COMPUTATIONAL METHODS IN APPLIED MATHEMATICS, Vol. 9 (2009), No. 2, pp. 154–164

A DEFECT CORRECTION METHOD FOR THE EVOLUTIONARY CONVECTION — DIFFUSION PROBLEM WITH INCREASED TIME ACCURACY

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Abstract — This work extends the results of [2] by presenting a defect correction method with increased time accuracy. The desired time accuracy is attained with no extra computational cost. The method is applied to the evolutionary transport problem, and is proven to be unconditionally stable. In the defect step, the artificial viscosity parameter is added to the Peclet number as a stability factor, and the system is antidiffused in the correction step. The time accuracy is also increased in the correction step by modifying the right hand side. Hence, we solve the transport problem twice, using the Crank-Nicolson scheme (with the artificial viscosity parameter added), and obtain the accuracy of $O(h^2 + k^4)$. Computational results verifying the claimed space and time accuracy of the approximate solution are presented.

2000 Mathematics Subject Classification: 65M12, 65M60, 76R99.

Keywords: defect correction, convection diffusion equation, artificial viscosity, increased time accuracy.

1. Introduction

Consider the evolutionary convection diffusion problem: find $\mathbf{u} : \Omega \times [0, T] \to \mathbb{R}^d \, (d = 2, 3)$ such that

$$\mathbf{u}_t - \epsilon\Delta\mathbf{u} + \mathbf{b}\cdot\nabla\mathbf{u} + g\mathbf{u} = \mathbf{f} \quad \text{for} \;\; \mathbf{x} \in \Omega, \quad 0 < t \leqslant T,$$
$$\mathbf{u} = 0 \quad \text{on} \;\; \partial\Omega \quad \text{for} \;\; 0 < t \leqslant T, \tag{1.1}$$

where $\mathbf{u}$ is the velocity field, ϵ is a diffusion coefficient, g is an absorbtion/reaction coefficient, and $\mathbf{f}$ is a forcing function.

In problems with a high Peclet number (i.e., $\epsilon \ll 1$), some iterative solvers fail to converge to a solution of (1.1). We propose a certain Defect Correction Method (DCM), that is stable, computes a solution to (1.1) for any ϵ with high space and time accuracy, and is computationally attractive.

The general theory of Defect Correction Methods was presented, e.g., by Böhmer, Hemker, Stetter [1]. In the late 1970s, Hemker (Böhmer, Stetter, Heinrichs, and others) discovered that the DCM, when properly interpreted, is also good for nearly singular problems. Examples, for which this has been successful, include equilibrium Euler equations (Koren, Lallemand [12]), high Reynolds number problems (Layton, Lee, Peterson [13]), and viscoelastic problems (Ervin, Lee [11]).

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There has been an extensive study and development of DC methods for equilibrium flow problems (see, e.g., Hemker[14], Koren[16], Heinrichs[15], Layton, Lee, Peterson[13], Ervin, Lee[11]). On the other hand, there is a parallel development of DCMs, for initial value problems in which no spacial stabilization is used, but the DCM is used to increase the accuracy of the time discretization. This work contains no reports of instabilities (see, e.g., Heywood, Rannacher[10], Hemker, Shishkin[6], Minion[17], Bourlioux, Layton, Minion [3]).

It was shown in [2] that the natural idea of time stepping combined with the DCM in space for the associated quasi-equilibrium problem gives an oscillatory computed solution of poor quality. Another DC method was introduced for an evolutionary PDE that was proven to be stable and accurate.

The method presented in this paper is a modification (aiming at higher a accuracy in time) of the DCM for the evolutionary PDEs presented in [2]. Compared to the method in [2], the right hand side of the system has been modified in the correction step resulting in higher time accuracy with no extra computational cost.

The method proceeds as follows: first we compute the artificial viscosity (AV) approximation $\mathbf{u}_1^h \in X^h$ via

$$L^h_{\epsilon+h}(\mathbf{u}_1^h) = \mathbf{f},$$

where

$$L^h_{\epsilon+h}(\mathbf{u}^h) = \mathbf{u}_t^h - (h+\epsilon)\Delta \mathbf{u}^h + \mathbf{b}\cdot\nabla\mathbf{u}^h + g\mathbf{u}^h.$$

The accuracy of the approximation is then increased by the correction step: compute $\mathbf{u}_2^h \in X^h$ satisfying

$$L^h_{\epsilon+h}(\mathbf{u}_2^h) - L^h_{\epsilon+h}(\mathbf{u}_1^h) = \mathbf{f} - L^h_{\epsilon}(\mathbf{u}_1^h) + B(\mathbf{u}_1^h).$$

Here $B(\cdot)$ is the time difference operator that increases the accuracy of the discrete time difference for $\mathbf{u}_t$.

The Crank — Nicolson time discretization combined with the two-step defect correction method in space leads to the following system of equations for $\mathbf{u}_1^{h,n+1}, \mathbf{u}_2^{h,n+1} \in \mathbf{X}^h, \forall \mathbf{v}^h \in \mathbf{X}^h$ at $t = t_{n+1}$, $n \geqslant 0$, with $k := \Delta t = t_{i+1} - t_i$:

$$\left(\frac{\mathbf{u}_1^{h,n+1} - \mathbf{u}_1^{h,n}}{k}, \mathbf{v}^h\right) + (h+\epsilon)\left(\nabla\left(\frac{\mathbf{u}_1^{h,n+1} + \mathbf{u}_1^{h,n}}{2}\right), \nabla\mathbf{v}^h\right) + \left(\mathbf{b}\cdot\nabla\left(\frac{\mathbf{u}_1^{h,n+1} + \mathbf{u}_1^{h,n}}{2}\right), \mathbf{v}^h\right) + g\left(\frac{\mathbf{u}_1^{h,n+1} + \mathbf{u}_1^{h,n}}{2}, \mathbf{v}^h\right) = (\mathbf{f}(t_{n+1/2}), \mathbf{v}^h), \tag{1.2a}$$

$$\left(\frac{\mathbf{u}_2^{h,n+1} - \mathbf{u}_2^{h,n}}{k}, \mathbf{v}^h\right) + (h+\epsilon)\left(\nabla\left(\frac{\mathbf{u}_2^{h,n+1} + \mathbf{u}_2^{h,n}}{2}\right), \nabla\mathbf{v}^h\right) + \left(\mathbf{b}\cdot\nabla\left(\frac{\mathbf{u}_2^{h,n+1} + \mathbf{u}_2^{h,n}}{2}\right), \mathbf{v}^h\right) + g\left(\frac{\mathbf{u}_2^{h,n+1} + \mathbf{u}_2^{h,n}}{2}, \mathbf{v}^h\right) = (\mathbf{f}(t_{n+1/2}), \mathbf{v}^h) + h\left(\nabla\left(\frac{\mathbf{u}_1^{h,n+1} + \mathbf{u}_1^{h,n}}{2}\right), \nabla\mathbf{v}^h\right) + B_n(\mathbf{u}_1^h, \mathbf{v}^h), \tag{1.2b}$$

where

$$B_0(u,v) = \frac{1}{12}k^2\left(\frac{u^4 - 5u^3 + 9u^2 - 7u^1 + 2u^0}{k^3}, v\right) - \frac{1}{16}k^2\left(\frac{f(t_3) - 5f(t_2) + 7f(t_1) - 3f(t_0)}{k^2}, v\right), \tag{1.3a}$$

$$B_n(u,v) = -\frac{1}{12}k^2\left(\frac{u^{n+2} - 3u^{n+1} + 3u^n - u^{n-1}}{k^3}, v\right) + \frac{1}{16}k^2\left(\frac{f(t_{n+2}) - f(t_{n+1}) - f(t_n) + f(t_{n-1})}{k^2}, v\right), \quad \text{for } n = 1, \ldots, N-2, \tag{1.3b}$$

$$B_{N-1}(u,v) = -\frac{1}{12}k^2\left(\frac{2u^N - 7u^{N-1} + 9u^{N-2} - 5u^{N-3} + u^{N-4}}{k^3}, v\right) +$$
$$\frac{1}{16}k^2\left(\frac{3f(t_N) - 7f(t_{N-1}) + 5f(t_{N-2}) - f(t_{N-3})}{k^2}, v\right). \tag{1.3c}$$

Depending on the current time level, we vary the templates - this demonstrates the resilience of the method. However, the condition $N \geqslant 4$ needs to be satisfied, where $N = T/k$ is the number of time levels.

Note that the operator B is chosen so that for any n the true solution $\mathbf{u}$ satisfies

$$(\frac{\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)}{k}, \mathbf{v}^h) + (h+\epsilon)(\nabla(\frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}), \nabla\mathbf{v}^h) + (\mathbf{b}\cdot\nabla(\frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}), \mathbf{v}^h) +$$
$$g(\frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}, \mathbf{v}^h) = (\mathbf{f}(t_{n+1/2}), \mathbf{v}^h) + h(\nabla(\frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}), \nabla\mathbf{v}^h) +$$
$$B_n(\mathbf{u}, \mathbf{v}^h) + k^4(\mathbf{u}^{(5)}(t_{n+\theta}), \mathbf{v}^h),$$

for some $\theta \in]0,1[$.

Also, the only extra computational cost for the correction step is due to the storage of a few vectors $\mathbf{u}_1^{h,i}$. The Crank-Nicolson scheme is used for computing both $\mathbf{u}_1^h$ and $\mathbf{u}_2^h$, but the time accuracy of the approximate solution $\mathbf{u}_2^h$ is increased to be $O(k^4)$.

The method is proven to be unconditionally stable over the finite time; it is also stable over all time under the assumption $g - (1/2)\nabla\cdot\mathbf{b} \geqslant \beta > 0$.

In section 2, we briefly describe the notation used and a few established results. Stability of the method is proven in Section 3. We conclude with the numerical results proving the error estimates for the method - this is presented in Section 4.

2. Notation and preliminaries

In this section, we present a few definitions, assumptions, and forms used. The variational formulation of (1.1) is naturally stated in

$$\mathbf{X} := H_0^1(\Omega)^d = \{\mathbf{v} : \Omega \to \mathbb{R}^d, \mathbf{v} \in L^2(\Omega)^d, \nabla\mathbf{v} \in L^2(\Omega)^d, \mathbf{v} = 0 \text{ on } \partial\Omega\}.$$

We use the standard L^2 norm, $\|\cdot\|$, and the usual norm on the Sobolev space H^k, namely $\|\cdot\|_k$.

We make several common assumptions.

Assumption 2.1. *There exists a constant β such that*

$$g - \frac{1}{2}\nabla\cdot\mathbf{b} \geqslant \beta > 0.$$

The method is proven to be stable over a finite time even if Assumption 2.1 does not hold. If it does, the method is stable over all time.

We shall assume that the velocity finite element spaces $\mathbf{X}^h \subset \mathbf{X}$ are conforming and have typical approximation properties of finite element spaces commonly in use. Namely, we take $\mathbf{X}^h$ to be spaces of continuous piecewise polynomials of degree m, with $m \geqslant 1$.

The interpolating properties of $\mathbf{X}^h$ are given by the following assumption.

Assumption 2.2. *For any function* $\mathbf{u} \in \mathbf{X}$

$$\inf_{\chi \in \mathbf{X}^h} \left\{ \|\mathbf{u} - \chi\| + h\|\nabla(\mathbf{u} - \chi)\| \right\} \leqslant ch^{r+1}\|\mathbf{u}\|_{r+1}, \quad 1 \leqslant r \leqslant m.$$

We conclude the preliminaries by formulating the discrete Gronwall's lemma (see, e.g., [10]).

Lemma 2.1. *Let k, B, and $a_\mu, b_\mu, c_\mu, \gamma_\mu$, for integers $\mu \geqslant 0$ be nonnegative numbers such that*

$$a_n + k\sum_{\mu=0}^{n} b_\mu \leqslant k\sum_{\mu=0}^{n} \gamma_\mu a_\mu + k\sum_{\mu=0}^{n} c_\mu + B \text{ for } n \geqslant 0.$$

Suppose that $k\gamma_\mu < 1$ for all μ, and set $\sigma_\mu = (1 - k\gamma_\mu)^{-1}$. Then

$$a_n + k\sum_{\mu=0}^{n} b_\mu \leqslant e^{k\sum_{\mu=0}^{n} \sigma_\mu \gamma_\mu} \left[k\sum_{\mu=0}^{n} c_\mu + B \right].$$

3. Stability of the method

In this section, we prove the unconditional stability of the discrete artificial viscosity approximation $\mathbf{u}_1^h$ and use this result to prove stability of the higher order approximation $\mathbf{u}_2^h$. The approximations $\mathbf{u}_1^h$ and $\mathbf{u}_2^h$ are shown to be bounded uniformly in ϵ.

Theorem 3.1. *Let $\mathbf{f} \in L^2(0,T;L^2(\Omega))$. Let $g - (1/2)\nabla\cdot\mathbf{b} \geqslant \beta > -\infty$. If $\beta < 0$, let the length of the time step satisfy $k|\beta| < 1/4$. Then the approximation $\mathbf{u}_1^h$ satisfying* (1.2a) *is stable over the finite time $T < \infty$. Specifically, there exist positive constants C_1, C_2 such that for any $n \leqslant N - 1$*

$$\|\mathbf{u}_1^{h,n+1}\|^2 + k\sum_{i=0}^{n}(h+\epsilon)\|\nabla(\frac{\mathbf{u}_1^{h,i+1} + \mathbf{u}_1^{h,i}}{2})\|^2 \leqslant e^{C_1 T}\left(\|\mathbf{u}_1^{h,0}\|^2 + \frac{1}{C_2}k\sum_{i=0}^{n}\|\mathbf{f}(t_{i+1/2})\|^2\right). \quad (3.1)$$

If Assumption 2.1 *is satisfied, then $\mathbf{u}_1^h$ is stable over all time and*

$$\|\mathbf{u}_1^{h,n+1}\|^2 + \beta k\sum_{i=0}^{n}\|\frac{\mathbf{u}_1^{h,i+1} + \mathbf{u}_1^{h,i}}{2}\|^2 + k\sum_{i=0}^{n}(h+\epsilon)\|\nabla(\frac{\mathbf{u}_1^{h,i+1} + \mathbf{u}_1^{h,i}}{2})\|^2 \leqslant$$

$$\|\mathbf{u}_1^{h,0}\|^2 + \frac{1}{\beta}k\sum_{i=0}^{n}\|\mathbf{f}(t_{i+1/2})\|^2. \quad (3.2)$$

Proof. Take $\mathbf{v}^h = (\mathbf{u}_1^{h,n+1} + \mathbf{u}_1^{h,n})/2 \in \mathbf{X}^h$ in (1.2a). Apply Green's theorem to the last two terms on the left hand side (with $\mathbf{u}_1^h = 0$ on $\partial\Omega$) and use the assumption $g - (1/2)\nabla\cdot\mathbf{b} \geqslant \beta$. The Cauchy-Schwartz and Young inequalities, applied to the left hand side yield

$$\frac{\|\mathbf{u}_1^{h,n+1}\|^2 - \|\mathbf{u}_1^{h,n}\|^2}{2k} + (h+\epsilon)\left\|\nabla(\frac{\mathbf{u}_1^{h,n+1} + \mathbf{u}_1^{h,n}}{2})\right\|^2 +$$

$$\beta\left\|\frac{\mathbf{u}_1^{h,n+1} + \mathbf{u}_1^{h,n}}{2}\right\|^2 \leqslant \left(\mathbf{f}(t_{n+1/2}), \frac{\mathbf{u}_1^{h,n+1} + \mathbf{u}_1^{h,n}}{2}\right). \quad (3.3)$$

If Assumption 2.1 is satisfied, we bound the right hand side of (3.3) by

$$\left|(\mathbf{f}(t_{n+1/2}), \frac{\mathbf{u}_1^{h,n+1}+\mathbf{u}_1^{h,n}}{2})\right| \leqslant \frac{1}{2\beta}\|\mathbf{f}(t_{n+1/2})\|^2 + \frac{\beta}{2}\left\|\frac{\mathbf{u}_1^{h,n+1}+\mathbf{u}_1^{h,n}}{2}\right\|^2. \tag{3.4}$$

Multiply (3.3) by $2k$ and sum over the time levels $i = 0, \dots, n+1$. Using (3.4), we obtain

$$\|\mathbf{u}_1^{h,n+1}\|^2 + k\sum_{i=0}^{n} 2(h+\epsilon)\left\|\nabla(\frac{\mathbf{u}_1^{h,i+1}+\mathbf{u}_1^{h,i}}{2})\right\|^2 + k\sum_{i=0}^{n}\beta\left\|\frac{\mathbf{u}_1^{h,i+1}+\mathbf{u}_1^{h,i}}{2}\right\|^2 \leqslant$$

$$\|\mathbf{u}_1^{h,0}\|^2 + k\sum_{i=0}^{n}\frac{1}{\beta}\|\mathbf{f}(t_{i+1/2})\|^2, \tag{3.5}$$

which proves stability over all time (provided that Assumption 2.1 is satisfied).

If $g - (1/2)\nabla\cdot\mathbf{b} \geqslant \beta$ with $-\infty < \beta < 0$, then we bound the right hand side of (3.3) by

$$\left|(\mathbf{f}(t_{n+1/2}), \frac{\mathbf{u}_1^{h,n+1}+\mathbf{u}_1^{h,n}}{2})\right| \leqslant \frac{1}{4|\beta|}\|\mathbf{f}(t_{n+1/2})\|^2 + |\beta|\left\|\frac{\mathbf{u}_1^{h,n+1}+\mathbf{u}_1^{h,n}}{2}\right\|^2. \tag{3.6}$$

It follows from (3.3), (3.6) and the triangle inequality that

$$\frac{\|\mathbf{u}_1^{h,n+1}\|^2 - \|\mathbf{u}_1^{h,n}\|^2}{2k} + (h+\epsilon)\left\|\nabla(\frac{\mathbf{u}_1^{h,n+1}+\mathbf{u}_1^{h,n}}{2})\right\|^2 \leqslant$$

$$\frac{1}{4|\beta|}\|\mathbf{f}(t_{n+1/2})\|^2 + |\beta|(\|\mathbf{u}_1^{h,n+1}\|^2 + \|\mathbf{u}_1^{h,n}\|^2). \tag{3.7}$$

Multiply (3.7) by $2k$ and sum over the time levels $i = 0, .., n+1$. Under the condition $k|\beta| < 1/4$ the discrete Gronwall's lemma yields

$$\|\mathbf{u}_1^{h,n+1}\|^2 + k\sum_{i=0}^{n} 2(h+\epsilon)\left\|\nabla(\frac{\mathbf{u}_1^{h,i+1}+\mathbf{u}_1^{h,i}}{2})\right\|^2 \leqslant e^{cT}\left(\|\mathbf{u}_1^{h,0}\|^2 + k\sum_{i=0}^{n}\frac{1}{2|\beta|}\|\mathbf{f}(t_{i+1/2})\|^2\right),$$

with $c > 0$. □

We now proceed to the proof of stability of $\mathbf{u}_2^h$. It follows from (1.3) that $|B_i(\mathbf{u}_1^h, \mathbf{v}^h)| \leqslant C\|\mathbf{v}^h\|$ for any $\mathbf{v}^h \in \mathbf{X}^h$, provided that the time difference $(\mathbf{u}_1^{h,i+1} - \mathbf{u}_1^{h,i})/k$ is bounded for any $i = 0, \dots, N-1$. Hence, we begin by establishing the bound for $(\mathbf{u}_1^{h,i+1} - \mathbf{u}_1^{h,i})/k$.

Lemma 3.1. *Let* $\mathbf{u}_1^{h,0} \in H^2(\Omega)$, $\mathbf{f}_t \in L^2(0,T;L^2(\Omega))$. *Let* $g - (1/2)\nabla\cdot\mathbf{b} \geqslant \beta > -\infty$. *If* $\beta < 0$, *let the length of the time step satisfy* $k|\beta| < 1/4$. *Then* $(\mathbf{u}_1^{h,n+1} - \mathbf{u}_1^{h,n})/k$ *is bounded for the finite time* $T < \infty$. *Specifically, there exist positive constants* c, $C = C(\mathbf{b}, g, \mathbf{f}, \mathbf{u}_1^{h,0})$ *such that*

$$\left\|\frac{\mathbf{u}_1^{h,n+1}-\mathbf{u}_1^{h,n}}{k}\right\|^2 + k\sum_{i=1}^{n}\frac{1}{2}(h+\epsilon)\left\|\nabla(\frac{\mathbf{u}_1^{h,i+1}-\mathbf{u}_1^{h,i-1}}{k})\right\|^2 \leqslant$$

$$e^{cT}\left(C + k\sum_{i=1}^{n}\frac{1}{2\beta}\left\|\frac{\mathbf{f}(t_{i+1/2})-\mathbf{f}(t_{i-1/2})}{k}\right\|^2\right).$$

If Assumption 2.1 *is satisfied, then* $(\mathbf{u}_1^{h,n+1} - \mathbf{u}_1^{h,n})/k$ *is bounded over all time: there exists* $C = C(\mathbf{b}, g, \mathbf{f}, \mathbf{u}_1^{h,0})$ *such that*

$$\left\|\frac{\mathbf{u}_1^{h,n+1}-\mathbf{u}_1^{h,n}}{k}\right\|^2 + k\sum_{i=1}^{n}\frac{1}{2}(h+\epsilon)\left\|\nabla(\frac{\mathbf{u}_1^{h,i+1}-\mathbf{u}_1^{h,i-1}}{k})\right\|^2 + k\sum_{i=1}^{n}\frac{1}{4}\beta\left\|\frac{\mathbf{u}_1^{h,i+1}-\mathbf{u}_1^{h,i-1}}{k}\right\|^2 \leqslant$$

$$C + k\sum_{i=1}^{n}\frac{1}{\beta}\left\|\frac{\mathbf{f}(t_{i+1/2}) - \mathbf{f}(t_{i-1/2})}{k}\right\|^2.$$

Proof. Consider (1.2a) at any time level $n \geqslant 1$. Take $\mathbf{v}^h = (\mathbf{u}_1^{h,n+1} - \mathbf{u}_1^{h,n-1})/k$ and make the same choice for $\mathbf{v}^h$ in (1.2a) at the previous time level. Subtracting the resulting equations leads to

$$\Big(\frac{\mathbf{u}_1^{h,n+1} - 2\mathbf{u}_1^{h,n} + \mathbf{u}_1^{h,n-1}}{k}, \frac{\mathbf{u}_1^{h,n+1} - \mathbf{u}_1^{h,n-1}}{k}\Big) + \frac{1}{2}(h+\epsilon)k\left\|\nabla(\frac{\mathbf{u}_1^{h,n+1} - \mathbf{u}_1^{h,n-1}}{k})\right\|^2 +$$

$$\frac{1}{2}k\Big(\mathbf{b}\cdot\nabla\Big(\frac{\mathbf{u}_1^{h,n+1} - \mathbf{u}_1^{h,n-1}}{k}\Big), \frac{\mathbf{u}_1^{h,n+1} - \mathbf{u}_1^{h,n-1}}{k}\Big) + \frac{1}{2}kg\left\|\frac{\mathbf{u}_1^{h,n+1} - \mathbf{u}_1^{h,n-1}}{k}\right\|^2 =$$

$$k\Big(\frac{\mathbf{f}(t_{n+1/2}) - f(t_{n-1/2})}{k}, \frac{\mathbf{u}_1^{h,n+1} - \mathbf{u}_1^{h,n-1}}{k}\Big).$$

Apply Green's theorem to the last two terms on the left hand side (with $(\mathbf{u}_1^{h,n+1} - \mathbf{u}_1^{h,n-1})/k = 0$ on $\partial\Omega$) and use the assumption $g - (1/2)\nabla\cdot b \geqslant \beta$. Rewrite the first term on the left hand side, using the identity

$$\Big(\frac{\mathbf{u}_1^{h,n+1} - 2\mathbf{u}_1^{h,n} + \mathbf{u}_1^{h,n-1}}{k}, \frac{\mathbf{u}_1^{h,n+1} - \mathbf{u}_1^{h,n-1}}{k}\Big) = \left\|\frac{\mathbf{u}_1^{h,n+1} - \mathbf{u}_1^{h,n}}{k}\right\|^2 - \left\|\frac{\mathbf{u}_1^{h,n} - \mathbf{u}_1^{h,n-1}}{k}\right\|^2.$$

This yields

$$\left\|\frac{\mathbf{u}_1^{h,n+1} - \mathbf{u}_1^{h,n}}{k}\|^2 - \|\frac{\mathbf{u}_1^{h,n} - \mathbf{u}_1^{h,n-1}}{k}\right\|^2 + \frac{1}{2}(h+\epsilon)k\left\|\nabla(\frac{\mathbf{u}_1^{h,n+1} - \mathbf{u}_1^{h,n-1}}{k})\right\|^2 +$$

$$\frac{1}{2}k\beta\left\|\frac{\mathbf{u}_1^{h,n+1} - \mathbf{u}_1^{h,n-1}}{k}\right\|^2 \leqslant k\Big(\frac{\mathbf{f}(t_{n+1/2}) - \mathbf{f}(t_{n-1/2})}{k}, \frac{\mathbf{u}_1^{h,n+1} - \mathbf{u}_1^{h,n-1}}{k}\Big). \tag{3.8}$$

Sum (3.8) over all time levels $i = 1, \dots, n$ and consider the cases $\beta > 0$ and $-\infty < \beta < 0$ separately, as in the proof of Theorem 3.1. If the Assumption 2.1 holds, this yields

$$\left\|\frac{\mathbf{u}_1^{h,n+1} - \mathbf{u}_1^{h,n}}{k}\right\|^2 + k\sum_{i=1}^{n}\frac{1}{2}(h+\epsilon)\left\|\nabla\Big(\frac{\mathbf{u}_1^{h,i+1} - \mathbf{u}_1^{h,i-1}}{k}\Big)\right\|^2 +$$

$$k\sum_{i=1}^{n}\frac{1}{4}\beta\left\|\frac{\mathbf{u}_1^{h,i+1} - \mathbf{u}_1^{h,i-1}}{k}\right\|^2 \leqslant \left\|\frac{\mathbf{u}_1^{h,1} - \mathbf{u}_1^{h,0}}{k}\right\|^2 + k\sum_{i=1}^{n}\frac{1}{\beta}\left\|\frac{\mathbf{f}(t_{i+1/2}) - \mathbf{f}(t_{i-1/2})}{k}\right\|^2. \tag{3.9}$$

If $-\infty < \beta < 0$ and $k|\beta| < 1/4$, it follows from the discrete Gronwall's lemma that

$$\left\|\frac{\mathbf{u}_1^{h,n+1} - \mathbf{u}_1^{h,n}}{k}\right\|^2 + k\sum_{i=1}^{n}\frac{1}{2}(h+\epsilon)\left\|\nabla\Big(\frac{\mathbf{u}_1^{h,i+1} - \mathbf{u}_1^{h,i-1}}{k}\Big)\right\|^2 \leqslant$$

$$e^{cT}\Big(\left\|\frac{\mathbf{u}_1^{h,1} - \mathbf{u}_1^{h,0}}{k}\right\|^2 + k\sum_{i=1}^{n}\frac{1}{2\beta}\left\|\frac{\mathbf{f}(t_{i+1/2}) - \mathbf{f}(t_{i-1/2})}{k}\right\|^2\Big). \tag{3.10}$$

To complete the proof, we need a bound on $\|(\mathbf{u}_1^{h,1} - \mathbf{u}_1^{h,0})/k\|^2$. Consider (1.2a) at $n = 0$ and take $v^h = (\mathbf{u}_1^{h,1} - \mathbf{u}_1^{h,0})/k$. This gives

$$\left\|\frac{\mathbf{u}_1^{h,1} - \mathbf{u}_1^{h,0}}{k}\right\|^2 + (h+\epsilon)\Big(\nabla\Big(\frac{\mathbf{u}_1^{h,1} + \mathbf{u}_1^{h,0}}{2}\Big), \nabla\Big(\frac{\mathbf{u}_1^{h,1} - \mathbf{u}_1^{h,0}}{k}\Big)\Big) +$$

$$\left(\mathbf{b}\cdot\nabla\left(\frac{\mathbf{u}_1^{h,1}+\mathbf{u}_1^{h,0}}{2}\right),\frac{\mathbf{u}_1^{h,1}-\mathbf{u}_1^{h,0}}{k}\right)+g\left(\frac{\mathbf{u}_1^{h,1}+\mathbf{u}_1^{h,0}}{2},\frac{\mathbf{u}_1^{h,1}-\mathbf{u}_1^{h,0}}{k}\right)=\left(\mathbf{f}(t_{1/2}),\frac{\mathbf{u}_1^{h,1}-\mathbf{u}_1^{h,0}}{k}\right). \tag{3.11}$$

Using the identity $(\mathbf{u}_1^{h,1}+\mathbf{u}_1^{h,0})/2=(\mathbf{u}_1^{h,1}-\mathbf{u}_1^{h,0})/2+\mathbf{u}_1^{h,0}$, we can rewrite the last three terms on the left hand side of (3.11). Applying Green's theorem as in the proofs above yields

$$\left\|\frac{\mathbf{u}_1^{h,1}-\mathbf{u}_1^{h,0}}{k}\right\|^2+(h+\epsilon)\left\|\nabla\left(\frac{\mathbf{u}_1^{h,1}-\mathbf{u}_1^{h,0}}{k}\right)\right\|^2+(h+\epsilon)\left(\Delta\mathbf{u}_1^{h,0},\frac{\mathbf{u}_1^{h,1}-\mathbf{u}_1^{h,0}}{k}\right)+$$

$$\frac{1}{2}k\beta\left\|\frac{\mathbf{u}_1^{h,1}-\mathbf{u}_1^{h,0}}{k}\right\|^2+\left(\mathbf{b}\cdot\nabla\mathbf{u}_1^{h,0},\frac{\mathbf{u}_1^{h,1}-\mathbf{u}_1^{h,0}}{k}\right)+g\left(\mathbf{u}_1^{h,0},\frac{\mathbf{u}_1^{h,1}-\mathbf{u}_1^{h,0}}{k}\right)\leqslant\left(\mathbf{f}(t_{1/2}),\frac{\mathbf{u}_1^{h,1}-\mathbf{u}_1^{h,0}}{k}\right).$$

The Cauchy — Schwartz and Young inequalities give

$$\frac{1}{2}\left\|\frac{\mathbf{u}_1^{h,1}-\mathbf{u}_1^{h,0}}{k}\right\|^2+(h+\epsilon)\left\|\nabla\left(\frac{\mathbf{u}_1^{h,1}-\mathbf{u}_1^{h,0}}{k}\right)\right\|^2+\frac{1}{2}k\beta\left\|\frac{\mathbf{u}_1^{h,1}-\mathbf{u}_1^{h,0}}{k}\right\|^2\leqslant C,$$

where $C=\|\mathbf{b}\cdot\nabla\mathbf{u}_1^{h,0}\|^2+2g^2\|\mathbf{u}_1^{h,0}\|^2+2(h+\epsilon)^2\|\Delta\mathbf{u}_1^{h,0}\|^2+\|\mathbf{f}(t_{1/2})\|^2<\infty$. Hence, if Assumption 2.1 is satisfied or if $-\infty<\beta<0$ and $k|\beta|<1/4$, we obtain the bound

$$\left\|\frac{\mathbf{u}_1^{h,1}-\mathbf{u}_1^{h,0}}{k}\right\|^2+(h+\epsilon)\left\|\nabla\left(\frac{\mathbf{u}_1^{h,1}-\mathbf{u}_1^{h,0}}{k}\right)\right\|^2\leqslant C<\infty. \tag{3.12}$$

Inserting the bound on $\|(\mathbf{u}_1^{h,1}-\mathbf{u}_1^{h,0})/k\|^2$ into (3.9) and (3.10) completes the proof. □

The unconditional stability of $\mathbf{u}_2^h$ (uniform in ϵ) follows from Theorem 3.1 and Lemma 3.1. The last term in the right hand side of (1.2b) is bounded by means of Lemma 3.1.

Theorem 3.2. *Let the assumptions of Theorem* 3.1 *and Lemma* 3.1 *be satisfied.*

Let $g-(1/2)\nabla\cdot\mathbf{b}\geqslant\beta>-\infty$. If $\beta<0$, let the length of the time step satisfy $k|\beta|<1/4$. Then the approximation $\mathbf{u}_2^h$ satisfying (1.2b) *is stable over the finite time $T<\infty$. Specifically, there exist positive constants c_1, C_2, C_3 such that for any $n\leqslant N-1$*

$$\|\mathbf{u}_2^{h,n+1}\|^2+k\sum_{i=0}^{n}(h+\epsilon)\left\|\nabla\left(\frac{\mathbf{u}_2^{h,i+1}+\mathbf{u}_2^{h,i}}{2}\right)\right\|^2\leqslant$$

$$e^{c_1T}\left(C_3+\|\mathbf{u}_2^{h,0}\|^2+\|\mathbf{u}_1^{h,0}\|^2+\frac{1}{C_2}k\sum_{i=0}^{n}\|\mathbf{f}(t_{i+1/2})\|^2\right).$$

If Assumption 2.1 *is satisfied, then $\mathbf{u}_1^h$ is stable over all time and*

$$\|\mathbf{u}_2^{h,n+1}\|^2+\beta k\sum_{i=0}^{n}\left\|\frac{\mathbf{u}_2^{h,i+1}+\mathbf{u}_2^{h,i}}{2}\right\|^2+k\sum_{i=0}^{n}(h+\epsilon)\left\|\nabla\left(\frac{\mathbf{u}_2^{h,i+1}+\mathbf{u}_2^{h,i}}{2}\right)\right\|^2\leqslant$$

$$C+\|\mathbf{u}_2^{h,0}\|^2+\|\mathbf{u}_1^{h,0}\|^2+\frac{1}{\beta}k\sum_{i=0}^{n}\|\mathbf{f}(t_{i+1/2})\|^2,$$

with $C=C(\mathbf{b},g,\mathbf{f},\mathbf{u}_1^{h,0})$.

Proof. The proof resembles that of Theorem 3.1. Take $\mathbf{v}^h = (\mathbf{u}_2^{h,n+1} + \mathbf{u}_2^{h,n})/2 \in X^h$ in (1.2b). Apply Green's theorem to the last two terms on the left hand side. This gives

$$\frac{\|\mathbf{u}_2^{h,n+1}\|^2 - \|\mathbf{u}_2^{h,n}\|^2}{2k} + (h+\epsilon)\left\|\nabla(\frac{\mathbf{u}_2^{h,n+1} + \mathbf{u}_2^{h,n}}{2})\right\|^2 + \beta\left\|\frac{\mathbf{u}_2^{h,n+1} + \mathbf{u}_2^{h,n}}{2}\right\|^2 \leqslant$$

$$\left(\mathbf{f}(t_{n+1/2}), \frac{\mathbf{u}_2^{h,n+1} + \mathbf{u}_2^{h,n}}{2}\right) + h\left(\nabla\left(\frac{\mathbf{u}_1^{h,n+1} + \mathbf{u}_1^{h,n}}{2}\right), \nabla\left(\frac{\mathbf{u}_2^{h,n+1} + \mathbf{u}_2^{h,n}}{2}\right)\right) +$$

$$B_n\left(\mathbf{u}_1^h, \frac{\mathbf{u}_2^{h,n+1} + \mathbf{u}_2^{h,n}}{2}\right). \tag{3.13}$$

It is easy to verify that for any n

$$\left|B_n\left(\mathbf{u}_1^h, \frac{\mathbf{u}_2^{h,n+1} + \mathbf{u}_2^{h,n}}{2}\right)\right| \leqslant C + \frac{1}{2}|\beta|\left\|\frac{\mathbf{u}_2^{h,n+1} + \mathbf{u}_2^{h,n}}{2}\right\|^2, \tag{3.14}$$

where

$$C = \frac{1}{|\beta|}\max_{0\leqslant i\leqslant N-1}\left\|\frac{\mathbf{u}_1^{h,i+1} - \mathbf{u}_1^{h,i}}{k}\right\|^2 + \frac{1}{|\beta|}k^2\max_{0\leqslant i\leqslant N-1}\left\|\frac{\mathbf{f}(t_{i+1}) - \mathbf{f}(t_i)}{k}\right\|^2.$$

It also follows from Lemma 3.1 that this constant is finite, $C < \infty$.

Using Cauchy — Schwartz and Young inequalities gives

$$h\left|\left(\nabla(\frac{\mathbf{u}_1^{h,n+1} + \mathbf{u}_1^{h,n}}{2}\right), \nabla\left(\frac{\mathbf{u}_2^{h,n+1} + \mathbf{u}_2^{h,n}}{2}\right)\right)\right| \leqslant$$

$$\frac{1}{2}(h+\epsilon)\left\|\nabla(\frac{\mathbf{u}_2^{h,n+1} + \mathbf{u}_2^{h,n}}{2})\right\|^2 + \frac{h^2}{2(h+\epsilon)^2}(h+\epsilon)\left\|\nabla(\frac{\mathbf{u}_1^{h,n+1} + \mathbf{u}_1^{h,n}}{2})\right\|^2. \tag{3.15}$$

Multiply (3.13) by $2k$, sum over the time levels and use (3.14), (3.15). Theorem 3.1 gives the bound on $k\sum_{i=0}^{n}(h+\epsilon)\|\nabla((\mathbf{u}_1^{h,i+1} + \mathbf{u}_1^{h,i})/2)\|^2$. The cases where Assumption 2.1 holds and where $-\infty < \beta < 0$, $k|\beta| < 1/4$ are treated as in the proof of Theorem 3.1. □

Thus, the method is unconditionally stable for all time, provided that Assumption 2.1 is satisfied. The approximate solutions $\mathbf{u}_1^h$ and $\mathbf{u}_2^h$ are bounded uniformly in h and ϵ. If the condition $g - (1/2)\nabla\cdot\mathbf{b} \geqslant \beta$ is satisfied with $-\infty < \beta < 0$, then the assumption $k|\beta| < 1/4$ is needed to conclude the stability and uniform boundedness of the approximate solutions over a finite time.

4. Computational results

Based on the results of [2] (and the general theory of Defect Correction Methods), we are expecting to obtain the following error estimates:

$$\|\mathbf{u} - \mathbf{u}_1^h\|_{L^2(0,T;L^2(\Omega))} \leqslant C(h + k^2),$$

$$\|\mathbf{u} - \mathbf{u}_2^h\|_{L^2(0,T;L^2(\Omega))} \leqslant C(h\|\mathbf{u} - \mathbf{u}_1^h\|_{L^2(0,T;L^2(\Omega))} + k^4) \leqslant C(h^2 + hk^2 + k^4).$$

Consider the following transport problem in $\Omega = [0,1]\times[0,1]$: find u satisfying (1.1) with $\epsilon = 10^{-5}$, $b = (1,1)^\top$, $g = 1$ and $f = [(2 + 2\epsilon\pi^2)\sin(\pi x)\sin(\pi y) + \pi\sin(\pi x + \pi y)]e^t$. This problem has a solution $u = \sin(\pi x)\sin(\pi y)e^t$.

The results presented in the following tables 4.1 and 4.2 are obtained by using the $FreeFEM++$ software [18]. In order to draw conclusions about the convergence rate, we take $k = h$ and $k = \sqrt{h}$. Note that the method needs the number of time steps $N \geqslant 4$.

We have chosen to take $k = h$ and $k = \sqrt{h}$ for the following reason. We are looking for constants α, β such that the error is bounded by $O(h^\alpha + k^\beta)$. Specifically, we are trying to show that for the approximation $\mathbf{u}_1$ one obtains $\alpha \geqslant 1$, $\beta \geqslant 2$. For the approximation $\mathbf{u}_2$ we want $\alpha \geqslant 2$, $\beta \geqslant 4$. In Table 4.1 we took $k = h$; therefore, the "rate" column gives $\min(\alpha, \beta) = 1$ for $\|\mathbf{u} - \mathbf{u}_1^h\|$ and $\min(\alpha, \beta) = 1.6$ for $\|\mathbf{u} - \mathbf{u}_2^h\|$. Similarly, from Table 4.2 we obtain that $\min(\alpha, \beta/2) = 1$ for $\|\mathbf{u} - \mathbf{u}_1^h\|$ and $\min(\alpha, \beta/2) = 1.6$ for $\|\mathbf{u} - \mathbf{u}_2^h\|$. This yields the result $\alpha \geqslant 1$, $\beta \geqslant 2$ for the approximation $\mathbf{u}_1^h$. However, for the approximation $\mathbf{u}_2^h$ the results from Tables 4.1, 4.2 only yield $\alpha \geqslant 3/2$, $\beta \geqslant 3$. The reason for the lower convergence rate is the error in the boundary layer.

Table 4.1. **Error estimates, $\epsilon = 10^{-5}$, $T = 1$, $k = h$**

h	$\|\mathbf{u} - \mathbf{u}_1^h\|_{L^2(0,T;L^2(\Omega))}$	rate	$\|\mathbf{u} - \mathbf{u}_2^h\|_{L^2(0,T;L^2(\Omega))}$	rate
1/4	0.648482		0.36992	
1/8	0.406708	0.6731	0.159371	1.2148
1/16	0.233742	0.7991	0.0590029	1.4335
1/32	0.126373	0.8872	0.0202292	1.5443

Table 4.2. **Error estimates, $\epsilon = 10^{-5}$, $T = 1$, $k = \sqrt{h}$**

h	$\|\mathbf{u} - \mathbf{u}_1^h\|_{L^2(0,T;L^2(\Omega))}$	*rate*	$\|\mathbf{u} - \mathbf{u}_2^h\|_{L^2(0,T;L^2(\Omega))}$	*rate*
1/16	0.267117		0.059804	
1/64	0.0717964	0.9477	0.00712605	1.5345
1/256	0.0179559	0.9997	0.00076384	1.6109

The defect correction methods are known to not resolve the problem of oscillations in the boundary layer; however, we were able to demonstrate that for our method the oscillations do not spread beyond the boundary layer. This is verified by the figure plots of the computed solution $\mathbf{u}_2^h$, as the mesh size and the time step are decreased.

The nonphysical oscillations in the upper right corner of Figs 4.1–4.3 are shown to be restricted to the boundary layer — they decrease in size as the mesh size h decreases. Hence, we compute the approximation error away from the boundary layer, namely in $(0, 0.75) \times (0, 0.75)$, in order to verify the true convergence rates of the method.

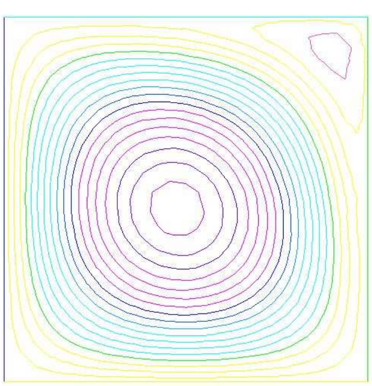

Fig. 4.1. Computed solution $\mathbf{u}_2^h$, $k = h = 1/8$

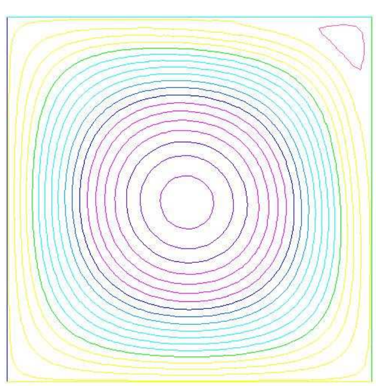

Fig. 4.2. Computed solution $\mathbf{u}_2^h$, $k = h = 1/16$

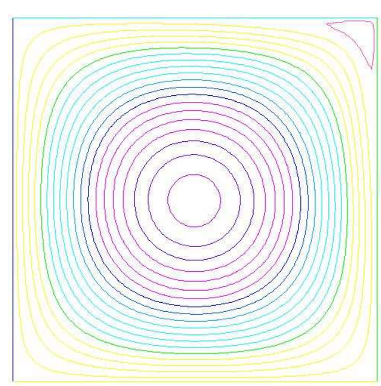

Fig. 4.3. Computed solution $\mathbf{u}_2^h$, $k = h = 1/32$

The first "rate" columns in tables 4.3, 4.4 show that we obtain $\alpha \geqslant 1$, $\beta \geqslant 2$ for the first approximation $\mathbf{u}_1$ and, therefore, we have $\|\mathbf{u} - \mathbf{u}_1^h\|_{L^2(0,T;L^2(\Omega))} \leqslant C(h + k^2)$. We obtain from the second "rate" columns that $\alpha \geqslant 2$, $\beta \geqslant 4$, and thus $\|\mathbf{u}-\mathbf{u}_2^h\|_{L^2(0,T;L^2(\Omega))} \leqslant C(h^2+hk^2+k^4)$.

Table 4.3. **Error estimates in $(0, 0.75) \times (0, 0.75)$, $\epsilon = 10^{-5}$, $k = h$**

h	$\|\mathbf{u} - \mathbf{u}_1^h\|_{L^2(0,T;L^2(\Omega))}$	rate	$\|\mathbf{u} - \mathbf{u}_2^h\|_{L^2(0,T;L^2(\Omega))}$	rate
1/4	0.545619		0.26598	
1/8	0.32111	0.7648	0.0844715	1.6548
1/16	0.172327	0.8979	0.02094	2.0122

Table 4.4. **Error estimates in $(0, 0.75) \times (0, 0.75)$, $\epsilon = 10^{-5}$, $k = \sqrt{h}$**

h	$\|\mathbf{u} - \mathbf{u}_1^h\|_{L^2(0,T;L^2(\Omega))}$	rate	$\|\mathbf{u} - \mathbf{u}_2^h\|_{L^2(0,T;L^2(\Omega))}$	rate
1/16	0.197961		0.0186758	
1/64	0.0491873	1.0044	0.0016107	1.7677
1/256	0.0120236	1.0162	0.000112279	1.9213

Hence, the computational results verify the claimed accuracy of the method away from boundaries. Also, the oscillations of the computed solution do not spread outside the boundary layer.

Acknowledgements. Part of this work was supported by NSF Grant DMS–0508260.

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