Distinct Angles in Three Dimensions

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Abstract

We consider the question of the minimum number of distinct angles in three dimensions. We discuss the difficulties of constructing a point configuration in general position with fewer than $O(n^2)$ distinct angles, and we examine explicit constructions of point configurations in $\mathbb{R}^3$, such as the cylindrical helix and the conchospiral, which use self-similarity to minimize the number of distinct angles. We then consider pinned variants of the question.

1. Introduction

The Question: There are n points distributed in space, where n is some large number. No three points are allowed on the same line, and no four points are allowed on the same circle. What is the smallest possible number of distinct angles formed?

There are $\binom{n}{3} \approx n^3/6$ ways to choose three of the n points. Three points form a triangle, which has three angles. So the total number of angles is $3 \cdot n^3/6 = n^3/2$. The goal is to repeat angles many times so that the number of *distinct* angles is as small as possible.

When points are placed in the two-dimensional plane, [FlHu] and [FlKo] showed that the minimum number of distinct angles is $\Omega(n)$, which means that we must have at least a constant times n distinct angles, and $O(n^2)$, which means that it is possible to have only a constant times n^2 distinct angles. They used a **logarithmic spiral** to get $O(n^2)$ distinct angles.

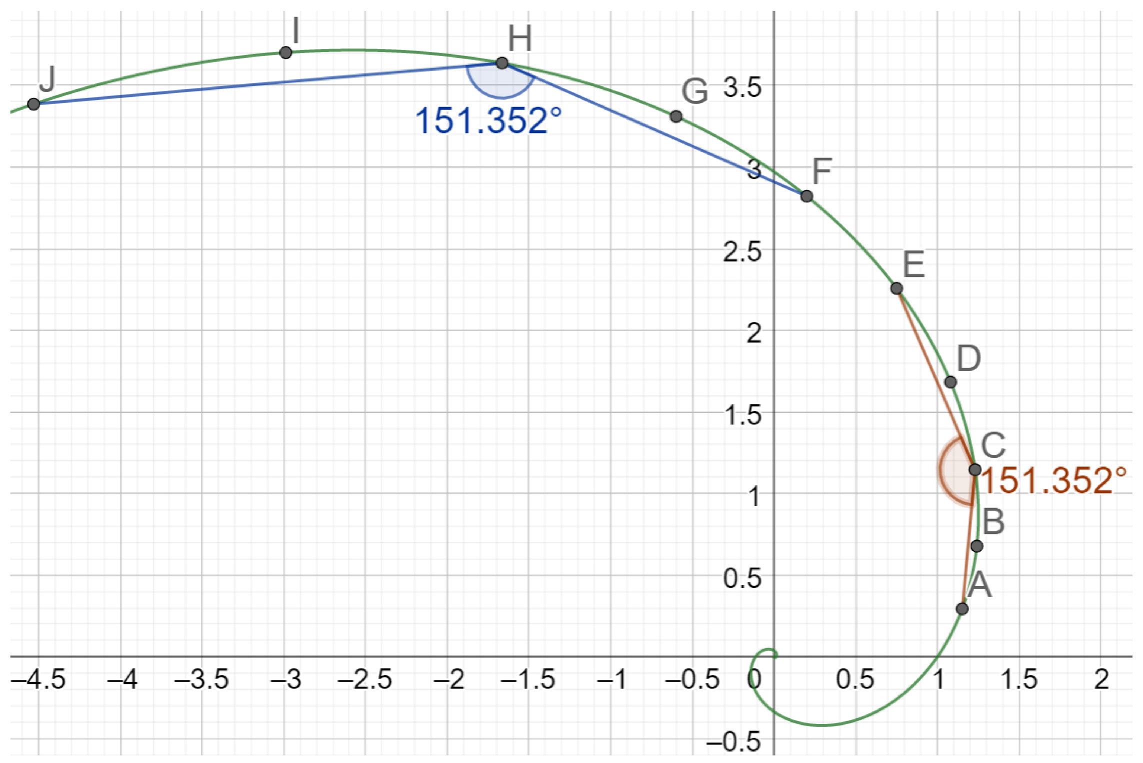

Figure 1: *Points distributed along the logarithmic spiral.*

Due to the geometric properties of the logarithmic spiral, any angle in this construction can be made again by "rotating" the three points along the spiral. This point configuration exhibits what we call **self-similarity**: any angle that can be made from three of the points can also be made with a special point A as one of the points. There are $\binom{n}{2} \approx n^2/2$ ways to choose the remaining two points, so any point configuration with self-similarity has only $O(n^2)$ angles.

2. Three-dimensional constructions

Minimizing the number of distinct angles seems easier in three dimensions: there is more space in which to move around. Consider the following question.

If we fix an endpoint and a center point, how many distinct angles must be formed by choosing the other endpoint out of the remaining points?

In two dimensions, the answer is at least $(n-2)/2$. Indeed, for any angle, there are only two lines from the fixed center point that form that angle with the fixed endpoint. We can't have three points on a line, so we can only put one point on each of these lines.

But in three dimensions, if we fix an endpoint and a center point, we might only form one distinct angle. This can be done by placing all the remaining points on a cone whose vertex is the center point and whose axis is the line formed by the center point and the middle point.

Nevertheless, no constructions have been found with fewer than n^2 angles in order of magnitude. We conjecture that no such construction exists.

> **Conjecture 2.1.** *For any configuration of n points in two or three dimensions that has no three points on a line and no three points on a circle, $\Omega(n^2)$ distinct angles must be formed.*

Still, it is worth studying uniquely three-dimensional constructions that use self-similarity to have only $O(n^2)$ distinct angles.

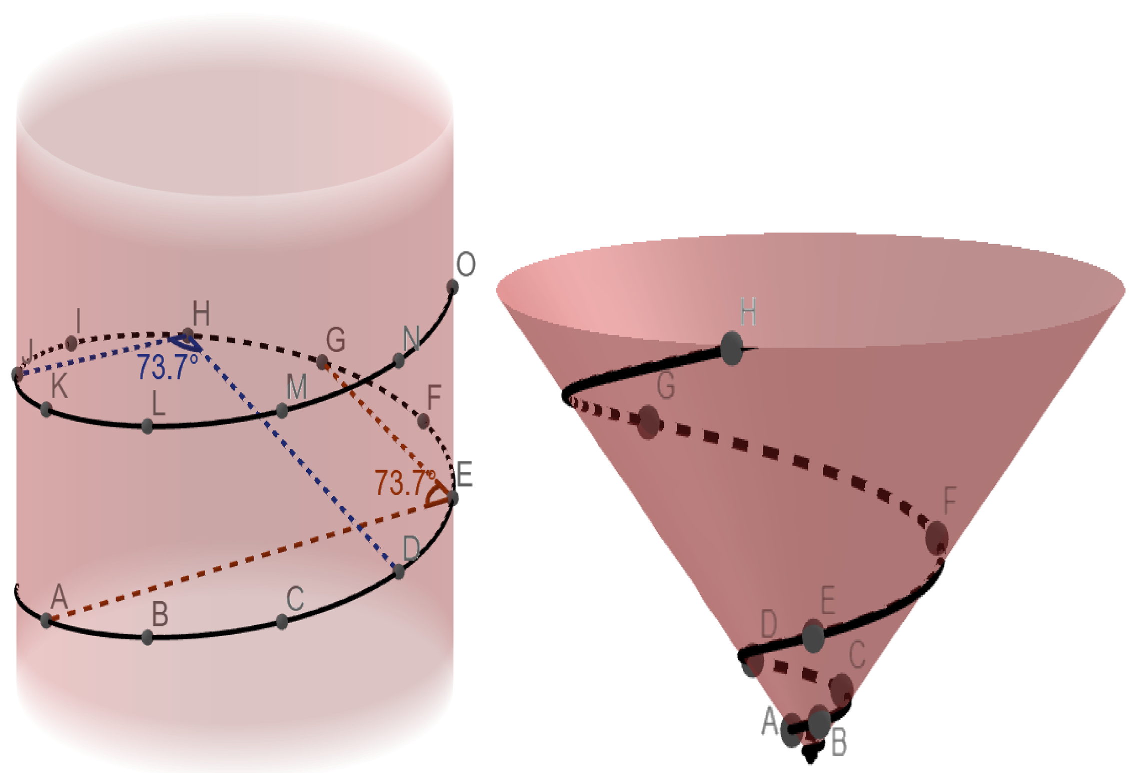

Figure 2: *To the left, points are distributed along a **cylindrical helix**, parametrized by $(\cos(t), \sin(t), t)$. To the right, points are distributed on a **conchospiral**, parametrized by $(e^t\cos(t), e^t\sin(t), e^t)$. Due to their symmetry, both of these point configurations exhibit self-similarity.*

The cylindrical helix has an additional layer of symmetry: flipping it upside down doesn't change the configuration! This cuts the number of angles in half compared to the conchospiral.

3. Pinned Variants

We can also ask "pinned" variants of the distinct angles question. For example, how many angles must be formed with a special point pinned as the center point, or as the endpoint?

3.1 Pinned Center Point and Distinct Distances on the Sphere

Consider pinning a point A. If B and C are arbitrary points, how many distinct angles of the form $\angle BAC$ are there? In other words, how many distinct angles are there when we pin A as the center point?

Notice that the angle $\angle BAC$ does not depend on the distance of B from A or on the distance of C from A. Therefore, we can manipulate the distance of each point from A, so that for any point B, the distance of B from A is 1. After this manipulation, each point lies on a sphere of radius 1 centered at A.

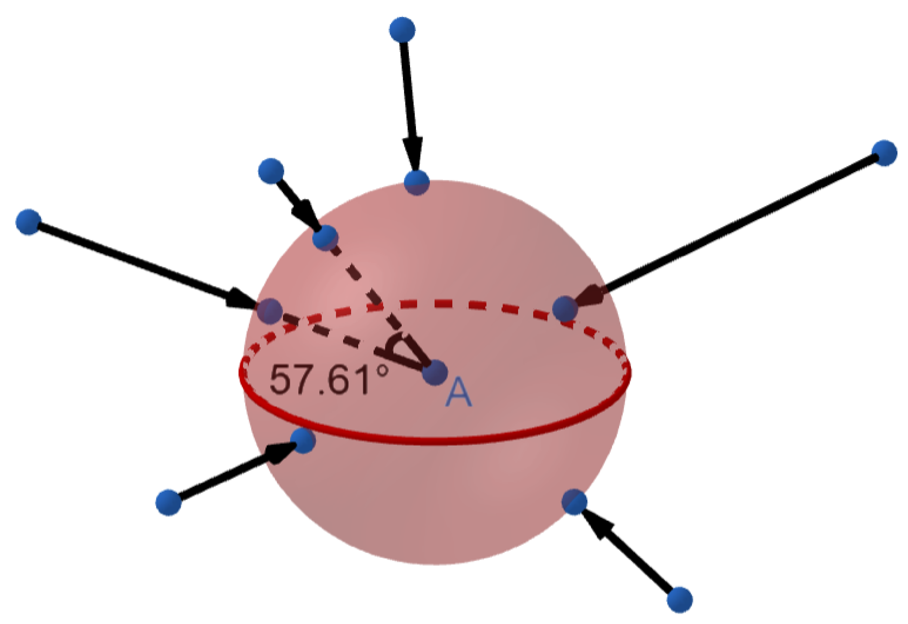

Figure 3: *The angular distance between B and C, which is measured by the angle $\angle BAC$, is a constant multiple of the spherical distance between B and C, the distance between B and C along a great circle of the sphere centered at A.*

Therefore, determining the number of distinct angles with fixed center point A is equivalent to determining the number of distinct distances for these points lying on a sphere of radius 1 centered at A. The best-known lower bound for the number of distinct distance on the sphere is $\Omega(\frac{n}{\log n})$, which is a generalization of the bound for the Erdős distinct distance conjecture proven by Guth and Katz ([GuKa]).

> **Corollary 3.1.** *The number of distinct angles for n points in $\mathbb{R}^3$ is $\Omega\left(\frac{n}{\log n}\right)$.*

3.2 One Pinned Endpoint

Suppose we are fixing point A as an endpoint. Now introduce another point B, and consider the angle formed with A and B as end points and the angle with B as the central point. The surfaces that preserve these 2 angles are a cone and a spindle torus, both with line AB as axis. The intersection of the two is a circle, so we are allowed no more than 3 points on it. So, the number of cones multiplied by the number of spindle tori must be at least $(n-2)/3$.

There are at least $(\#\{\text{cones}\} + \#\{\text{spindle tori}\})/2$ distinct angles. To minimize this, $\#\{\text{cones}\} = \#\{\text{spindle tori}\} = \sqrt{(n-2)/3}$.

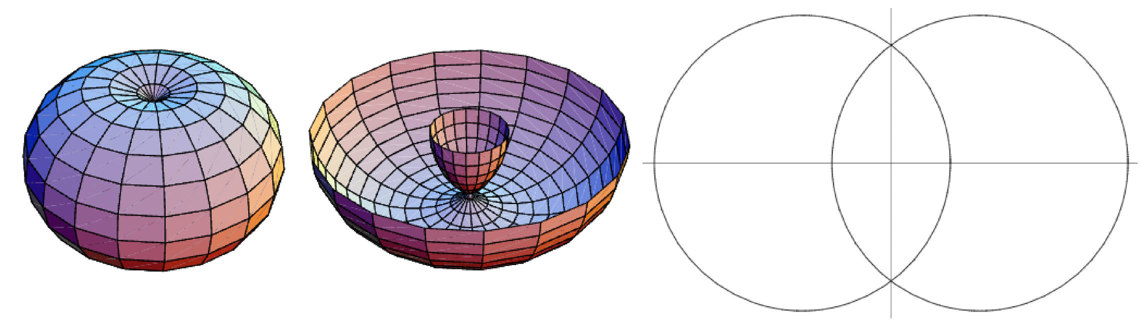

Figure 4: *A **spindle torus** (a circle rotated around a chord other than the diameter), cutaway view, and cross section. Credit: [WoAl].*

> **Theorem 3.2.** *Consider a configuration of n points in general position in three dimensions, and fix a point A. The number of angles formed with A as an endpoint is $\Omega(\sqrt{n})$.*

This gives us a lower bound, but it is likely a substantial undercount, because we are neglecting angles formed with A as an endpoint that have center points on the cone/torus intersections. In fact, we conjecture that having a pinned endpoint provides no asymptotic improvement in terms of lower bound compared to the unpinned case.

> **Conjecture 3.3.** *The number of distinct angles with a pinned endpoint in general position in $\mathbb{R}^3$ is $\Omega(n^2)$.*

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References

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[FlKo] Henry L. Fleischmann, Sergei V. Konyagin, Steven J. Miller, Eyvindur A. Palsson, Ethan Pesikoff, and Charles Wolf, *Distinct Angles in General Position*, preprint, 2022.

[GuKa] L. Guth and N.H. Katz, *On the Erdős distinct distances problem in the plane*, Annals Math. **181** (2015), 155–190.

[WoAl] Spindle torus images from Wolfram Alpha, `https://mathworld.wolfram.com/SpindleTorus.html`