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A POSTERIORI ERROR ESTIMATES FOR APPROXIMATE SOLUTIONS OF THE BARENBLATT-BIOT POROELASTIC MODEL

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Abstract — The paper is concerned with the Barenblatt-Biot model in the theory of poroelasticity. We derive a guaranteed estimate of the difference between exact and approximate solutions in a combined norm that encompasses errors for the pressure fields computed from the diffusion part of the model and errors related to stresses (strains) of the elastic part. Estimates do not contain generic (mesh-dependent) constants and are valid for any conforming approximation of the pressure and stress fields.

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1. Introduction

The standard mathematical model for diffusive flow in an elastic porous medium is the Biot diffusion-deformation model of poroelasticity [8] based on the coupling between the pore-fluid potential and the solid stress fields. The basic constitutive equations relate the total stress to both the effective stress given by the strain of the structure and to the potential arising from the pore fluid. The model consists of a momentum balance equation combined with Hooke law for elastic deformation, and a continuity equation combined with the Darcy law. Originally, the Biot model was designed for homogeneous porous media or single porosity media. The representation of porosity and permeability in naturally occurring materials often requires several distinct spatial scales. As for instance, in a reservoir model, the presence of heterogeneities as highly permeable channels has a significant impact on the flow properties of the reservoir rock. Two or more scales of permeability are usually observed, which are also referred to as dual permeability models.

Studies show that even for a single-phase flow in relatively simple porous media, such as sandstone, the fluid flows through a very small portion of the pore space, while its major portion remains stagnant. A system of connected highly permeable channels characterized by a relatively simple pore space geometry provides a fluid flow through the reservoir. The remainder of the reservoir characterized by tortuous pores and pore throats is significantly less permeable. The highly permeable channel component of the reservoir is relatively small, and the remainder of the reservoir contains most of the fluid. This contrast leads to the dual medium model of reservoir rock, originally proposed by Barenblatt et al. [6] in the rigid case.

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According to this model, the fluid flow in matrix blocks is local, and only the local exchange of fluid between individual blocks and the surrounding high permeable channels is supported. This model contains a system of two diffusion equations, one for each component, coupled by a distributed exchange term that, in its simplest form, is proportional to the potential difference between fluids in the two components.

A combination of the Barenblatt double–diffusion approach and the Biot diffusion–deformation theory leads to what we call the Barenblatt - Biot poroelastic model representing the double diffusion in elastic porous media. It takes the form

$$\begin{aligned} -\nabla\cdot(\mathbb{L}\,\varepsilon(\mathbf{u}))+\alpha_1\nabla p_1+\alpha_2\nabla p_2 &= \mathbf{f}(x,t),\\ c_1\dot{p}_1-\nabla\cdot(k_1\nabla p_1)+\alpha_1\nabla\cdot\dot{\mathbf{u}}+\kappa(p_1-p_2) &= h_1(x,t),\\ c_2\dot{p}_2-\nabla\cdot(k_2\nabla p_2)+\alpha_2\nabla\cdot\dot{\mathbf{u}}+\kappa(p_2-p_1) &= h_2(x,t), \end{aligned} \tag{1.1}$$

where $\mathbf{u}$ is the displacement of the solid skeleton and p_1 and p_2 are the fluid potentials in the respective components. With the vector gradient operator ∇, the linear Green strain tensor $\varepsilon(\cdot)$ is written as

$$\varepsilon(u) := \frac{1}{2}\left(\nabla\mathbf{u}+(\nabla\mathbf{u})^T\right). \tag{1.2}$$

The fourth-order elastic stiffness tensor $\mathbb{L}$ defines the stress tensor σ using the Hook law

$$\sigma := \mathbb{L}\,\varepsilon(\mathbf{u}).$$

In general, the permeabilities k_1 and k_2 may be heterogeneous and anisotropic tensors, which may be functions of the deformation. Herein, we will neglect this dependence and only consider constant, scalar, and homogeneous permeabilities. The constants α_1 and α_2 measure changes of porosities due to the applied volumetric strain. Mathematical analysis of this model based on the theory of implicit evolution equations in Hilbert spaces was performed in [21].

We note that multiple continua models are applicable to several other porous media problems. We mention two cases in particular. Firstly, contaminant transport experiments clearly indicate that the particle dispersion is non-Fickian, as reviewed in [7]. This makes both dual and multiple continua models of interest, with dual media approaches already common in applications. The use of more than two flowing continua was discussed by Gwo et al. [11], and a single flowing continuum coupled to multiple non-flowing continua (traps) was reviewed in [7]. The second application is the heat transfer in fractured rocks, in particular, related to modelling of geothermal heat extraction. Here, the slow interaction of diffusive heat transfer in the rock has to be modeled together with the fast fluid flow in fractures. The approach is to use multiple continua, frequently as many as four or more [15].

The aim of this paper is not to develop a numerical solver for the Barenblatt-Biot poroelastic model but to derive a reliable a posteriori estimate. In the last few decades, a posteriori error estimates for linear elliptic and parabolic problems have been intensively investigated. The reader will find discussions of the main approaches to the a posteriori error estimation of finite element approximations (such as residual or gradient averaging methods) in monographs [1, 4, 5, 23] and papers [2, 3, 10, 12, 13] and in the literature cited therein.

Here, we extend the techniques described in books [14, 18] and derive functional a posteriori estimates for the static case of the Barenblatt-Biot system

$$\begin{aligned} -\nabla\cdot(\mathbb{L}\,\varepsilon(\mathbf{u}))+\alpha_1\nabla p_1+\alpha_2\nabla p_2 &= \mathbf{f}(x),\\ -\nabla\cdot(k_1\nabla p_1)+\kappa(p_1-p_2) &= h_1(x),\\ -\nabla\cdot(k_2\nabla p_2)+\kappa(p_2-p_1) &= h_2(x), \end{aligned} \tag{1.3}$$

which is considered in a bounded connected domain $\Omega \subset \mathcal{R}^d$ with the Lipschitz continuous boundary Γ.

There are various boundary conditions motivated by hydrological applications, among which four boundary conditions, applicable to different parts of the boundary $\Gamma = \bigcup \Gamma_i$ represent the most typical cases.
1. A saturated land surface, Γ_1*, with infiltration and evaporation* is modeled as

$$\sigma(\mathbf{u})\mathbf{n} = 0 \qquad \text{(normal stress − free condition)}, \tag{1.4}$$
$$\psi_{\Gamma_1} = \mathbf{n} \cdot (-k_1 \nabla p_1 - k_2 \nabla p_2) \qquad \text{(normal fluid flux)}, \tag{1.5}$$

where $\mathbf{n}$ is the unit outward normal vector and ψ_{Γ_1} is a given function. We fulfil this boundary condition by specifying that the normal component of the potential gradients at the boundary is

$$\mathbf{n} \cdot \nabla(p_1 - p_2) = 0. \tag{1.6}$$

In the case of constant k (considered in this paper), the condition (1.5) reads

$$\psi_{\Gamma_1} = -(k_1 + k_2)\mathbf{n} \cdot \nabla p_1 = -(k_1 + k_2)\mathbf{n} \cdot \nabla p_2, \tag{1.7}$$

which is in fact a version of the Darcy law at the boundary.
2. Boundary to sea with a constant fluid potential (we call this boundary Γ_2*).* The porous medium is in contact with the ocean (or any other body of water with constant depth). This is often the case for underground porous media on islands (sand to ocean) or coasts (aquifers to ocean). Here we impose the normal stress as in (1.4), but the boundary conditions for the potentials are of the Dirichlet type, i.e.,

$$p_1 = p_2 = p_{\Gamma_2}. \tag{1.8}$$

3. Internal boundary with a known head (Γ_3*).* This may represent either a fixed potential pumping well or the potential at some measurement point. We model this as a no displacement boundary with Dirichlet conditions for the potentials as at Γ_3, i.e.,

$$\mathbf{u} = 0, \tag{1.9}$$
$$p_1 = p_2 = p_{\Gamma_3}. \tag{1.10}$$

4. Impermeable bedrock, Γ_4. Here, we impose no displacement (as for Γ_3) and a zero normal flux (as for Eq. (1.5) with $\psi_{\Gamma_4} = 0$).

For the unique solvability of the diffusion problem, one has to assume that

$$\text{meas}(\Gamma_2 \cup \Gamma_3) \neq \emptyset.$$

In this paper, we restrict ourselves to the case of Dirichlet type boundary conditions. The outline of the paper is as follows. In Section 2, we present and analyze the double diffusion problem and in Section 3 we provide its functional a posteriori error estimate. In Section 4, we obtain the estimate of the static Barenblatt-Biot model in terms of combined elasticity-diffusion norm.

2. Variational formulation of the double diffusion system

Since the displacement $\mathbf{u}$ is only involved in the first equation of system (1.3), the double-diffusion problem

$$-\nabla \cdot (k_1 \nabla p_1) + \kappa(p_1 - p_2) = h_1(x), \tag{2.1}$$

$$-\nabla \cdot (k_2 \nabla p_2) + \kappa(p_2 - p_1) = h_2(x) \tag{2.2}$$

is studied separately. It describes the steady flow of a slightly compressible fluid in a generally heterogeneous medium consisting of two components. Henceforth, we consider this problem with the Dirichlet boundary conditions $p_1 = p_2 = p_\Gamma$ on Γ. Let $\bar{p}$ be a function with square summable coefficients that satisfies this boundary condition. It is convenient to rewrite the problem in terms of the new functions

$$\mathsf{p}_1 := p_1 - \bar{p}, \quad \mathsf{p}_2 := p_2 - \bar{p}.$$

Then, a weak formulation of (2.1)-(2.2) leads to

Problem 1. *Assume that* $(h_1, h_2) \in L^2(\Omega, \mathcal{R}^2)$. *Find* $\mathbf{p} = (\mathsf{p}_1, \mathsf{p}_2) \in H^1_0(\Omega, \mathcal{R}^2)$ *satisfying the system of variational equalities*

$$\begin{aligned}
\int\limits_\Omega k_1 \nabla \mathsf{p}_1 \cdot \nabla \mathsf{q}_1 + \int\limits_\Omega \kappa(\mathsf{p}_1 - \mathsf{p}_2)\mathsf{q}_1 \, dx &= \int\limits_\Omega (h_1(x)\mathsf{q}_1 - k_1 \nabla \bar{\mathsf{p}} \cdot \nabla \mathsf{q}_1) \, dx \\
\int\limits_\Omega k_2 \nabla \mathsf{p}_2 \cdot \nabla \mathsf{q}_2 + \int\limits_\Omega \kappa(\mathsf{p}_2 - \mathsf{p}_1)\mathsf{q}_2 \, dx &= \int\limits_\Omega (h_2(x)\mathsf{q}_2 - k_2 \nabla \bar{\mathsf{p}} \cdot \nabla \mathsf{q}_2) \, dx
\end{aligned} \tag{2.3}$$

for all testing functions $\mathbf{q} = (\mathsf{q}_1, \mathsf{q}_2) \in H^1_0(\Omega, \mathcal{R}^2)$.

This problem can be represented in general form (which also encompasses other, more complicated models of porous media). For this purpose, we introduce the spaces

$$Q := H^1_0(\Omega, \mathcal{R}^2), \quad Y := L^2(\Omega, \mathcal{R}^{2d}), \tag{2.4}$$

and the corresponding dual spaces

$$Q^* := H^{-1}(\Omega, \mathcal{R}^2), \quad Y^* := L^2(\Omega, \mathcal{R}^{2d}). \tag{2.5}$$

Hereafter L_2 norms of all functions in Ω are denoted by $\|\cdot\|_\Omega$. Duality pairings of (Q, Q^*) and (Y, Y^*) are denoted by $\langle \cdot, \cdot \rangle$ and $\langle\langle \cdot, \cdot \rangle\rangle$, respectively. Also, we introduce a bounded linear operator $\Lambda \in \mathcal{L}(Q, Y)$ and its adjoint operator $\Lambda^* \in \mathcal{L}(Y^*, Q^*)$ by the relations

$$\Lambda \mathbf{q} := (\nabla \mathsf{q}_1, \nabla \mathsf{q}_2), \quad \Lambda^* \mathbb{Y}^* = (-\operatorname{div} y_1^*, -\operatorname{div} y_2^*)^T. \tag{2.6}$$

The operators Λ and Λ^* satisfy the relation representing integration by parts

$$\langle\langle \mathbb{Y}^*, \Lambda \mathbf{q} \rangle\rangle = \langle \Lambda^* \mathbb{Y}^*, \mathbf{q} \rangle \quad \text{for all } \mathbb{Y}^* \in Y^*, \mathbf{q} \in Q,$$

which can be written componentwise as

$$\int\limits_\Omega (\mathbf{Y}_1^* \cdot \nabla \mathsf{q}_1 + \mathbf{Y}_2^* \cdot \nabla \mathsf{q}_2) \, dx = -\int\limits_\Omega (\mathsf{q}_1 \operatorname{div} \mathbf{Y}_1^* + \mathsf{q}_2 \operatorname{div} \mathbf{Y}_2^*) \, dx, \tag{2.7}$$

where $\mathbf{q} = (\mathsf{q}_1, q_2)$ and $\mathbb{Y}^* = (\mathbf{Y}_1^*, \mathbf{Y}_2^*)$. Now Problem 1 can be represented in the form: Find $\mathbf{p} \in Q$ such that the equality

$$a(\mathbf{p}, \mathbf{q}) = l(\mathbf{q}) \tag{2.8}$$

holds for all $\mathbf{q} \in Q$. The bilinear form $a(\cdot, \cdot)$ and the linear form $l(\cdot)$ are defined as

$$\begin{aligned} a(\mathbf{p}, \mathbf{q}) &:= \int_\Omega (\Lambda \mathbf{p} : (\mathbb{A}\Lambda \mathbf{q}) + \mathbf{p} \cdot \mathbb{B}\mathbf{q})\, dx, \\ l(\mathbf{q}) &:= \int_\Omega (\mathbf{h} \cdot \mathbf{q} - \mathbb{C}\Lambda \mathbf{q})\, dx, \end{aligned}$$

$\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ are matrices formed by the material-dependent constants k_1, k_2, κ,

$$\mathbb{A} := \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}, \quad \mathbb{B} := \begin{pmatrix} \kappa & -\kappa \\ -\kappa & \kappa \end{pmatrix}, \quad \mathbb{C} := \begin{pmatrix} k_1 \nabla \bar{p} & 0 \\ 0 & k_2 \nabla \bar{p} \end{pmatrix}$$

and $\mathbf{h}$ is the right-hand side vector

$$\mathbf{h} := \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}.$$

Remark 2.1. We note that the symmetric matrix $\mathbb{A}$ is a positive definite matrix if k_1 and k_2 are positive (since $\mathbb{A}\xi \cdot \xi \geqslant \min\{k_1, k_2\} \|\xi\|^2$ for all $\xi \in \mathcal{R}^d$). However, $\mathbb{B}$ is symmetric but only positive semi-definite in the case of the positive parameter κ, and its one-dimensional kernel is generated by the vector $(1, 1)^T$.

Remark 2.2. If $\bar{p}$ is sufficiently regular (so that $\Lambda^*\mathbb{C}$ belongs to Y^*), then

$$l(\mathbf{q}) := \int_\Omega (\mathbf{h} \cdot \mathbf{q} - \Lambda^* \mathbb{C}\mathbf{q})\, dx = \int_\Omega \widehat{\mathbf{h}} \cdot \mathbf{q}\, dx,$$

where

$$\widehat{\mathbf{h}} := \begin{pmatrix} h_1 - \operatorname{div} k_1 \nabla \bar{p} \\ h_2 - \operatorname{div} k_2 \nabla \bar{p} \end{pmatrix}.$$

It is easy to verify that (2.8) is the necessary condition for the minimizer of the following convex variational problem.

Problem 2. *Find* $\mathbf{p} \in Q$ *satisfying*

$$F(\mathbf{p}) + G(\Lambda \mathbf{p}) = \inf_{\mathbf{q} \in Q} \{F(\mathbf{q}) + G(\Lambda \mathbf{q})\}, \tag{2.9}$$

where

$$F : Q \to \mathcal{R}, \qquad F(\mathbf{q}) := \frac{1}{2} \int_\Omega \mathbf{q} \cdot \mathbb{B}\mathbf{q}\, dx - l(\mathbf{q}), \tag{2.10}$$

and

$$G : Y \to \mathcal{R}, \qquad G(\Lambda \mathbf{q}) := \frac{1}{2} \int_\Omega \Lambda \mathbf{q} : (\mathbb{A}\Lambda \mathbf{q})\, dx. \tag{2.11}$$

Theorem 1 existence of a unique solution. *Assume that* $k_1, k_2 > 0$ *and* $\kappa \geqslant 0$. *Then, there exists a unique solution* $\mathbf{p} \in Q$ *of Problem 2, which also represents the solution of Problem 1.*

Proof. The existence of the unique minimizer follows from the known results in the calculus of variations. Indeed, under the given assumptions, the functional $F(\cdot) + G(\Lambda\cdot)$ is strictly convex and coercive in the reflexive space Q. □

3. A posteriori error estimate of the double diffusion system

In this section, we derive guaranteed and directly computable bounds of the difference between exact and approximate solutions. Our analysis is based upon a posteriori error estimation methods suggested in [14, 18]. Following chapters 6 and 7 in [14], we first need to find the explicit forms of the dual functionals

$$\begin{aligned} &F^* : Q^* \to \mathcal{R}, \qquad F^*(\Lambda^*\mathbb{Y}^*) := \sup_{\mathbf{q}\in Q}\{\langle \Lambda^*\mathbb{Y}^*, \mathbf{q}\rangle - F(\mathbf{q})\}, \\ &G^* : Y^* \to \mathcal{R}, \qquad G^*(\mathbb{Y}^*) := \sup_{\Lambda\mathbf{q}\in Y}\{\langle\langle \mathbb{Y}^*, \Lambda\mathbf{q}\rangle\rangle - G(\Lambda\mathbf{q})\}, \end{aligned} \tag{3.1}$$

and the corresponding compound functionals

$$\begin{aligned} &D_F : Q \times Q^* \to \mathcal{R}, \qquad D_F(\mathbf{q}, \Lambda^*\mathbb{Y}^*) := F(\mathbf{q}) + F^*(\Lambda^*\mathbb{Y}^*) - \langle \Lambda^*\mathbb{Y}^*, \mathbf{q}\rangle\,, \\ &D_G : Y \times Y^* \to \mathcal{R}, \qquad D_G(\Lambda\mathbf{q}, \mathbb{Y}^*) := G(\Lambda\mathbf{q}) + G^*(\mathbb{Y}^*) - \langle\langle \mathbb{Y}^*, \Lambda\mathbf{q}\rangle\rangle\,. \end{aligned} \tag{3.2}$$

By the the sum of D_F and D_G, we obtain the functional error majorant

$$M(\mathbf{q}, \mathbb{Y}^*) := D_F(\mathbf{q}, \Lambda^*\mathbb{Y}^*) + D_G(\Lambda\mathbf{q}, \mathbb{Y}^*), \tag{3.3}$$

which provides a guaranteed upper bound of the error

$$\frac{1}{2}a(\mathbf{p}-\mathbf{q}, \mathbf{p}-\mathbf{q}) \leqslant M(\mathbf{q}, \mathbb{Y}^*) \quad \text{for all } \mathbb{Y}^* \in Y^*. \tag{3.4}$$

The majorant is fully computable and depends only on the approximation $\mathbf{q} \in Q$ and arbitrary variable $\mathbb{Y}^* \in Y^*$.

Lemma 3.1 dual functionals. *For* $k_1, k_2 > 0$ *and* $\kappa > 0$,

$$G^*(\mathbb{Y}^*) = \frac{1}{2}\int_\Omega \mathbb{A}^{-1}\mathbb{Y}^* : \mathbb{Y}^*\,dx, \tag{3.5}$$

$$F^*(\Lambda^*\mathbb{Y}^*) = \begin{cases} \frac{1}{4\kappa}\int_\Omega (\Lambda^*\mathbb{Y}^* + \mathbf{h})^2\,dx & \text{if } \Lambda^*\mathbf{Y}_1^* + h_1 + \Lambda^*\mathbf{Y}_2^* + h_2 = 0, \\ +\infty & \text{otherwise} \end{cases} \quad \textit{hold.} \tag{3.6}$$

Proof. The derivation of $G^*(\mathbb{Y}^*)$ is straightforward (see [14]). The singularity of the matrix $\mathbb{B}$ makes the computation of $F^*(\Lambda^*\mathbb{Y}^*)$ more technical.

$$\begin{aligned}
F^*(\Lambda^*\mathbb{Y}^*) &= \sup_{\mathbf{q}\in Q}\{\langle \mathbf{q}, \Lambda^*\mathbb{Y}^*\rangle - F(\mathbf{q})\} \\
&\geqslant \sup_{\mathbf{q}\in Q:\mathsf{q}_1=\mathsf{q}_2}\{\langle \mathbf{q}, \Lambda^*\mathbb{Y}^*\rangle - F(\mathbf{q})\} \\
&= \sup_{\mathsf{q}_1\in H^1_0(\Omega)}\{\langle \mathsf{q}_1, \Lambda^*\mathbf{Y}^*_1 + \Lambda^*\mathbf{Y}^*_2\rangle - F(\mathsf{q}_1,\mathsf{q}_1)\} \\
&= \sup_{\mathsf{q}_1\in H^1_0(\Omega)}\{\langle \mathsf{q}_1, \Lambda^*\mathbf{Y}^*_1 + h_1 + \Lambda^*\mathbf{Y}^*_2 + h_2\rangle\} \\
&= \begin{cases} 0 & \text{if } \Lambda^*\mathbf{Y}^*_1 + h_1 + \Lambda^*\mathbf{Y}^*_2 + h_2 = 0, \\ +\infty & \text{otherwise.}\end{cases}
\end{aligned}$$

Thus, the finite values of $F^*(\Lambda^*\mathbb{Y}^*)$ are attained only on the subspace

$$\Lambda^*\mathbf{Y}^*_1 + h_1 + \Lambda^*\mathbf{Y}^*_2 + h_2 = 0, \tag{3.7}$$

and we must specially consider this case. It holds

$$\begin{aligned}
F^*(\Lambda^*\mathbb{Y}^*) &= \sup_{\mathbf{q}\in Q}\{\langle \mathbf{q}, \Lambda^*\mathbb{Y}^*\rangle - F(\mathbf{q})\} = \sup_{\mathbf{q}\in Q}\{\langle \mathbf{q}, \Lambda^*\mathbb{Y}^* + \mathbf{h}\rangle - \frac{1}{2}\int_\Omega \mathbb{B}\mathbf{q}\cdot\mathbf{q}\,dx\} \\
&\quad (\text{use the constraint } \Lambda^*\mathbf{Y}^*_2 + h_2 = -(\Lambda^*\mathbf{Y}^*_1 + h_1)) \\
&= \sup_{(\mathsf{q}_1,\mathsf{q}_2)\in Q}\{\langle \mathsf{q}_1 - \mathsf{q}_2, \Lambda^*\mathbf{Y}^*_1 + h_1\rangle - \frac{1}{2}\int_\Omega \kappa(\mathsf{q}_1-\mathsf{q}_2)^2\,dx\} \\
&\quad (\text{supremum is obtained for } \mathsf{q}_1 - \mathsf{q}_2 = (\Lambda^*\mathbf{Y}^*_1 + h_1)/\kappa) \\
&= \frac{1}{2\kappa}\int_\Omega (\Lambda^*\mathbf{Y}^*_1 + h_1)^2\,dx = \frac{1}{4\kappa}\int_\Omega \left[(\Lambda^*\mathbf{Y}^*_1 + h_1)^2 + (\Lambda^*\mathbf{Y}^*_2 + h_2)^2\right]\,dx \\
&= \frac{1}{4\kappa}\int_\Omega (\Lambda^*\mathbb{Y}^* + \mathbf{h})^2\,dx.
\end{aligned}$$

□

Remark 3.1. We note that (3.7) is a weaker restriction than the sum of two equilibrium relations $\Lambda^*\mathbf{Y}^*_1 + h_1 = 0$ and $\Lambda^*\mathbf{Y}^*_2 + h_2 = 0$, which one would expect from the general theory. In other words, our analysis shows that the strict equilibrium of the dual variables in the componentwise sense is not required in the coupled system.

After the substitution of (3.5) and (3.6) into (3.2), we obtain explicit expressions for the compound functionals

$$D_G(\Lambda\mathbf{q}, \mathbb{Y}^*) = \frac{1}{2}\int_\Omega \mathbb{A}(\Lambda\mathbf{q} - \mathbb{A}^{-1}\mathbb{Y}^*) : (\Lambda\mathbf{q} - \mathbb{A}^{-1}\mathbb{Y}^*)\,dx, \tag{3.8}$$

$$D_F(\mathbf{q}, \Lambda^*\mathbb{Y}^*) = \begin{cases} \frac{1}{2}\int_\Omega \mathbb{B}\mathbf{q}\cdot\mathbf{q}\,dx + \frac{1}{4\kappa}\int_\Omega (\Lambda^*\mathbb{Y}^* + \mathbf{h})^2\,dx \\ \qquad \text{if } \Lambda^*\mathbf{Y}^*_1 + h_1 + \Lambda^*\mathbf{Y}^*_2 + h_2 = 0, \\ +\infty \quad \text{otherwise.}\end{cases} \tag{3.9}$$

In view of (3.4), the sharpest bound of $a(\mathbf{p}-\mathbf{q}, \mathbf{p}-\mathbf{q})$ is provided by the estimate

$$\frac{1}{2}a(\mathbf{p}-\mathbf{q}, \mathbf{p}-\mathbf{q}) \leqslant \inf_{\mathbb{Y}^*\in Y^*} M(\mathbf{q}, \mathbb{Y}^*). \tag{3.10}$$

Since $M(\mathbf{q}, \mathbb{Y}^*) = +\infty$ if $\mathbb{Y}^*$ does not satisfy (3.7), we must restrict ourselves to arguments $\mathbb{Y}^* \in Y_h^*$, where

$$Y_h^* := \{(y_1^*, y_2^*) \in Y^* : \Lambda^* \mathbf{Y}_1^* + h_1 + \Lambda^* \mathbf{Y}_2^* + h_2 = 0 \text{ a.e. in } \Omega\}. \tag{3.11}$$

Constructing an element of Y_h^* requires an exact equilibration procedure which has been studied for a Poisson problem in [9]. Below, we show a way to avoid the constraint (3.11) by a special penalty term added to the functional majorant. We define

$$Y_{div}^* := \{(\mathbf{Y}_1^*, \mathbf{Y}_2^*) \in Y^* : \Lambda^* \mathbf{Y}_1^* + \Lambda^* \mathbf{Y}_2^* \in L^2(\Omega)\}, \tag{3.12}$$

and note that $Y_h^* \subset Y_{div}^*$ (since $h_1, h_2 \in L^2(\Omega)$). Further we decompose

$$\mathbb{Y}^* = \hat{\mathbb{Y}}^* + (\mathbb{Y}^* - \hat{\mathbb{Y}}^*)$$

with $\hat{\mathbb{Y}}^* \in Y_{div}^*$ and extend the dual functionals D_G and D_F by the new variable $\hat{\mathbb{Y}}^*$. We rewrite (3.8) as

$$\begin{aligned} D_G(\Lambda\mathbf{q}, \mathbb{Y}^*) &= \frac{1}{2} \int\limits_\Omega \mathbb{A}(\Lambda\mathbf{q} - \mathbb{A}^{-1}\hat{\mathbb{Y}}^*) : (\Lambda\mathbf{q} - \mathbb{A}^{-1}\hat{\mathbb{Y}}^*)\, dx \\ &+ \int\limits_\Omega (\Lambda\mathbf{q} - \mathbb{A}^{-1}\hat{\mathbb{Y}}^*) : (\mathbb{Y}^* - \hat{\mathbb{Y}}^*)\, dx + \frac{1}{2} \int\limits_\Omega \mathbb{A}^{-1}(\mathbb{Y}^* - \hat{\mathbb{Y}}^*) : (\mathbb{Y}^* - \hat{\mathbb{Y}}^*)\, dx, \end{aligned}$$

and use the inequality $2\mathbb{M}_1 : \mathbb{M}_2 \leqslant \beta_1 M_1 : M_1 + \frac{1}{\beta_1}\mathbb{M}_2 : \mathbb{M}_2$ valid for all matrices $\mathbb{M}_1, \mathbb{M}_2$ and for all $\beta_1 > 0$ to bound the middle term as

$$\begin{aligned} (\Lambda\mathbf{q} - \mathbb{A}^{-1}\hat{\mathbb{Y}}^*) : (\mathbb{Y}^* - \hat{\mathbb{Y}}^*) &= \mathbb{A}^{1/2}(\Lambda\mathbf{q} - \mathbb{A}^{-1}\hat{\mathbb{Y}}^*) : \mathbb{A}^{-1/2}(\mathbb{Y}^* - \hat{\mathbb{Y}}^*) \\ &\leqslant \frac{\beta_1}{2} \mathbb{A}(\Lambda\mathbf{q} - \mathbb{A}^{-1}\hat{\mathbb{Y}}^*) : (\Lambda\mathbf{q} - \mathbb{A}^{-1}\hat{\mathbb{Y}}^*) + \frac{1}{2\beta_1}\mathbb{A}^{-1}(\mathbb{Y}^* - \hat{\mathbb{Y}}^*) : (\mathbb{Y}^* - \hat{\mathbb{Y}}^*). \end{aligned} \tag{3.13}$$

Obviously, the middle term adds to the left and the right terms in $D_G(\Lambda\mathbf{q}, \mathbb{Y}^*)$ above, and the modified compound functional reads

$$\begin{aligned} D_G(\Lambda\mathbf{q}, \mathbb{Y}^*, \hat{\mathbb{Y}}^*) &:= \frac{1+\beta_1}{2} \int\limits_\Omega \mathbb{A}(\Lambda\mathbf{q} - \mathbb{A}^{-1}\hat{\mathbb{Y}}^*) : (\Lambda\mathbf{q} - \mathbb{A}^{-1}\hat{\mathbb{Y}}^*)\, dx \\ &+ (\frac{1}{2} + \frac{1}{2\beta_1}) \int\limits_\Omega \mathbb{A}^{-1}(\mathbb{Y}^* - \hat{\mathbb{Y}}^*) : (\mathbb{Y}^* - \hat{\mathbb{Y}}^*)\, dx. \end{aligned} \tag{3.14}$$

It also contains a scalar factor $\beta_1 > 0$ whose value can be chosen arbitrarily. A similar technique is used to modify the compound functional $D_F(\mathbf{q}, \Lambda^*\mathbb{Y}^*)$. For the second integral in (3.9), we have

$$\int\limits_\Omega (\Lambda^*\mathbb{Y}^* + \mathbf{h})^2\, dx \leqslant (1+\beta_2) \int\limits_\Omega (\Lambda^*\hat{\mathbb{Y}}^* + \mathbf{h})^2\, dx + (1 + \frac{1}{\beta_2}) \int\limits_\Omega (\Lambda^*(\mathbb{Y}^* - \hat{\mathbb{Y}}^*))^2\, dx,$$

where $\beta_2 > 0$. Therefore, the modified dual functional reads

$$\begin{aligned} D_F(\mathbf{q}, \Lambda^*\mathbb{Y}^*, \Lambda^*\hat{\mathbb{Y}}^*) &:= \frac{1}{2} \int\limits_\Omega \mathbb{B}\mathbf{q} \cdot \mathbf{q}\, dx + \frac{1}{4\kappa}(1+\beta_2) \int\limits_\Omega (\Lambda^*\hat{\mathbb{Y}}^* + \mathbf{h})^2\, dx \\ &+ \frac{1}{4\kappa}(1 + \frac{1}{\beta_2}) \int\limits_\Omega (\Lambda^*(\mathbb{Y}^* - \hat{\mathbb{Y}}^*))^2\, dx. \end{aligned} \tag{3.15}$$

By adding (3.14) and (3.15), we extend the functional majorant (3.3) to

$$M(\mathbf{q}, \mathbb{Y}^*, \hat{\mathbb{Y}}^*) := D_F(\mathbf{q}, \Lambda^*\mathbb{Y}^*, \Lambda^*\hat{\mathbb{Y}}^*) + D_G(\Lambda\mathbf{q}, \mathbb{Y}^*, \hat{\mathbb{Y}}^*), \tag{3.16}$$

in which the arbitrary variables satisfy the constraint

$$(\mathbb{Y}^*, \hat{\mathbb{Y}}^*) \in Y_h^* \times Y_{div}^*.$$

Clearly, the original and extended majorants satisfy the inequality

$$\frac{1}{2}a(\mathbf{p}-\mathbf{q}, \mathbf{p}-\mathbf{q}) \leqslant M(\mathbf{q}, \mathbb{Y}^*) \leqslant M(\mathbf{q}, \mathbb{Y}^*, \hat{\mathbb{Y}}^*) \tag{3.17}$$

for all $\hat{\mathbb{Y}}^* \in Y_{div}^*, \beta_1 > 0, \beta_2 > 0$. This estimate is sharp in the sense that there are no irremovable gaps in the inequalities. Indeed, if we set $\mathbb{Y}^* = \hat{\mathbb{Y}}^* = \Lambda\mathsf{p}$ and tend β_1 and β_2 to zero, then $M(\mathbf{q}, \mathbb{Y}^*, \hat{\mathbb{Y}}^*)$ tends to $M(\mathbf{q}, \mathbb{Y}^*)$ (and even to the exact error $\frac{1}{2}a(\mathbf{p}-\mathbf{q}, \mathbf{p}-\mathbf{q})$, cf. (3.10)).

3.1. Upper estimate of $M(\mathbf{q}, \mathbb{Y}^*, \hat{\mathbb{Y}}^*)$

Let us denote $\mathbb{Y}^* = (\mathbf{Y}_1^*, \mathbf{Y}_2^*)$ and $\hat{\mathbb{Y}}^* = (\hat{\mathbf{Y}}_1^*, \hat{\mathbf{Y}}_2^*)$ and consider a particular subspace

$$(\mathbb{Y}^*, \hat{\mathbb{Y}}^*) \in \{Y_h^* \times Y_{div}^* : \Lambda^*\mathbf{Y}_1^* + h_1 = 0, \mathbf{Y}_2^* = \hat{\mathbf{Y}}_2^* \quad \text{a.e. in } \Omega\}. \tag{3.18}$$

In this subspace,

$$\int\limits_\Omega (\Lambda^*(\mathbb{Y}^* - \hat{\mathbb{Y}}^*))^2\,dx = \int\limits_\Omega (\operatorname{div}(\hat{\mathbf{Y}}_1^* - \mathbf{Y}_1^*))^2\,dx = \int\limits_\Omega (\operatorname{div}\hat{\mathbf{Y}}_1^* - h_1)^2\,dx$$

holds (cf. (2.6)). Therefore, $D_F(\mathbf{q}, \Lambda^*\mathbb{Y}^*, \Lambda^*\hat{\mathbb{Y}}^*)$ defined in (3.15) is simplified as $\mathbb{Y}^*$-independent

$$\begin{aligned} D_F(\mathbf{q}, \Lambda^*\hat{\mathbb{Y}}^*) \;&:=\; \frac{1}{2}\int\limits_\Omega \mathbb{B}\mathbf{q}\cdot\mathbf{q}\,dx + \frac{1}{4\kappa}(1+\beta_2)\int\limits_\Omega (\Lambda^*\hat{\mathbb{Y}}^* + \mathbf{h})^2\,dx \\ &+\; \frac{1}{4\kappa}(1+\frac{1}{\beta_2})\int\limits_\Omega (\operatorname{div}\hat{\mathbf{Y}}_1^* - h_1)^2\,dx, \end{aligned} \tag{3.19}$$

and only $\mathbb{Y}^*$-dependent functional in $D_G(\Lambda\mathbf{q}, \mathbb{Y}^*, \hat{\mathbb{Y}}^*)$ defined in (3.14) writes

$$\int\limits_\Omega \mathbb{A}^{-1}(\mathbb{Y}^* - \hat{\mathbb{Y}}^*) : (\mathbb{Y}^* - \hat{\mathbb{Y}}^*)\,dx = \int\limits_\Omega k_1^{-1}(\mathbf{Y}_1^* - \hat{\mathbf{Y}}_1^*)\cdot(\mathbf{Y}_1^* - \hat{\mathbf{Y}}_1^*)\,dx. \tag{3.20}$$

Lemma 3.2. *Let us define the space*

$$Y_{h_1} := \{\mathbf{Y}_1^* \in L^2(\Omega)^d : \Lambda^*\mathbf{Y}_1^* + h_1 = 0 \text{ a.e. in } \Omega\}.$$

Then, for all $\hat{\mathbf{Y}}_1^* \in H(\operatorname{div};\Omega)$,

$$\inf_{\mathbf{Y}_1^* \in Y_{h_1}} \int\limits_\Omega \left\|\mathbf{Y}_1^* - \hat{\mathbf{Y}}_1^*\right\|^2\,dx \leqslant C^2\left\|\operatorname{div}\hat{\mathbf{Y}}_1^* + h_1\right\|^2$$

holds, where $C > 0$ *satisfies the Friedrichs inequality* $\|w\|_{L^2(\Omega)} \leqslant C\,\|\nabla w\|_{L^2(\Omega)}$ *valid for all* $w \in H_0^1(\Omega)$.

Proof. It follows from Theorem 6.1 in [20] by the modification related to the fact that we consider vector arguments. □

The application of Lemma 3.2 to (3.20) and the substitution back into (3.14) defines the $\mathbb{Y}^*$-independent dual functional

$$D_G(\Lambda\mathbf{q}, \hat{\mathbb{Y}}^*) := \frac{1+\beta_1}{2}\int\limits_\Omega \mathbb{A}(\Lambda\mathbf{q} - \mathbb{A}^{-1}\hat{\mathbb{Y}}^*) : (\Lambda\mathbf{q} - \mathbb{A}^{-1}\hat{\mathbb{Y}}^*)\,dx$$
$$+ k_1^{-1}(\frac{1}{2} + \frac{1}{2\beta_1})C^2 \left\|\operatorname{div}\hat{\mathbf{Y}}_1^* + h_1\right\|^2, \tag{3.21}$$

which provides the upper estimate of the quantity

$$\inf_{\mathbb{Y}^*\in Y_h^*} D_G(\Lambda\mathbf{q}, \mathbb{Y}^*, \hat{\mathbb{Y}}^*).$$

Therefore, the sum of (3.19) and (3.21) defines the $\mathbb{Y}^*$-independent functional

$$M_{\beta_1,\beta_2}(\mathbf{q}, \hat{\mathbb{Y}}^*) := D_F(\mathbf{q}, \Lambda^*\hat{\mathbb{Y}}^*) + D_G(\Lambda\mathbf{q}, \hat{\mathbb{Y}}^*) \tag{3.22}$$

that serves as an upper bound of $M(\mathbf{q}, \mathbb{Y}^*, \hat{\mathbb{Y}}^*)$, and provides a computable estimate

$$\frac{1}{2}a(\mathbf{p}-\mathbf{q}, \mathbf{p}-\mathbf{q}) \leqslant M_{\beta_1,\beta_2}(\mathbf{q}, \hat{\mathbb{Y}}^*) \quad \text{for all } \hat{\mathbb{Y}}^* \in Y_{div}^*. \tag{3.23}$$

Remark 3.2 symmetric form of D_G. If we replace subspace (3.18) by

$$(\mathbb{Y}^*, \hat{\mathbb{Y}}^*) \in \{Y_h^* \times Y_{div}^* : \Lambda^*\mathbf{Y}_2^* + h_2 = 0, \mathbf{Y}_1^* = \hat{\mathbf{Y}}_1^* \quad \text{a.e. in } \Omega\}, \tag{3.24}$$

then, instead of (3.21), we obtain

$$D_G(\Lambda\mathbf{q}, \hat{\mathbb{Y}}^*) := \frac{1+\beta_1}{2}\int\limits_\Omega \mathbb{A}(\Lambda\mathbf{q} - \mathbb{A}^{-1}\hat{\mathbb{Y}}^*) : (\Lambda\mathbf{q} - \mathbb{A}^{-1}\hat{\mathbb{Y}}^*)\,dx$$
$$+ k_2^{-1}(\frac{1}{2} + \frac{1}{2\beta_1})C^2 \left\|\operatorname{div}\hat{\mathbf{Y}}_2^* + h_2\right\|^2. \tag{3.25}$$

4. A posteriori error estimate for approximations of the coupled system (1.1)

Assume that the fluid pressures p_1 and p_2 are resolved exactly and substituted into the elasticity equation (cf. (1.1))

$$-\nabla\cdot(\mathbb{L}\,\varepsilon(u)) = \mathbf{f}(x,t) + \alpha_1\nabla\mathsf{p}_1 + \alpha_2\nabla\mathsf{p}_2.$$

Let $\mathbf{v}$ be an approximation of $\mathbf{u}$ (this problem is considered in the same domain Ω as problem (2.1)-(2.2)). We define the Dirichlet boundary condition by the function $\mathbf{u}_0 \in H^1(\Omega; \mathcal{R}^d)$ and assume

$$\mathbf{v} \in \mathbf{u}_0 + H_0^1(\Omega; \mathcal{R}^d).$$

Lemma 4.1. *For every function $\tau \in Q := \{\sigma \in L^2(\Omega;\mathbb{R}^{d\times d}_{sym}) : \operatorname{div}\sigma \in L^2(\Omega;\mathcal{R}^d)\}$,*

$$\|\varepsilon(\mathbf{u}-\mathbf{v})\|_{\mathbb{L};\Omega} \leqslant \left\|\varepsilon(\mathbf{v}) - \mathbb{L}^{-1}\tau\right\|_{\mathbb{L};\Omega} + C\,\|\operatorname{div}\tau + \mathbf{f} - \alpha_1\nabla\mathsf{p}_1 - \alpha_2\nabla\mathsf{p}_2\|_{\Omega}\,, \tag{4.1}$$

holds, where the constant $C > 0$ satisfies the inequality

$$\|\mathbf{w}\|_{\Omega} \leqslant C\,\|\varepsilon(\mathbf{w})\|_{\mathbb{L};\Omega} \quad \textit{for all } \mathbf{w} \in H^1_0(\Omega;\mathcal{R}^d) \tag{4.2}$$

and the norm $\|\cdot\|$ is defined as

$$\|\varepsilon\|^2_{\mathbb{L};\Omega} := \int\limits_{\Omega} \mathbb{L}\,\varepsilon : \varepsilon\, dx.$$

Proof. The estimates in chapter 6.5 in [18] are applied to the linear elasticity problem with the right-hand side $\mathbf{f} - \alpha_1\nabla\mathsf{p}_1 - \alpha_2\nabla\mathsf{p}_2$. The existence of constant C follows from the Korn and Friedrichs inequalities. □

Remark 4.1. Estimate (4.1) is sharp with respect to the parameter τ. Indeed, the choice of $\tau = \mathbb{L}\,\varepsilon(u)$ satisfies the equilibrium condition

$$\operatorname{div}\tau + \mathbf{f} = \alpha_1\nabla\mathsf{p}_1 + \alpha_2\nabla\mathsf{p}_2, \tag{4.3}$$

and reduces therefore (4.1) to the equality.

Let q_1 and q_2 be approximations of the exact pressure fields p_1 and p_2 respectively. By triangle inequalities, we obtain

$$\begin{aligned}\|\operatorname{div}\tau + \mathbf{f} - \alpha_1\nabla\mathsf{p}_1 - \alpha_2\nabla\mathsf{p}_2\|_{\Omega} &\leqslant \|\operatorname{div}\tau + \mathbf{f} - \alpha_1\nabla\mathsf{q}_1 - \alpha_2\nabla\mathsf{q}_2\|_{\Omega} \\ &\quad + \|\nabla(\mathsf{p}_1 - \mathsf{q}_1)\|_{\Omega} + \|\nabla(\mathsf{p}_2 - \mathsf{q}_2)\|_{\Omega}\,.\end{aligned} \tag{4.4}$$

Use (4.4) and square both parts of (4.1) to obtain

$$\begin{aligned}\|\varepsilon(\mathbf{u}-\mathbf{v})\|^2_{\mathbb{L};\Omega} &\leqslant \big(\left\|\varepsilon(\mathbf{v}) - \mathbb{L}^{-1}\tau\right\|_{\mathbb{L};\Omega} \\ &+ C\,\|\operatorname{div}\tau + \mathbf{f} - \alpha_1\nabla\mathsf{q}_1 - \alpha_2\nabla\mathsf{q}_2\|_{\Omega} \\ &+ C\,\|\nabla(\mathsf{p}_1 - \mathsf{q}_1)\|_{\Omega} + C\,\|\nabla(\mathsf{p}_2 - \mathsf{q}_2)\|_{\Omega}\big)^2.\end{aligned} \tag{4.5}$$

By the algebraic inequality

$$(a+b+c)^2 \leqslant (1+\beta_4+\beta_5)\ a^2 + (1+\frac{1}{\beta_4}+\beta_6)\ b^2 + (1+\frac{1}{\beta_5}+\frac{1}{\beta_6})\ c^2$$

valid for all scalars a, b, c and for all $\beta_4, \beta_5, \beta_6 > 0$, inequality (4.5) and the following inequality (β_3 is an arbitrary positive constant) yield

$$\begin{aligned}&(\|\nabla(\mathsf{p}_1 - \mathsf{q}_1)\|_{\Omega} + \|\nabla(\mathsf{p}_2 - \mathsf{q}_2)\|_{\Omega})^2 \\ &\qquad \leqslant (1+\beta_3)\,\|\nabla(\mathsf{p}_1 - \mathsf{q}_1)\|^2_{\Omega} + (1+\frac{1}{\beta_3})\,\|\nabla(\mathsf{p}_2 - \mathsf{q}_2)\|^2_{\Omega} \\ &\qquad \leqslant \max\{\frac{1+\beta_3}{k_1}, \frac{1+\beta_3}{k_2\beta_3}\}\ a(\mathbf{p}-\mathbf{q}, \mathbf{p}-\mathbf{q}) \\ &\qquad \leqslant 2\max\{\frac{1+\beta_3}{k_1}, \frac{1+\beta_3}{k_2\beta_3}\}\ M_{\beta_1,\beta_2}(\mathbf{q}, \hat{\mathbb{Y}}^*).\end{aligned} \tag{4.6}$$

Now we obtain the final estimate in terms of the coupled error norm

$$a(\mathbf{p}-\mathbf{q},\mathbf{p}-\mathbf{q}) + \|\varepsilon(\mathbf{u}-\mathbf{v})\|^2_{\mathbb{L};\Omega} \leqslant (1+\beta_4+\beta_5)\left\|\varepsilon(\mathbf{v})-\mathbb{L}^{-1}\tau\right\|^2_{\mathbb{L};\Omega} \\ + \left(1+\frac{1}{\beta_4}+\beta_6\right) C^2 \|\operatorname{div}\tau + \mathbf{f} - \alpha_1\nabla \mathsf{q}_1 - \alpha_2 \nabla \mathsf{q}_2\|^2_\Omega + 2\widehat{C}\ \ M_{\beta_1,\beta_2}(\mathbf{q},\hat{\mathbb{Y}}^*), \quad (4.7)$$

where

$$\widehat{C} = 1 + C^2\left(1+\frac{1}{\beta_5}+\frac{1}{\beta_6}\right)\max\left\{\frac{1+\beta_3}{k_1}, \frac{1+\beta_3}{k_2\beta_3}\right\}.$$

This estimate holds for all $\tau \in Q$, $\hat{\mathbb{Y}}^* \in Y^*_{div}$ and all $\beta_1,\ldots,\beta_6 > 0$.

Remark 4.2. Finally, we comment on how this estimate can be used in practical computations. Assume that numerical solutions of the Barenblatt-Biot system (1.3) are obtained on a certain finite dimensional subspace generated by the mesh $\mathcal{T}_h$. We denote them by q_h and $\mathbf{v}_h$. In the simplest case, we need to postprocess the functions $\mathbf{q}_h := \nabla \mathsf{q}_h$ and $\tau_h := \mathbb{L}\varepsilon(\mathbf{v}_h)$ in such a way that their post-processed images $\widetilde{\mathbf{q}}_h$ and $\widetilde{\tau}_h$ belong to Q and Y^*_{div}, respectively. Then a guaranteed upper bound follows from (4.7) by direct substitution and optimization with respect to the parameters $\beta_1,\ldots,\beta_6 > 0$. A sharper estimate can be obtained if the majorant is further minimized with respect to $\mathbf{q}_h$ and τ_h with the help of some direct minimization procedure (e.g., gradient descent). Examples of majorant minimization techniques can be found in [19, 22]. Another way may be efficient if the problem is solved on a sequence of refined meshes. Then, we can use the above procedure based on a relatively simple postprocessing procedure for $\mathbf{q}_h$ and τ_h but with one-step retardation, i.e., averaging is performed on the mesh h_k, but it is used in the error estimate for approximate solutions computed on the mesh h_{k-1}.

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