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Efficient Algorithms for Monroe and CC Rules in Multi-Winner Elections with (Nearly) Structured Preferences

Jiehua Chen[a], Christian Hatschka[a] and Sofia Simola[a]

[a]TU Wien, Austria
ORCiD ID: Jiehua Chen https://orcid.org/0000-0002-8163-1327,
Christian Hatschka https://orcid.org/0000-0002-0881-8259, Sofia Simola https://orcid.org/0000-0001-7941-0018

Abstract. We investigate winner determination for two popular proportional representation systems: the Monroe and Chamberlin-Courant (abbrv. CC) systems. Our study focuses on (nearly) single-peaked resp. single-crossing preferences. We show that for single-crossing approval preferences, winner determination of the Monroe rule is polynomial, and for both rules, winner determination mostly admits FPT algorithms with respect to the number of voters to delete to obtain single-peaked or single-crossing preferences. Our results answer some complexity questions from the literature [19, 29, 22].

1 Introduction

In multi-winner elections, a preference profile consists of a set of alternatives and voters, each with preferences over the alternatives. The objective is to select a committee of fixed size k, representing voters' preferences optimally. Two well-known voting rules for multi-winner elections are the Monroe rule and the Chamberlin-Courant (CC) rule [7, 23], both aiming to proportionally represent different voter groups. The Monroe rule minimizes the overall misrepresentation (i.e., voters' dissatisfaction) while ensuring each committee member represents roughly the same number of voters. In contrast, the CC rule also seeks to minimize the overall misrepresentation, but does not require equal representation among committee members.

Both the Monroe and CC rules are popular proportional voting rules that have been studied extensively from both social choice and computational perspectives [12, 26, 4, 27, 20, 2, 29, 28, 17, 22, 10, 30, 9]. Unfortunately, the winner determination problems for these rules, MONROE-MW and CC-MW, are NP-hard [20, 27], making it difficult to find an optimal solution. However, the NP-hardness reduction may result in instances with unstructured preferences. Consequently, researchers have explored the computational complexity of instances with nice preference structures, such as single-peaked (SP) [3], single-crossing (SC) [21], or nearly SP or SC preferences [6, 15] (also see Section 2 for the definition), as oftentimes the preferences of voters align with these structures.

Betzler et al. [2] designed polynomial-time algorithms for both MONROE-MW and CC-MW when voters have either approval or linear preferences, which are SP. Skowron et al. [29] continued this line of research on linear and SC preferences, demonstrating that in this case CC-MW is also polynomial-time solvable, while MONROE-MW with linear and SC preferences remains NP-hard. They left open the complexity of MONROE-MW for SC approval preferences or when minimizing the maximum misrepresentation. Elkind and Lackner [14] give polynomial-time algorithms for weighted proportional approval voting (weighted PAV) rules for SC approval preferences, which includes the CC rule. Recently, Constantinescu and Elkind [10] and Sornat et al. [30] improved existing positive results by providing more efficient algorithms for SP and nearly SP preferences. As for nearly structured preferences, Misra et al. [22] examined the parameterized complexity of CC-MW concerning the distance to SPness or SCness. The distance is measured by the number of voters (resp. alternatives) to delete to obtain SPness or SCness. They provide fixed-parameter (FPT) algorithms for the distance measure of deleting alternatives, but leave open the question of whether the same holds for deleting voters; we answer this question positively. For other distance measure, Skowron et al. [29] show that their algorithm for CC-MWcan be modified into an FPT algorithm wrt. the single-crossing width. The MONROE and CC rules have also been studied on other extensions of SPness. Peters et al. [25] study CC-MW on preferences that are SP on trees. Peters and Lackner [24] study Ordered Weight Average voting, which is an extension of the CC rule, on preferences that are SP on a circle. Godziszewski et al. [18] study multi-winner elections, including CC-MW$^+$ when the voters and alternatives can be embedded in 2-Euclidean space.

Our contributions. In this paper, we contribute to the algorithmic research of the Monroe and CC rules under (nearly) structure preferences, and provide several efficient algorithms. All algorithms are based on dynamic programming (DP). Foremost, in Section 3, we develop a novel polynomial-time algorithm for MONROE-MW with SC approval preferences, using DP; this answers an open question [29]. Note that under SC preferences, a key building block in standard DP algorithms for CC-MW is the continuous block property that an optimal solution may satisfy. A major challenge when developing algorithms for MONROE-MW however is the absence of such continuous block property even under SC approval preferences. We overcome this challenge by introducing a structural concept called *maximally good voter intervals* and show that they one-to-one correspond to the alternatives in an optimal solution. We observe that there exists an optimal solution where we can greedily assign to each alternative in the committee a voter interval (which may be potentially larger than the voter set that it represents). This approach allows us to effectively combine partial solutions and solve the MONROE-MW problem with

SC approval preferences efficiently.

Then, in Section 4, we show that CC-MW and MONROE-MW (except one case) are fixed-parameter tractable (FPT) for the distance measure of deleting t voters to achieve SPness or SCness, i.e., the corresponding problems can be solved in $f(t) \cdot (n+m)^{O(1)}$ time, where n and m denote the number of voters and alternatives, respectively. The results for the CC rule answer an open issue by Misra et al. The basic idea is to guess in FPT time how different deleted voters are going to be represented by the same alternatives and in which order these representing alternatives. We then combine the guessed structure into a DP which moves along the SC order and a specific order of the alternatives and finds a partial solution that additionally covers this guessed structure in FPT time. Finally, we present straightforward polynomial-time algorithms for the Monroe rule when only a constant number t of alternatives need to be deleted to achieve SPness (resp. SCness), demonstrating that the corresponding problems are in XP with respect to t. See Table 1 for an overview.

2 Preliminaries

(Full) proofs for results marked by $\star$ are deferred to the full version of the paper [8]. Given a non-negative integer $t \in \mathbb{N}$, let $[t]$ denote the set $\{1, \ldots, t\}$. We assume basic knowledge of parameterized complexity and refer to the textbook by Cygan et al. [11] for more details. For a more general introduction to multi-winner elections, see for example the book chapter by Faliszewski et al. [16]. For further methods and topics on multi-winner elections with approval preferences we refer to a recent book by Lackner and Skowron [19].

Preference profiles and structured preferences. A *preference profile* (*profile* in short) is a triple $(\mathcal{C}, \mathcal{V}, \mathcal{R})$, where $\mathcal{C}$ denotes a set $\mathcal{C}$ of m alternatives, $\mathcal{V}$ denotes a set of n voters with $\mathcal{V} = \{v_1, \ldots, v_n\}$, and $\mathcal{R}$ is a collection $\mathcal{R} = (\succeq_1, \ldots, \succeq_n)$ of preference orders such that each $\succeq_i$, is either a linear order or a subset of $\mathcal{C}$ and shall represents the preferences of voter v_i over $\mathcal{C}$, $i \in [n]$. For instance, for $\mathcal{C} = \{1, 2, 3, 4\}$, a preference order can be $3 \succ 1 \succ 2 \succ 4$ or a subset $\{2, 3\}$. The former means that 3 is preferred to 1, 1 to 2, and 2 to 4, while the latter means that 2 and 3 are approved while 1 and 4 not. Note that an approval set can also be considered as a dichotomous weak order, so having approval set $\{2, 3\}$ is equivalent to $\{2, 3\} \succ \{1, 4\}$, and we say that 2 is tied with 3, while 1 is tied with 4. For the sake of clarity, we will call a preference profile an *approval profile* if the preference orders are approval sets; otherwise it is a *linear preference profile*. Given a preference order $\succ$ and two alternatives x and y, we write $x \succeq y$ to mean that x is preferred to or tied with y. For instance, for $3 \succ 1 \succ 2$, we have that $3 \succeq 1$, and for the approval set $\{2, 3\}$, we have that $2 \succeq 3$ and $3 \succeq 2$. Given a voter $v_i \in \mathcal{V}$ and an alternative $a \in \mathcal{C}$, we define the rank of a in the preferences of v_i as $\mathsf{rk}_i(a) = |\{b \in \mathcal{C} \mid b \succ_i a\}|$. For approval preferences, we also use $\mathsf{A}(a)$ to denote the set of voters who each approve of a.

Definition 1 (Single-peaked and single-crossing)**.** Let $\mathcal{P}$ be a preference profile (with either linear or approval preferences). We say that $\mathcal{P}$ is *single-peaked* (SP) wrt. a linear order $\rhd$ of the alternatives $\mathcal{C}$ if for each voter $v_i \in \mathcal{V}$ and three distinct alternatives $a, b, c \in \mathcal{C}$ with $a \rhd b \rhd c$ or $c \rhd b \rhd a$ it holds that "$a \succeq_i b$" implies "$b \succeq_i c$". Accordingly, we say that $\mathcal{P}$ is *SP* if there exists a linear order $\rhd$ on $\mathcal{C}$ such that $\mathcal{P}$ is SP wrt. $\rhd$.

We say that $\mathcal{P}$ with *linear preferences* is *single-crossing* (SC) wrt. a linear order $\rhd$ of the voters $\mathcal{V}$ if for each pair $\{x, y\} \subseteq \mathcal{C}$ of alternatives and each three voters $v_i, v_j, v_k \in \mathcal{V}$ with $v_i \rhd v_j \rhd v_k$ it holds that "$x \succ_i y$ and $x \succ_k y$" implies "$x \succ_j y$". We say that $\mathcal{P}$ with *approval preferences* is *single-crossing* (SC) wrt. a linear order $\rhd$ of the voters $\mathcal{V}$ if for each alternative x the set of voters approving of x form an interval in $\rhd$.

It is polynomial-time solvable to check whether a given preference profile is single-peaked [1] (resp. single-crossing [5]). We say that a preference profile $\mathcal{P}$ is *t-alternatives nearly SP* (resp. *nearly SC*) if it is possible to delete at most t alternatives from $\mathcal{P}$ (and update the preference orders accordingly) to obtain an SP (resp. SC) profile. We define *t-voters nearly SP* (resp. *nearly SC*) profiles analogously. Determining the smallest t for a given profile to be t-alternatives nearly SP or t-voters nearly SC can be done in polynomial-time, while it is NP-hard to determine the smallest t for a given preference profile to be t-voters nearly SP or t-alternatives nearly SC [6, 15]. Nevertheless, the latter two problems are FPT with respect to t [13].

Multi-winner election with proportional misrepresentations. Let $\mathcal{P}$ denote a profile and $k \in \mathbb{N}$ be a number. An *assignment* σ is a function that maps voters to alternatives, i.e. $\sigma\colon \mathcal{V} \to \mathcal{C}$. We say that an assignment $\sigma\colon \mathcal{V} \to \mathcal{C}$ is a *k-assignment* if $|\sigma(V)| = k$, and that it is *proportional* if each assigned alternative represents roughly the same portion of the voters, i.e., $\lfloor n/k \rfloor \le |\{v \in V \mid \sigma^{-1}(a)\}| \le \lceil n/k \rceil$ holds for all $a \in \sigma(V)$. A *partial assignment* σ is a partial function that maps voters to alternatives.

A *misrepresentation function* $\rho\colon \mathcal{V} \times \mathcal{C} \to \mathbb{R}_{\ge 0}$ specifies the dissatisfaction of a voter towards an alternative. For linear preference profiles we consider Borda misrepresentation function, where $\rho(v_i, a) = \mathsf{rk}_i(a)$. For approval profiles we use the approval misrepresentation function, where $\rho(v_i, a) = 0$ if v_i approves a and 1 otherwise. There are two common ways of measuring the overall misrepresentations, one is the sum of misrepresentations of the voters and the other the maximum misrepresentation among all voters.

For a fixed number k, the MONROE rule finds a proportional k-assignment that minimizes either the sum or maximum of all misrepresentations. Under the CC rule, we do not have such a requirement–each alternative may represent an arbitrary number of voters. Consequently, we obtain four multi-winner (MW) election problems.

MONROE-MW^{+}(resp. MONROE-MW$^{\max}$)
Input: A preference profile $\mathcal{P}$, misrepresentation function ρ, a committee size k.
Question: Find a *proportional* k-assignment $\sigma\colon \mathcal{V} \to \mathcal{C}$ with minimum $\sum_{v_i \in \mathcal{V}} \rho(v_i, \sigma(v_i))$ (resp. $\max_{v_i \in \mathcal{V}} \rho(v_i, \sigma(v_i))$).

If we drop the proportionality requirement, we obtain the CC-MW^{+} and CC-MW$^{\max}$ problems. For the sake of brevity, we also use the same names to refer to the decision variants of these problems where a misrepresentation bound is given as input. Moreover, for each problem $\Pi \in \{$CC-MW^{+}, CC-MW$^{\max}$, MONROE-MW^{+}, MONROE-MW$^{\max}\}$, we use LINEAR-Π and APPROVAL-Π to refer to the problem Π with linear preferences and approval preferences, respectively. For instance, LINEAR-CC-MW^{+} denotes the problem CC-MW^{+} with linear preferences.

Let $I = ((\mathcal{C}, \mathcal{V}, \mathcal{R}), \rho, k)$ denote an instance of multi-winner election. We say that a tuple (W, σ) is a *solution of I* if σ is a k-assignment and $\sigma(V) = W$. Under the MONROE rule, we additionally require that a solution is proportional. Finally, if σ minimizes the corresponding misrepresentation measure, then we call (W, σ) an *optimal solution*.

As already noted by Betzler et al. [2], the decision variant of LINEAR-MW$^{\max}$ can be considered as a restriction of the decision variant APPROVAL-MW^{+} by letting each voter approve of those alternatives for whom the misrepresentation is within the bound β.

Table 1: Overview of the complexity for Monroe and CC (columns) with (nearly) SP/SC preferences (rows). Results in bold text are new. Note that "linear" means that the voters have linear preferences and the misrepresentation is based on Borda. MW+ refers to the problem of minimizing the *sum* of misrepresentations of all voters, while $\text{MW}^{\max}$ refers to the problem of minimizing the *maximum* misrepresentation of all voters. Next to the results one can see the source, or the corresponding corollary or theorem.

Preference structure	CC-MW+		$\text{CC-MW}^{\max}$		MONROE-MW+				$\text{MONROE-MW}^{\max}$			
	approval/linear		approval/linear		approval		linear		approval		linear	
Single-peaked	P	[2]	P	[2]	P	[2]	?		P	[2]	P	[2]
Single-crossing	P	[29, 14]	P	[29, 14]	**P**	(T1)	NP-hard	[29]	**P**	(T1)	?	
t-voters SP	**FPT**	(T2)	**FPT**	(C1)	**FPT**	(T4)	?		**FPT**	(T4)	**FPT**	(T4)
t-voters SC	**FPT**	(T3)	**FPT**	(C1)	**FPT**	(T4)	NP-hard	[29]	**FPT**	(T4)	?	
t-alternatives SP	FPT	[22]	FPT	[22]	**XP**	(T5)	?		**XP**	(T5)	**XP**	(T5)
t-alternatives SC	FPT	[22]	FPT	[22]	**XP**	(T5)	NP-hard	[29]	**XP**	(T5)	?	

Proposition 1 ([2]). $\text{CC-MW}^{\max}$ *(resp.* $\text{MONROE-MW}^{\max}$*) can be reduced to* APPROVAL-CC-MW+*(resp.* APPROVAL-MONROE-MW+*) in linear time.*

This reduction preserves the SP property, as was shown by Betzler et al., however the SC property is not necessarily satisfied.

3 Monroe for approval and SC preferences

In this section, we consider profiles with SC approval preferences and show that under the MONROE rule we can find an optimal solution in polynomial-time, using dynamic programming (DP). In Subsection 3.1 we describe some structural definitions that are in Subsection 3.2 used to show that there is always an optimal solution where the happy voters of the alternatives are ordered as certain types of voter intervals. This ordering allows us to build and combine partial solutions from the bottom up. Using our knowledge of this ordering, we will build and describe the DP in Subsection 3.3.

3.1 Additional definitions

To ease notation, we assume that the voters in $\mathcal{V}$ are named $1, 2, \ldots, n$ such that $1 > \cdots > n$ is an SC order. Before we show the DP approach, we first introduce necessary concepts and notations.

Let $\mathcal{P} = (\mathcal{C}, \mathcal{V}, \mathcal{R})$ be an SC approval preference profile. We call an assignment σ an *$[\hat{\imath}, \hat{\jmath}]$-assignment* if it is an assignment that maps the voter interval $[\hat{\imath}, \hat{\jmath}]$ to the set of alternatives, i.e. $\sigma\colon [\hat{\imath}, \hat{\jmath}] \to \mathcal{C}$.

Given an $[\hat{\imath}, \hat{\jmath}]$-assignment σ for $\mathcal{P}$, we derive a partial assignment called *$[\hat{\imath}, \hat{\jmath}]$-happy-assignment $\hat{\sigma}$* from σ which only considers happy voters. Therefore, $\hat{\sigma}(v) = \sigma(v)$ if $\rho(v, \sigma(v)) = 0$ and voters that are not happy under σ are left unassigned. If $[\hat{\imath}, \hat{\jmath}] = [n]$ we will just refer to it as a happy-assignment.

In the context of SC approval profiles, we say that (W, σ) is a *partial solution wrt. to an interval of voters $[\hat{\imath}, \hat{\jmath}]$* if σ is a partial $[\hat{\imath}, \hat{\jmath}]$-assignment with $\sigma([\hat{\imath}, \hat{\jmath}]) \subseteq W$ and $|\sigma^{-1}(a)| \leq \lceil \frac{n}{k} \rceil$ for all $a \in \sigma(\mathsf{A})$. Note that a solution is a partial solution wrt. $[n]$. We define the *misrepresentation of a partial solution* to be $\rho(\sigma) = \sum_{i \in [\hat{\imath}, \hat{\jmath}]} \rho(i, \sigma(i))$. Let (W, σ) be a (partial) solution. We say that a voter v is *happy with c* if he is assigned to c and has zero misrepresentation for c, i.e., $\sigma(v) = c$ and $\rho(v, c) = 0$. We say a voter v is *h-assignable* wrt. a (partial) solution (W, σ) if there exists an alternative $a \in \hat{\sigma}(V)$ such that $v \in \mathsf{A}(a)$.

Next, we define some relations between alternatives.

Definition 2 (Domination, incomparable sets, earlier sets)**.** Let $\mathcal{P}$ be an SC approval profile and let $\rhd$ be an SC order. For an arbitrary alternative $a \in \mathcal{C}$, we use $\text{fi}(a)$ and $\text{la}(a)$ to denote the first and the last voter in the order $\rhd$ that approve of a.

(a) $a_1 \in \mathsf{Earlier}(a_2)$. (b) $b_1 \in \mathsf{Dom}(b_2)$.

Figure 1: Illustration of relations between two alternatives in an SC approval profile (see Definition 2). Throughout all figures, the columns correspond to the voters such that the left-to-right order is SC, while the rows correspond to the alternatives. The squares in each row specify which voters approve the corresponding alternative. Due to the SC property, the squares in each row are consecutive.

Let a and b be two alternatives. We say that a *dominates* b if $\mathsf{A}(a) \supset \mathsf{A}(b)$, and call a (resp. b) the *dominator* (resp. *subordinate*) of b (resp. a). We use $\mathsf{Dom}(a)$ (resp. $\mathsf{Sub}(a)$) to denote the set consisting of all dominators (resp. subordinates) of a, i.e., $\mathsf{Dom}(a) = \{b \in \mathcal{C} \mid \mathsf{A}(a) \subset \mathsf{A}(b)\}$ and $\mathsf{Sub}(a) = \{b \in \mathcal{C} \mid \mathsf{A}(b) \subset \mathsf{A}(a)\}$. We say that a and b are *comparable* if $b \in \mathsf{Dom}(a) \cup \mathsf{Sub}(a) \cup \{a\}$, and they are *incomparable* if not. We will use the set $\mathsf{Incom}(a) = \mathcal{C} \setminus (\mathsf{Dom}(a) \cup \mathsf{Sub}(a) \cup \{a\})+$ to denote the set of alternatives that are incomparable to a.

We say that a *starts earlier* (resp. *starts later*) than b if $a \in \mathsf{Incom}(b)$ and $\text{fi}(a) < \text{fi}(b)$ (resp. $\text{fi}(b) < \text{fi}(a)$), and use $\mathsf{Earlier}(a)$ (resp. $\mathsf{Later}(a)$) to denote the set consisting of all alternatives that start earlier than a, i.e., $\mathsf{Earlier}(a) = \{b \in \mathsf{Incom}(a) \mid \text{fi}(b) < \text{fi}(a)\}$ (resp. $\mathsf{Later}(a) = \{b \in \mathsf{Incom}(a) \mid \text{fi}(a) < \text{fi}(b)\}$).

Figure 1 illustrates the relation between two alternatives.

The next definition gives a grouping of the alternatives based on how "dominated" they are and provides an ordering of the alternatives that is relevant for the DP. An alternative that is undominated is on level 1, an alternative that is only dominated by level 1 alternatives is on level 2 and so on.

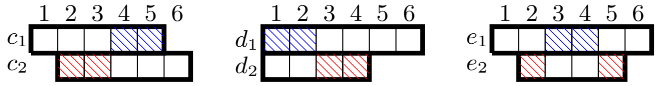
Definition 3 (Levels and canonical ordering)**.** Let $A \subseteq \mathcal{C}$ be a set of alternatives. We define the level sets inductively:

$$\begin{aligned} \text{level}_A(1) :=& \{c \in A \mid \mathsf{Dom}(c) = \emptyset\} \\ \text{level}_A(i+1) :=& \{c \in A \mid (\text{level}_A(i) \cap \mathsf{Dom}(c) \neq \emptyset) \\ & \wedge (\mathsf{Dom}(c) \cap A \subseteq \text{level}_A(1) \cup \ldots \cup \text{level}_A(i))\}. \end{aligned}$$

If A is clear from context, we will omit it from the subscript. The *canonical* ordering of A is defined recursively as follows: It starts with the alternatives in $\text{level}(1)$ ordered with alternatives that start earlier being earlier in the ordering, then $\text{level}(2)$ ordered based on the first voter that approves them and so on.

For the ease of notation, throughout the whole section, we assume

(a) The pair (a_1, a_2) is NT. (b) The pair (b_1, b_2) is NT.



(c) The pair (c_1, c_2) is not neatly ordered due to violating (i). (d) The pair (d_1, d_2) is not neatly ordered due to violating (ii). (e) The pair (e_1, e_2) is not neatly ordered due to violating (ii).

Figure 2: Illustration of neatly ordered and not neatly ordered (partial) solutions. The hatched parts indicate the assignment.

that $W = \{a_1, \ldots, a_t\}$ so that the order $a_1, \ldots, a_t$ respects the canonical ordering defined above.

We now go over desirable properties of a solution, based on which we then derive a corresponding nice structure of voter intervals to compute an optimal solution efficiently. The first property assumes that we can select the best alternatives among all that satisfy a group of voters, whereas the second property assumes that happy voters are assigned in an intuitive way.

Definition 4 (Monotone and neatly ordered solution)**.** Let (W, σ) be a partial solution wrt. a voter interval $[\hat{i}, \hat{j}]$. We say that (W, σ) is *monotone wrt. a set of alternatives* $A \subseteq \mathcal{C}$, if it is inclusion-wise maximal with relation to the domination relation. Formally it means:

- (i) For every $c \in W$ it holds that $\mathsf{Dom}(c) \cap A \subseteq W$ and
- (ii) for all $a, b \in A$ with $a \in \mathsf{Dom}(b)$ it holds that if there exists a voter $u \in [\hat{i}, \hat{j}]$ who is happy with b, then there exists a voter $v \in [\hat{i}, \hat{j}]$ who is happy with a.

We say a solution is *monotone* if it is monotone wrt. $\mathcal{C}$.

Let $a, b \in \hat{\sigma}([\hat{i}, \hat{j}])$ be a pair of alternatives. We say that *a precedes* b in (W, σ) if $\max(\hat{\sigma}^{-1}(a)) < \min(\hat{\sigma}^{-1}(b))$. Note that we identify the voter by its index in the single-crossing order, and take max and min over those. We say that a and b are *neatly ordered (NT)* in (W, σ) if the following two statements hold, where we assume, without loss of generality, that $a \in \mathsf{Earlier}(b) \cup \mathsf{Dom}(b)$:

- (i) If $a \in \mathsf{Earlier}(b)$, then a precedes b.
- (ii) If $a \in \mathsf{Dom}(b)$, then every voter $i \in \hat{\sigma}^{-1}(a)$ has $i < \mathrm{fi}(b)$ or $i > \max(\hat{\sigma}^{-1}(b))$.

We say a solution is neatly ordered (NT) if every pair $a, b \in \hat{\sigma}([\hat{i}, \hat{j}])$ is NT. We use MNT to abbreviate monotone and neatly ordered.

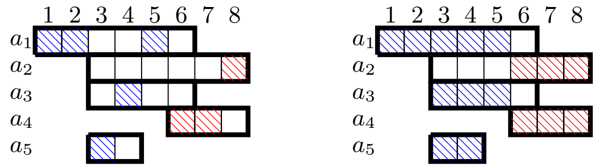
See Figure 2 for an illustration of monotonicity and neat ordering. Next, we define the voter interval structure which helps us find an optimal solution; also see Figure 3 for an illustration.

Definition 5 (Good and maximally good intervals)**.** Let (W, σ) be a (partial) solution and $a \in W$ an alternative. A voter interval $[i, j]$ is called *good* for a with respect to (W, σ) if it satisfies the following:

- (i) If $\hat{\sigma}^{-1}(a) = \emptyset$, then $[i, j] = \emptyset$.
- (ii) $\hat{\sigma}^{-1}(a) \subseteq [i, j] \subseteq \mathsf{A}(a)$.
- (iii) For all $\ell \in [i, j] \cap \hat{\sigma}^{-1}(W)$, we have $\hat{\sigma}(l) \cap \mathsf{Incom}(a) = \emptyset$.
- (iv) If (W, σ) is a partial sol. wrt. $[\hat{i}, \hat{j}]$, then $[i, j] \subseteq [\hat{i}, \hat{j}]$.

Note that the third requirement above states that every alternative that satisfies some voter from the range is comparable with a.

Let $\mathcal{I} := ([i_1, j_1], \ldots, [i_t, j_t])$ be a collection of voter intervals; recall that $W = \{a_1, \ldots, a_t\}$, where $a_1, \ldots, a_t$ are ordered according to the canonical ordering. We say that $\mathcal{I}$ is *good* for (W, σ) if for all $\ell \in [1, t]$, it holds that $[i_\ell, j_\ell]$ is good for a_ℓ and for all pairs $\{a_r, a_s\}$ with $a_r, a_s \in W$ it holds that



(a) An example for a (partial) solution that is neatly ordered. (b) The corresponding maximally good intervals.

Figure 3: Illustration of a maximally good collection of intervals corresponding to a solution; see Definition 5. The hatched parts indicate the assignment.

- (v) if $a_s \in \mathsf{Incom}(a_r)$, then $[i_s, j_s] \cap [i_r, j_r] = \emptyset$ and
- (vi) if $a_s \in \mathsf{Sub}(a_r)$, then $[i_s, j_s] \subseteq [i_r, j_r]$ or $[i_s, j_s] \cap [i_r, j_r] = \emptyset$.

We now define a signature to compare two good collections. Let $\mathcal{I} := \{[i_1, j_1], \ldots, [i_t, j_t]\}$ be a good collection of intervals with respect to (W, σ), then the signatures of $\mathcal{I}$ is defined as $\mathrm{Sig}(\mathcal{I}) = (j_1, \ldots, j_t, -i_1, \ldots, -i_t)$. We say that $\mathcal{I}$ is *maximally good* if $\mathrm{Sig}(\mathcal{I})$ is lexicographically maximal.

Next, we define what it means to be inbetween two alternatives and what it means to be usable.

Definition 6 (Inbetween and usable alternatives)**.** We start by defining subsets of incomparable alternatives that are only approved by voters from a given range: $\mathsf{Inbet}(a, b) = \{c \in \mathsf{Incom}(a) \cap \mathsf{Incom}(b) \mid \mathsf{A}(c) \subseteq [\mathrm{fi}(a), \mathrm{la}(b)]\}$ and $\widehat{\mathsf{Inbet}}(a, b)$ consists of all alternatives from $\mathsf{Inbet}(a, b)$ that are not dominated by any other alternative in $\mathsf{Inbet}(a, b)$. Formally, $\widehat{\mathsf{Inbet}}(a, b) = \mathrm{level}_{\mathsf{Inbet}(a,b)}(1)$.

Let c, c' be two (not necessarily distinct) alternatives and $[i, j]$ and $[i', j']$ two voter intervals such that $i' \leq i \leq j \leq j'$, $[i, j] \subseteq \mathsf{A}(c)$, and $[i', j'] \subseteq \mathsf{A}(c')$. We say that a third alternative a is *usable* for the six-tuple (c, i, j, c', i', j') if $a \in \mathsf{Sub}(c)$, $[i, j] \cap \mathsf{A}(a) \neq \emptyset$, and one of the following three conditions holds.

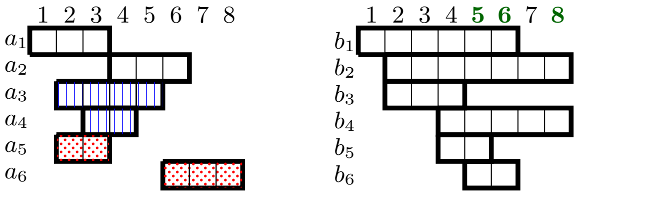
- (i) $\mathrm{fi}(a) \geq i$, or
- (ii) $\mathrm{fi}(a) \in [i', i-1]$ and for each alternative $b \in \mathsf{Dom}(a) \cap \mathsf{Sub}(c') \cap \mathsf{Earlier}(c)$, it holds that $\mathrm{fi}(b) < i'$ and there exists an alternative $b' \in \mathsf{Earlier}(c') \cap \mathsf{Dom}(b)$, or
- (iii) $\mathrm{fi}(a) < i'$ and for each alternative $b \in \mathsf{Dom}(a) \cap \mathsf{Incom}(c)$ it holds that $c \in \mathsf{Earlier}(b)$.

If an alternative b leads to the second or third condition being violated we say that *b blocks a from being usable* (wrt. the six-tuple (c, i, j, c', i', j')). We use $\mathcal{U}(c, i, j, c', i', j')$ to denote the set consisting of all useful alternatives:

$$\mathcal{U}(c, i, j, c', i', j') = \begin{cases} \{a \in \mathsf{Sub}(c) \mid a \text{ is usable for } (c, i, j, c', i', j')\}, & \\ \quad \text{if } c' \in \mathsf{Dom}(c) \cup \{c\} \text{ and } [i, j] \subseteq \mathsf{A}(c) \cap [i', j'] & \\ \quad \text{and } [i', j'] \subseteq \mathsf{A}(c') & \\ \{a \in \mathsf{Sub}(c) \mid a \text{ is usable for } (c, i, j, c, i, j)\}, & \\ \quad \text{if } c' = i' = j' = 0 & \\ \emptyset, \quad \text{otherwise.} & \end{cases}$$

Finally, let $\hat{\mathcal{U}}(c, i, j, c', i', j')$ denote the set consisting of all alternatives from $\mathcal{U}(c, i, j, c', i', j')$ that are not dominated by others in $\mathcal{U}(c, i, j, c', i', j')$, i.e., $\hat{\mathcal{U}}(c, i, j, c', i', j') = \mathrm{level}_{\mathcal{U}(c,i,j,c',i',j')}(1)$.

By the above definition it holds that $\mathsf{Inbet}(a, a) = \emptyset$. Intuitively the alternatives in $\mathcal{U}(c, i, j, c', i', j')$ will be the alternatives domi-



(a) $a_3 \in \widehat{\mathsf{Inbet}}(a_1, a_2)$, $a_4 \in \mathsf{Inbet}(a_1, a_2)$, $a_5, a_6 \notin \mathsf{Inbet}(a_1, a_2)$.

(b) $b_1, b_2, b_5 \notin \mathcal{U}(b_4, 6, 8, b_2, 5, 8)$ as $b_1, b_3 \notin \mathsf{Sub}(b_4)$ and although $b_5 \in \mathsf{Sub}(b_4)$, the existence of b_1 destroys the third condition given in Definition 6. Finally $b_6 \in \mathcal{U}(b_4, 6, 8, b_2, 5, 8)$.

Figure 4: Illustration of the inbetween and usable alternatives from Definition 6.

nated by c that can be assigned voters on the interval $[i, j]$ in a MNT solution where neither an alternative that is incomparable to c is assigned voters on $[i, j]$ nor an alternative that is incomparable to c' is assigned alternatives on $[i', j']$. Figure 4 illustrates Definition 6.

3.2 Technical lemmas

In this section we introduce some lemmas that relate to the structure of an optimal solution. The main observation is that any optimal solution can be translated into another optimal solution that is MNT and admits a maximally good collection of voter intervals.

Lemma 1 ($\star$). *Let $((\mathcal{C}, \mathcal{V}, \mathcal{R}), \rho, k)$ be an instance of* MONROE-MW$^+$*, $A \subseteq \mathcal{C}$ a subset of alternatives, and (W, σ) a partial solution wrt. $[\hat{\imath}, \hat{\jmath}]$. Then, we can construct another partial solution wrt. $[\hat{\imath}, \hat{\jmath}]$ that is* NT*and monotone with respect to A such that the misrepresentation is at most $\rho(\sigma)$.*

Observation 1 ($\star$). *Let (W, σ) be an* NT *partial solution wrt. $[\hat{\imath}, \hat{\jmath}]$. Then there exists a unique maximally good collection of intervals $\mathcal{I}$.*

Intuitively, the next lemma guarantees that every voter that could be part of some interval is part of at least one interval.

Lemma 2 ($\star$). *Let (W, σ) be an* NT *partial solution wrt. $[\hat{\imath}, \hat{\jmath}]$ and $\mathcal{I} = ([i_1, j_1], \dots, [i_t, j_t])$ the corresponding maximally good collection of intervals. Then, for each h-assignable voter $v \in [\hat{\imath}, \hat{\jmath}]$ there exists an alternative $a_x \in \hat{\sigma}([\hat{\imath}, \hat{\jmath}])$ such that $v \in [i_x, j_x]$.*

If a dominated alternative is not included in the interval of its dominator, the following lemma tells us that it must have another dominating alternative that contains it. Moreover, this alternative must be assigned happy voters on a specific interval. Contrapositively, we can use this lemma to show that no other dominator can contain the interval of the dominated alternative.

Lemma 3 ($\star$). *Let (W, σ) be an* NT *partial solution wrt. $[\hat{\imath}, \hat{\jmath}]$ and $\mathcal{I} = ([i_1, j_1], \dots, [i_t, j_t])$, the corresponding maximally good collection of intervals. For all pairs of alternatives $a_x, a_y \in \hat{\sigma}([\hat{\imath}, \hat{\jmath}])$ such that $a_y \in \mathsf{Dom}(a_x)$ and $[i_x, j_x] \nsubseteq [i_y, j_y]$, there exists an $a_z \in \mathsf{Dom}(a_x) \setminus \mathsf{Sub}(a_y)$ which satisfies $[i_x, j_x] \subseteq [i_z, j_z], [i_z, j_z] \cap [i_y, j_y] = \emptyset$ and the following:*
(1) If $j_y < i_x$, then $\hat{\sigma}^{-1}(a_z) \cap [j_y + 1, i_x - 1] \neq \emptyset$.
(2) If $j_x < i_y$, then $\hat{\sigma}^{-1}(a_z) \cap [1, i_y - 1] \neq \emptyset$.

3.3 The DP table

For ease of reasoning, in this section we will aim to find an optimal solution that maximizes the overall satisfaction of the voters rather than minimizes their dissatisfactions.

Intuition. As already mentioned, we will search for a maximally good collection of voter intervals that completely characterizes an MNT optimal solution. The idea is to iterate over the possible good voter-intervals for each alternative. The alternatives that can be assigned voters in these intervals are described by the usable sets (see Definition 6). These sets are disjoint for alternatives with disjoint intervals, which allows us to combine them to build bigger partial solutions from the bottom up.

The table. Table T has an entry for every tuple $(a, b, i, j, \hat{\mathsf{k}}, \check{\mathsf{k}}, n_a, n_b, n^*, B, c', i', j')$, where a and b are two incomparable alternatives, $[i, j]$ defines a voter interval, $\hat{\mathsf{k}}, \check{\mathsf{k}}, n_a, n_b, n^*$, are five non-negative integers with $0 \le \hat{\mathsf{k}}, \check{\mathsf{k}} \le m$, $0 \le n_a, n_b \le \lceil n/k \rceil$, and $0 \le n^* \le n$, $B \in \{0, 1\}$ being a binary value, and (c', i', j') being a promise to be specified shortly.

Informally, the table entries store the number of maximum happy voters on the interval $[i, j]$ of a partial MNT solution, if a is the first and b the last undominated alternative of the partial solution. We can show that in such a solution, we can only assign alternatives that are usable of a, b or alternatives between them (see Definition 6). The arguments c', i', j' tell us that we assume there is some so far uncounted alternative c' whose interval is $[i', j']$. This affects the usable sets of a, b and all other alternatives. The value B indicates whether we can use alternatives whose first approving voter is before i. The remaining arguments count the assigned and unassigned voters.

Formally, each table entry stores the maximum number of voters from the range $[i, j]$ that can be satisfied by an NT partial solution (W', σ) wrt. $[i, j]$, where $\sigma \colon [i, j] \to W'$ under the following seven conditions, where $W'_R := W' \cap \big(\mathsf{Inbet}(a, b) \cup \bigcup_{c \in \{a,b\} \cup \mathsf{Inbet}(a,b)} \mathcal{U}(c, \max\{i, \mathrm{fi}(c)\}, \min\{j, \mathrm{la}(c)\}, c', i', j')\big)$:

(T1) Every happy voter from $[i, j]$ is assigned to an alternative that is either a, or b, or is "inbetween" a and b, or is usable for a or b. Formally,

$$\hat{\sigma}([i, j]) \subseteq W' \subseteq \{a, b\} \cup \mathsf{Inbet}(a, b) \bigcup_{c \in \{a,b\} \cup \mathsf{Inbet}(a,b)} \mathcal{U}(c, \max\{i, \mathrm{fi}(c)\}, \min\{j, \mathrm{la}(c)\}, c', i', j')$$

(T2) W'_R contains $\hat{\mathsf{k}}$ (resp. $\check{\mathsf{k}}$) many alternatives that are each assigned to $\lceil n/k \rceil$ (resp. at most $\lfloor n/k \rfloor$) happy voters from $[i, j]$.

(T3) Alternative a (resp. b) is assigned n_a (resp. n_b) happy voters from $[i, j]$, i.e., $|\sigma^{-1}(a) \cap [i, j]| = n_a$ (resp. $|\sigma^{-1}(b) \cap [i, j]| = n_b$).

(T4) It holds that $|[i, j] \setminus \hat{\sigma}^{-1}(W')| \ge n^*$.

(T5) If $B = 0$, then there exists no alternative $c \in W'_R \cap \hat{\mathcal{U}}(a, i, \min(\mathrm{la}(a), j), c', i', j')$, such that $\mathrm{fi}(c) < i$.

(T6) (W', σ) is monotone with respect to $\big(\mathsf{Inbet}(a, b) \bigcup_{c \in \{a,b\} \cup \mathsf{Inbet}(a,b)} \mathcal{U}(c, \max\{i, \mathrm{fi}(c)\}, \min\{j, \mathrm{la}(c)\}, c', i', j')\big)$.

(T7) For each voter $v \in [i, j]$ that is unassigned under $\hat{\sigma}$ and each alternative $c \in \hat{\sigma}([i, j])$, it holds that if $v \in \mathsf{A}(c)$, then $v > \max(\hat{\sigma}^{-1}(c))$. Informally, we want voters that are assignable to an alternative to be only unassigned if they are later than the voters already assigned to that alternative.

We say that a table entry is *correct* if it computes the maximal number of happy voters for an NT partial solution that satisfies (T1)-(T7). Similarly we will say T is *correct* if every table entry is correct. In the following, we describe how to compute $T(a, b, i, j, \hat{\mathsf{k}}, \check{\mathsf{k}}, n_a, n_b, n^*, B, c', i', j')$. We assume that the arguments of the configuration fulfill the following conditions in the configuration.

(CT1) $[i, j] \subseteq [\text{fi}(a), \text{la}(b)]$.
(CT2) $a \in \mathsf{Earlier}(b) \cup \{b\}$.
(CT3) If $a \neq b$, then $j - i + 1 \geq n_a + n_b + n^*$; otherwise $n_a = n_b$ and $j - i + 1 \geq n_a + n^*$.
(CT4) If $c' \neq 0$, then $[i, j] \subseteq [i', j']$ and $i' \leq \text{fi}(a)$; otherwise $i' = j' = 0$.
(CT5) If $c' \neq 0, \mathcal{U}(c', i', j', c', i', j') \cap \mathsf{Dom}(a) \cap \mathsf{Earlier}(b) = \emptyset$.
(CT6) If $c' = 0$, then $\mathsf{Dom}(a) \cap \mathsf{Earlier}(b) = \emptyset$.
(CT7) If $a \neq b$, then $n_a \neq 0$ and $n_b \neq 0$.

Intuitively, (CT1)-(CT4) ensure that the chosen arguments make sense, e.g. the interval does not exceed the range of voters that could be assigned to the alternatives. Similarly, (CT5) and (CT6) are meant to prevent the combinations of table configurations that could not lead to an MNT solution. (CT7) is meant to ensure monotonicity.

For the case when at least one of the above conditions is violated, the corresponding entry is set to $-\infty$. We start by initializing the table as:

$$T(a, a, i, j, 0, 0, n_a, n_a, n^*, 0, c', i', j') := n_a \quad (1)$$

We update the table, distinguishing between three cases, either "$a = b$ and $B = 0$" (Case 1), "$a = b$ and $B = 1$" (Case 2), or "$a \neq b$" (Case 3). We first define two auxiliary functions

$$f_1(c_1, c_2, n_1, n_2) = \begin{cases} 2 & \text{if } c_1 \neq c_2 \wedge n_1 = n_2 = \lceil n/k \rceil \\ 0 & \text{if } n_1 \leq \lfloor n/k \rfloor \wedge n_2 \leq \lfloor n/k \rfloor \\ 1 & \text{else} \end{cases}$$

$$f_2(c_1, c_2, n_1, n_2) = |\{c_1, c_2\}| - f_1(c_1, c_2, n_1, n_2)$$

We compute the table in the following order, where the order holds only subject to the previous steps, e.g., 2 is only relevant subject to 1 and so on:

1. Order the entries by $\hat{\mathsf{k}} + \check{\mathsf{k}}$ in a non-decreasing order.
2. Order the entries with $a = b$ before the entries where $a \neq b$.
3. Order the entries by $j - i$ in a non-decreasing order.
4. Order the entries by $n_a + n_b$ in a non-decreasing order.
5. Order the entries by n^* in an increasing order.
6. Order the entries by B in an increasing order.

For the optimal solution, we return $\max_{\substack{a,b \in \text{level}_{\mathcal{C}}(1), i \leq j \\ \hat{\mathsf{k}} \leq n \mod k - f_1(a,b,n_a,n_b) \\ \hat{\mathsf{k}} + \check{\mathsf{k}} + |\{a,b\}| \leq k \\ n_a, n_b \leq \lceil n/k \rceil, n^* \leq n, B \in \{0,1\}}} T(a, b, i, j, \hat{\mathsf{k}}, \check{\mathsf{k}}, n_a, n_b, n^*, B, 0, 0, 0)$.

Theorem 1 (⋆). *For SC profiles,* Approval-Monroe-MW$^+$ *and* Approval-Monroe-MW$^{\max}$ *can be solved in polynomial time.*

Proof idea. We briefly discuss why the DP given in (2)–(4) is correct. Each DP entry corresponds to an optimal partial solution on interval $[i, j]$ when only alternatives that are usable for a, b, alternatives between them and alternatives usable for them may be assigned happy voters. By Lemma 1 we can always find an optimal solution which is MNT. In such a solution, we can construct a collection of good intervals (Definition 5). If the interval of a voter $a \in W$ is $[i, j]$, we can show that apart from a, the only alternatives that are usable for a (Definition 6) can be assigned happy voters on the interval $[i, j]$. Thus each partial solution considers all the alternatives that could be used for it and ignores the rest.

In Operations (2)–(3) we use partial solutions from lower level alternatives (Definition 3) to build a solution for the dominating alternative. In the first $\max$-operation we assume b and c are in the usable set of a, and we build a new committee that also assigns some voters to a. Because we keep track of unassigned voters with n^*, we know that there are enough unassigned voters to assign them to a. We can show that all the alternatives that are usable for b, c and the alternatives inbetween must also be usable for a, and thus the table requirements are preserved. In the second $\max$-operation we combine two committees that use alternatives usable for a but assign voters on disjoint intervals. With the parameters counting the number of assigned voters, we can be sure that we do not violate proportionality. The value $B = 0$ in the second entry enforces that the alternatives we use do not overlap. In Operation (4) we combine two disjoint table entries on the same level (Definition 3) to obtain a "wider" committee. By Lemma 3, we can show that the usable sets must be disjoint; the proof is in the full version of the paper. Therefore, as long as we take care to not assign too many voters, we will obtain a valid partial solution.

The final calculation goes over all intervals of level-1 alternative and all possible voter intervals they could cover. As Lemma 2 states that good intervals must cover all the h-assignable voters, we do not ignore any voters when we compute the final optimal solution.

It is easy to see that the algorithm for MW$^+$ can be adapted to solve MW$^{\max}$. □

4 Parameterized algorithms

In this section, we show parameterized results for CC-MW, Approval-Monroe-MW$^+$ and Monroe-MW$^{\max}$ for nearly SP and SC profiles wrt. the number of alternatives or voters to delete. We achieve all the results in this section by extending DP approaches from the literature and Section 3. Note that one can find a set of t voters (resp. alternatives) deleting which yields an SP (resp. SC) profile in FPT time wrt. t. Hence we assume in this section we are given such a set as part of the input.

Theorem 2 (⋆). CC-MW$^+$ *is* FPT *wrt. the number t of voters to delete to obtain an SP profile.*

Proof idea. For approval based voter deletion, the basic idea is to group the alternatives together that are approved by the same set of deleted voters, and observe that for each group we need at most one alternative of each group to represent the deleted voters. Hence, we can extend the DP algorithm by Betzler et al. [2] by trying all possible subsets of groups.

The idea for linear voter deletion is a bit more involved. We instead guess in FPT time an ordered partition of the deleted voters which corresponds to the ordering of the alternatives according to the SP ordering. This allows us to extend the DP algorithm by Betzler et al. [2] by trying all possible ordered partitions. □

Before we continue with our next FPT result, we observe that a similar continuous block property utilized by Skowron et al. [29, Lemma 5] also holds for SC approval preferences. This property is crucial for designing FPT result for the parameter number of voters (resp. alternatives) to delete to obtain SC preferences, and hence may be of independent interest. It is worth of noting that Elkind et al. [14] give an algorithm for Approval-CC-MW$^+$ that does not explicitly use continuous block property. This property is however the foundation of our FPT algorithm.

Lemma 4 (⋆). *Let $I = ((\mathcal{C}, \mathcal{V}, \mathcal{R}), \rho, k)$ be an instance of* Approval-CC-MW$^+$ *with SC preferences such that $\mathcal{C} = \{c_1, \ldots, c_m\}$ and $\mathcal{V} = [n]$, and $1 \rhd \ldots \rhd n$ is an SC order of the voters. Further, let $c_1 \blacktriangleright \ldots \blacktriangleright c_m$ be an order of the alternatives that orders the alternatives non-decreasingly according to*

Case 1: $T(a,a,i,j,\hat{\mathsf{k}},\check{\mathsf{k}},n_a,n_a,n^*,0,c',i',j') :=$

$$\max\left\{\begin{array}{l} \max\limits_{\substack{b,c\in\hat{\mathcal{U}}(a,i,j,c',i',j') \\ \hat{\mathsf{k}}'=\hat{\mathsf{k}}-f_1(b,c,n_b,n_c),\check{\mathsf{k}}'=\check{\mathsf{k}}-f_2(b,c,n_b,n_c) \\ \mathrm{fi}(b)\in[i,j],0<n_b,n_c\leq\lceil n/k\rceil,B\in\{0,1\}}} \left\{\begin{array}{l} T(b,c,i,j,\hat{\mathsf{k}}',\check{\mathsf{k}}',n_b,n_c,n_a+n^*,B,c',i',j')+n_a, \\ T(b,c,i,j,\hat{\mathsf{k}}',\check{\mathsf{k}}',n_b,n_c,n_a+n^*,B,a,i,j)+n_a \end{array}\right\}, \\ \max\limits_{\substack{n_1^*+n_2^*=n^*,b_1+b_2=n_a \\ i\leq i^*<j^*\leq j,\hat{\mathsf{k}}_1+\hat{\mathsf{k}}_2=\hat{\mathsf{k}},\check{\mathsf{k}}_1+\check{\mathsf{k}}_2=\check{\mathsf{k}}}} \left\{ T(a,a,i,i^*,\hat{\mathsf{k}}_1,\check{\mathsf{k}}_1,b_1,b_1,n_1^*,0,c',i',j'),+T(a,a,j^*,j,\hat{\mathsf{k}}_2,\check{\mathsf{k}}_2,b_2,b_2,n_2^*,0,c',i',j') \right\} \end{array}\right\}. \tag{2}$$

Case 2: $T(a,a,i,j,\hat{\mathsf{k}},\check{\mathsf{k}},n_a,n_a,n^*,1,c',i',j') :=$

$$\max\left\{\begin{array}{l} \max\limits_{\substack{b,c\in\hat{\mathcal{U}}(a,i,j,c',i',j') \\ \hat{\mathsf{k}}=\hat{\mathsf{k}}'+f_1(b,c,n_b,n_c),\check{\mathsf{k}}=\check{\mathsf{k}}'+f_2(b,c,n_b,n_c) \\ 0<n_b,n_c\leq\lceil n/k\rceil,B\in\{0,1\}}} \left\{\begin{array}{l} T(b,c,i,j,\hat{\mathsf{k}}',\check{\mathsf{k}}',n_b,n_c,n_a+n^*,B,c',i',j')+n_a, \\ T(b,c,i,j,\hat{\mathsf{k}}',\check{\mathsf{k}}',n_b,n_c,n_a+n^*,B,a,i,j)+n_a \end{array}\right\}, \\ \max\limits_{\substack{n_1^*+n_2^*=n^*,b_1+b_2=n_a \\ i\leq i^*<j^*\leq j,\hat{\mathsf{k}}_1+\hat{\mathsf{k}}_2=\hat{\mathsf{k}},\check{\mathsf{k}}_1+\check{\mathsf{k}}_2=\check{\mathsf{k}}}} \left\{ T(a,a,i,i^*,\hat{\mathsf{k}}_1,\check{\mathsf{k}}_1,b_1,b_1,n_1^*,1,c',i',j')+T(a,a,j^*,j,\hat{\mathsf{k}}_2,\check{\mathsf{k}}_2,b_2,b_2,n_2^*,0,c',i',j') \right\} \end{array}\right\}. \tag{3}$$

Case 3: $T(a,b,i,j,\hat{\mathsf{k}},\check{\mathsf{k}},n_a,n_b,n^*,B,c',i',j') :=$

$$\max\left\{\begin{array}{l} \max\limits_{\substack{c\in\widehat{\mathsf{Inbet}}(a,b) \\ i\leq i^*<j^*\leq j,0<n_c\leq\lfloor n/k\rfloor \\ \hat{\mathsf{k}}_1+\hat{\mathsf{k}}_2=\hat{\mathsf{k}},\check{\mathsf{k}}_1+\check{\mathsf{k}}_2+1=\check{\mathsf{k}} \\ n_1^*+n_2^*=n^*,B'\in\{0,1\}}} \left\{ T(a,c,i,i^*,\hat{\mathsf{k}}_1,\check{\mathsf{k}}_1,n_a,n_c,n_1^*,B,c',i',j')+T(b,b,j^*,j,\hat{\mathsf{k}}_2,\check{\mathsf{k}}_2,n_b,n_b,n_2^*,B',c',i',j') \right\}, \\ \max\limits_{\substack{c\in\widehat{\mathsf{Inbet}}(a,b) \\ i\leq i^*<j^*\leq j,n_c=\lceil n/k\rceil \\ \hat{\mathsf{k}}_1+\hat{\mathsf{k}}_2+1=\hat{\mathsf{k}},\check{\mathsf{k}}_1+\check{\mathsf{k}}_2=\check{\mathsf{k}} \\ n_1^*+n_2^*=n^*,B'\in\{0,1\}}} \left\{ T(a,c,i,i^*,\hat{\mathsf{k}}_1,\check{\mathsf{k}}_1,n_a,n_c,n_1^*,B,c',i',j')+T(b,b,j^*,j,\hat{\mathsf{k}}_2,\check{\mathsf{k}}_2,n_b,n_b,n_2^*,B',c',i',j') \right\}, \\ \max\limits_{\substack{i\leq i^*<j^*\leq j \\ \hat{\mathsf{k}}_1+\hat{\mathsf{k}}_2=\hat{\mathsf{k}},\check{\mathsf{k}}_1+\check{\mathsf{k}}_2=\check{\mathsf{k}} \\ n_1^*+n_2^*=n^*,B'\in\{0,1\}}} \left\{ T(a,a,i,i^*,\hat{\mathsf{k}}_1,\check{\mathsf{k}}_1,n_a,n_a,n_1^*,B,c',i',j')+T(b,b,j^*,j,\hat{\mathsf{k}}_2,\check{\mathsf{k}}_2,n_b,n_b,n_2^*,B',c',i',j') \right\} \end{array}\right\}. \tag{4}$$

their first approving voter $\mathrm{fi}()$, *subject to that non-increasingly according to their last approving voter* $\mathrm{la}()$ *in* $\rhd$. *Then, for each solution* (W,σ^*), *there exists another one* (W,σ) *for the same committee* W *with* $\rho(\sigma)\leq\rho(\sigma^*)$ *such that the following holds:*

(i) For each alternative $c\in W$ *the assigned voters* $\sigma^{-1}(c)$ *defines an interval in* $\rhd$.

(ii) For each two voters $u,v\in\mathcal{V}$ *with* $u\rhd v$, $\sigma(u)=c_i$ *and* $\sigma(v)=c_j$ *it holds that* $i<j$.

Theorem 3 ($\star$). *For* t*-voters nearly SP profiles,* CC-MW$^+$ *is* FPT *wrt.* t.

Together with Proposition 1, we obtain the following.

Corollary 1. *For* t*-voters nearly SP (resp. SC) profiles,* CC-MW$^{\max}$ *is* FPT *wrt.* t.

Using analogous methods to Theorems 2 and 3, we extend the DP approaches for APPROVAL-MONROE-MW$^+$ for SP (resp. SC) profiles to account for the deleted voters in $\hat{V}$.

Theorem 4 ($\star$). *For* t*-voters nearly SP profiles,* APPROVAL-MONROE-MW$^+$ *and* MONROE-MW$^{\max}$ *are* FPT *wrt.* t. *For* t*-voters nearly SC profiles,* APPROVAL-MONROE-MW$^+$ *and* APPROVAL-MONROE-MW$^{\max}$ *are* FPT *wrt.* t.

While the parameter "number of voters to delete to obtain an SP (resp. SC) profile" turned out to be very useful for APPROVAL-MONROE-MW, similar to CC-MW, the parameter "number of alternatives to delete", for which CC-MW was FPT, does not immediately yield FPT-result for APPROVAL-MONROE. By guessing how many voters each deleted alternative will be assigned, we can extend the existing DP approaches and obtain the following XP result:

Theorem 5 ($\star$). *For* t*-alternatives nearly SP profiles,* APPROVAL-MONROE-MW$^+$ *and* MONROE-MW$^{\max}$ *are in XP wrt.* t. *For* t*-alternatives nearly SC profiles,* APPROVAL-MONROE-MW$^+$ *and* APPROVAL-MONROE-MW$^{\max}$ *are in XP wrt.* t.

5 Conclusion and open questions

We provide several efficient algorithms for MONROE-MW and CC-MW under (nearly) SP and SC preferences. Our work leads to some immediate open questions. First, it remains open whether our XP results from Theorem 5 could be improved to FPT algorithms. Perhaps the flow network approach from Betzler et al. [2] could be useful here. Second, our polynomial-time algorithm for Theorem 1 is rather complicated and has a high running time. It would be interesting to know whether it can be improved, and if so, how.

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