundedly (as $\mathcal{O}(x^2)$), which leads to an unbounded growth of the solution at infinity (as $\mathcal{O}(\Psi(x))$, where $\Psi(x) = x^2 + 1$). For small values of the parameter $\varepsilon$, a parabolic boundary layer arises in a neighbourhood of the lateral part of the boundary. In this problem, the error of a discrete solution in the maximum norm grows without bound as $x \to \infty$ even for fixed values of the parameter $\varepsilon$. In the present paper, the proximity of solutions of the initial-boundary value problem and of its numerical approximations is considered in the weight maximum norm $\| \cdot \|^w$ with the weight function $\Psi^{-1}(x)$; in this norm the solution of the initial-boundary value problem is $\varepsilon$-uniformly bounded. Using the method of special grids condensing in a neighbourhood of the boundary layer, a special finite difference scheme converging $\varepsilon$-uniformly in the weight maximum norm has been constructed.

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## 1. Introduction

Difficulties are well known in solving boundary value problems for singularly perturbed equations in *bounded domains* in the case of sufficiently smooth data (see, e.g., [1, 3, 4, 9, 10, 21] and the bibliography therein). Such singularly perturbed problems call for special numerical methods are required enabling us to approximate solutions with errors independent of the perturbation parameter $\varepsilon$, i.e., $\varepsilon$-uniformly convergent methods. As a rule, in constructing and studying $\varepsilon$-uniformly convergent difference schemes, it is assumed that the data of the boundary value problem under consideration are sufficiently smooth and satisfy the compatibility conditions [7] providing the required smoothness of the problem. The difficulties in the approximation of the solution increase when, except for the boundary layers, *additional difficulties* due to both the *unboundedness of the domain* and the *unbounded growth of the solution* arise.

Special difference schemes for singularly perturbed problems for elliptic and parabolic equations in unbounded domains were studied in [15, 16] and [18], respectively; solutions

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of such problems were assumed to be bounded. In some singularly perturbed problems in unbounded domains that arise in mathematical modelling, the problem solution grows without bound at infinity (see, e.g., [6] and the bibliography therein). However, investigations of special schemes for such problems are unknown in the literature.

Thus, the development and study of $\varepsilon$-uniformly convergent difference schemes for the above wide classes of singularly perturbed problems with singularities is an important problem.

In the present paper, an initial-boundary value problem is considered in an unbounded domain on the $x$-axis for a singularly perturbed parabolic reaction-diffusion equation; the highest-order derivative of the equation is multiplied by the parameter $\varepsilon^2$. The right-hand side of the equation and the initial function increase unboundedly similarly to $\mathcal{O}\big(x^2\big)$ as $|x| \to \infty$, which leads to an unbounded growth of the problem solution. When the parameter $\varepsilon$ tends to zero, a parabolic boundary layer arises.

For fixed values of the parameter $\varepsilon$, the solution of the initial-boundary value problem and its derivatives grow as $\mathcal{O}\big(\Psi(x)\big)$, where $\Psi(x) = x^2 + 1$. This yields an *unbounded growth of pointwise errors* in the solutions of grid approximations with increasing $|x|$ (see estimates (3.1), (3.6) and (4.6), (4.7) in Sections 3 and 4, respectively).

In the present research, the proximity of solutions of the problem and its grid approximations is considered in the *weight maximum norm* $\|\cdot\|^w$ defined as the maximum of relation of the error of the grid solution at the point $(x, t)$ to the function $\Psi(x)$. In this norm, the *solutions* of the initial-boundary value problem and the difference scheme are *$\varepsilon$-uniformly bounded*. It has been established that in the case of problem (2.2), (2.1) (see Section 2), i.e., a problem with sufficiently smooth data satisfying the compatibility conditions, the known implicit difference scheme on a *piecewise-uniform grid* condensing in the *boundary layer* converges $\varepsilon$-uniformly in the weight maximum norm at the rate $\mathcal{O}\big(N^{-2}\ln^2 N + N_0^{-1}\big)$, where $N+1$ is the maximal number of mesh points on the $x$-axis per unit length and $N_0+1$ is the number of mesh points in $t$.

Note that under the construction and justification of $\varepsilon$-uniformly convergent difference schemes, mainly the approaches and methods developed by A.A. Samarskii are used (see, e.g., [11, 12] and the bibliography therein). Difference schemes inheriting the monotonicity property of differential problems are constructed. To justify the convergence of the difference schemes constructed, the majorant function technique is applied. The convergence of $\varepsilon$-uniformly convergent difference schemes is considered in the maximum norm adequate for describing the solutions of problems with boundary and interior layers. Such basic principles allow us to develop pioneer approaches [1, 2, 4] to the construction of $\varepsilon$-uniformly convergent difference schemes for wide classes of singularly perturbed problems for *partial differential equations* [21].

## 2. Problem Formulation. Aim of the Research

**2.1.** On the set $\overline{G}$, where

$$\overline{G} = G \bigcup S, \quad G = D \times (0, T], \quad D = \{x :\ x \in (d, \infty)\}, \quad d \in (-\infty, \infty), \tag{2.1}$$

and the value of $|d|$ can be arbitrarily large, we consider the initial-boundary value problem for the singularly perturbed parabolic equation[1]

$$L_{(2.2)}\, u(x,t) = f(x,t), \quad (x,t) \in G,$$
$$u(x,t) = \varphi(x,t), \quad (x,t) \in S. \tag{2.2}$$

Here

$$L_{(2.2)} \equiv \varepsilon^2\, a(x,t)\, \frac{\partial^2}{\partial x^2} - c(x,t) - p(x,t)\, \frac{\partial}{\partial t},$$

the coefficients $a(x,t)$, $c(x,t)$, $p(x,t)$ and the right-hand side $f(x,t)$ are assumed to be sufficiently smooth on the set $\overline{G}$, moreover,

$$a_0 \leqslant a(x,t) \leqslant a^0, \quad 0 \leqslant c(x,t) \leqslant c^0, \quad p_0 \leqslant p(x,t) \leqslant p^0, \quad (x,t) \in \overline{G}; \quad a_0, p_0 > 0; \tag{2.3}$$

the parameter $\varepsilon$ takes arbitrary values in the open-closed interval (0, 1].

The initial-boundary function $\varphi(x,t)$ is continuous on $S$ and is piecewise-smooth on the sets $S_0$ and $\overline{S}^L$. Here $S = S_0 \bigcup S^L$, $S_0 = \overline{S}_0$ and $S^L$ are the lower and the lateral parts of the boundary $S$, $S^L = \Gamma \times (0,\, T]$, $\Gamma = \overline{D} \setminus D$. The functions $f(x,t)$, $(x,t) \in \overline{G}$, and $\varphi(x,t)$, $(x,t) \in S_0$ grow without bound as $|x| \to \infty$; assume that the following condition holds:

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} f(x,t) \right| \leqslant M\Psi(x), \quad (x,t) \in \overline{G}, \quad k + 2k_0 \leqslant K_f;$$

$$\left| \frac{\partial^{k_0}}{\partial t^{k_0}}\, \varphi(x,t) \right|_{\overline{S}^L}, \ \left| \frac{\partial^{k}}{\partial x^{k}}\, \varphi(x,t) \right|_{S_0} \leqslant M\Psi(x), \quad (x,t) \in S, \quad k, 2k_0 \leqslant K_\varphi, \tag{2.4}$$

where $\Psi(x) = 1 + x^2$ and $K_f$, $K_\varphi > 0$ are constants specified below. For simplicity, we assume that on the set of corner points $S^c = S_0 \bigcap \overline{S}^L$ the compatibility conditions are fulfilled to guarantee the required smoothness of the problem solution [7].

By a solution of the problem (2.2), (2.1), is meant the function $u \in C(\overline{G}) \bigcap C^{2,1}(G)$ that satisfies the differential equation on $G$ and the boundary condition on $S$.

**2.2.** We point out some problems that arise in solving the boundary value problem in the case of condition (2.4).

When $\varepsilon$ tends to zero, a boundary layer with a typical scale $\varepsilon$ in a neighbourhood of the set $S^L$ appears. It is known that in the case of a singularly perturbed problem on bounded domains for sufficiently smooth data, solutions of classical difference schemes do not converge $\varepsilon$-uniformly (see, e.g., [3, 5]).

In the present paper, even for fixed values of the parameter $\varepsilon$ the solution of problem (2.2), (2.1) has singularities generated by the unboundedness of the domain, and also by the right-hand side of the equation and the initial function that grows unboundedly. The solution of the problem grows without bound as $|x| \to \infty$; this solution and its derivatives in $x$ and $t$ are of order $\Psi(x)$, $x \in \overline{D}$ (see the estimates (3.1), (3.6) and (3.2) in Section 3).

Unlike the problems considered earlier in [15, 16, 18], an unbounded growth of derivatives to the problem solution leads to an unbounded growth of errors in the approximation of difference schemes on the problem solution [12]. As a result, the error of the numerical solution in the maximum norm grows as $x \to \infty$ even for fixed values of the parameter $\varepsilon$. For the problem under consideration, the relative error of the solution (with respect to the

[1] The notation $L_{(j.k)}$ ($\overline{G}_{(j.k)}$, $M_{(j.k)}$, $m_{(j.k)}$) means that these operators (domains, constants) are introduced in formula $(j.k)$.

absolute value of the solution in the case when the solution is not too small) does not tend to zero as the mesh step-size tends to zero even for fixed values of the parameter $\varepsilon$ (see discussions in 4.2. in Section 4).

Taking into account such a specific behaviour of the error for solutions of difference schemes, we shall consider the approximation of solutions to problem (2.2), (2.1) in *the weight maximum norm* $\|\cdot\|^w$ defined by the relation

$$\|v\|^w = \sup_{\overline{G}} |v(x,t)|^w, \tag{2.5a}$$

where

$$|v(x,t)|^w = \Psi^{-1}(x)|v(x,t)|, \quad (x,t) \in \overline{G}, \qquad v \in C(\overline{G}). \tag{2.5b}$$

In the case of problems in bounded domains and also in unbounded domains under the condition that the problem data are bounded, the weight maximum norm $\|\cdot\|^w$ is equivalent to the maximum norm.

**Our aim** for the problem (2.2), (2.1) is to construct a difference scheme converging $\varepsilon$-uniformly in the weight maximum norm $\|\cdot\|^w$.

## 3. *A priori* estimates of solutions

We give some estimates for the solutions of problem (2.2), (2.1) and their derivatives needed for the construction

These estimates are obtained similarly to those in [6, 18–20].

**3.1.** Using majorant functions of the type

$$w(x,t) = \Psi(x)\exp(\alpha\, t), \quad (x,t) \in \overline{G},$$

where the value $\alpha$ is chosen sufficiently large, we find the following estimate for the solution:

$$|u(x,t)| \leqslant M\,\Psi(x), \quad (x,t) \in \overline{G}. \tag{3.1}$$

In the case when the solutions of the boundary value problem are sufficiently smooth on the set $\overline{G}$, we consider these solutions in bounded subdomains covering $\overline{G}$. Using the results of [21] (see also [6, 18–20]) developed for problems in bounded domains and taking into account estimate (3.1), we establish the estimate

$$\left|\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, u(x,t)\right| \leqslant M\,\varepsilon^{-k}\,\Psi(x), \quad (x,t) \in \overline{G}, \quad k + 2k_0 \leqslant K. \tag{3.2}$$

The $K$ value is specified by the smoothness of the problem data.

**3.2.** Write the solution of problem (2.2), (2.1) as the sum of the functions

$$u(x,t) = U(x,t) + V(x,t), \quad (x,t) \in \overline{G}, \tag{3.3}$$

where $U(x,t)$ and $V(x,t)$ are the regular and singular parts of the solution. The function $U(x,t)$ is the restriction of the function $U^e(x,t)$, $(x,t) \in \overline{G}^{\,e}$ to the set $\overline{G}$. Here $U^e(x,t)$ is the solution of the problem

$$L^e\, U^e(x,t) = f^e(x,t), \quad (x,t) \in G^e,$$

$$U^e(x,t) = \varphi^e(x,t), \quad (x,t) \in S^e, \tag{3.4}$$

where $\overline{G}^{\,e} = G^e \bigcup S^e$. The domain $G^e$ is an extension of the domain $G$ beyond the boundary $S^L$. The operator $L^e$ and the right-hand side $f^e(x,t)$ of the equation in (3.4) are smooth continuations of the operator $L_{(2.2)}$ and the function $f(x,t)$ on $\overline{G}^{\,e}$, preserving their properties (2.3), (2.4). The function $\varphi^e(x,t)$ is smooth on each piecewise-smooth part of the set $S^e$, and it coincides with the function $\varphi(x,t)$ on the set $S_0$.

Note that in the expansion of the component $U^e(x,t)$, $(x,t) \in \overline{G}^{\,e}$, with respect to powers of the parameter $\varepsilon$ (see, e.g., [18])

$$U^e(x,t) = \sum_{k=0}^{n} \varepsilon^{2k} U_k^e(x,t) + v_U^{[n]e}(x,t) \equiv U^{[n]e}(x,t) + v_U^{[n]e}(x,t), \quad (x,t) \in \overline{G}^e, \quad n \geqslant 0,$$

unlike the paper [18], the functions $U_k^e(x,t)$ and the remainder term $v_U^{[n]e}(x,t)$ grow unboundedly as $|x| \to \infty$.

The function $V(x,t)$, $(x,t) \in \overline{G}$ is the solution of the problem

$$L_{(2.2)}\, V(x,t) = 0, \quad (x,t) \in G, \quad V(x,t) = \begin{cases} \varphi(x,t) - U(x,t), & (x,t) \in S^L, \\ 0, & (x,t) \in S_0. \end{cases}$$

The function $V(x,t)$, $(x,t) \in \overline{G}$, is bounded for bounded values of $d$, and it decreases exponentially as $x - d \to \infty$. The function $\max |V(x,t)|$, $(x,t) \in \overline{G}$, increases without bound as $|d| \to \infty$.

We assume that the compatibility conditions are fulfilled on the set $S^c = S_0 \bigcap \overline{S}^{\,L}$ that ensures the local smoothness of the solution for fixed values of $\varepsilon$ [7]; we suppose also that on the set $\overline{G}^{\,\delta}$, i.e., in the $\delta$-neighbourhood of the set $S_c$, one has

$$u \in C^{l+\alpha,(l+\alpha)/2}(\overline{G}^{\,\delta}), \quad l \geqslant 2, \quad \alpha \in (0,\,1), \tag{3.5}$$

where $\delta$ is a sufficiently small constant.

Under condition (3.5), for the components $U(x,t)$ and $V(x,t)$ in (3.3), we have the estimates

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, U(x,t) \right| \leqslant M \Psi(x), \tag{3.6a}$$

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, V(x,t) \right| \leqslant M\, \varepsilon^{-k}\, \Psi(d) \exp\Big( -m\, \varepsilon^{-1}\, r\big(x,\,\Gamma\big)\Big), \quad (x,t) \in \overline{G}, \quad k + 2k_0 \leqslant K. \tag{3.6b}$$

Here $r\big(x,\,\Gamma\big)$ is the distance from the point $x$ to the set $\Gamma$, and $m$ in (3.6b) is an arbitrary constant.

Thus, we have the following theorem (see [18]).

**Theorem 3.1.** *Let the data of the initial-boundary value problem* (2.2), (2.1) *satisfy the conditions* $a,\ c,\ p,\ f \in C^{l_1, l_1/2}(\overline{G})$, $\varphi \in C(S) \bigcap \{C^{l_1}(S_0) \bigcap C^{l_1/2}(\overline{S}^{\,L})\}$ *for* $l_1 = l + \alpha$ *with* $l = K$ *and* $\alpha \in (0,1)$, *and also let the problem solution satisfy condition* (3.5). *Then the problem solution and its components in the decomposition* (3.3) *satisfy estimates* (3.1) *and* (3.6).

## 4. Classical approximations of the problem on uniform grids

We construct a finite difference scheme based on classical approximations of the initial-boundary value problem (2.2), (2.1) and study its convergence for not too small values of the parameter $\varepsilon$ compared to the stepsize of a uniform mesh in $x$. To construct and investigate the schemes, we apply a technique similar to that used in [19] for a parabolic convection-diffusion equation (see also [21]).

**4.1.** On the set $\overline{G}_{(2.1)}$, we introduce the rectangular grid

$$\overline{G}_h = \overline{D}_h \times \overline{\omega}_0 = \overline{\omega} \times \overline{\omega}_0, \tag{4.1}$$

where $\overline{\omega}$ and $\overline{\omega}_0$ are meshes on the sets $\overline{D}$ and $[0, T]$, respectively; $\overline{\omega}$ is a mesh with an arbitrary distribution of nodes satisfying only the condition $h \leqslant MN^{-1}$, where $h = \max_i h^i$ with $h^i = x^{i+1} - x^i$ for $x^i$, $x^{i+1} \in \overline{\omega}$, and $\overline{\omega}_0$ is a uniform mesh with stepsize $h_0 = TN_0^{-1}$. Here $N + 1$ is the maximum number of nodes in the mesh $\overline{\omega}$ on the unit interval in $\overline{D}$ and $N_0 + 1$ is the number of nodes in the mesh $\overline{\omega}_0$.

We approximate problem (2.2), (2.1) by the finite difference scheme [12]

$$\begin{aligned} \Lambda_{(4.2)}\, z(x,t) &= f(x,t), \quad (x,t) \in G_h, \\ z(x,t) &= \varphi(x,t), \quad (x,t) \in S_h. \end{aligned} \tag{4.2}$$

Here

$$\Lambda_{(4.2)} \equiv \varepsilon^2\, a(x,t)\, \delta_{\overline{x}\widehat{x}} - c(x,t) - p(x,t)\, \delta_{\overline{t}},$$

$$\delta_{\overline{x}\widehat{x}}\, z(x,t) = z_{\overline{x}\widehat{x}}(x,t) = 2(h^i + h^{i-1})^{-1}[\delta_x\, z(x,t) - \delta_{\overline{x}}\, z(x,t)], \quad (x,t) = (x^i,t) \in G_h,$$

is the second-order difference derivative on the nonuniform mesh, $\delta_x z(x,t)$ and $\delta_{\overline{x}} z(x,t)$, $\delta_{\overline{t}}\, z(x,t)$ are the first-order (forward and backward) difference derivatives, $\delta_x z(x,t) = (h^i)^{-1} \times (z(x^{i+1},t) - z(x^i,t))$, $\delta_{\overline{x}} z(x,t) = (h^{i-1})^{-1}(z(x^i,t) - z(x^{i-1},t))$, $\delta_{\overline{t}} z(x,t) = \tau^{-1}\,(z(x^i,t) - z(x^i, t-\tau))$.

The difference scheme (4.2), (4.1) is $\varepsilon$-uniformly monotone [12].

Let the subset $\overline{G}{}^1_h \subseteq \overline{G}_h$ be formed by elementary rectangular cells generated by the nodes of the grid $\overline{G}_h$. Let $\overline{G}{}^1_h = G^1_h \bigcup S^1_h$, where $S^1_h$ is the boundary of the set $\overline{G}{}^1_h$.

The following version of the comparison theorem holds.

**Theorem 4.1.** *Let the functions $z^1(x,t)$ and $z^2(x,t)$, $(x,t) \in \overline{G}_h$, on the subset $\overline{G}{}^1_h$ satisfy the conditions*

$$\Lambda\, z^1(x,t) < \Lambda\, z^2(x,t), \quad (x,t) \in G^1_h, \quad z^1(x,t) > z^2(x,t), \quad (x,t) \in S^1_h.$$

*Then $z^1(x,t) > z^2(x,t)$, $(x,t) \in \overline{G}{}^1_h$.*

**4.2.** In the case of smooth solutions to problem (2.2), (2.1), by virtue of estimate (3.2), the derivatives of the solutions grow unboundedly as $|x| \to \infty$ even for fixed values of the parameter $\varepsilon$. Because of this, the pointwise errors of numerical solutions also increase without bound as $|x|$ grows that leads to an unboundedness of errors in the discrete maximum norm on $\overline{G}$. Thus, for the problem under consideration it is of interest to find an adequate form for the exposition of the discrete solution errors.

It seems to be quite natural to consider the error of the discrete solution as the absolute one in these parts of the domain $\overline{G}$, where the solution of the boundary value problem is bounded, and as the relative one (with respect to the module of the solution to the boundary value problem) in the remaining parts of the domain $\overline{G}$.

Let $z(x,t)$, $(x,t) \in \overline{G}_h$ be the solution of the difference scheme (4.2), (4.1). We shall consider the value of $|u(x,t) - z(x,t)|$, $(x,t) \in \overline{G}_h$, i.e., the pointwise error of the discrete solution, in the metric $\|\cdot\|^r$ defined by the relation

$$\|v\|^r = \|v\|^r_{\overline{G}_h} = \sup_{\overline{G}_h} |v(x,t)|^r, \tag{4.3a}$$

where

$$|v(x,t)|^r = |u(x,t) - z(x,t)| \, \left[1 + |u(x,t)|\,\right]^{-1}, \quad (x,t) \in \overline{G}_h. \tag{4.3b}$$

We call the value $\|u - z\|^r = \|u - z\|^r_{\overline{G}_h}$ *the relative error of the solution* to the difference scheme (4.2), (4.1).

In the case of problems in bounded domains and also in unbounded domains under the condition that the problem data are bounded, the metric $\|\cdot\|^r$ is equivalent to the maximum norm.

To verify the efficiency of the approach based on the relative errors for the description of the discrete solution error in the case of problem (2.2), (2.1), (2.4), we consider the difference scheme (4.2) on the uniform grid

$$\overline{G}_h = \overline{G}^u_h = \overline{\omega} \times \overline{\omega}_0. \tag{4.4}$$

We assume that the data of the initial-boundary value problem (2.2), (2.1), (2.4) satisfy the conditions

$$a(x,t) = p(x,t) = 1, \quad c(x,t) = 0, \quad (x,t) \in \overline{G}; \quad \varepsilon = 1, \tag{4.5a}$$

i.e., the problem (2.2), (2.1), (2.4) is regular with respect to the parameter $\varepsilon$, and let the solution of this problem be the function

$$u(x,t) = x^2 \left[ 2 - \cos(2^{-1}\pi\, x) \right], \quad (x,t) \in \overline{G}, \tag{4.5b}$$

that defines the function $f(x,t)$, $(x,t) \in \overline{G}$.

The solution of problem (2.2), (2.1), (4.5) equals zero for $x = x_0$, $x_0 \equiv 4\,n$, where $n$ is an integer, moreover, under the condition $|x - x_0| \leqslant m_1$ the function $u(x,t)$ is of order $\mathcal{O}\big(x_0^2\,(x - x_0)^2\big)$. The derivative $(\partial^4/\partial x^4)\,u(x,t)$ for $x = x_0$ is of order $\mathcal{O}\big(x^2\big)$.

We consider the error $|u(x,t) - z(x,t)|$ in an $m_1$-neighbourhood of the values $x = x_0$, $(x,t) \in \overline{G}_h$; the function $z(x,t)$, $(x,t) \in \overline{G}_h$ is the solution of the difference scheme (4.2), (4.5) on the uniform grid (4.4). The error in the approximation of the difference scheme on the problem solution is of order $\mathcal{O}\big(x_0^2\,N^{-2}\big)$. Taking into account Theorem 4.1 applied for the estimate $|u(x,t) - z(x,t)|$ on the subdomains from an $m_1$-neighbourhood of the values $x = x_0$, $(x,t) \in \overline{G}_h$, and using majorant functions, we find the estimate

$$\max \big| u(x,t) - z(x,t) \big|^r \geqslant m, \quad (x,t) \in \overline{G}_h, \quad \big| x - x_0 \big| \leqslant m_1.$$

Hence

$$\big\| u - z \big\|^r_{\overline{G}_h} \geqslant m.$$

Thus, the solution of the difference scheme (4.2), (4.5) on the uniform grid (4.4) does not converge on the set $\overline{G}_h$ in the metric $\|\cdot\|^r$.

**Theorem 4.2.** *The solution of the difference scheme* (4.2) *on the uniform grid* (4.4) *considered in the metric* $\|\cdot\|^r$ *does not converge for fixed values of the parameter* $\varepsilon$ *even in the case of the initial-boundary value problem* (2.2), (2.1), (2.4) *with a sufficiently smooth solution.*

**Remark 4.1.** For problem (2.2), (2.1), (2.4) the metric $\|\cdot\|^r$ does not allow us to reveal convergence of the solution to the difference scheme (4.2) on the uniform grid (4.4).

**Remark 4.2.** It could be possible to invent a metric, considering only the pointwise ratio of the error of the discrete solution to the differential problem solution. But such an approach also turns out to be confusing. So, in the case of problem (2.2), (2.1), (4.5) for $x = x_0$ the error of the discrete solution is of order $\mathcal{O}\big(x^2 N^{-2}\big)$, and the ratio of the numerical solution to the exact solution equals infinity.

**Remark 4.3.** In the case of problem (2.2), (2.1), (2.4), (4.5), for the solution of the difference scheme (4.2), (4.5), (4.4) we have the estimate

$$\big|u(x,t) - z(x,t)\big| \leqslant M\left(1 + x^2\right) N^{-2}, \quad (x,t) \in \overline{G}_h,$$

i.e., for fixed values of the parameter $\varepsilon$ the scheme converges on $\overline{G}_h$ in the maximum norm under the condition $x = o(N)$. In the norm $\big\|\cdot\big\|^w_{\overline{G}_h(2.5)}$ we have the estimate

$$\big\|u - z\big\|^w_{\overline{G}_h} \leqslant M\, N^{-2},$$

i.e., for fixed values of $\varepsilon$ the scheme converges in the norm $\big\|\cdot\big\|^w_{\overline{G}_h(2.5)}$ at the rate $\mathcal{O}\big(N^{-2}\big)$.

**4.3.** We now study the convergence of the difference scheme (4.2) on the uniform grid (4.4) in the norm $\|\cdot\|^w$.

Taking into account the *a priori* estimates and the monotonicity of the difference scheme [12]), we obtain the estimate

$$\big|u(x,t) - z(x,t)\big| \leqslant M\,\Psi(x)\left[(\varepsilon + N^{-1})^{-2} N^{-2} + N_0^{-1}\right], \quad (x,t) \in \overline{G}_h. \tag{4.6}$$

For the interpolant $\overline{z}(x,t)$, $(x,t) \in \overline{G}$, which is linear (in $(x,t)$) on partition triangular elements generated by the grid $\overline{G}_h$ (see, e.g., [8])), we have the estimate

$$\big|u(x,t) - \overline{z}(x,t)\big| \leqslant M\,\Psi(x)\left[(\varepsilon + N^{-1})^{-2} N^{-2} + N_0^{-1}\right], \quad (x,t) \in \overline{G}. \tag{4.7}$$

Estimates (4.6) and (4.7) are unimprovable with respect to $N, N_0, \varepsilon$. The error of the discrete solution increases without bound as $x$ grows.

Keeping in mind estimate (3.1), we shall consider the proximity of the functions $u(x,t)$ and $\overline{z}(x,t)$ on the set $\overline{G}$ in the weight maximum norm $\big\|\cdot\big\|^w_{(2.5)}$.

For the solution of the difference scheme (4.2), (4.4) we have the estimate

$$\big|u(x,t) - \overline{z}(x,t)\big|^w \leqslant M\left[(\varepsilon + N^{-1})^{-2} N^{-2} + N_0^{-1}\right], \quad (x,t) \in \overline{G}, \tag{4.8a}$$

where $|\cdot|^w = |\cdot|^w_{(4.3b)}$. Thus, for the solution of the difference scheme (4.2), (4.4) in the weight maximum norm we have also the estimate

$$\big\|u - \overline{z}\big\|^w \leqslant M\left[(\varepsilon + N^{-1})^{-2} N^{-2} + N_0^{-1}\right]. \tag{4.8b}$$

In the weight maximum norm $\|\cdot\|^w$, the difference scheme converges under condition

$$N^{-1} = o(\varepsilon), \quad N_0^{-1} = o(1), \quad \varepsilon \in (0,1]; \tag{4.9}$$

the condition (4.9) is unimprovable.

**Theorem 4.3.** *Let the components in the decomposition* (3.3) *of the solution to the initial-boundary value problem* (2.2), (2.1) *satisfy estimate* (3.6), *where* $K = 4$. *Then the solution of the difference scheme* (4.2), (4.4) *under condition* (4.9) *converges in the norm* $\|\cdot\|^w$. *For the solution of the difference scheme the estimate* (4.8) *holds.*

## 5. A difference scheme on piecewise uniform grids

For the initial-boundary value problem (2.2), (2.1), (2.4) we consider the difference scheme (4.2) on a piecewise uniform grid.

On the set $\overline{G}$, we introduce a grid that condenses in a neighbourhood of the boundary layer similar to that used in [19, 21]

$$\overline{G}_h = \overline{D}_h \times \overline{\omega}_0 = \overline{\omega}^* \times \overline{\omega}_0, \tag{5.1a}$$

where $\overline{\omega}_0 = \overline{\omega}_{0(4.1)}$ and $\overline{\omega}^* = \overline{\omega}^*(\sigma)$ is a piecewise-uniform mesh on $[d, \infty)$; $\sigma$ is a parameter depending on $\varepsilon$ and $N$. We choose $\sigma$ so as to satisfy the condition

$$\sigma = \sigma(N,\, \varepsilon) = \min\,[\,\beta,\, 2\,m^{-1}\,\varepsilon\,\ln N\,], \tag{5.1b}$$

where $\beta$ is an arbitrary number in the interval $(0, 1)$ and $m = m_{(3.6)}$). The set $[d, \infty)$ is divided into two parts: $[d,\, d+\sigma]$ and $[d+\sigma,\, \infty)$; in each part, the mesh step-size is constant and equal to $h^{(1)} = \sigma\,\beta^{-1}\,N^{-1}$ on the interval $[d, d+\sigma]$ and $h^{(2)} = (\,1-\sigma)\,(\,1-\beta)^{-1}\,N^{-1}$ on the set $[d+\sigma,\, \infty)$; $\sigma = \sigma_{(5.1)}$.

We call the difference scheme (4.2) on the piecewise-uniform grid (5.1) the *basic scheme* for problem (2.2), (2.1), (2.4).

With regard to estimate (3.6), for the solution of the basic scheme (4.2), (5.1), we obtain the $\varepsilon$-depending estimate

$$\big|u(x,t) - \overline{z}(x,t)\big|^w \leqslant M\,\Big[\min\big[\varepsilon^{-2},\, \ln^2 N\big]\,N^{-2} + N_0^{-1}\Big], \quad (x,t) \in \overline{G}, \tag{5.2a}$$

and also the $\varepsilon$-uniform estimate

$$\big|u(x,t) - \overline{z}(x,t)\big|^w \leqslant M\,\big[N^{-2}\,\ln^2 N + N_0^{-1}\big], \quad (x,t) \in \overline{G}. \tag{5.2b}$$

Thus, the basic scheme (4.2), (5.1) converges $\varepsilon$-uniformly in the norm $\|\cdot\|^w$ at the rate $\mathcal{O}\big(N^{-2}\,\ln^2 N + N_0^{-1}\big)$, i.e.,

$$\big\|u - \overline{z}\big\|^w \leqslant M\,\big[N^{-2}\,\ln^2 N + N_0^{-1}\big]. \tag{5.2c}$$

From the $\varepsilon$-depending estimate

$$\big\|u - \overline{z}\big\|^w \leqslant M\,\Big[\min\big[\varepsilon^{-2},\, \ln^2 N\big]\,N^{-2} + N_0^{-1}\Big] \tag{5.2d}$$

it follows that for fixed values of the parameter $\varepsilon$ the basic scheme converges in the weight maximum norm $\|\cdot\|^w$ at the rate $\mathcal{O}\big(N^{-2} + N_0^{-1}\big)$

**Theorem 5.1.** *Let in the case of the initial-boundary value problem* (2.2), (2.1), (2.4), *the hypothesis of Theorem* 4.3 *be fulfilled. Then the solution of the basic scheme* (4.2), (5.1) *converges in the weight maximum norm* $\|\cdot\|^w$ $\varepsilon$*-uniformly. The solution of the basic scheme satisfies estimate* (5.2).

## 6. Remarks and generalizations

**6.1.** In the problems under consideration, a reduction either in smoothness to the problem data or in order in the compatibility conditions for the corner point $S^c$ in the case of problem (2.2), (2.1) leads to a reduction in smoothness to the problem solution and the appearance of inner layers. As a result, the $\varepsilon$-uniform convergence rate in $\|\cdot\|^w$ decreases for the schemes constructed (see, e.g., [6, 13, 19, 20] in the case of a problem on bounded domain).

Note that the $\varepsilon$-uniform convergence rate $\mathcal{O}\big(N^{-2}\ln^2 N + N_0^{-1}\big)$ in $\|\cdot\|^w$ of scheme (4.2), (5.1) is preserved in the case of the piecewise-smooth boundary function $\varphi(x,t)$ if on the boundary at the points of nonsmoothness of the function $\varphi(x,t)$ their derivatives in $x$ and in $t$, respectively, up to the second and to the first orders are continuous. Only the furthest reduction in the smoothness of the boundary function $\varphi(x,t)$ at the points of its nonsmoothness leads to a decrease and even to a loss of the $\varepsilon$-uniform convergence rate in $\|\cdot\|^w$.

**6.2.** For the problems studied in unbounded domains, to construct schemes that converge $\varepsilon$-uniformly in $\|\cdot\|^w$ with the improved order of accuracy it is possible to apply the technique developed for problems in bounded domains (see, e.g., [17] and the bibliography therein).

**6.3.** The difference schemes constructed in Sections 4 and 5 belong to formal schemes because their solutions are defined on grids with an infinite number of nodes. The technique from [14, 18] allows us to construct constructive difference schemes (schemes on grids with a finite number of nodes) that converge $\varepsilon$-uniformly in the weight maximum norm $\|\cdot\|^w$ in prescribed bounded domains.

**6.4.** Let the hypothesis of Theorem 4.3 be fulfilled. Then the solution of the difference scheme (4.2) on grid (4.1), where $\overline{D}_h$ is a Bakhvalov grid [1] that condenses in neighbourhoods of the endpoints to the interval $\overline{D}$, converges $\varepsilon$-uniformly in the weight maximum norm $\|\cdot\|^w$. Here, unlike scheme (4.2), (5.1), the convergence rate in $\|\cdot\|^w$ is $\mathcal{O}\big(N^{-2} + N_0^{-1}\big)$.

**6.5.** Likewise, for problem (2.2), (2.1) in the case of condition (2.4), where

$$\Psi(x) = \mathcal{O}\big(\exp(M\,x)\big), \quad x \in \overline{D}, \tag{6.1}$$

schemes are constructed that converge in the norm $\|\cdot\|^w_{(2.5)}$, where $\Psi(x) = \Psi(x)_{(6.1)}$.

**6.6.** Note that the $\varepsilon$-entropy of the solution to problem (2.2), (2.1), (2.4) considered on the unit interval in $\overline{D}$ in the maximum norm grows unboundedly as $x$ increases while in the weight maximum norm it is bounded (the definition of the $\varepsilon$-entropy and its estimates can be found, e.g., in [2]).

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