Green Hackenbush

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Combinatorial Games

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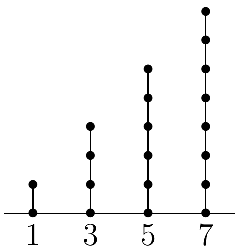
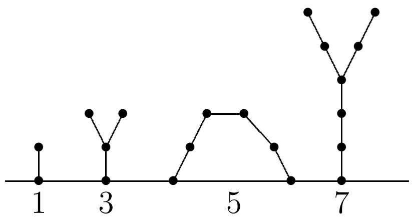
Introduction In this paper, we will explore the game of Green Hackenbush. Green Hackenbush is a combinatorial game two people take turns playing. Each person takes their turn by taking an edge (line), from a grounded graph called a landscape. These edges can be arranged to form many figures. The number of possible edges is infinite. Let's begin with a simple game of Bouton's Nim. This game is directly equivalent to a corresponding game of Green Hackenbush. We will begin with pile sizes of 1, 3, 5, and 7 counters. At each player's turn, they are able to take as many as desired from any one pile . We will designate their game with four-tuples (a, b, c, d) of pile sizes. The game begins with $(1, 3, 5, 7)$. For example, the game might proceed $(1, 3, 5, 7) \xrightarrow{1} (1, 1, 5, 7) \xrightarrow{2} (1, 1, 1, 7) \xrightarrow{1} (1, 1, 1, 1) \xrightarrow{2} (1, 1, 1, 0) \xrightarrow{1} (1, 1, 0, 0) \xrightarrow{2} (1, 0, 0, 0) \xrightarrow{1} (0, 0, 0, 0)$.

In this game player number 1 was the winner. This game of nim can be represented in Green Hackenbush.

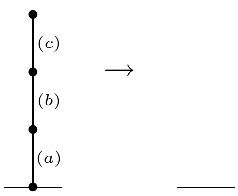
In this particular setup, the Green Hackenbush game can be played the same as nim.

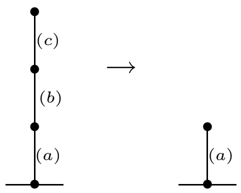


However, Green Hackenbush can also be set up in many different ways as such:

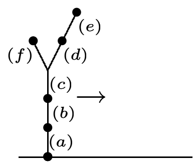


In the game of Green Hackenbush, when one takes a bottom edge such as (a), all edges that are disconnected from the ground are removed.





When one takes an edge such as (b) this is also true, but the edges under the one taken remain.



This rule applies to all games of Green Hackenbush. Here is another example.

Suppose we take (d), picture (e) will also be taken.

Now we will take (b), which will also take (C) and (f) with it. picture

This move will leave (a). picture

We will now show how to compute the Grundy value of a Green Hackenbush game. Suppose we have a string of three edges. We look at the possible moves we could make from the string.

picture

From edge (r) the only move that can be made is to an empty landscape whose value is zero. Thus, the value of (r) is 0. Zero is the Grundy value for (r). From edge (s), the only move one can make is to (r) or to zero. Using the MEX system (minimum excludant), one would know that the value for (s) is 1.This method is true for (t) also, and so (t)'s value is 2. When we take these numbers we have 0,1,and 2. The least non-negative number that is missing (MEX) is the number 3, so the value of the whole game is 3. When one has a string such as the one above, there is an easier way to find the value. In the case above the value is 3 because there is a string of 3 edges all in a line. The value of the line is equivalent to the number of edges in it. To find a winning strategy we will first learn the relation of n to this game. In a game of consecutive edges or a game in a "y" shape the number of edges in that string is denoted by n.

picture picture picture picture

In these pictures n is equivalent to the Grundy value of the game. The value of n is needed when you are trying to compute the overall value of a Green Hackenbush figure. Below is a formula for such Green Hackenbush figures.

picture picture

These values were found according to a formula. Below is a demonstration of how the Grundy values are calculated.

picture

When (a) is left out (b), (d), and (e) form a line. Since there are three edges the value is three. picture

When we take (b), it takes all of the other edges along with it, so it has value 0. picture

When we take the edge (c) it takes (d) with it. When these two edges are gone we are left with a line of two edges which has value 2. picture

When (d) is left out (a), (b), and (c) form a " y " shape. We learned earlier that the value of a "y" shape is n and in this case n = 1 therefore picture (d) = 1.

Now that we have figured the values for the figure above we see the edges of the figure have values of 0,1,2, and 3. When looking to find the value of the overall game, we use the MEX system as we did earlier. One should look at the numbers and find the least number that is missing . In this case the value of the game is 4.

When playing Green Hackenbush with arcs there is a very simple way to find the overall value of the game. The number of edges corresponds with the games value. If the number of edges is odd, the value is 1. Similarly, if the number of edges is even the value is 0.

Since the game has three edges, the value of the game is 1. picture

This game has 4 edges and so its value is 0. picture

If an extra line is added to the particular game (b) to create an odd number of edges the value becomes one. This is not true in all cases but in this particular one it is.

Put together these 5 edges create an odd number and give an overall value of picture 1.

Once one knows the value of an individual game the values for each game are nim added to determine the total value consequently, the player is able to make a safe move. Earlier we found the value of (a) to be 1, and the value of (b) to be zero.

picture picture Nim Picture

When these numbers are nim added the result is 1. In this case you would want to make the first move so that you could make the sum 0. If the sum had been 0 to start with one would have wanted to go second so that you would not disturb the symmetry (zero). When one makes the move as the first player one desires to create a zero, so in this case we would take the middle edge of (a). By taking this edge we have created a zero and made the game symmetric. Player one can now just mimic the moves of player. Thus, player one will make the last move.

picture

In our game after our first move we had two single edges left. We were then able to create a zero because canceled one another out to create a zero. Say for instance there were three ones, the total value would be 1. The result is 1 because each pair cancels themselves out and since there was only one pair there was a 1 left over, which in turn became our result. This method is called nim addition, as I mentioned earlier. We use this addition to find a safe move in the game. Explanation of the safe and unsafe movements in a game if very important. From every safe position there is an unsafe position. A safe position can never lead to another safe position, but an unsafe position may lead to another unsafe position or a safe position.

safe picture

With this in mind, apply this idea to the game that is being played. The first player took the middle edge, which then left two single edges and a four-sided arc. From this safe position there is no other safe position. Thus, the second player must make an unsafe move.

picture

Suppose the second player took a side of the arc. This leaves three single edges and a string of two edges. When we nim add these values we find the value to be three.

picture nim picture

To create a zero we must change the line of two edges to a single edge.

picture nim picture

In doing this we have created a zero and have therefore made a safe move. When our opponent makes the next move they must take a single edge, leaving an odd number which is unsafe. This game continues until all edges are gone. Since there were four single edges left it is inevitable that the first player will win because he can mimic the moves of the second player since he has mad it symmetrical. In the game that has just been played one would want to be the first player because the game is not symmetric and the first player would want to restore the symmetry. As has been exhibited in this paper, the Green Hackenbush game is an altered, very complex way of playing nim. One must be able to find the Grundy values, apply those values to nim addition, and know how to make a game symmetric. Once these skills are acquired one must build upon them to become a skilled master at the game of Green Hackenbush.