Distinct Angles and Angle Chains in $\mathbb{R}^3$

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History

- Erdős, 1946: For a configuration of n points in the plane, what is the minimum number of distinct distances between pairs of points?
  - Conjecture: The $\sqrt{n} \times \sqrt{n}$ lattice is the best configuration, with $c\frac{n}{\sqrt{\log(n)}}$ distinct distances.
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- Erdős also asked the question of how many distinct angles there must be, but the question depended a lot on what constraints are placed on the point configuration.
- **SMALL 2021**: The question of distinct angles in a plane is most interesting when we disallow three points on a line or four points on a circle (also called general position).

Notation

- $f(n) = O(g(n))$ means that for some constant c,
  $f(n) \leq c \cdot g(n)$ for all n sufficiently large.
- $f(n) = \Omega(g(n))$ means that for some constant c,
  $f(n) \geq c \cdot g(n)$ for all n sufficiently large.

Distinct Angles in Two Dimensions

SMALL 2021: Let A_{gen} be the minimum number of angles formed by n points on a plane, with no three points on a line and no four points on a circle. Then, $A_{\text{gen}} = O(n^2)$.

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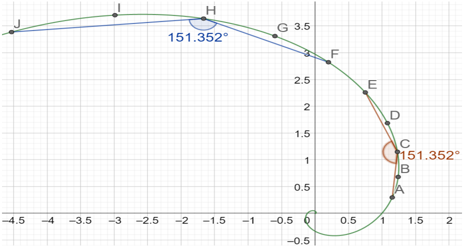
- To prove this, they distributed points on a *logarithmic spiral*.



Rotate three points along the spiral to repeat the same angle!

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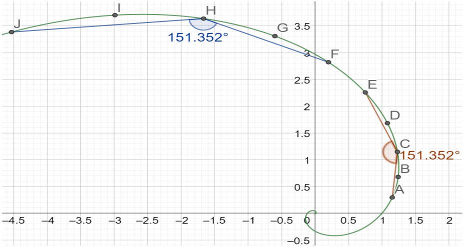

Rotate three points along the spiral to repeat the same angle!

- **Self-similarity**: Any angle formed by three of the points can also be formed using a special point A as one of the points.
- $\binom{n}{2}$ ways to choose the remaining two points, so $O(n^2)$ angles.

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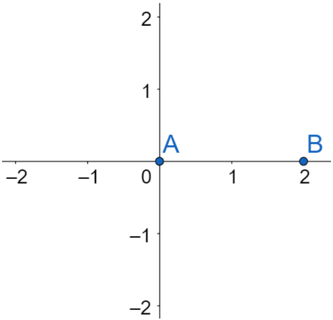
- Fix two points A and B and consider only angles with A as an endpoint and B as the center point.



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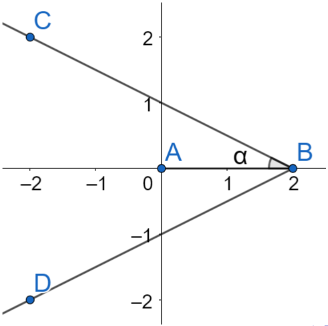
- Fix two points A and B and consider only angles with A as an endpoint and B as the center point.
- We can only form a given angle twice without putting three points on a line.



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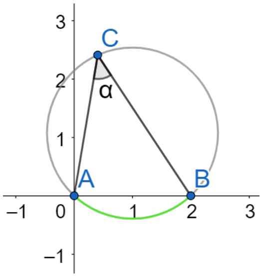

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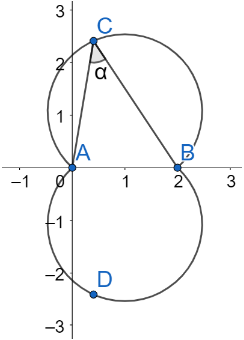

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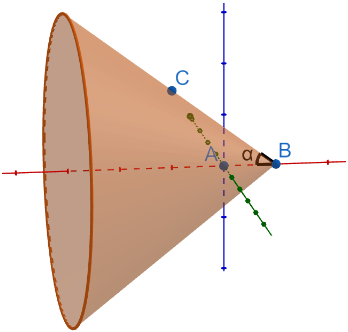

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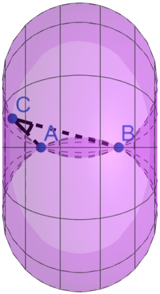
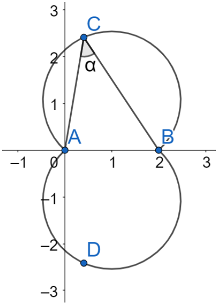
- Now if we fix A as an endpoint and B as the center point, we can put all remaining points on a cone to form only one distinct angle.
- If we fix A and B as endpoints, we can put all remaining points on a spindle torus to form only one distinct angle.





Questions we can ask

1. What lower bound can we get on the number of distinct angles in three dimensions with no three points on a line and no four points on a circle?
2. Using the extra space that we have in 3D, can we find a construction with fewer than $O(n^2)$ distinct angles?

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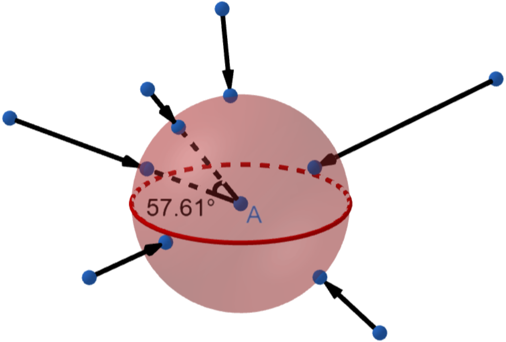

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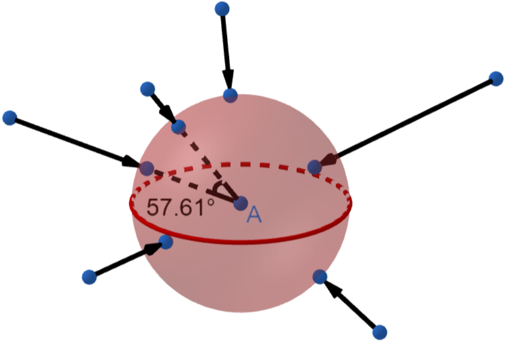
- The measure of $\angle BAC$ is a constant multiple of the spherical distance between B and C.



Pinned Center Point

Theorem (Guth and Katz, 2015)

A set of n points in the plane determines $\Omega\left(\frac{n}{\log n}\right)$ distinct distances.

Generalizing to sphere (Tao)

A set of n points on a sphere determines $\Omega\left(\frac{n}{\log n}\right)$ distinct distances.

Pinned Center Point

Determining the number of distinct angles with fixed center point A is equivalent to determining the number of distinct distances for these points lying on a sphere of radius 1 centered at A.

Corollary

The number of distinct angles for n points in $\mathbb{R}^3$ with a fixed center point is $\Omega\left(\frac{n}{\log n}\right)$.

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Note: By distributing points along a circle on a sphere, we get an $O(n)$ upper bound on the minimum number of distinct angles with a fixed center point. The lower and upper bounds are very close together!

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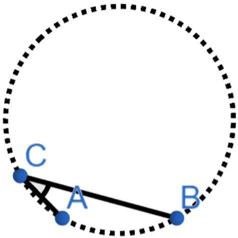

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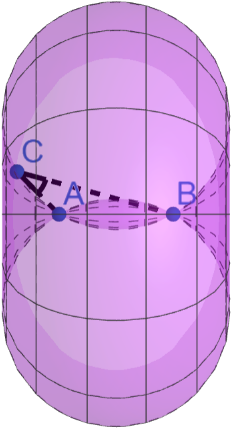

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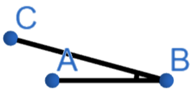

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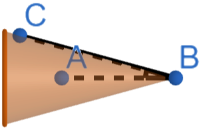

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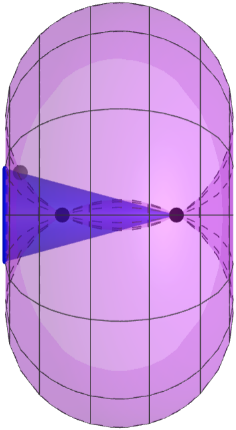

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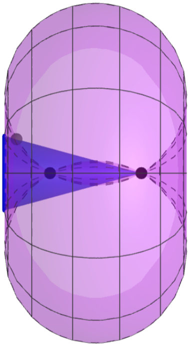
- The intersection of a cone and spindle torus with the same axis is a circle. We only allow 3 points on this circle!



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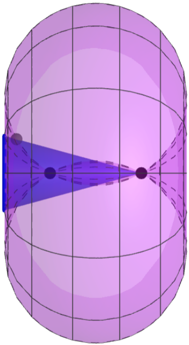

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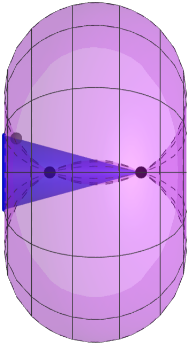

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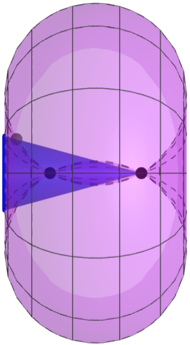
- There are at least $\max(\#\{\text{cones}\}, \#\{\text{s. tori}\})$ distinct angles.
- To minimize this, $\#\{\text{cones}\} = \#\{\text{s. tori}\} = \sqrt{(n-2)/3}$.



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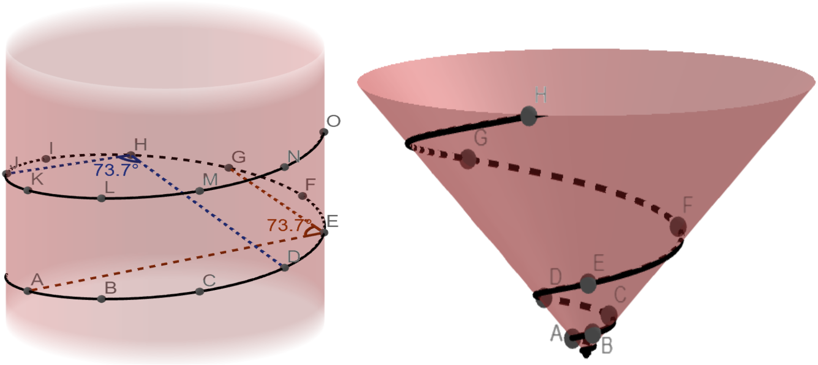
Conjecture

For any construction in general position in $\mathbb{R}^3$, there are $\Omega(n^2)$ distinct angles formed when an endpoint is pinned (the same order as with no pinned points).

- Even without proving this conjecture, it's clear the extent to which pinning an endpoint and pinning a center point lead to radically different results.

3D Constructions



To the left, points are distributed along a **cylindrical helix**, parametrized by $(\cos(t), \sin(t), t)$. To the right, points are distributed on a **conchospiral**, parametrized by $(e^t \cos(t), e^t \sin(t), e^t)$. Due to their symmetry, both of these point configurations exhibit self-similarity and thus have $O(n^2)$ distinct angles.

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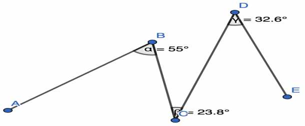
Distinct Angle Chains

- A *k-chain* is a $(k+2)$-tuple of points $(x_1, \ldots, x_{k+2})$ along with the associated k-tuple of angles

$$(\alpha_1, \ldots, \alpha_k) = (\angle x_1x_2x_3, \ldots, \angle x_kx_{k+1}x_{k+2}).$$



A sample three-chain in $\mathbb{R}^2$

- There are n points in space with no three points on a line and no four points on a circle. For a given k, what is the minimum number of distinct k-tuples such that there exists a k-chain with those angles?
- If $k = 1$, this is just the question we already asked.

Distinct Angle Chains in 2D

- Recall: In 2D, if one endpoint and the center point of the angle are fixed, we get $\Omega(n)$ angles since no three points are on a line.
- So, adding one leg to the chain must multiply the number of distinct angle chains by $\Omega(n)$.

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- So, adding one leg to the chain must multiply the number of distinct angle chains by $\Omega(n)$.
- By induction:

Theorem

For n points in general position in two dimensions, there are $\Omega(n^k)$ distinct k-tuples of angles with associated k-chains.

Distinct Angle Chains in 2D

- The logarithmic spiral provides the best upper bound we could hope for in two dimensions.

Theorem

With points distributed on the logarithmic spiral, there are $O(n^{k+1})$ distinct k-tuples of angles with associated k-chains.

Distinct Angle Chains in 3D

- In 3D, it is no longer true that adding a leg to the chain creates n choices for the new angle.
- We have the following weaker lower bound on the number of distinct angle k-chains.

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Theorem

In three dimensions, the number of distinct k-tuples of angles with associated k-chains is bounded below by:

$$\begin{cases} \Omega\left(\frac{n^{(k+2)/3}}{(\log n)^{(k+2)/3}}\right) & \text{if } k = 1 \mod 3; \\ \Omega\left(\frac{n^{(k+1)/3}}{(\log n)^{(k-2)/3}}\right) & \text{if } k = 2 \mod 3; \\ \Omega\left(\frac{n^{k/3+1/2}}{(\log n)^{k/3}}\right) & \text{if } k = 0 \mod 3. \end{cases}$$

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- Takeaway: The lower bound gets multiplied by $n/\log(n)$ every time the chain gets 3 longer (compared to n^3 for 2D).

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Future Work

- Going forward, we hope to make progress in raising the lower bound for distinct angles in $\mathbb{R}^3$ with a pinned endpoint.
- This would improve bounds for the number of distinct angle chains in 3D, as would any further improvements on A_{gen}.
- The ultimate goal would be to come up with explicit constructions that minimize the number of distinct k-chains for a given k and n.

Future Work

- We also want to look into what happens when we relax the general position requirement, for example allowing $O(\sqrt{n})$ points on a line or on a circle.

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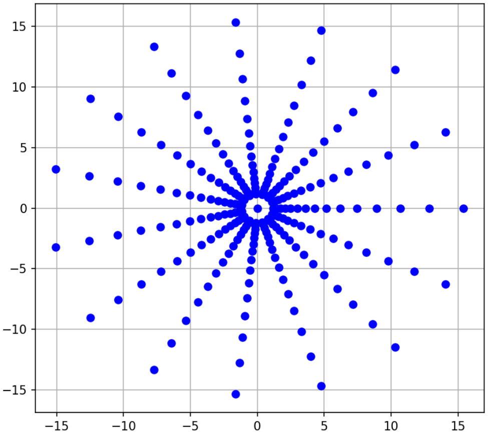

Permitting $O(\sqrt{n})$ points on lines and circles allows for a configuration with $O(n)$ distinct angles with a pinned endpoint.

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