

Original citation:

Karni, Edi and Safra, Zvi. (2015) Continuity, completeness, betweenness and conomonotonicity. *Mathematical Social Sciences*, 74 . pp. 68-72.

Permanent WRAP url:

<http://wrap.warwick.ac.uk/74542>

Copyright and reuse:

The Warwick Research Archive Portal (WRAP) makes this work by researchers of the University of Warwick available open access under the following conditions. Copyright © and all moral rights to the version of the paper presented here belong to the individual author(s) and/or other copyright owners. To the extent reasonable and practicable the material made available in WRAP has been checked for eligibility before being made available.

Copies of full items can be used for personal research or study, educational, or not-for-profit purposes without prior permission or charge. Provided that the authors, title and full bibliographic details are credited, a hyperlink and/or URL is given for the original metadata page and the content is not changed in any way.

Publisher's statement:

© 2015, Elsevier. Licensed under the Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International <http://creativecommons.org/licenses/by-nc-nd/4.0/>

A note on versions:

The version presented here may differ from the published version or, version of record, if you wish to cite this item you are advised to consult the publisher's version. Please see the 'permanent WRAP url' above for details on accessing the published version and note that access may require a subscription.

For more information, please contact the WRAP Team at: publications@warwick.ac.uk

warwick**publications**wrap

highlight your research

<http://wrap.warwick.ac.uk>

Continuity, Completeness, Betweenness and Cone-monotonicity*

Edi Karni[†] and Zvi Safra[‡]

December 16, 2014

Abstract

A non-trivial, transitive and reflexive binary relation on the set of lotteries satisfying independence that also satisfies any two of the following three axioms satisfies the third: completeness, Archimedean and mixture continuity (Dubra (2011)). This paper generalizes Dubra's result in two ways: First, by replacing independence with a weaker betweenness axiom. Second, by replacing independence with a weaker cone-monotonicity axiom. The latter is related to betweenness and, in the case in which outcomes correspond to real numbers, implies monotonicity with respect to first-order stochastic dominance.

·
·

Keywords: Archimedean, mixture continuity, completeness, betweenness, first-order stochastic dominance, decision making under risk

·
·

JEL classification numbers: D01, D81

*We thank Juan Dubra and an anonymous Associate Editor for very useful and insightful comments.

[†]Department of Economics, Johns Hopkins University and Warwick Business School, University of Warwick

[‡]Warwick Business School, University of Warwick (and Tel Aviv University, Emeritus)

1 Introduction

Building on a theorem of Schmeidler (1971), Dubra (2011) proved that a non-trivial, transitive and reflexive binary relation on the set of lotteries satisfying independence that also satisfies any two of the following three axioms satisfies the third: completeness, Archimedean and mixture continuity. In this paper we generalize Dubra's result by replacing independence with the weaker betweenness axiom.¹ In addition, we show that if outcomes correspond to real numbers (e.g., monetary prizes) then Dubra's result still holds even if instead of independence we only assume monotonicity with respect to first-order stochastic dominance. In fact, we prove the result replacing independence with a weaker axiom dubbed cone-monotonicity. Cone-monotonicity axiom is weaker than monotonicity with respect to first-order stochastic dominance and independence, and is related to (but not implied by) the betweenness axiom.

In the next section we introduce the analytical framework and the axioms whose interrelation constitute the focal point of this work. The relations among the continuity conditions and completeness with betweenness are analyzed in section 3. In section 4 we introduce the cone-monotonicity axiom and analyze the relations among the continuity conditions and completeness under cone-monotonicity. In section 5 we discuss the relations between cone-monotonicity and betweenness and the relations between cone-monotonicity and monotonicity with respect to first order stochastic dominance. Concluding remarks appear in section 6.

2 The Analytical Framework

Let X be a finite set of k *outcomes*, denote by $\Delta(X)$ the set of all probability measures on X , and by $\text{aff}\Delta(X)$ the affine hull of $\Delta(X)$. For each $p, q \in \Delta(X)$ and $\alpha \in [0, 1]$ define $\alpha p + (1 - \alpha)q \in \Delta(X)$ by $(\alpha p + (1 - \alpha)q)(x) = \alpha p(x) + (1 - \alpha)q(x)$, for all $x \in X$.

Let \succsim be a binary relation on $\Delta(X)$ and denote by \succ and by \sim the asymmetric and symmetric parts of \succsim , respectively. We list below some

¹Safra (2014) studied the representations of axiomatic theories that include betweenness and depart from the completeness axiom. Karni and Zhou (2014) examined representations of weighted utility theory (which satisfies betweenness) without completeness.

well-known properties that \succsim might satisfy.

(A.1) **Non-trivial partial order** \succsim is reflexive and transitive with a non-empty asymmetric part.

The relations among the next three axioms are the focal point of our analysis.

(A.2) **Archimedean** For all $p, q, r \in \Delta(X)$, if $p \succ q$ then there exist $\alpha, \beta \in (0, 1)$ such that $\alpha p + (1 - \alpha)r \succ q$ and $p \succ \beta q + (1 - \beta)r$.

(A.3) **Mixture continuity** For all $p, q, r \in \Delta(X)$ the sets

$$\{\alpha \in [0, 1] \mid \alpha p + (1 - \alpha)r \succsim q\} \text{ and } \{\alpha \in [0, 1] \mid q \succsim \alpha p + (1 - \alpha)r\}$$

are closed.

(A.4) **Completeness** For all $p, q \in \Delta(X)$, either $p \succsim q$ or $q \succsim p$.

3 Continuity, Completeness and Betweenness

The next axiom is a weakening of the independence axiom.²

(A.5) **Betweenness** For all $p, q \in \Delta(X)$, $r \in \{p, q\}$ and $\alpha \in (0, 1)$,

$$p \succsim q \iff \alpha p + (1 - \alpha)r \succsim \alpha q + (1 - \alpha)r$$

Clearly, this implies that similar equivalences hold for \succ and \sim .

This statement of the betweenness axiom demonstrates to what extent it is a weakening of the independence axiom (as r is not restricted to the set $\{p, q\}$). Moreover, Axiom (A.5) implies the following more common statements of the betweenness property (see Chew 1989 and Dekel 1986): For all $p, q \in \Delta(X)$, $\alpha \in (0, 1)$

$$p \succ q \implies p \succ \alpha p + (1 - \alpha)q \succ q$$

and

$$p \succ q \implies p \succ \alpha p + (1 - \alpha)q \succ q.$$

²For clarity, here is the version of the independence we refer to: For all $p, q, r \in \Delta(X)$ and $\alpha \in (0, 1)$,

$$p \succ q \iff \alpha p + (1 - \alpha)r \succ \alpha q + (1 - \alpha)r.$$

When \succsim is complete (A.5) is implied by them.

Our first result generalizes the theorem of Dubra (2011) by replacing independence with betweenness. Formally,

Theorem 1 *Suppose that \succsim is a non-trivial partial order on $\Delta(X)$ satisfying betweenness. Then any two of the three axioms (A.2)-(A.4) imply the third.*

Proof (a) Suppose that \succsim satisfies Archimedean and completeness. Let $p, q, r \in \Delta(X)$ and consider the set $A = \{\alpha \in [0, 1] \mid \alpha p + (1 - \alpha)r \succsim q\}$. Without loss of generality assume that $p \succsim r$ (by completeness, either $p \succsim r$ or $r \succsim p$) and note that, by betweenness, $\alpha p + (1 - \alpha)r \succsim \beta p + (1 - \beta)r$ for all $1 \geq \alpha \geq \beta \geq 0$. If $A = \emptyset$ then we are done. Otherwise, define $\alpha^* = \inf\{\alpha \in A\}$. If $\alpha^* \in A$ then, by betweenness, $A = [\alpha^*, 1]$ and hence A is closed. Assume that $\alpha^* \notin A$ and note that, by definition, for every $\varepsilon' > 0$ there exists $\varepsilon \in (0, \varepsilon')$ satisfying $(\alpha^* + \varepsilon)p + (1 - (\alpha^* + \varepsilon))r \succsim q$. By betweenness, this implies $[\alpha^* + \varepsilon, 1] \subset A$ and hence $(\alpha^*, 1] \subset A$. Next note that if $\alpha^*p + (1 - \alpha^*)r \succ q$ does not hold then, by completeness, $q \succ \alpha^*p + (1 - \alpha^*)r$ and hence, by Archimedean, there exists $\beta \in (0, 1)$ such that $q \succ \beta(\alpha^*p + (1 - \alpha^*)r) + (1 - \beta)p$. But

$$\begin{aligned} \beta(\alpha^*p + (1 - \alpha^*)r) + (1 - \beta)p &= (\alpha^* + (1 - \beta)(1 - \alpha^*))p \\ &\quad + (1 - (\alpha^* + (1 - \beta)(1 - \alpha^*)))r \end{aligned}$$

while $\alpha^* + (1 - \beta)(1 - \alpha^*) \in (\alpha^*, 1]$; a contradiction.

The proof that $\{\alpha \in [0, 1] \mid q \succsim \alpha p + (1 - \alpha)r\}$ is closed follows by the same argument.

(b) Suppose that \succsim satisfies mixture continuity and completeness. Let $p, q, r \in \Delta(X)$ and suppose that $p \succ q$. Mixture continuity implies that the set $A^q = \{\beta \in [0, 1] \mid q \succsim \beta p + (1 - \beta)r\}$ is closed and $p \succ q$ implies that $1 \notin A^q$. Hence, $[0, 1] \setminus A^q$ is a non-empty open subset of $[0, 1]$. Take $\alpha \in [0, 1] \setminus A^q$ and note that, by completeness, $\alpha p + (1 - \alpha)r \succ q$.

The proof that there exists $\beta \in (0, 1) \setminus A^p$ for which $p \succ \beta q + (1 - \beta)r$ is similar.³

³Note that betweenness was not used in this part. Under completeness, mixture continuity is stronger than Archimedean.

(c) Suppose that \succsim satisfies Archimedean and mixture continuity. We show first that, for all $p \in \Delta(X)$, the sets $\bar{B}(p) = \{q \in \Delta(X) \mid q \succsim p\}$ and $\bar{W}(p) = \{q \in \Delta(X) \mid p \succsim q\}$ are closed and the sets $B(p) = \{q \in \Delta(X) \mid q \succ p\}$ and $W(p) = \{q \in \Delta(X) \mid p \succ q\}$ are open relative to $\Delta(X)$.

Fix $p \in \Delta(X)$. If $\bar{B}(p)$ is a singleton there is nothing to prove. We therefore assume it is not and start by showing that $\bar{B}(p)$ is convex. Take $p^1, p^2 \in \bar{B}(p)$ and let $p^\alpha = \alpha p^2 + (1 - \alpha)p^1$ for some $\alpha \in (0, 1)$. Note that we cannot assume that either $p^1 \succ p^2$ or $p^2 \succ p^1$ hold. If for some $i \in \{1, 2\}$ $p^i \sim p$ then, for $j \neq i$ and by transitivity, $p^j \succ p^i$ and hence, by betweenness, $p^\alpha \succ p$. Otherwise, assume that $p^i \succ p$, for both $i = 1, 2$, and consider the set $A = \{\beta \in [0, 1] \mid \beta p^1 + (1 - \beta)p^\alpha \succ p\}$ ($A \neq \emptyset$ by construction). Denote $\beta^* = \inf\{\beta \in A\}$, $p^* = \beta^* p^1 + (1 - \beta^*)p^\alpha$ and note that, by mixture continuity, $\beta^* \in A$. If $p^* = p^\alpha$ (that is, if $\beta^* = 0$) then $p^\alpha \succ p$ and we are done. Assume $p^* \neq p^\alpha$ and note that, by an argument similar to one used in part (a), Archimedean implies that p^* is not strictly preferred to p . Hence $p^* \sim p$. Finally by transitivity, both $p^1 \succ p^*$ and $p^2 \succ p^*$ hold and, by betweenness, $p^\gamma \succ p$ for all $\gamma \in [0, 1]$. Hence $p^\alpha \succ p$ and $\bar{B}(p)$ is convex.

Next choose $q \in \Delta(X)$ in the boundary of $\bar{B}(p)$, let $\{q^n\} \subset \bar{B}(p)$ be a sequence that converges to q , let r belong to the relative interior of $\bar{B}(p)$ and let $N_\varepsilon(r)$ be an open ε -ball around r whose intersection with $\text{aff}(\bar{B}(p))$ is a subset of $\bar{B}(p)$. By construction, for a fixed $\varepsilon \in (0, \bar{\varepsilon})$ $r^n = r + \frac{\varepsilon}{\|q^n - q\|}(q - q^n) \in N_\varepsilon(r) \cap \bar{B}(p)$ and, by the convexity of $\bar{B}(p)$,

$$\bar{q}^n = \frac{\varepsilon}{\varepsilon + \|q^n - q\|}q^n + \frac{\|q^n - q\|}{\varepsilon + \|q^n - q\|}r^n \succ p$$

Next observe that, as $\bar{q}^n = \frac{\varepsilon}{\varepsilon + \|q^n - q\|}q + \frac{\|q^n - q\|}{\varepsilon + \|q^n - q\|}r$, \bar{q}^n belongs to the line segment connecting r and q and, by construction, the sequence \bar{q}^n converges to q . By mixture continuity, $q \succ p$ and hence $\bar{B}(p)$ is closed.

The proof that $\bar{W}(p)$ is closed follows by a similar argument.

Consider next the set $B(p)$.⁴ If $B(p) = \emptyset$ then there is nothing to prove. Suppose that it is not and observe that, by arguments similar to the above, $B(p)$ is convex. Take $q \in B(p)$ and $r^1, \dots, r^{k-1} \in \Delta(X)$ such that $\{r^i - q\}_{i=1}^{k-1}$ span $\text{aff}(\Delta(X))$. By Archimedean, for each r^i there exists $\alpha^i \in (0, 1)$ satisfying $q + \alpha^i(r^i - q) \succ p$. Next, if there exists $\gamma^i >$

⁴In this part of the proof we follow in the footsteps of Dubra (2011).

0 such that $q - \gamma^i (r^i - q) \in \Delta(X)$ then, by Archimedean, there exists $\beta^i \in (0, \gamma^i)$ satisfying $q - \beta^i (r^i - q) \succ p$. If such γ^i does not exist (this happens if q belongs to the boundary of $\Delta(X)$) then for this i we define $\beta^i = 1$. Let $\alpha = \min \{\alpha^i, \beta^i\}_{i=1}^{k-1}$ and note that, by convexity, the convex hull of $\{q + \alpha (r^i - q), q - \alpha (r^i - q)\}_{i=1}^{k-1}$ contains the intersection of $\text{aff}(\Delta(X))$ with an open ball $N_\varepsilon(q)$ around q . Hence, q is an interior point of $B(p)$ and $B(p)$ is open.

By similar argument, $W(p)$ is open.

Since $\Delta(X)$ is a connected topological space these observations, in conjunction with the theorem of Schmeidler (1971), imply that \succ is complete. \square

4 Continuity, Completeness and Cone-monotonicity

For our second result, we replace the betweenness axiom with an axiom dubbed cone-monotonicity. Although this axiom seems weaker than betweenness (and indeed is satisfied by some betweenness relations), this implication does not always hold (see the discussion below). As we explain later, all relations that are monotone with respect to the partial relation of first-order stochastic dominance satisfy cone-monotonicity.⁵

(A.6) Cone-monotonicity Every $p \in \text{int}\Delta(X)$ has a non-empty cone $C^p \subset \text{aff}\Delta(X) - \{p\}$, open relative to $\text{aff}\Delta(X) - \{p\}$, such that for all $r \in \Delta(X)$,

$$\begin{aligned} r - p \in C^p &\implies r \succ p, \\ p - r \in C^p &\implies p \succ r \\ \text{and } r - p \in C^p &\iff r - p \in C^r \end{aligned}$$

Let $r \succ_{C^p} p$ denote $r - p \in C^p$.

Theorem 2 *Suppose that \succ is a non-trivial partial order on $\Delta(X)$ satisfying cone-monotonicity. Then, on $\text{int}\Delta(X)$, any two of the three axioms (A.2)-*

⁵All our cones are assumed to be ‘nonnegative’ (i.e., closed under nonnegative scalar multiplications).

(A.4) imply the third. If, in addition, mixture continuity holds on $\Delta(X)$ then completeness holds on $\Delta(X)$.

Note that non-triviality is implied by cone-monotonicity. It is left in the statement of the theorem for ease of exposition.

Proof (a) Suppose that \succcurlyeq satisfies Archimedean and completeness on $\text{int}\Delta(X)$. Let $p, q, r \in \text{int}\Delta(X)$ and consider the set $A = \{\alpha \in [0, 1] \mid \alpha p + (1 - \alpha)r \succcurlyeq q\}$. If A is either empty or finite then we are done. Otherwise let α^* be an accumulation point of A (that is, there exists a sequence $\{\alpha^n\} \subset A \setminus \{\alpha^*\}$ that converges to α^*) and denote $p^n = \alpha^n p + (1 - \alpha^n)r$ and $p^* = \alpha^* p + (1 - \alpha^*)r$. Assume, by way of negation, that $p^* \not\succeq q$ does not hold. By completeness, $q \succ p^*$. As $p^* \in \text{int}\Delta(X)$, cone-monotonicity implies the existence of $\bar{p} \in \text{int}\Delta(X)$ satisfying $\bar{p} >_{C^p} p^*$. By Archimedean, there exists $\beta \in (0, 1)$ such that $q \succ \beta\bar{p} + (1 - \beta)p^*$ and, since C^p is a cone, $\beta\bar{p} + (1 - \beta)p^* >_{C^p} p^*$. By cone-monotonicity, $\beta\bar{p} + (1 - \beta)p^* >_{C^{\beta\bar{p} + (1 - \beta)p^*}} p^*$ and hence, by the openness of $C^{\beta\bar{p} + (1 - \beta)p^*}$, there exists an open ε -ball $N_\varepsilon(p^*)$ satisfying $N_\varepsilon(p^*) \cap \text{aff}\Delta(X) \subset \Delta(X)$ such that $\beta\bar{p} + (1 - \beta)p^* >_{C^{\beta\bar{p} + (1 - \beta)p^*}} p'$ for all $p' \in N_\varepsilon(p^*) \cap \text{aff}\Delta(X)$. By cone-monotonicity $\beta\bar{p} + (1 - \beta)p^* \succ p'$ and, by transitivity, $q \succ p'$, for all $p' \in N_\varepsilon(p^*) \cap \text{aff}\Delta(X)$. But, for sufficiently large n , $p^n \in N_\varepsilon(p^*) \cap \text{aff}\Delta(X)$ and hence $q \succ p^n$; a contradiction.

The proof that $\{\alpha \in [0, 1] \mid q \succcurlyeq \alpha p + (1 - \alpha)r\}$ is closed follows by the same argument.

(b) The proof that Archimedean is implied by mixture continuity and completeness is identical to that of Theorem 1 part (b) (note that the betweenness property is not needed there and that the proof holds for the entire $\Delta(X)$).

(c) Suppose that \succcurlyeq satisfies Archimedean and mixture continuity on $\text{int}\Delta(X)$. We begin by showing that, for all $p \in \text{int}\Delta(X)$, the sets $\bar{B}^{int}(p) = \{q \in \text{int}\Delta(X) \mid q \succcurlyeq p\}$ and $\bar{W}^{int}(p) = \{q \in \text{int}\Delta(X) \mid p \succcurlyeq q\}$ are closed relative to $\text{int}\Delta(X)$, and the sets $B^{int}(p) = \{q \in \text{int}\Delta(X) \mid q \succ p\}$ and $W^{int}(p) = \{q \in \text{int}\Delta(X) \mid p \succ q\}$ are open relative to $\text{int}\Delta(X)$. Then we use Schmeidler's theorem to derive completeness of \succcurlyeq on $\text{int}\Delta(X)$. Finally, we show that, if mixture continuity holds on $\Delta(X)$, \succcurlyeq on $\Delta(X)$ is also complete.

Fix $p \in \text{int}\Delta(X)$ and note that, by cone-monotonicity, the intersection of C^p and $\text{int}\Delta(X)$ is relatively open and non-empty and hence $\bar{B}^{int}(p)$

is not a singleton. If q is an interior boundary point of $\bar{B}^{int}(p)$, that is if $q \in \partial \bar{B}^{int}(p) \cap \text{int}\Delta(X)$ (∂ denotes the boundary), then let $\{q^n\} \subset \bar{B}^{int}(p)$ be a sequence that converges to q . By cone-monotonicity the intersection of C^q and $\text{int}\Delta(X)$ is non-empty and there exists r satisfying $r >_{C^q} q$. Using cone-monotonicity again, $r >_{C^r} q$ and, by the relative openness of C^r , there exists an open ε -ball $N_\varepsilon(q)$ satisfying $N_\varepsilon(p^*) \cap \text{aff}\Delta(X) \subset \Delta(X)$ such that $r >_{C^r} q'$, for all $q' \in N_\varepsilon(q) \cap \text{aff}\Delta(X)$. This implies that for n sufficiently large, $r >_{C^r} q^n$. Hence, by cone-monotonicity $r \succ q^n$ and, by transitivity, $r \succcurlyeq p$. Since C^q is a cone, for all $\alpha \in (0, 1]$, $r^\alpha = \alpha r + (1 - \alpha)q >_{C^q} q$ and hence, by similar arguments, $r^\alpha \succcurlyeq p$. Therefore, the set $\{\alpha \in [0, 1] \mid \alpha r + (1 - \alpha)q \succcurlyeq p\}$ contains the interval $(0, 1]$ and hence, by mixture continuity, includes 0. Therefore $q \in \bar{B}^{int}(p)$ and $\bar{B}^{int}(p)$ is closed in $\text{int}\Delta(X)$.

The proof that $\bar{W}^{int}(p)$ is closed in $\text{int}\Delta(X)$ follows by similar arguments.

Next consider the set $B^{int}(p)$ and note, again, that $p \in \text{int}\Delta(X)$ implies $B^{int}(p) \neq \emptyset$. Choose $q \in B^{int}(p)$ and r satisfying $q >_{C^q} r$. By Archimedean there exists $\alpha \in (0, 1)$ such that $q^\alpha = \alpha q + (1 - \alpha)r \succ p$. Since C^q is a cone, $q >_{C^q} q^\alpha$ which implies $q >_{C^{q^\alpha}} q^\alpha$. Hence there exists an open ε -ball $N_\varepsilon(q)$ satisfying $N_\varepsilon(p^*) \cap \text{aff}\Delta(X) \subset \Delta(X)$ such that $q' >_{C^{q^\alpha}} q^\alpha$ for all $q' \in N_\varepsilon(q) \cap \text{aff}\Delta(X)$. By cone-monotonicity and transitivity $q' \succ p$. Hence $N_\varepsilon(q) \cap \text{aff}\Delta(X) \subset B^{int}(p)$ and $B^{int}(p)$ is open in $\text{int}\Delta(X)$.

By similar arguments, $W^{int}(p)$ is open in $\text{int}\Delta(X)$.

Since $\text{int}\Delta(X)$ is a connected topological space, by Schmeidler (1971), these observations imply that \succcurlyeq is complete on $\text{int}\Delta(X)$.

Finally, we show that if mixture continuity holds on $\Delta(X)$ then completeness extends to the entire set $\Delta(X)$.

Let $q \in \partial\Delta(X)$, $p, r \in \text{int}\Delta(X)$ and consider the set $A = \{\alpha \in [0, 1] \mid \alpha q + (1 - \alpha)r \succcurlyeq p\}$. If 1 is an accumulation point of A then, by mixture continuity, $q \succcurlyeq p$. Otherwise, by completeness on $\text{int}\Delta(X)$, there exists $\bar{\alpha} < 1$ such that $p \succcurlyeq \alpha q + (1 - \alpha)r$, for all $\alpha \in (\bar{\alpha}, 1)$. By mixture continuity, $p \succcurlyeq q$. Hence either $q \succcurlyeq p$ or $p \succcurlyeq q$.

Next let $q, p \in \partial\Delta(X)$, $r \in \text{int}\Delta(X)$ and consider the set $A = \{\alpha \in [0, 1] \mid \alpha q + (1 - \alpha)r \succcurlyeq p\}$. If 1 is an accumulation point of A then, by mixture continuity, $q \succcurlyeq p$. Otherwise, since all points $\alpha q + (1 - \alpha)r$ are interior points when $\alpha \in [0, 1)$ then, by the preceding argument, there exists $\bar{\alpha} < 1$ such that $p \succcurlyeq \alpha q + (1 - \alpha)r$ for all $\alpha \in (\bar{\alpha}, 1)$ and again, by mixture continuity, $p \succcurlyeq q$. Hence \succcurlyeq is complete on $\Delta(X)$. \square

5 Cone-Monotonicity, Betweenness and Stochastic Dominance

To analyze the relationships between cone-monotonicity and betweenness we make the following definition. A binary relation \succsim on $\Delta(X)$ is *non-trivial at* $p \in \Delta(X)$ if there exists $q \in \Delta(X)$ such that either $p \succ q$ or $q \succ p$ hold. The following proposition serves to clarify the relation between cone-monotonicity and betweenness.

Proposition 1 *Suppose that \succsim is a non-trivial partial order on $\Delta(X)$ satisfying betweenness, mixture continuity and Archimedean. Let $p \in \text{int}\Delta(X)$ and denote*

$$C(p) = \{\lambda(q - p) \mid \lambda > 0, q \in \Delta(X) \text{ and } q \succ p\}$$

If \succsim is non-trivial at p then the cone $C(p)$ is non-empty, open relative to $\text{aff}\Delta(X) - \{p\}$, and satisfies

$$\begin{aligned} B(p) &= (p + C(p)) \cap \Delta(X) \\ W(p) &= (p - C(p)) \cap \Delta(X) \end{aligned}$$

where $B(p) = \{q \in \Delta(X) \mid q \succ p\}$ and $W(p) = \{q \in \Delta(X) \mid p \succ q\}$.

Proof Fix $p \in \text{int}\Delta(X)$ at which \succsim is non-trivial. First note that, by betweenness, if p belongs to the open line segment (q', q'') then $q' \succ p \Leftrightarrow p \succ q''$. Hence $B(p)$ is non-empty.

It can be shown (see Safra 2014 for details) that, by betweenness, $r \in B(p)$ if and only if $p + \lambda(r - p) \in B(p)$ for all $\lambda > 0$ and $p + \lambda(r - p) \in \Delta(X)$. Then, from the proof of Theorem 1 part (c) it follows that $B(p)$ is convex and open. Hence, the equivalences

$$B(p) = (p + C(p)) \cap \Delta(X), \quad W(p) = (p - C(p)) \cap \Delta(X)$$

are satisfied. □

Proposition 1 implies that a partial order that satisfies betweenness, Archimedean and is non-trivial at every $p \in \text{int}\Delta(X)$ also satisfies cone-monotonicity (just define $C^p = C(p)$). It is immediate to verify that, for such

relations (on $\text{int}\Delta(X)$), Theorem 1 is implied by Theorem 2.⁶ In addition, it is easy to verify that, for non-trivial partial orders, independence implies cone-monotonicity (independence implies that non-triviality at a given point extends to the entire $\Delta(X)$ and that all cones $C(p)$ are identical).

The next example demonstrates that non-triviality at every $p \in \text{int}\Delta(X)$ does not follow from the betweenness property.

Example Let $X = \{0, 1, 2\}$ and consider the incomplete partial order satisfying betweenness defined by

$$p \succsim q \iff V_j(p) \geq V_j(q) \text{ for } j = 1, 2$$

where

$$V_1(p) = \frac{\sum_i p(i) w(i) i}{\sum_i p(i) w(i)}, \quad w(1) = 1, \quad w(0) = w(2) = 0.5$$

is a weighted utility function that ranks 2 at the top and 0 at the bottom and

$$V_2(p) = \sum_i p(i) (2 - i)$$

in an expected utility function that ranks 0 at the top and 2 at the bottom. As can be seen in Figure 1, \succsim is not non-trivial at p for all $p \in \{\alpha\delta_1 + (1 - \alpha)(0.5\delta_2 + 0.5\delta_0) \mid \alpha \in [0, 1]\}$.

Place Figure 1 here

To analyze the relationships between cone-monotonicity and monotonicity with respect to first-order stochastic dominance, we assume that $X \subset \mathbb{R}$ and, without loss of generality, let $x_1 < x_2 < \dots < x_k$. For each $p \in \Delta(X)$, we denote by p_i the probability the lottery p assigns to x_i , $i = 1, \dots, k$. Then $p \in \Delta(X)$ is said to *strongly dominate* $q \in \Delta(X)$ with respect to first-order stochastic dominance if, for all $j=1, \dots, k-1$

$$\sum_{i=1}^j p_i < \sum_{i=1}^j q_i$$

⁶Since (given completeness) mixture continuity is stronger than Archimedean, part (b) in the proof of Theorem 1, the only one that does not assume Archimedean, requires neither betweenness nor cone-monotonicity.

We denote this relation by $p >_1 q$ and say that a preference relation \succsim on $\Delta(X)$ satisfies *monotonicity* if, for all $p, q \in \Delta(X)$,

$$p >_1 q \implies p \succ q$$

To see that monotonicity implies cone-monotonicity, let

$$C = \left\{ h \in \mathbb{R}^k \mid \sum_{i=1}^k h_i = 0, \sum_{i=1}^j h_i < 0, j = 1, \dots, k-1 \right\}$$

and note that cone-monotonicity is satisfied for $C^p \equiv C$, for all $p \in \Delta(X)$. Hence, when $X \subset \mathbb{R}$, Theorem 2 applies to partial orders that satisfy monotonicity. Note that such orders are quite common, as monotonicity is usually assumed in most economic applications. It should also be mentioned that the partial relation of first-order stochastic dominance satisfies independence (indeed, this was utilized in the proof of Theorem 2, parts (a) and (c)). More on this can be found in Dubra and Ok (2002).

6 Concluding Remarks

The implication of continuity for completeness, as appeared in the theorem of Schmeidler (1971) is inherited by the theorem of Dubra (2011) and the two theorems in this paper. Karni (2011) showed that, starting with a strict preference relation, these results depend crucially on the definition of the weak preference relation. In particular, if the weak preference relation is defined as in Galaabaatar and Karni (2013) (that is, for all $q, p \in \Delta(x)$, q is weakly preferred over p if every $r \in \Delta(X)$ that is strictly preferred over q is strictly preferred over p), then Archimedean and mixture monotonicity no longer imply completeness, regardless of whether the preference relation satisfies independence. If we start with a weak preference relation, as we do in this note, a different definition of the strict preference relation is required to overcome the difficulty posed by incompleteness. Consider, for instance, a reflexive and transitive preference relation \succsim with a non-empty asymmetric part that satisfies the independence axiom. Such preference relation has

a representation as follows:⁷ For all $p, q \in \Delta(X)$, $p \succsim q$ if and only if $\sum_{x \in X} u(x)p(x) \geq \sum_{x \in X} u(x)q(x)$, for all $u \in \mathcal{U}$, where \mathcal{U} is a set of real-valued functions on X . For each $u \in \mathcal{U}$, define an induced preference relation on $\Delta(X)$ as follows: $p \succsim_u q$ if and only if $\sum_{x \in X} u(x)p(x) \geq \sum_{x \in X} u(x)q(x)$ and let \succsim_u be the asymmetric part of \succsim_u . Then $\succsim = \bigcap_{u \in \mathcal{U}} \succsim_u$ and $\hat{\succ} = \bigcap_{u \in \mathcal{U}} \succ_u$. For all $p \in \text{int}\Delta(X)$, the set $\{q \in \Delta(X) \mid q \hat{\succ} p\}$ is equal to the relative interior of $\{q \in \Delta(X) \mid q \succsim p\}$ and, as a result, Archimedean and mixture monotonicity no longer imply completeness. Similar argument applies if the independence axiom is replaced by betweenness or cone monotonicity.

⁷See, for example, Shapley and Baucells (2008).

References

- [1] Chew, S. H., 1983. A generalization of the quasilinear mean with application to the measurement of income inequality and decision theory resolving the Allais paradox. *Econometrica* 51(4) 1065-1092.
- [2] Chew, S. H., 1989. Axiomatic utility theories with the betweenness property. *Annals of Operations Research* 19(1) 273-298.
- [3] Dekel, E., 1986. An axiomatic characterization of preferences under uncertainty: Weakening the Independence Axiom,” *Journal of Economic Theory* 40, 304–318.
- [4] Dubra, J., 2011. Continuity and completeness under risk. *Mathematical Social Sciences*, 61, 80-81.
- [5] Dubra, J. and Ok, E. A., 2002. A model of procedural decision making in the presence of risk. *International Economic Review*, 43.
- [6] Galaabaatar T. and Karni, E., 2013. Subjective expected utility theory with incomplete preferences. *Econometrica* 81 255-284.
- [7] Karni, E., 2011. Continuity, completeness and the definition of weak preferences. *Mathematical Social Sciences*, 62, 123-125.
- [8] Karni, E. and Zhou, N., 2014. Weighted utility theory with incomplete preferences: parsimonious representations. Working Paper.
- [9] Safra, Z., 2014. On betweenness, completeness and transitivity. Working Paper.
- [10] Schmeidler, D., 1971. A condition for the completeness of partial preference relations *Econometrica*, 39.
- [11] Shapley, Lloyd S. and Baucells, Manel J. (2008) “Multiperson Utility,” *Games and Economic Behavior*, 62, 329-347.

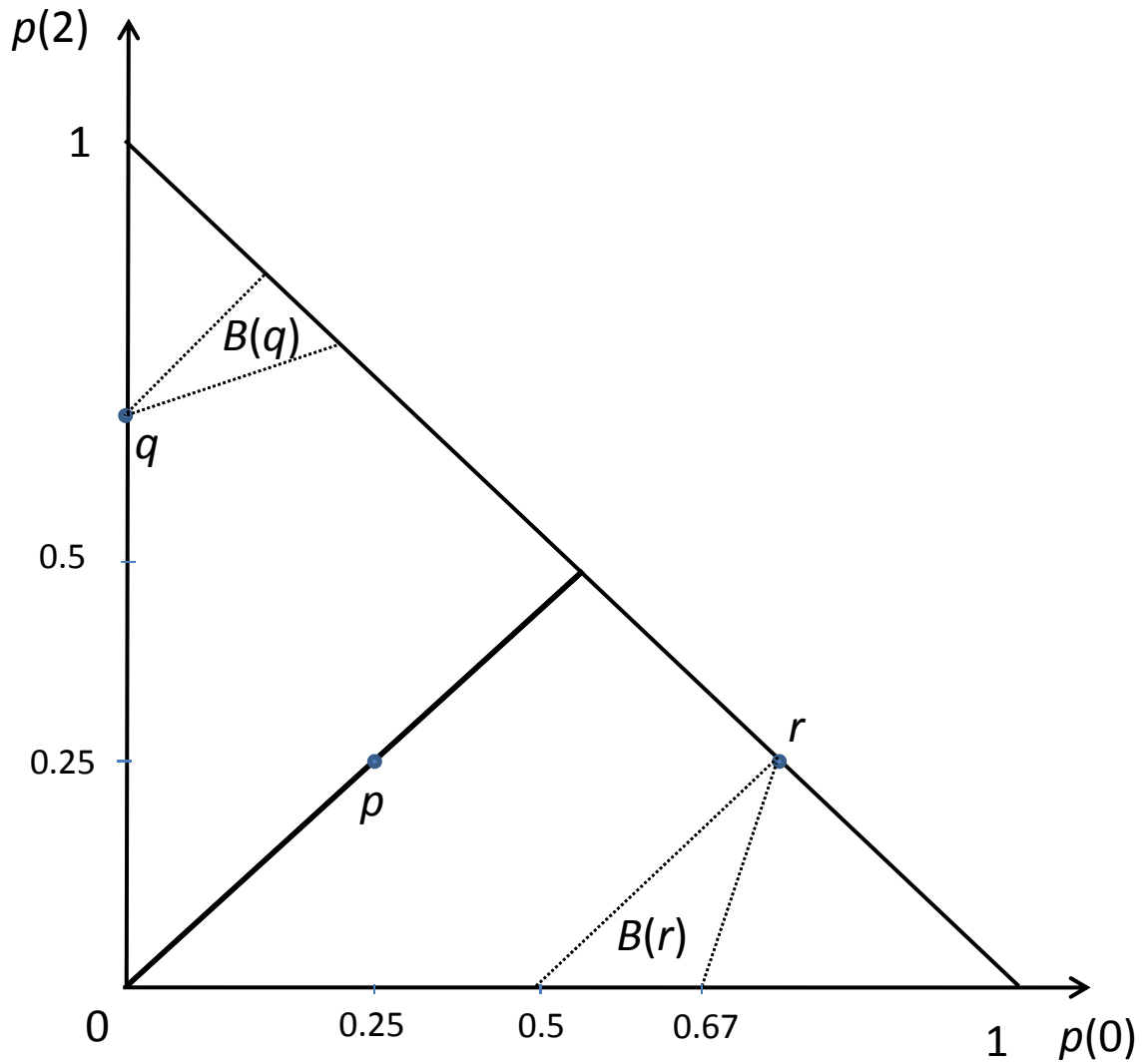


Figure 1 No lottery is strictly preferred to p . The better-than sets of q and r , $B(q)$ and $B(r)$, respectively, are depicted