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Research Article

Multi-Attribute Decision-Making Using Hesitant Fuzzy Dombi–Archimedean Weighted Aggregation Operators

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ABSTRACT

Multi-attribute decision-making (*MADM*) has been receiving great attention in recent years due to two major issues which are basically to describe attribute values and secondly to aggregate the described information to generate a ranking of alternatives. For the first case it entails the hesitant fuzzy elements (*HFEs*) as a more flexible and general tool in comparison to fuzzy set theory and for the second one, we allow the aggregation operator (*AO*) as an effective tool. Having said that there is not yet reported an *AO* which can provide desirable generality and flexibility in aggregating attribute values under hesitant fuzzy (*HF*) environment, although many *AOs* have been developed earlier to attempt to meet above such eventualities. So, the primary objective of this paper is to develop some general as well as flexible *AOs* that can be exploited to solve *MADM* problems with the *HF* information. From this perspective, at the very beginning, we develop some operations between *HFEs* by uniting the features of Dombi and Archimedean operations. Next, we bring up some *HF* weighted *AOs* based on Dombi and Archimedean operations. We discuss in detail some intriguing properties of the proposed *AOs*. Secondly, we emphasize establishing a procedure of *MADM* endowed by the proposed operators under the *HF* environment. Finally, we present a practical example concerning human resource selection to gloss the decision steps of the proposed method and at the same time, we explore the feasibility of the new method.

1. INTRODUCTION

In most of the real-life decision circumstances, the prevalence of multi-attribute decision-making (*MADM*) problems is vividly known. The primary objective of the *MADM* approach is to determine the best-suited alternative(s) from a set of available alternatives with respect to multiple attributes, both qualitative and quantitative. It has been observed the significant success rate of many *MADM* approaches in the case of web 2.0 [1], social network [2], and recommendation system [3,4]. Various regulating factors viz. a shortfall of information and consciousness, hardness in information expulsion, incertitude of the decision-making ambiance, etc. will come in limelight especially in many real-world situations on account of the rising perplexity of the socio-economic conditions. As a result, for decision-makers, it comes true a formidable task to unveil numerically to the values of attributes. In lieu, representation of the preferences using fuzzy numbers or extended fuzzy numbers [5–8] is more appropriate indeed.

For the last few years, it has been noticed about the extensive theoretical study of numerous *MADM* problems. The various approaches about *MADM* have been put forward in the framework of the fuzzy set (*FS*) theory [9–13]. The notion of the hesitant fuzzy set (*HFS*) that was first reported by Torra [14] is reckoned as an extension of the *FS* which allows the membership grades presuming a set of possible values instead of a single one. In case of dealing the fuzzy uncertain information, such *HFS* can contribute as a strong tool. The connection between the *HFS* and other extended *FS*s were investigated by Torra and Narukawa [15] in which an intuitionistic fuzzy set (*IFS*) is appeared to be an envelope of *HFS* and all possible *HFSs* are lying in the category of the type-2 fuzzy set (*T2FS*). Xu and Xia [16] suggested a class of distance and similarity measures for HFS and subsequently, they proposed further the hesitant ordered weighted distance measures and the hesitant ordered weighted similarity measures. The concept of generalized hesitant fuzzy synergetic weighted distance (*GHFSWD*) measure and also, some obligate properties along with their feasible applications was studied by Peng *et al.* [17]. The strong connection among similarity measure and distance measure for *HFS*, the interval-valued hesitant fuzzy set (*IVHFS*), and entropy was reported by Farhadinia [18]. Qian *et al.* [19] were able to explore *HFS* through *IFS* and utilized it in the decision support system. Chen *et al.* [20] exercised interval-valued

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hesitant preference relations and successfully applied it to group decision-making problems. The idea of the dual hesitant fuzzy set (*DHFS*) was put forth by Zhu *et al.* [21]. Rodríguez *et al.* [22] brought the notion of hesitant fuzzy (*HF*) linguistic term set which explaining the supremacy of linguistic excerption with the help of a fuzzy linguistic approach. The forcible VIKOR method can be utilized to solve a series of *MADM* problems having *HF* information which was propounded by Zhang and Wei [23]. We note that Zhang and Xu [24] expanded the tools of TODIM to adapt the *HF* environment and they were able to provide two *HF* decision-making approaches. Dong *et al.* [25] formulated the efficient consensus model adopting hesitant linguistic assessments in group decision-making. For *HF* linguistic preference relations, Zhu and Xu [26] yield some compatibility measures. In succession, Wang and Xu [27] presented in their study, the idea of extensive *HF* linguistic preference relations. It has been observed the widespread of the *HFS* and some varieties in its extension especially in the fields of group decision-making [28–34] preference relations [35], clustering analysis [36], and other applications [37–40]. Aiming at transparent aspects of the varied ideas, tools, and affinities in connection with the extent of FS, lately, Rodríguez *et al.* [41] summarized the HFSs. Over and above, Xu [42] devoted toward review in detail about the progress of aggregation techniques and infliction for HFS and side by side indicated several active trends in this domain.

We note that although in general, Aggregation operators (*AOs*) are considered just as functions mathematically but confronting as very common techniques, these suffice to integrate all the input individual data into a single one. Over the last three decades, both researchers and practitioners have been giving attention toward inquisition on *AOs* on account of their large impact in the vicinity of wide ranges of information processing, such as decision-making, pattern recognition, information retrieval, medical diagnosis, data mining, machine learning, etc. We may recall here that the study of *HF AOs* has been drawing significant attention to the researchers due to its importance for information fusion. Xia and Xu [43] provided a series of *AOs* and developed their enviable features. In this context, we also mention here that Xia *et al.* [44] employed quasi-arithmetic means to constitute several series *AOs* in the regime of *HF* environment. Wei [45] proposed some preference operators and used them to deal with *MADM* problems where the attributes perform various levels of priority under a *HF* environment. Furthermore, many hesitant interval-valued *AOs* were put forth by Wei *et al.* [46], and even they developed a procedure to solve interval-valued hesitant fuzzy *MADM* problems. Having paid heed to all the ideas of fuzzy integral theory, Wei *et al.* [47] attempted to utilize the Choquet integral to have two *AOs* adhering to the *HF* information: hesitant fuzzy Choquet ordered averaging (*HFCOA*) operator and hesitant fuzzy Choquet ordered geometric (*HFCOG*) operator. In this context, it may be pointed out that these two aforementioned operators are not only taken it granted based on the importance of attributes but also another aspect of its consideration is solely due to an active role in determining the correlations among the independent attributes. It was possible to solve *MADM* problems by applying both the *HF* Bonferroni mean, introduced by Zhu *et al.* [48] and geometric Bonferroni mean. To aggregate *HF* information, Zhang [49] interpreted a large extent of *HF* power *AOs* and in turn, examined different fascinating properties and relationships among those operators. Zhang and Wu [50] suggested two weighted *HF AOs* invoking the Archimedean t-conorms and t-norms, leading to aggregate weighted *HF* information. Zhang *et al.* [51] considered some induced generalized *HF* operators and utilized them to MADM problems. It was Liao and Xu [52] who lead a significant contribution for searching some *HF* hybrid weighted *AOs* and at the same time formed a decision algorithm to assist *MADM* problems utilizing *HF* information. Surprisingly, Zhou and Xu [53] put forward two new aggregation approaches-first one is an optimal discrete fitting *AO* and the second one is, simplified optimal discrete fitting *AO* to analyze group decision-making problems. In the meantime, two *AOs* for hesitant fuzzy linguistic term set (*HFLTS*) were defined by Wei *et al.* [54] which are respectively the *HFLWA* (Hesitant fuzzy linguistic weighted aggregation) operator and the *HFLOWA* (Hesitant fuzzy linguistic ordered weighted aggregation) operator and on the other hand, utilized these operators to solve MADM problems based on the conditions in which importance weights of criteria or experts are known or unknown. Based on Einstein's operational laws, Yu [55] established a few more *HF AOs*. Some special kinds of operators called *HF* Hamacher *AOs* were introduced by Tan *et al.* [56] for handling the MADM problems. It is equally important to mention here that Qin *et al.* [57] with their work stimulated the idea by presenting some new *HF* operators associating the Maclaurin symmetric mean and also considered a variety of allied eligible properties. We recall that upon a state of being more or less with *HF* Bonferroni mean operators [58], the notable characteristics of *HF* Maclaurin symmetric mean is that it can record the entire interrelationships between the arguments; hence it is more usual and appropriate for serving the real-world decision-making problems. We also refer here that it is possible to determine new some *HF AOs* using Frank operations which exclusively achieved by Qin *et al.* [59] together with solving the *MADM* problems. Liao and Xu [60] developed a VIKOR method for solving *MADM* under *HF* environment. The notion of the cubic *HFS* was developed by Mahmood *et al.* [61] to deal with MCDM problems. He [62] successfully carried out the operations of Dombi operators to Typhoon disaster assessment under a *HF* environment. Xu and Zhou [63] implemented hesitant probabilistic fuzzy operations and used them to solve group decision-making problems. In this context, it may be pointed out that as an augmentation of the hesitant probabilistic fuzzy weighted averaging operator and hesitant probabilistic fuzzy weighted geometric operator, Park *et al.* [64] made their remarkable contribution by introducing hesitant probabilistic fuzzy Einstein weighted averaging operator and hesitant probabilistic fuzzy Einstein weighted geometric operator respectively. It is worthy to say that various theoretical and practical reasoning can be simplified with the help of a uniformly typical hesitant fuzzy set (*UTHFS*) which was articulated by Alcantud and Torra [65]. Fairly recently, Wang and Li [66] introduced operational laws of picture hesitant fuzzy elements (*PHFEs*) and attributed them to define generalized picture *HF AOs* which genuinely helped to tackle the varied situations while we cope up with *MCDM* in the framework of picture *HF* environment. Interestingly, an improved A* algorithm based on hesitant *FS* theory for multi-criteria Arctic Route planning was developed by Wang *et al.* [67]. We do refer here that Lioa *et al.* [68] practiced *HF* linguistic preference utility set in the selection process of fire rescue plans. To serve the personnel selection problem, Yalcin and Pehlivan [69] were able to construct a methodology in which the fuzzy CODAS method gets connected with the fuzzy envelope of *HFLTSs* concerning the comparative linguistic expressions. Moreover, Wu *et al.* [70] formulated a unique dynamic emergency decision-making method with probabilistic *HF* information underlying the GM (1, 1) model for prophecy of the decision-making information at the subsequent stage. Alcantud *et al.* [71] first reported the notion of dual extended *HFSs* and even employed them to develop an algorithm that involves a weight score function. Furthermore, Liu *et al.* [72] initiated the *HF* cognitive maps and showed how to use it for the exploration of the risk factors that arise in an electric power system. Exploiting the regret theory, Liu *et al.* [73] submitted an approach for probabilistic *HF MADM* in view of the selection process of venture capital investment projects.

The Dombi operations, after the name of Dombi [74] are nothing but the operations of t-norm and t-conorm, bearing the nature of good flexibility with the general parameter. On the contrary, Archimedean t-norms and t-conorms [75–77] are the generalizations of many other t-norms and t-corms like algebraic t-norm and t-conorm, Einstein t-norm and t-conorm, Hamacher t-norm and t-corm, etc. The operations of Archimedean t-norms and t-conorms are treated as Archimedean operations. The growing capacity of decision complexity induces the real-life decision-making problems that indulge both generality of the operations used and flexibility of the parameters that appeared. In this context, it is more appropriate to suggest a tool obtained through merging the Dombi and Archimedean operations to serve the purpose. Due to different characteristics, AOs obtained through Archimedean operations will not generate the same ranking order deduced from Dombi operations based AOs. Thus, optimal alternative(s) may not be the same in these two cases. Since the fusion of Dombi and Archimedean operations will retain their individual characteristics, so it is logical to combine them for the purpose of aggregation. As far as we know, no such study has been carried out related to *AOs* based on the simultaneous act of the Dombi and Archimedean operations when we deal with the *HF* decision-making problems. Getting inspired and provoked by that fact, in this paper, we have tried to investigate a family of *AOs* with the coalescence of Dombi operations and Archimedean operations that regard *HF* information and successfully conjoin them to solve *MADM* problems. The aims in this article are pursued below:

1. Suggest some operational laws and *AOs* under *HF* environment with the help of the simultaneous act of Dombi and Archimedean operations.
2. Construct a novel *MADM* method with *HF* information based on the proposed *AOs* and illustrate it with a numerical example.

In such scenarios, we propose in this study, some Dombi–Archimedean operational laws for hesitant fuzzy elements (*HFEs)* alongside some *HF* Dombi–Archimedean *AO*s, such as hesitant fuzzy Dombi–Archimedean weighted arithmetic aggregation operator (*HFDAWAA*), hesitant fuzzy Dombi–Archimedean ordered weighted arithmetic aggregation operator (*HFDAOWAA*), hesitant fuzzy Dombi–Archimedean hybrid arithmetic aggregation operator (*HFDAHAA*), hesitant fuzzy Dombi–Archimedean weighted geometric aggregation operator (*HFDAWGA*), hesitant fuzzy Dombi–Archimedean ordered weighted geometric aggregation operator (*HFDAOWGA*), hesitant fuzzy Dombi–Archimedean hybrid geometric aggregation operator (*HFDAHGA*). In addition to this, we require to initiate a procedure for the *MADM* approach after interposed by the proposed *AOs* in the regime of *HF* environment.

We summarize the rest part of this paper as follows. In Section 2, we introduce in brief some important and vital concepts of *HFSs*, Archimedean t-norm (t-conorm) and Dombi t-norm (t-conorm). In Section 3, we define the Dombi–Archimedean operational laws for *HFEs*. In Sections 4, we develop some *HF* Dombi–Archimedean *AO*s, such as *HFDAWAA, HFDAOWAA, HFDAHAA, HFDAWGA, HFDAOWGA,* and *HFDAHGA*. All the essential properties of these operators are also demonstrated. In Section 5, we present a novel approach using the proposed *AOs* beneficial for stirring the various kinds of *MADM* problems in which the attribute values take the form of *HF* information. In Section 6, we allow an example of personnel selection to gloss the proposed method. Section 7 is solely devoted to a comparative study to confirm the superiority of the proposed method. In the end, in Section 8, we make some conclusions upon this entire study.

2. PRELIMINARIES

In this section, we briefly review some basic concepts of *HFS*, Archimedean and Dombi operations, which will be directly used in the next sections.

2.1. Hesitant Fuzzy Sets

Definition 1. [14] Let U be a finite universe of discourse. Then a *HFS* ξ on U is defined as:

$$\xi = \left\{ < x, \mu_{\xi}^{h}(x) >: x \in U \right\}$$

where $\mu_{\xi}^{h} : U \to 2^{[0,1]}$ is a mapping and $\mu_{\xi}^{h}(x)$ denotes the set (finite) of possible membership degrees of the element $x \in U$.

Given $x \in U$, μ_{ξ}^{h} is termed as a *HFE* [43]. For sake of simplicity, the *HFS* ξ is represented by $\left\langle \mu_{\xi}^{h} \right\rangle$. The set of all *HFEs* on U is denoted by HFE^{U}.

Definition 2. [43] For two *HFEs* $\mu_{\xi_1}^{h}$ and $\mu_{\xi_2}^{h}$ on U, some basic operations between them can be described as:

1. $\mu_{\xi_1}^{h} \cup \mu_{\xi_2}^{h} = \bigcup_{\alpha_1 \in \mu_{\xi_1}^{h}, \alpha_2 \in \mu_{\xi_2}^{h}} \max\{\alpha_1, \alpha_2\}$

2. $\mu^h_{\xi_1} \cap \mu^h_{\xi_2} = \underset{\alpha_1 \in \mu^h_{\xi_1}, \alpha_2 \in \mu^h_{\xi_2}}{\cup} \min\{\alpha_1, \alpha_2\}$

3. $\left(\mu^h_{\xi_1}\right)^c = \underset{\alpha_1 \in \mu^h_{\xi_1}}{\cup} \{1 - \alpha_1\}$

4. $\mu^h_{\xi_1} \oplus \mu^h_{\xi_2} = \underset{\alpha_1 \in \mu^h_{\xi_1}, \alpha_2 \in \mu^h_{\xi_2}}{\cup} \{\alpha_1 + \alpha_2 - \alpha_1\alpha_2\}$

5. $\mu^h_{\xi_1} \otimes \mu^h_{\xi_2} = \underset{\alpha_1 \in \mu^h_{\xi_1}, \alpha_2 \in \mu^h_{\xi_2}}{\cup} \{\alpha_1\alpha_2\}$

6. $\lambda\mu^h_{\xi_1} = \underset{\alpha_1 \in \mu^h_{\xi_1}}{\cup} \left\{1 - (1 - \alpha_1)^\lambda\right\} \quad (\lambda > 0)$

7. $\left(\mu^h_{\xi_1}\right)^\lambda = \underset{\alpha_1 \in \mu^h_{\xi_1}}{\cup} \left\{(\alpha_1)^\lambda\right\} \quad (\lambda > 0)$

Theorem 1. *[43] Let $\mu^h_{\xi_1}$ and $\mu^h_{\xi_2}$ be two HFEs defined on U and $\lambda, \lambda_1, \lambda_2 > 0$. Then*

i. $\mu^h_{\xi_1} \oplus \mu^h_{\xi_2} = \mu^h_{\xi_2} \oplus \mu^h_{\xi_1}$

ii. $\mu^h_{\xi_1} \otimes \mu^h_{\xi_2} = \mu^h_{\xi_2} \otimes \mu^h_{\xi_1}$

iii. $\lambda\left(\mu^h_{\xi_1} \oplus \mu^h_{\xi_2}\right) = \left(\lambda\mu^h_{\xi_1}\right) \oplus \left(\lambda\mu^h_{\xi_2}\right)$

iv. $\left(\mu^h_{\xi_1} \otimes \mu^h_{\xi_2}\right)^\lambda = \left(\mu^h_{\xi_1}\right)^\lambda \otimes \left(\mu^h_{\xi_2}\right)^\lambda$

v. $(\lambda_1 + \lambda_2)\mu^h_{\xi_1} = \left(\lambda_1\mu^h_{\xi_1}\right) \oplus \left(\lambda_2\mu^h_{\xi_1}\right)$

vi. $\left(\mu^h_{\xi_1}\right)^{\lambda_1+\lambda_2} = \left(\mu^h_{\xi_1}\right)^{\lambda_1} \otimes \left(\mu^h_{\xi_1}\right)^{\lambda_2}$

Definition 3. [43] Let μ^h_ξ be a *HFE* on *U*. Then the score value of $\mu^h_\xi$is defined as:

$$S\left(\mu^h_\xi\right) = \frac{1}{\#\mu^h_\xi} \sum_{\alpha \in \mu^h_\xi} \alpha$$

where $\#\mu^h_\xi$ denotes the number of elements in μ^h_ξ.

Based on the score values of *HFEs*, a comparison method of *HFEs* is described below:

Definition 4. [43] Suppose $\mu^h_{\xi_1}$ and $\mu^h_{\xi_2}$ be two *HFEs* on *U*. Then

1. if $S\left(\mu^h_{\xi_1}\right) > S\left(\mu^h_{\xi_2}\right)$, then $\mu^h_{\xi_1} \succ \mu^h_{\xi_2}$
2. if $S\left(\mu^h_{\xi_1}\right) < S\left(\mu^h_{\xi_2}\right)$, then $\mu^h_{\xi_1} \prec \mu^h_{\xi_2}$
3. if $S\left(\mu^h_{\xi_1}\right) = S\left(\mu^h_{\xi_2}\right)$, then $\mu^h_{\xi_1} = \mu^h_{\xi_2}$

2.2. Archimedean t-norm and t-conorm

Definition 5. [75,76] A fuzzy *t*-norm $f: [0,1] \times [0,1] \to [0,1]$ is a function which satisfies the following axioms:

i. $f(x, 1) = x$ for $x \in [0, 1]$

ii. $f(x, y) \le f(x', y')$ *provided* $x \le x', y \le y'$ for $x, x', y, y' \in [0, 1]$

iii. $f(x, y) = f(y, x)$ for $x, y, \in [0, 1]$

iv. $f(x, f(y, z)) = f(f(x, y), z)$ for $x, y, \in [0, 1]$

Definition 6. [75,76] A fuzzy t-conorm $g : [0,1] \times [0,1] \to [0,1]$ is a function which satisfies the following axioms:

i. $g(x,0) = x$ for $x \in [0,1]$

ii. $g(x,y) \le g(x',y')$ *provided* $x \le x', y \le y'$ for $x, x', y, y' \in [0,1]$

iii. $g(x,y) = g(y,x)$ for $x, y, \in [0,1]$

iv. $g(x, g(y,z)) = g(g(x,y),z)$ for $x, y, \in [0,1]$

Definition 7. [75,76] A t-norm function $f(x,y)$ is called a strictly Archimedean t-norm if it is continuous, $f(x,x) < x\ \forall x \in (0,1)$ and strictly increasing for $x, y \in (0,1)$.

Definition 8. [75,76] A t-conorm function $g(x,y)$ is called a strictly Archimedean t-conorm if it is continuous, $g(x,x) > x\ \forall x \in (0,1)$ and strictly increasing for $x, y \in (0,1)$.

Definition 9. [77] Suppose $\theta : (0,1] \to R$ is a continuous function such that θ is strictly decreasing and $\theta(1) = 0$. Then a strictly Archimedean t-norm is expressed by:

$$\delta(x,y) = \theta^{-1}\left(\theta(x) + \theta(y)\right)\ for\, x, y \in (0,1]$$

Definition 10. [77] Suppose $\psi : [0,1) \to R$ is a continuous function such that $\psi(l) = \theta(1-l), l \in [0,1)$ and ψ is strictly increasing. Then a strictly Archimedean t-conorm is expressed by:

$$\rho(x,y) = \psi^{-1}(\psi(x) + \psi(y))\, for\, x, y \in [0,1)$$

2.3. Dombi Operations

The operations of t-norm and t-conorm, developed by Dombi [74] are generally known as Dombi operations described below:

Definition 11. [74] For any two real numbers x and y in [0, 1], the Dombi t-norm and Dombi t-conorm can be defined as follows:

$$Dom(x,y) = \frac{1}{1 + \left\{\left(\frac{1-x}{x}\right)^k + \left(\frac{1-y}{y}\right)^k\right\}^{\frac{1}{k}}},\ Dom^c(x,y) = 1 - \frac{1}{1 + \left\{\left(\frac{x}{1-x}\right)^k + \left(\frac{y}{1-y}\right)^k\right\}^{\frac{1}{k}}} (k \ge 1)$$

Dombi operations have good precedence of change w.r.t values of the parameter "k." Based on the Dombi operations, He [62] developed a few operations between *HFEs* given below:

Definition 12. [62] Let $\mu^h_{\xi_1}$ and $\mu^h_{\xi_2}$ be two HFEs on U.

i. $$\mu^h_{\xi_1} \oplus_D \mu^h_{\xi_2} = \bigcup_{\alpha_1 \in \mu^h_{\xi_1}, \alpha_2 \in \mu^h_{\xi_2}} \left\{1 - \left(1 + \left\{\left(\frac{\alpha_1}{1-\alpha_1}\right)^k + \left(\frac{\alpha_2}{1-\alpha_2}\right)^k\right\}^{\frac{1}{k}}\right)^{-1}\right\}$$

ii. $$\mu^h_{\xi_1} \otimes_D \mu^h_{\xi_2} = \bigcup_{\alpha_1 \in \mu^h_{\xi_1}, \alpha_2 \in \mu^h_{\xi_2}} \left\{\left(1 + \left\{\left(\frac{1-\alpha_1}{\alpha_1}\right)^k + \left(\frac{1-\alpha_2}{\alpha_2}\right)^k\right\}^{\frac{1}{k}}\right)^{-1}\right\}$$

iii. $$\lambda *_D \mu^h_{\xi_1} = \bigcup_{\alpha_1 \in \mu^h_{\xi_1}} \left\{1 - \left(1 + \left\{\lambda\left(\frac{\alpha_1}{1-\alpha_1}\right)^k\right\}^{\frac{1}{k}}\right)^{-1}\right\}\ (\lambda > 0)$$

iv. $$\lambda \circ_D \mu^h_{\xi_1} = \bigcup_{\alpha_1 \in \mu^h_{\xi_1}} \left\{\left(1 + \left\{\lambda\left(\frac{1-\alpha_1}{\alpha_1}\right)^k\right\}^{\frac{1}{k}}\right)^{-1}\right\}\ (\lambda > 0)$$

3. DOMBI–ARCHIMEDEAN OPERATIONS ON *HFEs*

In this section, we develop a few operations between *HFEs* using Dombi and Archimedian operations and study the underlying properties of these proposed operations.

Definition 13. Let $\mu^h_{\xi_1}$ and $\mu^h_{\xi_2}$ be two *HFEs* on *U*. Suppose $\Delta_k(\vartheta) = \left(\frac{\vartheta}{1-\vartheta}\right)^k$, $\nabla_k(\vartheta') = \left(\frac{1-\vartheta'}{\vartheta'}\right)^k$, $\vartheta \in [0,1), \vartheta' \in (0,1]$ and $k \geq 1$. Then we define the Dombi–Archimedian operations on *HFEs* as below:

1. $$\mu^h_{\xi_1} \oplus_{DA} \mu^h_{\xi_2} = \bigcup_{\alpha_1\in\mu^h_{\xi_1},\alpha_2\in\mu^h_{\xi_2}} \left\{1 - \left(1 + \{\psi\left(\psi^{-1}\left(\Delta_k(\alpha_1)\right) + \psi^{-1}\left(\Delta_k(\alpha_2)\right)\right)\}^{\frac{1}{k}}\right)^{-1}\right\}$$

2. $$\mu^h_{\xi_1} \otimes_{DA} \mu^h_{\xi_2} = \bigcup_{\alpha_1\in\mu^h_{\xi_1},\alpha_2\in\mu^h_{\xi_2}} \left\{\left(1 + \{\theta\left(\theta^{-1}\left(\nabla_k(\alpha_1)\right) + \theta^{-1}\left(\nabla_k(\alpha_2)\right)\right)\}^{\frac{1}{k}}\right)^{-1}\right\}$$

3. $$\lambda *_{DA} \mu^h_{\xi_1} = \bigcup_{\alpha_1\in\mu^h_{\xi_1}} \left\{1 - \left(1 + \{\psi\left(\lambda\psi^{-1}\left(\Delta_k(\alpha_1)\right)\right)\}^{\frac{1}{k}}\right)^{-1}\right\} \quad (\lambda > 0)$$

4. $$\lambda \circ_{DA} \mu^h_{\xi_1} = \bigcup_{\alpha_1\in\mu^h_{\xi_1}} \left\{\left(1 + \{\theta\left(\lambda\theta^{-1}\left(\nabla_k(\alpha_1)\right)\right)\}^{\frac{1}{k}}\right)^{-1}\right\} \quad (\lambda > 0)$$

Example 1. Suppose $\mu^h_{\xi_1} = \{0.5\}$ and $\mu^h_{\xi_2} = \{0.2, 0.3\}$ be two *HFEs* on *U*. Then for $k = 2, \lambda = 0.3, \psi(t) = -\ln(1-t), \psi^{-1}(t) = 1-e^{-t}, \theta(t) = -\ln(t), \theta^{-1}(t) = e^{-t}$; we have,

i. $\mu^h_{\xi_1} \oplus_{DA} \mu^h_{\xi_2}$

$$= \bigcup_{\alpha_1\in\mu^h_{\xi_1},\alpha_2\in\mu^h_{\xi_2}} \left\{1 - \left(1 + \left\{\psi\left(1 - e^{-\left(\frac{\alpha_1}{1-\alpha_1}\right)^2} + 1 - e^{-\left(\frac{\alpha_2}{1-\alpha_2}\right)^2}\right)\right\}^{\frac{1}{2}}\right)^{-1}\right\}$$

$$= \bigcup_{\alpha_1\in\mu^h_{\xi_1},\alpha_2\in\mu^h_{\xi_2}} \left\{1 - \left(1 + \left\{-\ln\left(1 - \left(1 - e^{-\left(\frac{\alpha_1}{1-\alpha_1}\right)^2}\right) - \left(1 - e^{-\left(\frac{\alpha_2}{1-\alpha_2}\right)^2}\right)\right)\right\}^{\frac{1}{2}}\right)^{-1}\right\}$$

$$= \left\{1 - \left(1 + \left\{-\ln\left(1 - \left(1 - e^{-\left(\frac{0.5}{1-0.5}\right)^2}\right) - \left(1 - e^{-\left(\frac{0.2}{1-0.2}\right)^2}\right)\right)\right\}^{\frac{1}{2}}\right)^{-1}, \right.$$

$$\left. 1 - \left(1 + \left\{-\ln\left(1 - \left(1 - e^{-\left(\frac{0.5}{1-0.5}\right)^2}\right) - \left(1 - e^{-\left(\frac{0.3}{1-0.3}\right)^2}\right)\right)\right\}^{\frac{1}{2}}\right)^{-1}\right\}$$

$$= \{0.5206, 0.5591\}$$

ii. $\mu^h_{\xi_1} \otimes_{DA} \mu^h_{\xi_2}$

$$= \bigcup_{\alpha_1\in\mu^h_{\xi_1},\alpha_2\in\mu^h_{\xi_2}} \left\{\left(1 + \left\{\theta\left(e^{-\left(\frac{1-\alpha_1}{\alpha_1}\right)^2} + e^{-\left(\frac{1-\alpha_2}{\alpha_2}\right)^2}\right)\right\}^{\frac{1}{2}}\right)^{-1}\right\}$$

$$= \bigcup_{\alpha_1\in\mu^h_{\xi_1},\alpha_2\in\mu^h_{\xi_2}} \left\{\left(1 + \left\{-\ln\left(e^{-\left(\frac{1-\alpha_1}{\alpha_1}\right)^2} + e^{-\left(\frac{1-\alpha_2}{\alpha_2}\right)^2}\right)\right\}^{\frac{1}{2}}\right)^{-1}\right\}$$

$$= \left\{\left(1 + \left\{-\ln\left(e^{-\left(\frac{1-0.5}{0.5}\right)^2} + e^{-\left(\frac{1-0.2}{0.2}\right)^2}\right)\right\}^{\frac{1}{2}}\right)^{-1}, \left(1 + \left\{-\ln\left(e^{-\left(\frac{1-0.5}{0.5}\right)^2} + e^{-\left(\frac{1-0.3}{0.3}\right)^2}\right)\right\}^{\frac{1}{2}}\right)^{-1}\right\}$$

$$= \{0.5000, 0.5014\}$$

iii. $\lambda *_{DA} \mu^h_{\xi_1}$

$$= \bigcup_{\alpha_1 \in \mu^h_{\xi_1}} \left\{ 1 - \left(1 + \left\{ \psi \left(0.3 \times \left(1 - e^{-\left(\frac{\alpha_1}{1-\alpha_1}\right)^2} \right) \right) \right\}^{\frac{1}{2}} \right)^{-1} \right\}$$

$$= \bigcup_{\alpha_1 \in \mu^h_{\xi_1}} \left\{ 1 - \left(1 + \left\{ -\ln \left(0.3 \times \left(1 - e^{-\left(\frac{\alpha_1}{1-\alpha_1}\right)^2} \right) \right) \right\}^{\frac{1}{2}} \right)^{-1} \right\}$$

$$= \left\{ 1 - \left(1 + \left\{ -\ln \left(0.3 \times \left(1 - e^{-\left(\frac{0.5}{1-0.5}\right)^2} \right) \right) \right\}^{\frac{1}{2}} \right)^{-1} \right\}$$

$$= \{0.3143\}$$

iv. $\lambda \circ_{DA} \mu^h_{\xi_1}$

$$= \bigcup_{\alpha_1 \in \mu^h_{\xi_1}} \left\{ \left(1 + \left\{ \theta \left(0.3 \times e^{-\left(\frac{1-\alpha_1}{\alpha_1}\right)^2} \right) \right\}^{\frac{1}{2}} \right)^{-1} \right\}$$

$$= \bigcup_{\alpha_1 \in \mu^h_{\xi_1}} \left\{ \left(1 + \left\{ -\ln \left(0.3 \times e^{-\left(\frac{1-\alpha_1}{\alpha_1}\right)^2} \right) \right\}^{\frac{1}{2}} \right)^{-1} \right\}$$

$$= \bigcup_{\alpha_1 \in \mu^h_{\xi_1}} \left\{ \left(1 + \left\{ -\ln \left(0.3 \times e^{-\left(\frac{1-0.5}{0.5}\right)^2} \right) \right\}^{\frac{1}{2}} \right)^{-1} \right\}$$

$$= \{0.4024\}$$

Theorem 2. *Let $\mu^h_{\xi_1}$ and $\mu^h_{\xi_2}$ be two HFEs defined on U and $\lambda, \lambda_1, \lambda_2 > 0$. Then we have,*

i. $\mu^h_{\xi_1} \oplus_{DA} \mu^h_{\xi_2} = \mu^h_{\xi_2} \oplus_{DA} \mu^h_{\xi_1}$

ii. $\mu^h_{\xi_1} \otimes_{DA} \mu^h_{\xi_2} = \mu^h_{\xi_2} \otimes_{DA} \mu^h_{\xi_1}$

iii. $\lambda *_{DA} \left(\mu^h_{\xi_1} \oplus_{DA} \mu^h_{\xi_2} \right) = \left(\lambda *_{DA} \mu^h_{\xi_1} \right) \oplus_{DA} \left(\lambda *_{DA} \mu^h_{\xi_2} \right)$

iv. $\lambda \circ_{DA} \left(\mu^h_{\xi_1} \otimes_{DA} \mu^h_{\xi_2} \right) = \left(\lambda \circ_{DA} \mu^h_{\xi_1} \right) \otimes_{DA} \left(\lambda \circ_{DA} \mu^h_{\xi_2} \right)$

v. $(\lambda_1 + \lambda_2) *_{DA} \mu^h_{\xi_1} = \left(\lambda_1 *_{DA} \mu^h_{\xi_1} \right) \oplus_{DA} \left(\lambda_2 *_{DA} \mu^h_{\xi_1} \right)$

vi. $(\lambda_1 + \lambda_2) \circ_{DA} \mu^h_{\xi_1} = \left(\lambda_1 \circ_{DA} \mu^h_{\xi_1} \right) \otimes_{DA} \left(\lambda_2 \circ_{DA} \mu^h_{\xi_1} \right)$

Proof: (i)-(ii) are straight forward.

(iii) $\lambda *_{DA} \left(\xi_1 \oplus_{DA} \xi_2 \right)$

$$= \lambda *_{DA} \left(\bigcup_{\alpha_1 \in \mu^h_{\xi_1}, \alpha_2 \in \mu^h_{\xi_2}} \left\{ 1 - \left(1 + \left\{ \psi \left(\psi^{-1} \left(\Delta_k (\alpha_1) \right) + \psi^{-1} \left(\Delta_k (\alpha_2) \right) \right) \right\}^{\frac{1}{k}} \right)^{-1} \right\} \right)$$

$$= \bigcup_{\alpha_1 \in \mu^h_{\xi_1}, \alpha_2 \in \mu^h_{\xi_2}} \left\{ 1 - \left(1 + \left\{ \psi \left(\lambda \psi^{-1} \left(\left(\frac{1 - \left(1 + \{\psi(\psi^{-1}(\Delta_k(\alpha_1)) + \psi^{-1}(\Delta_k(\alpha_2)))\}^{\frac{1}{k}} \right)^{-1}}{\left(1 + \{\psi(\psi^{-1}(\Delta_k(\alpha_1)) + \psi^{-1}(\Delta_k(\alpha_2)))\}^{\frac{1}{k}} \right)^{-1}} \right)^k \right) \right) \right\}^{\frac{1}{k}} \right)^{-1} \right\}$$

$$= \bigcup_{\alpha_1 \in \mu^h_{\xi_1},\, \alpha_2 \in \mu^h_{\xi_2}} \left\{ 1 - \left(1 + \left\{ \psi \left(\lambda \left(\psi^{-1} \left(\Delta_k (\alpha_1) \right) + \psi^{-1} \left(\Delta_k (\alpha_2) \right) \right) \right) \right\}^{\frac{1}{k}} \right)^{-1} \right\}$$

On the other hand, $(\lambda *_{DA} \xi_1) \oplus_{DA} (\lambda *_{DA} \xi_2)$

$$= \left(\bigcup_{\alpha_1 \in \mu^h_{\xi_1}} \left\{ 1 - \left(1 + \left\{ \psi \left(\lambda \psi^{-1} \left(\Delta_k(\alpha_1) \right) \right) \right\}^{\frac{1}{k}} \right)^{-1} \right\} \right) \oplus_{DA} \left(\bigcup_{\alpha_2 \in \mu^h_{\xi_2}} \left\{ 1 - \left(1 + \left\{ \psi \left(\lambda \psi^{-1} \left(\Delta_k(\alpha_2) \right) \right) \right\}^{\frac{1}{k}} \right)^{-1} \right\} \right)$$

$$= \bigcup_{\alpha_1 \in \mu^h_{\xi_1}, \alpha_2 \in \mu^h_{\xi_2}} \left\{ 1 - \left(1 + \left\{ \psi \left(\psi^{-1} \left(\left(\frac{1 - \left(1 + \{ \psi(\lambda \psi^{-1}(\Delta_k(\alpha_1))) \}^{\frac{1}{k}} \right)^{-1}}{\left(1 + \{ \psi(\lambda \psi^{-1}(\Delta_k(\alpha_1))) \}^{\frac{1}{k}} \right)^{-1}} \right)^k \right) + \psi^{-1} \left(\left(\frac{1 - \left(1 + \{ \psi(\lambda \psi^{-1}(\Delta_k(\alpha_2))) \}^{\frac{1}{k}} \right)^{-1}}{\left(1 + \{ \psi(\lambda \psi^{-1}(\Delta_k(\alpha_2))) \}^{\frac{1}{k}} \right)^{-1}} \right)^k \right) \right) \right\}^{\frac{1}{k}} \right)^{-1} \right\}$$

$$= \bigcup_{\alpha_1 \in \mu^h_{\xi_1}, \alpha_2 \in \mu^h_{\xi_2}} \left\{ 1 - \left(1 + \left\{ \psi \left(\lambda \psi^{-1} \left(\Delta_k(\alpha_1) \right) + \lambda \psi^{-1} \left(\Delta_k(\alpha_2) \right) \right) \right\}^{\frac{1}{k}} \right)^{-1} \right\}$$

Hence $\lambda *_{DA} (\xi_1 \oplus_{DA} \xi_2) = (\lambda *_{DA} \xi_1) \oplus_{DA} (\lambda *_{DA} \xi_2)$.

(iv) The proof is similar to (iii).

(vi) By Dombi–Archimedean operational laws (1 and 3) in Definition 13, we have, $(\lambda_1 + \lambda_2) *_{DA} \xi_1 =$

$$\bigcup_{\alpha_1 \in \mu^h_{\xi_1}} \left\{ 1 - \left(1 + \left\{ \psi \left((\lambda_1 + \lambda_2) \psi^{-1} \left(\Delta_k(\alpha_1) \right) \right) \right\}^{\frac{1}{k}} \right)^{-1} \right\}$$

Next, using Dombi–Archimedean operational laws (3 and 1) in Definition 13, we have,

$(\lambda_1 *_{DA} \xi_1) \oplus_{DA} (\lambda_2 *_{DA} \xi_1)$

$$= \left(\bigcup_{\alpha_1 \in \mu^h_{\xi_1}} \left\{ 1 - \left(1 + \left\{ \psi \left(\lambda_1 \psi^{-1} \left(\Delta_k(\alpha_1) \right) \right) \right\}^{\frac{1}{k}} \right)^{-1} \right\} \right) \oplus_{DA} \left(\bigcup_{\alpha_1 \in \mu^h_{\xi_1}} \left\{ 1 - \left(1 + \left\{ \psi \left(\lambda_2 \psi^{-1} \left(\Delta_k(\alpha_1) \right) \right) \right\}^{\frac{1}{k}} \right)^{-1} \right\} \right)$$

$$= \bigcup_{\alpha_1 \in \mu^h_{\xi_1}} \left\{ 1 - \left(1 + \left\{ \psi \left(\psi^{-1} \left(\left(\frac{1 - \left(1 + \{ \psi(\lambda_1 \psi^{-1}(\Delta_k(\alpha_1))) \}^{\frac{1}{k}} \right)^{-1}}{\left(1 + \{ \psi(\lambda_1 \psi^{-1}(\Delta_k(\alpha_1))) \}^{\frac{1}{k}} \right)^{-1}} \right)^k \right) + \psi^{-1} \left(\left(\frac{1 - \left(1 + \{ \psi(\lambda_2 \psi^{-1}(\Delta_k(\alpha_1))) \}^{\frac{1}{k}} \right)^{-1}}{\left(1 + \{ \psi(\lambda_2 \psi^{-1}(\Delta_k(\alpha_1))) \}^{\frac{1}{k}} \right)^{-1}} \right)^k \right) \right) \right\}^{\frac{1}{k}} \right)^{-1} \right\}$$

$$= \bigcup_{\alpha_1 \in \mu^h_{\xi_1}} \left\{ 1 - \left(1 + \left\{ \psi \left(\lambda_1 \psi^{-1} \left(\Delta_k(\alpha_1) \right) + \lambda_2 \psi^{-1} \left(\Delta_k(\alpha_1) \right) \right) \right\}^{\frac{1}{k}} \right)^{-1} \right\}$$

Hence, $(\lambda_1 + \lambda_2) *_{DA} \xi_1 = (\lambda_1 *_{DA} \xi_1) \oplus_{DA} (\lambda_2 *_{DA} \xi_1)$.

(vi) The proof is similar to (v).

4. *HF* DOMBI–ARCHIMEDEAN WEIGHTED *AO*S

In this section, we develop some hesitant fuzzy Dombi–Archimedean weighted *AO*s with the help of the Dombi–Archimedean operations.

4.1. *HF* Dombi–Archimedean Arithmetic *AO*s

In this sub-section, we propose Dombi–Archimedean arithmetic *AO*s with *HFEs* such as *HFDAWAA*, *HFDAOWAA*, and *HFDAHAA*.

Definition 14. Suppose $\mu^h_{\xi_1}(j = 1, 2, 3, \ldots\ldots, n)$be a collection of *HFEs* on U. Then *HF* Dombi–Archimedean weighted arithmetic aggregation operator (*HFDAWAA*) is a function $HFDAWAA : HFE^U \to HFE^U$ which is defined as follows:

$$HFDAWAA\left(\mu^h_{\xi_1}, \mu^h_{\xi_2}, \mu^h_{\xi_2}, \ldots\ldots, \mu^h_{\xi_n}\right) = \underset{j=1\,DA}{\overset{n}{\oplus}} \left(w_j *_{DA} \mu^h_{\xi_1}\right)$$

where w_j $(j = 1, 2, 3, \ldots\ldots, n)$ is the weight of $\mu^h_{\xi_j}(j = 1, 2, 3, \ldots\ldots, n)$ with $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$.

The following theorem follows from Definition 14.

Theorem 3. *The aggregated value* $HFDAWAA\left(\mu^h_{\xi_1}, \mu^h_{\xi_2}, \mu^h_{\xi_3}, \ldots\ldots, \mu^h_{\xi_n}\right)$ *is also a HFE and*

$$HFDAWAA\left(\mu^h_{\xi_1}, \mu^h_{\xi_2}, \mu^h_{\xi_3}, \ldots\ldots, \mu^h_{\xi_n}\right) = \bigcup_{\substack{\alpha_1 \in \mu^h_{\xi_1}, \alpha_2 \in \mu^h_{\xi_2}, \\ \ldots\ldots, \alpha_n \in \mu^h_{\xi_n}}} \left\{1 - \left(1 + \left\{\psi\left(\sum_{j=1}^{n} w_j \psi^{-1}\left(\Delta_k(\alpha_j)\right)\right)\right\}^{\frac{1}{k}}\right)^{-1}\right\} \tag{1}$$

where $w_j(j = 1, 2, 3, \ldots\ldots, n)$ *is the weight of* $\mu^h_{\xi_j}(j = 1, 2, 3, \ldots\ldots, n)$ *with* $w_j > 0$ *and* $\sum_{j=1}^{n} w_j = 1$.

Proof: The first result holds immediately from Definition 14. Now to show the rest part, we use the method of mathematical induction on n which are summarized as follows:

For $n = 1$, the result is obvious.

For $n = 2$, we have, $HFDAWAA(\xi_1, \xi_2)$

$$\begin{aligned}
&= (w_1 *_{DA} \xi_1) \oplus_{DA} (w_2 *_{DA} \xi_2) \\
&= \left(\bigcup_{\alpha_1 \in \mu^h_{\xi_1}} \left\{1 - \left(1 + \left\{\psi\left(w_1 \psi^{-1}\left(\Delta_k(\alpha_1)\right)\right)\right\}^{\frac{1}{k}}\right)^{-1}\right\}\right) \oplus_{DA} \left(\bigcup_{\alpha_2 \in \mu^h_{\xi_2}} \left\{1 - \left(1 + \left\{\psi\left(w_2 \psi^{-1}\left(\Delta_k(\alpha_2)\right)\right)\right\}^{\frac{1}{k}}\right)^{-1}\right\}\right) \\
&= \bigcup_{\substack{\alpha_1 \in \mu^h_{\xi_1}, \\ \alpha_2 \in \mu^h_{\xi_2}}} \left\{1 - \left(1 + \left\{\psi\left(\psi^{-1}\left(\left(\frac{1 - \left(1 + \left\{\psi\left(w_1 \psi^{-1}\left(\Delta_k(\alpha_1)\right)\right)\right\}^{\frac{1}{k}}\right)^{-1}}{\left(1 + \left\{\psi\left(w_1 \psi^{-1}\left(\Delta_k(\alpha_1)\right)\right)\right\}^{\frac{1}{k}}\right)^{-1}}\right)^{k}\right) + \psi^{-1}\left(\left(\frac{1 - \left(1 + \left\{\psi\left(w_2 \psi^{-1}\left(\Delta_k(\alpha_2)\right)\right)\right\}^{\frac{1}{k}}\right)^{-1}}{\left(1 + \left\{\psi\left(w_2 \psi^{-1}\left(\Delta_k(\alpha_2)\right)\right)\right\}^{\frac{1}{k}}\right)^{-1}}\right)^{k}\right)\right)\right\}^{\frac{1}{k}}\right)^{-1}\right\} \\
&= \bigcup_{\alpha_1 \in \mu^h_{\xi_1}, \alpha_2 \in \mu^h_{\xi_2}} \left\{1 - \left(1 + \left\{\psi\left(w_1 \psi^{-1}\left(\Delta_k(\alpha_1)\right) + w_2 \psi^{-1}\left(\Delta_k(\alpha_2)\right)\right)\right\}^{\frac{1}{k}}\right)^{-1}\right\} \\
&= \bigcup_{\alpha_1 \in \mu^h_{\xi_1}, \alpha_2 \in \mu^h_{\xi_2}} \left\{1 - \left(1 + \left\{\psi\left(\sum_{j=1}^{2} w_j \psi^{-1}\left(\Delta_k(\alpha_j)\right)\right)\right\}^{\frac{1}{k}}\right)^{-1}\right\}
\end{aligned}$$

Thus Equation (1) holds good for $n = 2$. Let us assume that Equation (1) holds for $n = r$. Then,

$$HFDAWAA\left(\mu^h_{\xi_1}, \mu^h_{\xi_2}, \mu^h_{\xi_3}, \ldots\ldots, \mu^h_{\xi_n}\right) = \bigcup_{\substack{\alpha_1 \in \mu^h_{\xi_1}, \alpha_2 \in \mu^h_{\xi_2}, \\ \ldots\ldots, \alpha_r \in \mu^h_{\xi_r}}} \left\{1 - \left(1 + \left\{\psi\left(\sum_{j=1}^{r} w_j \psi^{-1}\left(\Delta_k(\alpha_j)\right)\right)\right\}^{\frac{1}{k}}\right)^{-1}\right\}$$

Now for $n = r + 1$, we have

$$HFDAWAA(\xi_1, \xi_2, \xi_3, \dots\dots, \xi_{r+1})$$

$$= \left(\bigcup_{\substack{\alpha_1 \in \mu^h_{\xi_1}, \alpha_2 \in \mu^h_{\xi_2}, \\ \dots\dots, \alpha_r \in \mu^h_{\xi_r}}} \left\{ 1 - \left(1 + \left\{ \psi \left(\sum_{j=1}^{r} w_j \psi^{-1} \left(\Delta_k(\alpha_j) \right) \right) \right\}^{\frac{1}{k}} \right)^{-1} \right\} \right) \oplus_{DA} \left(w_{k+1} *_{DA} \xi_{r+1} \right)$$

$$= \bigcup_{\substack{\alpha_1 \in \mu^h_{\xi_1}, \alpha_2 \in \mu^h_{\xi_2}, \\ \dots\dots, \alpha_{r+1} \in \mu^h_{\xi_{r+1}}}} \left\{ 1 - \left(1 + \left\{ \psi \left(\psi^{-1} \left(\left(\frac{1 - \left(1 + \left\{ \psi \left(\sum_{j=1}^{r} w_j \psi^{-1} \left(\Delta_k(\alpha_j) \right) \right) \right\}^{\frac{1}{k}} \right)^{-1}}{\left(1 + \left\{ \psi \left(\sum_{j=1}^{r} w_j \psi^{-1} \left(\Delta_k(\alpha_j) \right) \right) \right\}^{\frac{1}{k}} \right)^{-1}} \right)^{k} \right) + \psi^{-1} \left(\left(\frac{1 - \left(1 + \left\{ \psi \left(w_{r+1} \psi^{-1} \left(\Delta_k(\alpha_{r+1}) \right) \right) \right\}^{\frac{1}{k}} \right)^{-1}}{\left(1 + \left\{ \psi \left(w_{r+1} \psi^{-1} \left(\Delta_k(\alpha_{r+1}) \right) \right) \right\}^{\frac{1}{k}} \right)^{-1}} \right)^{k} \right) \right) \right\}^{\frac{1}{k}} \right)^{-1} \right\}$$

$$= \bigcup_{\substack{\alpha_1 \in \mu^h_{\xi_1}, \alpha_2 \in \mu^h_{\xi_2}, \\ \dots\dots, \alpha_{r+1} \in \mu^h_{\xi_{r+1}}}} \left\{ 1 - \left(1 + \left\{ \psi \left(\sum_{j=1}^{r} w_j \psi^{-1} \left(\Delta_k(\alpha_j) \right) + w_{r+1} \psi^{-1} \left(\Delta_k(\alpha_{r+1}) \right) \right) \right\}^{\frac{1}{k}} \right)^{-1} \right\}$$

$$= \bigcup_{\substack{\alpha_1 \in \mu^h_{\xi_1}, \alpha_2 \in \mu^h_{\xi_2}, \\ \dots\dots, \alpha_{r+1} \in \mu^h_{\xi_{r+1}}}} \left\{ 1 - \left(1 + \left\{ \psi \left(\sum_{j=1}^{r+1} w_j \psi^{-1} \left(\Delta_k(\alpha_j) \right) \right) \right\}^{\frac{1}{k}} \right)^{-1} \right\}$$

Thus, Equation (1) also holds good for $n = r + 1$. Thus, Equation (1) is true for all natural number n.

Theorem 4. *(Shift invariance) Suppose* $\mu^h_{\xi_j} (j = 1, 2, 3, \dots\dots, n)$ *be a collection of HFEs on U and* $\mu^h_{\xi_0} \in HFE^U$. *Then* $HFDAWAA \left(\mu^h_{\xi_0} \oplus_{DA} \mu^h_{\xi_1}, \mu^h_{\xi_0} \oplus_{DA} \mu^h_{\xi_2}, \dots\dots, \mu^h_{\xi_0} \oplus_{DA} \mu^h_{\xi_n} \right) = \mu^h_{\xi_0} \oplus_{DA} HFDAWAA \left(\mu^h_{\xi_1}, \mu^h_{\xi_2}, \dots\dots, \mu^h_{\xi_n} \right)$.

Proof: Since $\mu^h_{\xi_0} \oplus_{DA} \mu^h_{\xi_j} = \bigcup_{\alpha_0 \in \mu^h_{\xi_1}, \alpha_j \in \mu^h_{\xi_2}} \left\{ 1 - \left(1 + \left\{ \psi \left(\psi^{-1} \left(\Delta_k(\alpha_0) \right) + \psi^{-1} \left(\Delta_k(\alpha_j) \right) \right) \right\}^{\frac{1}{k}} \right)^{-1} \right\}$, we have,

$$HFDAWAA \left(\mu^h_{\xi_0} \oplus_{DA} \mu^h_{\xi_1}, \mu^h_{\xi_0} \oplus_{DA} \mu^h_{\xi_2}, \dots\dots, \mu^h_{\xi_0} \oplus_{DA} \mu^h_{\xi_n} \right)$$

$$= \bigcup_{\substack{\alpha_1 \in \mu^h_{\xi_1}, \alpha_2 \in \mu^h_{\xi_2}, \\ \dots\dots, \alpha_n \in \mu^h_{\xi_n}, \alpha_0 \in \mu^h_{\xi_0}}} \left\{ 1 - \left(1 + \left\{ \psi \left(\sum_{j=1}^{n} w_j \psi^{-1} \left(\left(\frac{1 - \left(1 + \left\{ \psi \left(\psi^{-1}(\Delta_k(\alpha_0)) + \psi^{-1} \left(\Delta_k(\alpha_j) \right) \right) \right\}^{\frac{1}{k}} \right)^{-1}}{\left(1 + \left\{ \psi \left(\psi^{-1}(\Delta_k(\alpha_0)) + \psi^{-1} \left(\Delta_k(\alpha_j) \right) \right) \right\}^{\frac{1}{k}} \right)^{-1}} \right)^{k} \right) \right) \right\}^{\frac{1}{k}} \right)^{-1} \right\}$$

$$= \bigcup_{\substack{\alpha_1 \in \mu^h_{\xi_1}, \alpha_2 \in \mu^h_{\xi_2}, \\ \dots\dots, \alpha_n \in \mu^h_{\xi_n}, \alpha_0 \in \mu^h_{\xi_0}}} \left\{ 1 - \left(1 + \left\{ \psi \left(\sum_{j=1}^{n} w_j \left(\psi^{-1} \left(\Delta_k(\alpha_0) \right) + \psi^{-1} \left(\Delta_k(\alpha_j) \right) \right) \right) \right\}^{\frac{1}{k}} \right)^{-1} \right\}$$

$$= \bigcup_{\substack{\alpha_1 \in \mu^h_{\xi_1}, \alpha_2 \in \mu^h_{\xi_2}, \\ \dots\dots, \alpha_n \in \mu^h_{\xi_n}, \alpha_0 \in \mu^h_{\xi_0}}} \left\{ 1 - \left(1 + \left\{ \psi \left(\psi^{-1} \left(\Delta_k(\alpha_0) \right) + \sum_{j=1}^{n} w_j \psi^{-1} \left(\Delta_k(\alpha_j) \right) \right) \right\}^{\frac{1}{k}} \right)^{-1} \right\}$$

Again, using Dombi–Archimedean operational laws, we have,

$$
\begin{aligned}
&\xi_0 \oplus_{DA} HFDAWAA(\xi_1, \xi_2, \xi_3, \ldots\ldots, \xi_n)\\
&= \left(\bigcup_{\alpha_0 \in \mu^h_{\xi_0}} \{\alpha_0\}\right) \oplus_{DA} \left(\bigcup_{\substack{\alpha_1 \in \mu^h_{\xi_1}, \alpha_2 \in \mu^h_{\xi_2},\\ \ldots\ldots, \alpha_n \in \mu^h_{\xi_n}}} \left\{1 - \left(1 + \left\{\psi\left(\sum_{j=1}^{n} w_j \psi^{-1}\left(\Delta_k(\alpha_j)\right)\right)\right\}^{\frac{1}{k}}\right)^{-1}\right\}\right)\\
&= \bigcup_{\substack{\alpha_1 \in \mu^h_{\xi_1}, \alpha_2 \in \mu^h_{\xi_2},\\ \ldots\ldots, \alpha_n \in \mu^h_{\xi_n}, \alpha_0 \in \mu^h_{\xi_0}}} \left\{1 - \left(1 + \left\{\psi\left(\psi^{-1}\left(\Delta_k(\alpha_0)\right) + \psi^{-1}\left(\left(\frac{1 - \left(1 + \left\{\psi\left(\sum_{j=1}^{n} w_j \psi^{-1}\left(\Delta_k(\alpha_j)\right)\right)\right\}^{\frac{1}{k}}\right)^{-1}}{\left(1 + \left\{\psi\left(\sum_{j=1}^{n} w_j \psi^{-1}\left(\Delta_k(\alpha_j)\right)\right)\right\}^{\frac{1}{k}}\right)^{-1}}\right)^{k}\right)\right)\right\}^{\frac{1}{k}}\right)^{-1}\right\}\\
&= \bigcup_{\substack{\alpha_1 \in \mu^h_{\xi_1}, \alpha_2 \in \mu^h_{\xi_2},\\ \ldots\ldots, \alpha_n \in \mu^h_{\xi_n}, \alpha_0 \in \mu^h_{\xi_0}}} \left\{1 - \left(1 + \left\{\psi\left(\psi^{-1}\left(\Delta_k(\alpha_0)\right) + \sum_{j=1}^{n} w_j \psi^{-1}\left(\Delta_k(\alpha_j)\right)\right)\right\}^{\frac{1}{k}}\right)^{-1}\right\}
\end{aligned}
$$

Hence, $HFDAWAA\left(\mu^h_{\xi_0} \oplus_{DA} \mu^h_{\xi_1}, \mu^h_{\xi_0} \oplus_{DA} \mu^h_{\xi_2}, \ldots\ldots, \mu^h_{\xi_0} \oplus_{DA} \mu^h_{\xi_n}\right) = \mu^h_{\xi_0} \oplus_{DA} HFDAWAA\left(\mu^h_{\xi_1}, \mu^h_{\xi_2}, \ldots\ldots, \mu^h_{\xi_n}\right)$.

Theorem 5. *(Idempotency) Suppose $\mu^h_{\xi_j} (j = 1, 2, 3, \ldots\ldots, n)$ be a collection of HFEs on U and $\mu^h_{\xi_0} \in HFE^U$ such that $\mu^h_{\xi_0} = \mu^h_{\xi_j}\ \forall j$. Then we have, $HFDAWAA\left(\mu^h_{\xi_1}, \mu^h_{\xi_2}, \mu^h_{\xi_3}, \ldots\ldots\ldots, \mu^h_{\xi_n}\right) = \mu^h_{\xi_0}$.*

Proof: we have,

$$
\begin{aligned}
&HFDAWAA\left(\mu^h_{\xi_1}, \mu^h_{\xi_2}, \mu^h_{\xi_3}, \ldots\ldots\ldots, \mu^h_{\xi_n}\right)\\
&= \bigcup_{\alpha_1 \in \mu^h_{\xi_1}, \alpha \in \mu^h_{\xi_2}, \ldots\ldots, \alpha_n \in \mu^h_{\xi_n}} \left\{1 - \left(1 + \left\{\psi\left(\sum_{j=1}^{n} w_j \psi^{-1}\left(\Delta_k(\alpha_j)\right)\right)\right\}^{\frac{1}{k}}\right)^{-1}\right\}\\
&= \bigcup_{\alpha_0 \in \mu^h_{\xi_0}} \left\{1 - \left(1 + \left\{\psi\left(\sum_{j=1}^{n} w_j \psi^{-1}\left(\Delta_k(\alpha_0)\right)\right)\right\}^{\frac{1}{k}}\right)^{-1}\right\}\\
&= \bigcup_{\alpha_0 \in \mu^h_{\xi_0}} \left\{1 - \left(1 + \left\{\psi\left(\psi^{-1}\left(\Delta_k(\alpha_0)\right) \sum_{j=1}^{n} w_j\right)\right\}^{\frac{1}{k}}\right)^{-1}\right\}\\
&= \bigcup_{\alpha_0 \in \mu^h_{\xi_0}} \left\{1 - \left(1 + \left\{\psi\left(\psi^{-1}\left(\Delta_k(\alpha_0)\right)\right)\right\}^{\frac{1}{k}}\right)^{-1}\right\}\\
&= \bigcup_{\alpha_0 \in \mu^h_{\xi_0}} \left\{1 - \left(1 + \left\{\Delta_k(\alpha_0)\right\}^{\frac{1}{k}}\right)^{-1}\right\}\\
&= \bigcup_{\alpha_0 \in \mu^h_{\xi_0}} \left\{1 - \left(1 + \frac{\alpha_0}{1 - \alpha_0}\right)^{-1}\right\}\\
&= \bigcup_{\alpha_0 \in \mu^h_{\xi_0}} \{\alpha_0\} = \mu^h_{\xi_0}.
\end{aligned}
$$

Theorem 6. *(Boundedness) Suppose* $\mu^h_{\xi_j}(j = 1,2,3,......,n)$ *be a collection of HFEs on U. Then,* $\left(\mu^h_{\xi}\right)^- \prec HFDAWAA\left(\mu^h_{\xi_1},\mu^h_{\xi_2},\mu^h_{\xi_3},........,\mu^h_{\xi_n}\right) \prec \left(\mu^h_{\xi}\right)^+$

where $\left(\mu^h_{\xi}\right)^+ = \bigcup_{\alpha_1\in\mu^h_{\xi_1},\alpha_2\in\mu^h_{\xi_2},......,\alpha_n\in\mu^h_{\xi_n}} \max\{\alpha_1,\alpha_2,.....,\alpha_n\}$ *and* $\left(\mu^h_{\xi}\right)^- = \bigcup_{\alpha_1\in\mu^h_{\xi_1},\alpha_2\in\mu^h_{\xi_2},......,\alpha_n\in\mu^h_{\xi_n}} \min\{\alpha_1,\alpha_2,.....,\alpha_n\}.$

Proof: For any $j\in\{1,2,3,......,n\}$, we have, $\min_j(\alpha_j) \le \alpha_j \le \max_j(\alpha_j)$ *where* $\alpha_j \in \mu^h_{\xi_j}$. This gives,

$$\psi\left(\sum_{j=1}^n w_j\psi^{-1}\left(\Delta_k(\alpha')\right)\right) \le \psi\left(\sum_{j=1}^n w_j\psi^{-1}\left(\Delta_k(\alpha_j)\right)\right) \le \psi\left(\sum_{j=1}^n w_j\psi^{-1}\left(\Delta_k(\alpha'')\right)\right) \text{ where } \alpha' \in \left(\mu^h_{\xi}\right)^-, \alpha'' \in \left(\mu^h_{\xi}\right)^+$$

$$\left(\text{since } \psi \text{ is monotonic increasing on } [0,1]\right)$$

$$\Rightarrow 1-\left(1+\left\{\psi\left(\sum_{j=1}^n w_j\psi^{-1}(\Delta_k(\alpha'))\right)\right\}^{\frac{1}{k}}\right)^{-1} \le 1-\left(1+\left\{\psi\left(\sum_{j=1}^n w_j\psi^{-1}(\Delta_k(\alpha_j))\right)\right\}^{\frac{1}{k}}\right)^{-1} \le 1-\left(1+\left\{\psi\left(\sum_{j=1}^n w_j\psi^{-1}(\Delta_k(\alpha''))\right)\right\}^{\frac{1}{k}}\right)^{-1}$$

$$\Rightarrow 1-\left(1+\left\{\psi\left(\psi^{-1}(\Delta_k(\alpha'))\sum_{j=1}^n w_j\right)\right\}^{\frac{1}{k}}\right)^{-1} \le 1-\left(1+\left\{\psi\left(\sum_{j=1}^n w_j\psi^{-1}(\Delta_k(\alpha_j))\right)\right\}^{\frac{1}{k}}\right)^{-1} \le 1-\left(1+\left\{\psi\left(\psi^{-1}(\Delta_k(\alpha''))\sum_{j=1}^n w_j\right)\right\}^{\frac{1}{k}}\right)^{-1}$$

$$\Rightarrow 1-\left(1+\left\{\psi\left(\psi^{-1}(\Delta_k(\alpha'))\right)\right\}^{\frac{1}{k}}\right)^{-1} \le 1-\left(1+\left\{\psi\left(\sum_{j=1}^n w_j\psi^{-1}(\Delta_k(\alpha_j))\right)\right\}^{\frac{1}{k}}\right)^{-1} \le 1-\left(1+\left\{\psi\left(\psi^{-1}(\Delta_k(\alpha''))\right)\right\}^{\frac{1}{k}}\right)^{-1}$$

$$\Rightarrow 1-\left(1+\left\{\Delta_k(\alpha')\right\}^{\frac{1}{k}}\right)^{-1} \le 1-\left(1+\left\{\psi\left(\sum_{j=1}^n w_j\psi^{-1}(\Delta_k(\alpha_j))\right)\right\}^{\frac{1}{k}}\right)^{-1} \le 1-\left(1+\left\{\Delta_k(\alpha'')\right\}^{\frac{1}{k}}\right)^{-1}$$

$$\Rightarrow 1-\frac{1}{1+\frac{\alpha'}{1-\alpha'}} \le 1-\left(1+\left\{\psi\left(\sum_{j=1}^n w_j\psi^{-1}(\Delta_k(\alpha_j))\right)\right\}^{\frac{1}{k}}\right)^{-1} \le 1-\frac{1}{1+\frac{\alpha''}{1-\alpha''}}$$

$$\Rightarrow \alpha' \le 1-\left(1+\left\{\psi\left(\sum_{j=1}^n w_j\psi^{-1}(\Delta_k(\alpha_j))\right)\right\}^{\frac{1}{k}}\right)^{-1} \le \alpha''$$

Therefore, by definition of score values of *HFE*s, we obtain $S\left(\left(\mu^h_{\xi}\right)^-\right) \le S\left(HFDAWAA\left(\mu^h_{\xi_1},\mu^h_{\xi_2},\mu^h_{\xi_3},........,\mu^h_{\xi_n}\right)\right) \le S\left(\left(\mu^h_{\xi}\right)^+\right)$.

Hence, by ranking rules of *HFE*s, we get, $\left(\mu^h_{\xi}\right)^- \prec HFDAWAA\left(\mu^h_{\xi_1},\mu^h_{\xi_2},\mu^h_{\xi_3},........,\mu^h_{\xi_n}\right) \prec \left(\mu^h_{\xi}\right)^+$.

Theorem 7. *(Monotonocity) Suppose* $\mu^h_{\xi_j}(j = 1,2,3,......,n)$ *and* $\mu^h_{\xi'_j}(j = 1,2,3,......,n)$ *be two collections of HFEs on U such that* $\alpha_j \le \alpha'_j$ *where* $\alpha_j \in \mu^h_{\xi_j}, \alpha'_j \in \mu^h_{\xi'_j}\ \forall(j = 1,2,3,......,n)$. *Then we have,* $HFDAWAA\left(\mu^h_{\xi_1},\mu^h_{\xi_2},\mu^h_{\xi_3},........,\mu^h_{\xi_n}\right) \prec HFDAWAA\left(\mu^h_{\xi'_1},\mu^h_{\xi'_2},\mu^h_{\xi'_3},........,\mu^h_{\xi'_n}\right)$.

Proof: We have from Theorem 3,

$$HFDAWAA\left(\mu^h_{\xi_1},\mu^h_{\xi_2},\mu^h_{\xi_3},........,\mu^h_{\xi_n}\right) = \bigcup_{\alpha_1\in\mu^h_{\xi_1},\alpha_2\in\mu^h_{\xi_2},......,\alpha_n\in\mu^h_{\xi_n}} \left\{1-\left(1+\left\{\psi\left(\sum_{j=1}^n w_j\psi^{-1}(\Delta_k(\alpha_j))\right)\right\}^{\frac{1}{k}}\right)^{-1}\right\},$$

$$HFDAWAA\left(\mu^h_{\xi'_1},\mu^h_{\xi'_2},\mu^h_{\xi'_3},........,\mu^h_{\xi'_n}\right) = \bigcup_{\alpha'_1\in\mu^h_{\xi'_1},\alpha'_2\in\mu^h_{\xi'_2},......,\alpha'_n\in\mu^h_{\xi'_n}} \left\{1-\left(1+\left\{\psi\left(\sum_{j=1}^n w_j\psi^{-1}(\Delta_k(\alpha'_j))\right)\right\}^{\frac{1}{k}}\right)^{-1}\right\}.$$

Since $\alpha_j \le \alpha_j'$ where $\alpha_j \in \mu^h_{\xi_j}, \alpha_j' \in \mu^h_{\xi_j'} (j = 1, 2, 3, ..., n)$, $\Delta_k(\alpha_j) \le \Delta_k(\alpha_j')$.

Since ψ is monotonic increasing on [0, 1], we get $\psi\left(\sum_{j=1}^{n} w_j \psi^{-1}\left(\Delta_k(\alpha_j)\right)\right) \le \psi\left(\sum_{j=1}^{n} w_j \psi^{-1}\left(\Delta_k(\alpha_j')\right)\right)$. which implies that $1 - \left(1 + \left\{\psi\left(\sum_{j=1}^{n} w_j \psi^{-1}\left(\Delta_k(\alpha_j)\right)\right)\right\}^{\frac{1}{k}}\right)^{-1} \le 1 - \left(1 + \left\{\psi\left(\sum_{j=1}^{n} w_j \psi^{-1}\left(\Delta_k(\alpha_j')\right)\right)\right\}^{\frac{1}{k}}\right)^{-1}$.

Consequently, $S\left(HFDAWAA\left(\mu^h_{\xi_1}, \mu^h_{\xi_2}, \mu^h_{\xi_3}, ..., \mu^h_{\xi_n}\right)\right) \le S\left(HFDAWAA\left(\mu^h_{\xi_1'}, \mu^h_{\xi_2'}, \mu^h_{\xi_3'}, ..., \mu^h_{\xi_n'}\right)\right)$ and so by ranking rules of *HFEs*, $HFDAWAA\left(\mu^h_{\xi_1}, \mu^h_{\xi_2}, \mu^h_{\xi_3}, ..., \mu^h_{\xi_n}\right) \prec HFDAWAA\left(\mu^h_{\xi_1'}, \mu^h_{\xi_2'}, \mu^h_{\xi_3'}, ..., \mu^h_{\xi_n'}\right)$.

Next, based on *HFDAWAA* operator, we shall develop the *HFDAOWAA* operator as follows:

Definition 15. Suppose $\mu^h_{\xi_j} (j = 1, 2, 3, ..., n)$ be a collection of *HFEs* on U. Then *HFDAOWAA* is a function $HFDAOWAA : HFE^U \to HFE^U$ which is defined as follows:

$$HFDAOWAA\left(\mu^h_{\xi_1}, \mu^h_{\xi_2}, \mu^h_{\xi_3}, ..., \mu^h_{\xi_n}\right) = \bigoplus_{j=1_{DA}}^{n} \left(w_j *_{DA} \mu^h_{\xi_{\sigma(j)}}\right)$$

where $(\sigma(1), \sigma(2), \sigma(3), ..., \sigma(n))$ is a permutation of $(1, 2, 3, ..., n)$ such that $\mu^h_{\xi_{\sigma(j-1)}} \ge \mu^h_{\xi_{\sigma(j)}}$ for all $j = 1, 2, 3, ..., n$ and $w_j (j = 1, 2, 3, ..., n)$ is the weight of $\mu^h_{\xi_j} (j = 1, 2, 3, ..., n)$ with $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$.

Theorem 8. *The aggregated value* $HFDAOWAA\left(\mu^h_{\xi_1}, \mu^h_{\xi_2}, \mu^h_{\xi_3}, ..., \mu^h_{\xi_n}\right)$ *is still a HFE and*

$$HFDAOWAA\left(\mu^h_{\xi_1}, \mu^h_{\xi_2}, ..., \mu^h_{\xi_n}\right) = \bigcup_{\substack{\alpha_{\sigma(1)} \in \mu^h_{\xi_{\sigma(1)}}, \alpha_{\sigma(2)} \in \mu^h_{\xi_{\sigma(2)}}, \\ ...., \alpha_{\sigma(n)} \in \mu^h_{\xi_{\sigma(n)}}}} \left\{1 - \left(1 + \left\{\psi\left(\sum_{j=1}^{n} w_j \psi^{-1}\left(\Delta_k(\alpha_{\sigma(j)})\right)\right)\right\}^{\frac{1}{k}}\right)^{-1}\right\}$$

where w_j *is the weight of* $\mu^h_{\xi_j} (j = 1, 2, 3, ..., n)$ *with* $w_j > 0$ *and* $\sum_{j=1}^{n} w_j = 1$.

Proof: Similar to Theorem 3.

In particular, if $w_j = \frac{1}{n} \forall j = 1, 2, 3, ..., n$;, then the operator *HFDAOWAA* reduces to the *HFDAWAA* operator.

Theorem 9. *(Shift invariance) Suppose* $\mu^h_{\xi_j} (j = 1, 2, 3, ..., n)$ *be a collection of HFEs on U and* $\mu^h_{\xi_0} \in HFE^U$. *Then* $HFDAOWAA\left(\mu^h_{\xi_0} \oplus_{DA} \mu^h_{\xi_1}, \mu^h_{\xi_0} \oplus_{DA} \mu^h_{\xi_2}, ..., \mu^h_{\xi_0} \oplus_{DA} \mu^h_{\xi_n}\right) = \mu^h_{\xi_0} \oplus_{DA} HFDAOWAA\left(\mu^h_{\xi_1}, \mu^h_{\xi_2}, ..., \mu^h_{\xi_n}\right)$.

Proof: Similar to Theorem 4.

Theorem 10. *(Idempotency) Suppose* $\mu^h_{\xi_j} (j = 1, 2, 3, ..., n)$ *be a collection of HFEs on U and* $\mu^h_{\xi_0} \in HFE^U$ *such that* $\mu^h_{\xi_0} = \mu^h_{\xi_j} \forall j$. *Then we have,* $HFDAOWAA\left(\mu^h_{\xi_1}, \mu^h_{\xi_2}, ..., \mu^h_{\xi_n}\right) = \mu^h_{\xi_0}$.

Proof: Similar to Theorem 5.

Theorem 11. *(Boundedness) Suppose* $\mu^h_{\xi_j} (j = 1, 2, 3, ..., n)$ *be a collection of HFEs on U. Then,* $\left(\mu^h_{\xi}\right)^- \prec HFDAOWAA\left(\mu^h_{\xi_1}, \mu^h_{\xi_2}, \mu^h_{\xi_3}, ..., \mu^h_{\xi_n}\right) \prec \left(\mu^h_{\xi}\right)^+$

where $\left(\mu^h_{\xi}\right)^+ = \bigcup_{\alpha_1 \in \mu^h_{\xi_1}, \alpha_2 \in \mu^h_{\xi_2}, ..., \alpha_n \in \mu^h_{\xi_n}} \max\{\alpha_1, \alpha_2, ..., \alpha_n\}$ *and* $\left(\mu^h_{\xi}\right)^- = \bigcup_{\alpha_1 \in \mu^h_{\xi_1}, \alpha_2 \in \mu^h_{\xi_2}, ..., \alpha_n \in \mu^h_{\xi_n}} \min\{\alpha_1, \alpha_2, ..., \alpha_n\}$

Proof: Similar to Theorem 6.

Theorem 12. *(Monotonocity) Suppose* $\mu^h_{\xi_j}(j = 1, 2, 3, ..., n)$ *and* $\mu^h_{\xi'_j}(j = 1, 2, 3, ..., n)$ *be two collections of HFEs on U such that* $\alpha_j \leq \alpha'_j$ *where* $\alpha_j \in \mu^h_{\xi_j}, \alpha'_j \in \mu^h_{\xi'_j} \forall (j = 1, 2, 3, ..., n)$. *Then we have,* $HFDAOWAA\left(\mu^h_{\xi_1}, \mu^h_{\xi_2}, \mu^h_{\xi_3}, ..., \mu^h_{\xi_n}\right) \prec HFDAOWAA\left(\mu^h_{\xi'_1}, \mu^h_{\xi'_2}, \mu^h_{\xi'_3}, ..., \mu^h_{\xi'_n}\right)$.

Proof: Similar to Theorem 7.

We can differentiate *HFDAWAA* and *HFDAOWAA* operators in terms of *HFE* assessment which means that the *HFDAWAA* operator specifies only the self-importance of each *HFE*, whereas the *HFDAOWAA* operator specifies the ordered position importance of each *HFE*. In many practical situations, we need to consider both these cases altogether. Because of this, by means of linking up the features of these two operators each other, we propose *HFDAHAA* operator in the ambiance of the Dombi–Archimedean operations as follows:

Definition 16. Suppose $\mu^h_{\xi_j}(j = 1, 2, 3, ..., n)$ be a collection of *HFEs* on *U*. Then *HFDAHAA* (for short) is a function $HFDAHAA : HFE^U \to HFE^U$ which is defined as follows:

$$HFDAHAA\left(\mu^h_{\xi_1}, \mu^h_{\xi_2}, \mu^h_{\xi_3}, ..., \mu^h_{\xi_n}\right) = \underset{j=1_{AD}}{\overset{n}{\oplus}} \left(\hat{w}_j *_{AD} \mu^h_{\hat{\xi}_{\sigma(j)}}\right)$$

Where $\hat{w} = (\hat{w}_1, \hat{w}_2, \hat{w}_3, ..., \hat{w}_n)^T$ is the weight vector associated with the *HFDAHAA* operator such that $\hat{w}_j > 0$ and $\sum_{j=1}^{n} \hat{w}_j = 1$, $(\sigma(1), \sigma(2), \sigma(3), ..., \sigma(n))$ is a permutation of $(1, 2, 3, ..., n)$, $\mu^h_{\hat{\xi}_{\sigma(j)}}$ is the *j*th largest of the weighted *HFE* of $\mu^h_{\hat{\xi}_j}\left(\mu^h_{\hat{\xi}_j} = (n\hat{w}_j) *_{DA} \mu^h_{\xi_j}, j = 1, 2, 3, ..., n\right)$ and w_j is the weight of $\mu^h_{\xi_j}(j = 1, 2, 3, ..., n)$ with $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$.

Theorem 13. *The aggregated value* $HFDAHAA\left(\mu^h_{\xi_1}, \mu^h_{\xi_2}, \mu^h_{\xi_3}, ..., \mu^h_{\xi_n}\right)$ *is still a HFE and*

$$HFDAHAA\left(\mu^h_{\xi_1}, \mu^h_{\xi_2}, ..., \mu^h_{\xi_n}\right) = \bigcup_{\hat{\alpha}_{\sigma(1)} \in \mu^h_{\hat{\xi}_{\sigma(1)}}, \hat{\alpha}_{\sigma(2)} \in \mu^h_{\hat{\xi}_{\sigma(2)}}, ..., \hat{\alpha}_{\sigma(n)} \in \mu^h_{\hat{\xi}_{\sigma(n)}}} \left\{1 - \left(1 + \left\{\psi\left(\sum_{j=1}^{n} \hat{w}_j \psi^{-1}\left(\Delta_k(\hat{\alpha}_{\sigma(j)})\right)\right)\right\}^{\frac{1}{k}}\right)^{-1}\right\}$$

where $\hat{w} = (\hat{w}_1, \hat{w}_2, \hat{w}_3, ..., \hat{w}_n)^T$ *is the weight vector associated with the HFDAHAA operator such that* $\hat{w}_j > 0$ *and* $\sum_{j=1}^{n} \hat{w}_j = 1$, $(\sigma(1), \sigma(2), \sigma(3), ..., \sigma(n))$ *is a permutation of* $(1, 2, 3, ..., n)$, $\mu^h_{\hat{\xi}_{\sigma(j)}}$ *is the jth largest of the weighted HFE of* $\mu^h_{\hat{\xi}_j}\left(\mu^h_{\hat{\xi}_j} = (n\hat{w}_j) *_{DA} \mu^h_{\xi_j}, j = 1, 2, 3, ..., n\right)$ *and* w_j *is the weight of* $\mu^h_{\xi_j}(j = 1, 2, 3, ..., n)$ *with* $w_j > 0$ *and* $\sum_{j=1}^{n} w_j = 1$.

In particular, if $w_j = \frac{1}{n} \forall j = 1, 2, 3, ..., n;$, then the operator *HFDAHAA* reduces to the *HFDAWAA* operator and if $\hat{w}_j = \frac{1}{n} \forall j = 1, 2, 3, ..., n;$, then the operator *HFDAHAA* reduces to the *HFDAOWAA*.

4.2. *HF* Dombi–Archimedean Geometric *AOs*

In this sub-section, we propose Dombi–Archimedean geometric *AOs* with *HFEs* such as *HFDAWGA*, *HFDAOWGA*, and *HFDAHGA*.

Definition 17. Suppose $\mu^h_{\xi_j}(j = 1, 2, 3, ..., n)$ be a collection of *HFEs* on *U*. Then *HFDAWGA* (for short) is a function $HFDAWGA : HFE^U \to HFE^U$ which is defined as follows:

$$HFDAWGA\left(\mu^h_{\xi_1}, \mu^h_{\xi_2}, \mu^h_{\xi_3}, ..., \mu^h_{\xi_n}\right) = \underset{j=1_{DA}}{\overset{n}{\otimes}} \left(w_j \circ_{DA} \mu^h_{\xi_j}\right)$$

where $w_j(j = 1, 2, 3, ..., n)$ is the weight of $\mu^h_{\xi_j}$ $(j = 1, 2, 3, ..., n)$ with $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$.

The following theorems readily follow from Definition 17 and Dombi–Archimedean operational laws for *HFEs*.

Theorem 14. *The aggregated value* $HFDAWGA\left(\mu^h_{\xi_1}, \mu^h_{\xi_2}, \mu^h_{\xi_3}, \ldots\ldots, \mu^h_{\xi_n}\right)$ *is also a HFE and*

$$HFDAWGA\left(\mu^h_{\xi_1}, \mu^h_{\xi_2}, \mu^h_{\xi_3}, \ldots\ldots, \mu^h_{\xi_n}\right) = \bigcup_{\alpha_1 \in \mu^h_{\xi_1}, \alpha_2 \in \mu^h_{\xi_2}, \ldots\ldots, \alpha_n \in \mu^h_{\xi_n}} \left\{ \left(1 + \left\{ \theta\left(\sum_{j=1}^{n} w_j \theta^{-1}\left(\nabla_k(\alpha_j)\right) \right) \right\}^{\frac{1}{k}} \right)^{-1} \right\}$$

where $w_j (j = 1, 2, 3, \ldots\ldots, n)$ *is the weight of* $\mu^h_{\xi_j} (j = 1, 2, 3, \ldots\ldots, n)$ *with* $w_j > 0$ *and* $\sum_{j=1}^{n} w_j = 1$.

Theorem 15. *(Shift invariance) Suppose* $\mu^h_{\xi_j} (j = 1, 2, 3, \ldots\ldots, n)$ *be a collection of HFEs on U and* $\mu^h_{\xi_0} \in HFE^U$. *Then* $HFDAWGA\left(\mu^h_{\xi_0} \otimes_{DA} \mu^h_{\xi_1}, \mu^h_{\xi_0} \otimes_{DA} \mu^h_{\xi_2}, \ldots\ldots, \mu^h_{\xi_0} \otimes_{DA} \mu^h_{\xi_n}\right) = \mu^h_{\xi_0} \otimes_{DA} HFDAWGA\left(\mu^h_{\xi_1}, \mu^h_{\xi_2}, \ldots\ldots, \mu^h_{\xi_n}\right)$.

Theorem 16. *(Idempotency) Suppose* $\mu^h_{\xi_j} (j = 1, 2, 3, \ldots\ldots, n)$ *be a collection of HFEs on U and* $\mu^h_{\xi_0} \in HFE^U$ *such that* $\mu^h_{\xi_0} = \mu^h_{\xi_j} \forall j$. *Then we have,* $HFDAWGA\left(\mu^h_{\xi_1}, \mu^h_{\xi_2}, \mu^h_{\xi_3}, \ldots\ldots, \mu^h_{\xi_n}\right) = \mu^h_{\xi_0}$.

Theorem 17. *(Boundedness) Suppose* $\mu^h_{\xi_j} (j = 1, 2, 3, \ldots\ldots, n)$ *be a collection of HFEs on U. Then*

$$\left(\mu^h_{\xi}\right)^{-} \prec HFDAWGA\left(\mu^h_{\xi_1}, \mu^h_{\xi_2}, \mu^h_{\xi_3}, \ldots\ldots, \mu^h_{\xi_n}\right) \prec \left(\mu^h_{\xi}\right)^{+}$$

where $\left(\mu^h_{\xi}\right)^{+} = \bigcup_{\alpha_1 \in \mu^h_{\xi_1}, \alpha_2 \in \mu^h_{\xi_2}, \ldots\ldots, \alpha_n \in \mu^h_{\xi_n}} \max\{\alpha_1, \alpha_2, \ldots\ldots, \alpha_n\}$ *and* $\left(\mu^h_{\xi}\right)^{-} = \bigcup_{\alpha_1 \in \mu^h_{\xi_1}, \alpha_2 \in \mu^h_{\xi_2}, \ldots\ldots, \alpha_n \in \mu^h_{\xi_n}} \min\{\alpha_1, \alpha_2, \ldots\ldots, \alpha_n\}$

Theorem 18. *(Monotonocity) Suppose* $\mu^h_{\xi_j} (j = 1, 2, 3, \ldots\ldots, n)$ *and* $\mu^h_{\xi'_j} (j = 1, 2, 3, \ldots\ldots, n)$ *be two collections of HFEs on U such that* $\alpha_j \leq \alpha'_j$ *where* $\alpha_j \in \mu^h_{\xi_j}, \alpha'_j \in \mu^h_{\xi'_j} (j = 1, 2, 3, \ldots\ldots, n)$. *Then we have,* $HFDAWGA\left(\mu^h_{\xi_1}, \mu^h_{\xi_2}, \mu^h_{\xi_3}, \ldots\ldots, \mu^h_{\xi_n}\right) \prec HFDAWGA\left(\mu^h_{\xi'_1}, \mu^h_{\xi'_2}, \mu^h_{\xi'_3}, \ldots\ldots, \mu^h_{\xi'_n}\right)$.

Next, based on HFDAWGA operator, we shall develop the HFDAOWGA operator as follows:

Definition 18. Suppose $\mu^h_{\xi_j} (j = 1, 2, 3, \ldots\ldots, n)$ be a collection of *HFEs* on *U*. Then *HFDAOWGA* is a function $HFDAOWGA : HFE^U \rightarrow HFE^U$ which is defined as follows:

$$HFDAOWGA\left(\mu^h_{\xi_1}, \mu^h_{\xi_2}, \mu^h_{\xi_3}, \ldots\ldots, \mu^h_{\xi_n}\right) = \underset{j=1}{\overset{n}{\otimes}}{}_{DA} \left(w_j \circ_{DA} \mu^h_{\xi_{\sigma(j)}}\right)$$

where $(\sigma(1), \sigma(2), \sigma(3), \ldots\ldots, \sigma(n))$ is a permutation of $(1, 2, 3, \ldots., n)$ such that $\mu^h_{\xi_{\sigma(j-1)}} \geq \mu^h_{\xi_{\sigma(j)}}$ for all $j = 1, 2, 3, \ldots\ldots, n$ and $w_j (j = 1, 2, 3, \ldots\ldots, n)$ is the weight of $\mu^h_{\xi_j} (j = 1, 2, 3, \ldots\ldots, n)$ with $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$.

Theorem 19. *The aggregated value* $HFDAOWGA\left(\mu^h_{\xi_1}, \mu^h_{\xi_2}, \mu^h_{\xi_3}, \ldots\ldots, \mu^h_{\xi_n}\right)$ *is still a HFE and*

$$HFDAOWGA\left(\mu^h_{\xi_1}, \mu^h_{\xi_2}, \ldots\ldots, \mu^h_{\xi_n}\right) = \bigcup_{\substack{\alpha_{\sigma(1)} \in \mu^h_{\xi_{\sigma(1)}}, \alpha_{\sigma(2)} \in \mu^h_{\xi_{\sigma(2)}}, \\ \ldots\ldots, \alpha_{\sigma(n)} \in \mu^h_{\xi_{\sigma(n)}}}} \left\{ \left(1 + \left\{ \theta\left(\sum_{j=1}^{n} w_j \theta^{-1}(\nabla_k(\alpha_{\sigma(j)})) \right) \right\}^{\frac{1}{k}} \right)^{-1} \right\}$$

where w_j *is the weight of* $\mu^h_{\xi_j} (j = 1, 2, 3, \ldots\ldots, n)$ *with* $w_j > 0$ *and* $\sum_{j=1}^{n} w_j = 1$.

In particular, if $w_j = \frac{1}{n} \forall j = 1, 2, 3, \ldots., n$;, then the *HFDAOWGA* operator reduces to the *HFDAWGA* operator.

Theorem 20. *(Shift invariance) Suppose* $\mu^h_{\xi_j} (j = 1, 2, 3, \ldots\ldots, n)$ *be a collection of HFEs on U and* $\mu^h_{\xi_0} \in HFE^U$. *Then,* $HFDAOWGA\left(\mu^h_{\xi_0} \otimes_{DA} \mu^h_{\xi_1}, \mu^h_{\xi_0} \otimes_{DA} \mu^h_{\xi_2}, \ldots\ldots, \mu^h_{\xi_0} \otimes_{DA} \mu^h_{\xi_n}\right) = \mu^h_{\xi_0} \otimes_{DA} HFDAOWGA\left(\mu^h_{\xi_1}, \mu^h_{\xi_2}, \ldots\ldots, \mu^h_{\xi_n}\right)$.

Theorem 21. *(Idempotency) Suppose $\mu^h_{\xi_j}(j = 1, 2, 3, ..., n)$ be a collection of HFEs on U and $\mu^h_{\xi_0} \in HFE^U$ such that $\mu^h_{\xi_0} = \mu^h_{\xi_j}\ \forall j$. Then we have, $HFDAOWGA\left(\mu^h_{\xi_1}, \mu^h_{\xi_2}, \mu^h_{\xi_3}, ..., \mu^h_{\xi_n}\right) = \mu^h_{\xi_0}$.*

Theorem 22. *(Boundedness) Suppose $\mu^h_{\xi_j}(j = 1, 2, 3, ..., n)$ be a collection of HFEs on U. Then, $\left(\mu^h_{\xi}\right)^- \prec HFDAOWGA\left(\mu^h_{\xi_1}, \mu^h_{\xi_2}, \mu^h_{\xi_3}, ..., \mu^h_{\xi_n}\right) \prec \left(\mu^h_{\xi}\right)^+$*

where $\left(\mu^h_{\xi}\right)^+ = \bigcup_{\alpha_1 \in \mu^h_{\xi_1}, \alpha_2 \in \mu^h_{\xi_2}, ..., \alpha_n \in \mu^h_{\xi_n}} \max\{\alpha_1, \alpha_2, ..., \alpha_n\}$

and $\left(\mu^h_{\xi}\right)^- = \bigcup_{\alpha_1 \in \mu^h_{\xi_1}, \alpha_2 \in \mu^h_{\xi_2}, ..., \alpha_n \in \mu^h_{\xi_n}} \min\{\alpha_1, \alpha_2, ..., \alpha_n\}$.

Theorem 23. *(Monotonocity) Suppose $\mu^h_{\xi_j}(j = 1, 2, 3, ..., n)$ and $\mu^h_{\xi'_j}(j = 1, 2, 3, ..., n)$ be two collections of HFEs on U such that $\alpha_j \leq \alpha'_j$ where $\alpha_j \in \mu^h_{\xi_j}, \alpha'_j \in \mu^h_{\xi'_j}(j = 1, 2, 3, ..., n)$. Then we have, $HFDAOWGA\left(\mu^h_{\xi_1}, \mu^h_{\xi_2}, \mu^h_{\xi_3}, ..., \mu^h_{\xi_n}\right) \prec HFDAOWGA\left(\mu^h_{\xi'_1}, \mu^h_{\xi'_2}, \mu^h_{\xi'_3}, ..., \mu^h_{\xi'_n}\right)$.*

According to the definitions of *HFDAWGA* and *HFDAOWGA* operators, the *HFDAWGA* operator can only contemplate the self-importance of each *HFE* and on the other hand, the *HFDAOWGA* operator indulges the ordered position importance of each *HFE*. In several real -world situations, it is required to consider simultaneously both these two categories. So, by combining the characteristics of these two operators, we propose *HFDAHGA* operator in the verge of Dombi–Archimedean operations as follows:

Definition 19. Suppose $\mu^h_{\xi_j}(j = 1, 2, 3, ..., n)$ be a collection of *HFEs* on *U*. Then *HFDAHGA* (for short) is a function $HFDAHGA : HFE^U \to HFE^U$ which is defined as follows:

$$HFDAHGA\left(\mu^h_{\xi_1}, \mu^h_{\xi_2}, \mu^h_{\xi_3}, ..., \mu^h_{\xi_n}\right) = \underset{j=1}{\overset{n}{\otimes}}{}_{DA} \left(\hat{w}_j \circ_{DA} \mu^h_{\dot{\xi}_{\sigma(j)}}\right)$$

where $\hat{w} = (\hat{w}_1, \hat{w}_2, \hat{w}_3, ..., \hat{w}_n)^T$ is the weight vector associated with the *HFDAHGA* operator such that $\hat{w}_j > 0$ and $\sum_{j=1}^{n} \hat{w}_j = 1$, $(\sigma(1), \sigma(2), \sigma(3), ..., \sigma(n))$ is a permutation of $(1, 2, 3, ..., n)$, $\mu^h_{\dot{\xi}_{\sigma(j)}}$ is the *j*th largest of the weighted *HFE* of $\mu^h_{\dot{\xi}_j}\left(\mu^h_{\dot{\xi}_j} = (n\hat{w}_j) \circ_{DA} \mu^h_{\xi_j}, j = 1, 2, 3, ..., n\right)$ and $w_j(j = 1, 2, 3, ..., n)$ is the weight of $\mu^h_{\xi_j}(j = 1, 2, 3, ..., n)$ with $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$.

Theorem 24. *The aggregated value $HFDAHGA\left(\mu^h_{\xi_1}, \mu^h_{\xi_2}, \mu^h_{\xi_3}, ..., \mu^h_{\xi_n}\right)$ is still a HFE and*

$$HFDAHGA\left(\mu^h_{\xi_1}, \mu^h_{\xi_2}, ..., \mu^h_{\xi_n}\right) = \bigcup_{\substack{\dot{\alpha}_{\sigma(1)} \in \mu^h_{\dot{\xi}_{\sigma(1)}}, \dot{\alpha}_{\sigma(2)} \in \mu^h_{\dot{\xi}_{\sigma(2)}}, \\ ...., \dot{\alpha}_{\sigma(n)} \in \mu^h_{\dot{\xi}_{\sigma(n)}}}} \left\{\left(1 + \left\{\theta\left(\sum_{j=1}^{n} \hat{w}_j \theta^{-1}\left(\nabla_k(\dot{\alpha}_{\sigma(j)})\right)\right)\right\}^{\frac{1}{k}}\right)^{-1}\right\}$$

where $\hat{w} = (\hat{w}_1, \hat{w}_2, \hat{w}_3, ..., \hat{w}_n)^T$ is the weight vector associated with the HFDAHGA operator such that $\hat{w}_j > 0$ and $\sum_{j=1}^{n} \hat{w}_j = 1$, $(\sigma(1), \sigma(2), \sigma(3), ..., \sigma(n))$ is a permutation of $(1, 2, 3, ..., n)$, $\mu^h_{\dot{\xi}_{\sigma(j)}}$ is the jth largest of the weighted HFE of $\mu^h_{\dot{\xi}_j}\left(\mu^h_{\dot{\xi}_j} = (n\hat{w}_j) \circ_{DA} \mu^h_{\xi_j}, j = 1, 2, 3, ..., n\right)$ and $w_j(j = 1, 2, 3, ..., n)$ is the weight of $\mu^h_{\xi_j}(j = 1, 2, 3, ..., n)$ with $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$.

In particular, if $w_j = \frac{1}{n}\ \forall j = 1, 2, 3, ..., n;$, then the operator *HFDAHGA* reduces to the *HFDAWGA* operator and if $\hat{w}_j = \frac{1}{n}\ \forall j = 1, 2, 3, ..., n;$, then the operator *HFDAHGA* reduces to the *HFDAOWGA* operator.

5. MULTI-ATTRIBUTE DECISION-MAKING

In this section, we shall utilize the *HFDAWAA* (or *HFDAWGA* or *HFDAOWAA* or *HFDAOWGA*) operator to develop an approach to solve *MADM* problems with *HF* information.

Let $U = \{A_1, A_2, A_3, \ldots\ldots, A_m\}$ be a set of alternatives, $A = \{C_1, C_2, C_3, \ldots\ldots, C_n\}$ be a set of attributes, and $w = \{w_1, w_2, w_3, \ldots\ldots, w_n\}$ be a set of weights (w_j is the weight of attribute c_j $(j = 1, 2, 3, \ldots\ldots, n)$) such that $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$. Suppose $\tilde{D} = [d_{ij}]_{m\times n}$ represents the *HF* decision matrix, where each d_{ij} is represented in the form of *HFE* $\mu^h_{\xi_{ij}}$ $(i = 1, 2, 3, \ldots\ldots, m; j = 1, 2, 3, \ldots\ldots, n)$.

Now, the proposed approach based on *HFDAWAA* (or *HFDAWGA* or *HFDAOWAA* or *HFDAOWGA*) operator to resolve the *MADM* problems with *HF* information involves the following steps:

ALGORITHM:

Step-1: Normalize the decision matrix.

In case of real-world decision-making situations, the attributes often divided into two categories, namely- benefit attributes and cost attributes. The first type is taken as profit index which renders positive impact on the decision-making result that means, the better evaluation result is directly proportional to the attribute values. On the other hand, the second one i.e., the cost attribute is treated as the cost index which is carrying the negative effect at the time of final assessment of the decision-making problem. In turn it means that the better evaluation result is inversely proportional to the attribute values. In this aspect, we shall keep in mind which is, it requires standardizing and transforming the cost type attributes into benefit type attributes in case of appearance of cost type attributes. If such scenario doesn't occur then we shall not entertain this step during this discussion.

The Normalized decision matrix is $\tilde{D} = [d_{ij}]_{m\times n}$, where

$$\tilde{d}_{ij} = \begin{cases} \mu^h_{\xi_{ij}} & \text{if } C_j \text{ is a benefit type attribute} \\ \left(\mu^h_{\xi_{ij}}\right)^c & \text{if } C_j \text{ is a cost type attribute} \end{cases}$$

Step-2: Compute the aggregation (overall preference) values $g_i (i = 1, 2, \ldots, m)$ of A_i $(i = 1, 2, \ldots, m)$ as;

$$g_i = HFDAWAA\left(\tilde{d}_{i1}, \tilde{d}_{i2}, \ldots\ldots, \tilde{d}_{in}\right) = \bigcup_{\alpha_{i1}\in\tilde{d}_{i1}, \alpha_{i2}\in\tilde{d}_{i2}, \ldots\ldots\ldots, \alpha_{in}\in\tilde{d}_{in}} \left\{1 - \left(1 + \left\{\psi\left(\sum_{j=1}^{n} w_j \psi^{-1}\left(\Delta_k(\alpha_{ij})\right)\right)\right\}^{\frac{1}{k}}\right)^{-1}\right\} \quad (i = 1, 2, \ldots, m)$$

or

$$g_i = HFDAWGA\left(\tilde{d}_{i1}, \tilde{d}_{i2}, \ldots\ldots, \tilde{d}_{in}\right) = \bigcup_{\alpha_{i1}\in\tilde{d}_{i1}, \alpha_{i2}\in\tilde{d}_{i2}, \ldots\ldots\ldots, \alpha_{in}\in\tilde{d}_{in}} \left\{\left(1 + \left\{\theta\left(\sum_{j=1}^{n} w_j \theta^{-1}\left(\nabla_k(\alpha_{ij})\right)\right)\right\}^{\frac{1}{k}}\right)^{-1}\right\} \quad (i = 1, 2, \ldots, m)$$

or

$$g_i = HFDAOWAA\left(\tilde{d}_{i1}, \tilde{d}_{i2}, \ldots\ldots, \tilde{d}_{in}\right) = \bigcup_{\substack{\alpha_{i\sigma(1)} \in \tilde{d}_{i\,\sigma(1)}, \\ \alpha_{i\sigma(2)} \in \tilde{d}_{i\,\sigma(2)}, \\ \ldots\ldots\ldots\ldots, \\ \alpha_{i\sigma(n)} \in \tilde{d}_{i\,\sigma(n)}}} \left\{1 - \left(1 + \left\{\psi\left(\sum_{j=1}^{n} w_j \psi^{-1}\left(\Delta_k(\alpha_{i\sigma(j)})\right)\right)\right\}^{\frac{1}{k}}\right)^{-1}\right\} \quad (i = 1, 2, \ldots, m)$$

or

$$g_i = HFDAOWGA\left(\tilde{d}_{i1}, \tilde{d}_{i2}, \ldots\ldots, \tilde{d}_{in}\right) = \bigcup_{\substack{\alpha_{i\sigma(1)} \in \tilde{d}_{i\,\sigma(1)}, \\ \alpha_{i\sigma(2)} \in \tilde{d}_{i\,\sigma(2)}, \\ \ldots\ldots\ldots\ldots, \\ \alpha_{i\sigma(n)} \in \tilde{d}_{i\,\sigma(n)}}} \left\{\left(1 + \left\{\theta\left(\sum_{j=1}^{n} w_j \theta^{-1}\left(\nabla_k(\alpha_{i\sigma(j)})\right)\right)\right\}^{\frac{1}{k}}\right)^{-1}\right\} \quad (i = 1, 2, \ldots, m)$$

Step-3: Calculate the score values of $g_i (i = 1, 2, \ldots, m)$ of $A_i (i = 1, 2, \ldots, m)$ based on Definition 3.

Step-4: Rank the alternatives A_i $(i = 1, 2, \ldots, m)$ using Definition 4.

6. NUMERICAL EXAMPLE

In this section, an illustrative example is provided to emphasize the application of the proposed method for "enterprise personnel selection."

6.1. Problem Description

It is quite natural that there should be a large impact of rising globalization and fast technological advancement upon the world markets which immediately start demanding companies to supply high-quality products as well as to provide quality service. To attain so, we need to make sure about the employability of reasonable personnel. Personnel selection is a procedure of opting individuals who fit in the qualifications needed to accomplish a defined job at best. It aims to decide the input quality of personnel and plays a crucial part in human resource management. Due to accretive competition in global markets, the organizations are being rebuked to focus considerably on personnel selection process. Formally, the personnel selection process and its recruitment get influenced by some major grounds such as changes in organizations, work, society, regulations, and marketing. A skillful decision is made by some enterprises to sort out the best candidate by taking advantage of stern and expensive selection procedures. This is required just because of the fact that during personnel selection process many individual attributes are taken into consideration which manifest the vagueness and imprecision and consequently the *HFS* theory appears to be a competent approach to come up with an infrastructure that includes dubious judgments intrinsic at the personnel selection procedure.

Now, let's think about a manufacturing company, which desires to appoint a sales manager against the unfilled post (adopted from Boran *et al.* [78]). We confer here that six candidates $A_i (i = 1, 2, ..., 6)$ get nominated for further evaluation after preelimination. In this regard, a decision has to make to someone upon consideration of four Attributes given below:

i. C_1: oral communication skill
ii. C_2: past experience
iii. C_3: general aptitude
iv. C_4: self-confidence.

The attribute weight is given by decision-maker as: $w = (0.35, 0.25, 0.25, 0.15)^T$. The decision-maker can start the evaluation process of the six candidates $A_i (i = 1, 2, ..., 6)$ subject to anonymous *HF* information under the aforementioned four attributes, which is presented in the following Table 1.

6.2. Problem Solution

To solve the *MADM* problem described above, we apply *HFDAWAA* operator for the evaluation of alternatives with *HF* information. The proposed method involves the following consecutive steps:

Step-1: Since all the given attributes $C_j (j = 1, 2, 3, 4)$ are benefit attribute, so, the attribute values of the alternatives $A_i (i = 1, 2, ..., 6)$ do not require normalization.

Step-2: Utilizing the decision information presented in Table 1 and using the *HFDAWAA* operator, we compute the aggregation (overall preference) values (taking $k = 2$) $g_i (i = 1, 2, ..., 6)$ of A_i $(i = 1, 2, ..., 6)$.

To do so, take $\psi(\beta) = \frac{\beta}{1-\beta}$, $\beta \in [0, 1)$; $\theta(\beta') = \frac{1-\beta'}{\beta'}, \beta' \in (0, 1]$. Then clearly, $\psi^{-1}(\beta) = \frac{\beta}{1+\beta}$ and $\theta^{-1}(\beta') = \frac{1}{1+\beta'}$.

Table 1 Hesitant fuzzy decision matrix $\tilde{D} = [d_{ij}]_{6\times4}$.

	C_1	C_2	C_3	C_4
A_1	<{0.2, 0.4}>	<{0.2, 0.6, 0.8}>	<{0.3, 0.6, 0.7}>	<{0.4, 0.6}>
A_2	<{0.4, 0.7, 0.9}>	<{0.2, 0.4}>	<{0.4, 0.6, 0.9}>	<{0.4, 0.6}>
A_3	<{0.3, 0.4}>	<{0.4, 0.8}>	<{0.3, 0.4, 0.7}>	<{0.2, 0.4, 0.7}>
A_4	<{0.3, 0.4, 0.6}>	<{0.1, 0.3}>	<{0.2, 0.4, 0.9}>	<{0.2, 0.3}>
A_5	<{0.4, 0.7}>	<{0.2, 0.3}>	<{0.3, 0.7, 0.8}>	<{0.3, 0.4, 0.8}>
A_6	<{0.7, 0.8, 0.9}>	<{0.4, 0.6, 0.7}>	<{0.4, 0.6}>	<{0.7, 0.8}>

To reduce the length of this article, alternative A_1 is provided as a representative example.

$$\begin{aligned} g_1 &= HFDAWAA\left(\tilde{d}_{11}, \tilde{d}_{12}, \tilde{d}_{13}, \tilde{d}_{14}\right) \\ &= \{0.250269475, 0.283480375, 0.314132359, 0.33138089, 0.325953704, 0.34093905, \\ &\quad 0.321805241, 0.337562841, 0.354272232, 0.364585279, 0.361269889, 0.370581059, \\ &\quad 0.338668082, 0.351429379, 0.365327952, 0.374084452, 0.371255031, 0.379235433, \\ &\quad 0.313749186, 0.334439886, 0.355154173, 0.367393055, 0.363500897, 0.374334358, \\ &\quad 0.360550882, 0.371870223, 0.384189522, 0.391938976, 0.38943631, 0.396491213, \\ &\quad 0.372675435, 0.382072224, 0.39250103, 0.399166213, 0.397004974, 0.403118578. \end{aligned}$$

Step-3: Calculate the score values $S(g_i)(i = 1, 2, ..., 6)$ of overall *HF* preference values $g_i(i = 1, 2, ..., 6)$ of the alternatives $A_i(i = 1, 2, ..., 6)$.

Here, $S(g_1) = 0.0181$, $S(g_2) = 0.0218$, $S(g_3) = 0.0193$, $S(g_4) = 0.0178$, $S(g_5) = 0.0207$, $S(g_6) = 0.0248$.

Step-4: Rank the alternatives $A_i(i = 1, 2, ..., 6)$ according to the score values $S(g_i)(i = 1, 2, ..., 6)$ of overall *HF* preference values $g_i(i = 1, 2, ..., 6)$.

Thus the ranking order of the alternatives is: $A_6 \succ A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4$ where the symbol "$\succ$" means "superior to." Hence, the most desirable person is A_6.

Now, if we apply the other proposed weighted *AO* namely, *HFDAWGA* or *HFDAOWAA* or *HFDAOWGA* instead of the operator *HFDAWAA*, then the above problem can be solved similarly as discussed and the final score values and the ranking order of the given alternatives are summarized in Table 2. We can conclude from Table 2 that although the ranking orders of the alternatives are slightly different; the most desirable alternative is still A_6 in all cases.

6.3. Analysis of the Effect of the Parameter *k* on Score Values

In order to diagnose the effect of the parameter "k," on the ranking of the alternatives, we utilize the operators *HFDAWAA*, *HFDAWGA*, *HFDAOWAA*, and *HFDAOWGA* for different values of k and we summarize the final score values and the ranking order of the given alternatives in Table 3 and Figure 1.

As we see from Table 3 and Figure 1, the ranking order of the alternatives for different values of k is mostly same and the best alternative is A_6 in all cases. Further analyzing Table 3, we observe that:

1. The score values of the alternatives $A_1, A_2, A_3, A_4, A_5, A_6$ obtained from *HFDAWAA* operator increase with the increasing value of k which varies from 1 to 8. The same is obeyed for the operator *HFDAOWAA*.
2. The score values of each of the alternatives A_2, A_3, A_5, A_6 received by the operator *HFADWGA* are found to decrease with the increase of k, starting from 1 to 8, whereas remarkably the score values of the alternative A_1 derived from the operator *HFDAWGA* are observed considerably to be decreased in the sub-intervals $1 \le k \le 4$ and increased in $4 < k \le 8$ respectively in the stipulated interval $1 \le k \le 8$. On the other hand, interestingly, the score values of the alternative A_4 determined by the operator *HFDAWAA* are found to occur decrease in the sub-interval $1 \le k \le 2$ and increase in $2 < k \le 8$ respectively. The *HFDAOWGA* operator complies with the same behavior.

Thus the above analysis makes it clear that the operators *HFDAWAA* and *HFDAOWAA* exhibit the similar response and on the other hand surprisingly the same behavior is found to observe for the case of *HFDAWGA* and *HFDAOWGA* operators while changing the values of the parameter k lying in [1,8] in our proposed *MADM* process. As a result, decision-makers can select any of the operators either from of the pair *HFDAWAA* operator-*HFDAOWAA* operator or from the pair *HFDAWGA* operator- *HFDAOWGA* operator to get the desired outcome.

6.4. Validity Test

To examine the legality and authenticity of the proposed method, some test criteria [79] are asserted below:

Table 2 Ranking order of alternatives (for $k = 2$).

Operators	Score Values						Ranking Order
	A_1	A_2	A_3	A_4	A_5	A_6	
HFDAWGA	0.0236	0.0281	0.0253	0.0209	0.0259	**0.0350**	$A_6 \succ A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4$
HFDAOWAA	0.0208	0.0221	0.0212	0.0187	0.0212	**0.0250**	$A_6 \succ A_2 \succ A_5 = A_3 \succ A_1 \succ A_4$
HFDAOWGA	0.0264	0.0292	0.0276	0.0215	0.0276	**0.0364**	$A_6 \succ A_2 \succ A_5 = A_3 \succ A_1 \succ A_4$

Table 3 Ranking order of alternatives for different values of k.

	Operators	Score Values						Ranking Order
		A_1	A_2	A_3	A_4	A_5	A_6	
$k=1$	*HFDAWAA*	0.0138	0.0179	0.0148	0.0139	0.0163	**0.0206**	$A_6 \succ A_2 \succ A_5 \succ A_3 \succ A_4 \succ A_1$
	HFDAWGA	0.0238	0.03	0.0258	0.0212	0.0263	**0.0363**	$A_6 \succ A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4$
	HFDAOWAA	0.0166	0.0181	0.0170	0.0155	0.0171	**0.0208**	$A_6 \succ A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4$
	HFDAOWGA	0.0266	0.0311	0.0280	0.0223	0.0283	**0.0377**	$A_6 \succ A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4$
$k=2$	*HFDAWAA*	0.0181	0.0218	0.0193	0.0178	0.0207	**0.0248**	$A_6 \succ A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4$
	HFDAWGA	0.0236	0.0281	0.0253	0.0209	0.0259	**0.0350**	$A_6 \succ A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4$
	HFDAOWAA	0.0208	0.0221	0.0212	0.0187	0.0212	**0.0250**	$A_6 \succ A_2 \succ A_5 = A_3 \succ A_1 \succ A_4$
	HFDAOWGA	0.0264	0.0292	0.0276	0.0215	0.0276	**0.0364**	$A_6 \succ A_2 \succ A_5 = A_3 \succ A_1 \succ A_4$
$k=3$	*HFDAWAA*	0.0198	0.0230	0.0208	0.0192	0.0221	**0.0260**	$A_6 \succ A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4$
	HFDAWGA	0.0235	0.0270	0.0249	0.0210	0.0255	**0.0339**	$A_6 \succ A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4$
	HFDAOWAA	0.0221	0.0233	0.0224	0.0197	0.0223	**0.0261**	$A_6 \succ A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4$
	HFDAOWGA	0.0262	0.0281	0.0270	0.0213	0.0269	**0.0352**	$A_6 \succ A_2 \succ A_3 \succ A_5 \succ A_1 \succ A_4$
$k=4$	*HFDAWAA*	0.0207	0.0236	0.0216	0.0199	0.0228	**0.0265**	$A_6 \succ A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4$
	HFDAWGA	0.0235	0.0265	0.0246	0.0213	0.0253	**0.0331**	$A_6 \succ A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4$
	HFDAOWAA	0.0227	0.0238	0.0229	0.0202	0.0229	**0.0266**	$A_6 \succ A_2 \succ A_5 = A_3 \succ A_1 \succ A_4$
	HFDAOWGA	0.0260	0.0274	0.0266	0.0214	0.0264	**0.0343**	$A_6 \succ A_2 \succ A_3 \succ A_5 \succ A_1 \succ A_4$
$k=5$	*HFDAWAA*	0.0214	0.0239	0.0221	0.0204	0.0232	**0.0267**	$A_6 \succ A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4$
	HFDAWGA	0.0237	0.0261	0.0245	0.0214	0.0252	**0.0325**	$A_6 \succ A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4$
	HFDAOWAA	0.0232	0.0241	0.0233	0.0205	0.0233	**0.0268**	$A_6 \succ A_2 \succ A_5 = A_3 \succ A_1 \succ A_4$
	HFDAOWGA	0.0258	0.0270	0.0262	0.0215	0.0261	**0.0336**	$A_6 \succ A_2 \succ A_3 \succ A_5 \succ A_1 \succ A_4$
$k=6$	*HFDAWAA*	0.0219	0.0242	0.0225	0.0207	0.0234	**0.0269**	$A_6 \succ A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4$
	HFDAWGA	0.0238	0.0259	0.0244	0.0216	0.0251	**0.0322**	$A_6 \succ A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4$
	HFDAOWAA	0.0235	0.0243	0.0235	0.0208	0.0235	**0.0270**	$A_6 \succ A_2 \succ A_5 = A_3 \succ A_1 \succ A_4$
	HFDAOWGA	0.0257	0.0267	0.0259	0.0216	0.0258	**0.0331**	$A_6 \succ A_2 \succ A_3 \succ A_5 \succ A_1 \succ A_4$
$k=7$	*HFDAWAA*	0.0223	0.0243	0.0228	0.0209	0.0237	**0.0270**	$A_6 \succ A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4$
	HFDAWGA	0.0239	0.0258	0.0244	0.0217	0.0251	**0.0319**	$A_6 \succ A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4$
	HFDAOWAA	0.0237	0.0244	0.0237	0.0209	0.0237	**0.0271**	$A_6 \succ A_2 \succ A_5 = A_3 \succ A_1 \succ A_4$
	HFDAOWGA	0.0256	0.0265	0.0258	0.0217	0.0257	**0.0327**	$A_6 \succ A_2 \succ A_3 \succ A_5 \succ A_1 \succ A_4$
$k=8$	*HFDAWAA*	0.0226	0.0244	0.0231	0.0211	0.0238	**0.0271**	$A_6 \succ A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4$
	HFDAWGA	0.0240	0.0257	0.0244	0.0217	0.0251	**0.0317**	$A_6 \succ A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4$
	HFDAOWAA	0.0238	0.0245	0.0238	0.0211	0.0238	**0.0272**	$A_6 \succ A_2 \succ A_5 = A_3 = A_1 \succ A_4$
	HFDAOWGA	0.0256	0.0263	0.0256	0.0218	0.0256	**0.0324**	$A_6 \succ A_2 \succ A_5 = A_3 = A_1 \succ A_4$

- **Test criterion 1**: If on replacing a nonoptimal alternative by another worse alternative without changing the relative importance of each decision-attribute, the indication of the best alternative remains the same, the *MCDM* method is effective.
- **Test criterion 2**: An effective *MCDM* method should follow the transitive property.
- **Test criterion 3:** We decompose the *MCDM* problem into smaller problems and apply the same MCDM method to these sub-problems for ranking the alternatives. If the overall ranking of the alternatives remains the same as the ranking of the original problem, then the *MCDM* method is effective.

Let us implement the above criteria on the proposed approach.

6.4.1. Test with criterion 1 (taking *k* = 2)

1. **Using *HFDAWAA* operator:** We replace the initial ratings of A_1 with A_2 and then execute the steps of the decision-making algorithm. The final score values $S(g_i)(i = 1, 2, ..., 6)$ of overall *HF* preference values $g_i(i = 1, 2, ..., 6)$ of the alternatives $A_i(i = 1, 2, ..., 6)$ are computed as:

$$S(g_1) = 0.0244, S(g_2) = 0.0226, S(g_3) = 0.0231, S(g_4) = 0.0211, S(g_5) = 0.0238, S(g_6) = 0.0271.$$

   Hence A_6 is the best alternative. Therefore, "*test criterion 1*" is validated for *HFDAWAA* operator.

2. **Using *HFDAWGA* operator:** We replace the initial ratings of A_1 with A_2 and then execute the steps of the decision-making algorithm. The final score values $S(g_i)(i = 1, 2, ..., 6)$ of overall *HF* preference values $g_i(i = 1, 2, ..., 6)$ of the alternatives $A_i(i = 1, 2, ..., 6)$ are computed as:

$$S(g_1) = 0.0257, S(g_2) = 0.0240, S(g_3) = 0.0244, S(g_4) = 0.0217, S(g_5) = 0.0250, S(g_6) = 0.0317.$$

   Hence A_6 is still the best alternative. Therefore, "*test criterion 1*" is validated for *HFDAWGA* operator.

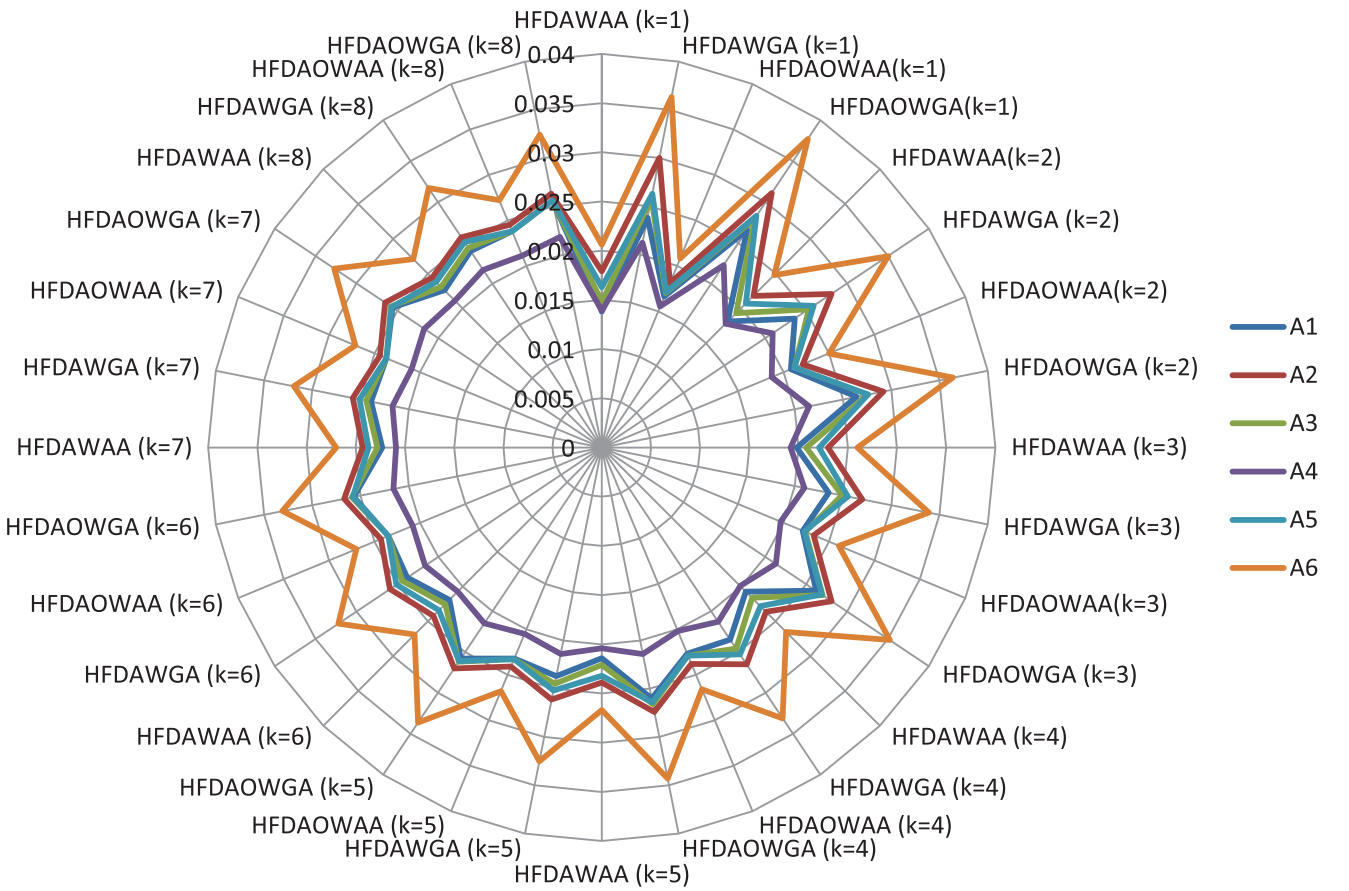

Figure 1 Scores of the alternatives based on different values of "*k*."

3. **Using *HFDAOWAA* operator:** We replace the initial ratings of A_1 with A_2 and then execute the steps of the decision-making algorithm. The final score values $S(g_i)(i = 1, 2, ..., 6)$ of overall *HF* preference values $g_i(i = 1, 2, ..., 6)$ of the alternatives $A_i(i = 1, 2, ..., 6)$ are computed as:

$$S(g_1) = 0.0181, S(g_2) = 0.0166, S(g_3) = 0.0170, S(g_4) = 0.0155, S(g_5) = 0.0171, S(g_6) = 0.0208.$$

Hence A_6 is the best alternative. Therefore, "*test criterion 1*" is validated for *HFDAOWAA* operator.

4. **Using *HFDAOWGA* operator:** We replace the initial ratings of A_1 with A_2 and then execute the steps of the decision-making algorithm. The final score values $S(g_i)(i = 1, 2, ..., 6)$ of overall *HF* preference values $g_i(i = 1, 2, ..., 6)$ of the alternatives $A_i(i = 1, 2, ..., 6)$ are computed as:

$$S(g_1) = 0.0311, S(g_2) = 0.0266, S(g_3) = 0.0280, S(g_4) = 0.0223, S(g_5) = 0.0283, S(g_6) = 0.0377.$$

Hence A_6 is the best alternative. Therefore, "*test criterion 1*" is validated for *HFDAOWGA* operator.

6.4.2. Test with criteria 2 and 3 (taking $k = 2$)

Assume that the given decision-making problem is split into four sub-problems by taking three groups of alternatives, namely-$\{A_1, A_2, A_3, A_5\}$, $\{A_1, A_3, A_4, A_6\}$ and $\{A_2, A_4, A_5, A_6\}$.

Now for each sub-problem, all the steps of proposed algorithms are executed. Then

1. utilizing *HFDAWAA* operator, the ranking orders are: $A_2 \succ A_5 \succ A_3 \succ A_1, A_6 \succ A_3 \succ A_1 \succ A_4, A_6 \succ A_2 \succ A_5 \succ A_4$ and hence overall ranking order is: $A_6 \succ A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4$ which validates the test criteria 2 and 3.
2. utilizing *HFDAWGA* operator, the ranking orders are: $A_2 \succ A_5 \succ A_3 \succ A_1, A_6 \succ A_3 \succ A_1 \succ A_4, A_6 \succ A_2 \succ A_5 \succ A_4$ and hence overall ranking order is: $A_6 \succ A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4$ which validates the test criteria 2 and 3.

3. utilizing *HFDAOWAA* operator, the ranking orders are: $A_2 \succ A_5 = A_3 \succ A_1, A_6 \succ A_3 \succ A_1 \succ A_4, A_6 \succ A_2 \succ A_5 \succ A_4$ and hence overall ranking order is: $A_6 \succ A_2 \succ A_5 = A_3 \succ A_1 \succ A_4$ which validates the test criteria 2 and 3.

4. utilizing *HFDAOWGA* operator, the ranking orders are: $A_2 \succ A_5 = A_3 \succ A_1, A_6 \succ A_3 \succ A_1 \succ A_4, A_6 \succ A_2 \succ A_5 \succ A_4$ and hence overall ranking order is: $A_6 \succ A_2 \succ A_5 = A_3 \succ A_1 \succ A_4$ which validates the test criteria 2 and 3.

7. COMPARATIVE STUDY

In pursuance of performance comparison of the eloquent method developed by us discussed here with some existing *MADM* methods under hesitant fuzzy environment, we have conducted an analysis with some of the existing methods namely- Yu's method [55] using Hesitant fuzzy Einstein weighted arithmetic aggregation operator (*HFEWA*), Hesitant fuzzy Einstein weighted geometric aggregation operator (*HFEWG*), Hesitant fuzzy Einstein ordered weighted arithmetic aggregation operator (*HFEOWA*), and Hesitant fuzzy Einstein ordered weighted geometric aggregation operator (*HFEOWG*) operators [55]; Xia and Xu's method [43] using Hesitant fuzzy weighted arithmetic aggregation operator (*HFWA*) and Hesitant fuzzy weighted geometric aggregation operator (*HFWG*) operators; Qin *et al.*'s method [59] by using Hesitant fuzzy Frank weighted averaging operator (*HFFWA*) and Hesitant fuzzy Frank weighted geometric operator (*HFFWG*) operators; Tan *et al.*'s method [56] using Hesitant fuzzy Hamachar weighted arithmetic aggregation operator (*HFHWA*), Hesitant fuzzy Hamachar weighted geometric aggregation operator (*HFHWG*), Hesitant fuzzy Hamachar ordered weighted arithmetic aggregation operator (*HFHOWA*), and Hesitant fuzzy Hamachar ordered weighted geometric aggregation operator (*HFHOWG*) operators; and He's method [62] using Hesitant fuzzy Dombi weighted arithmetic aggregation operator (*HFDWA*), Hesitant fuzzy Dombi weighted geometric aggregation operator (*HFDWG*), Hesitant fuzzy Dombi ordered weighted arithmetic aggregation operator (*HFDOWA*), and Hesitant fuzzy Dombi ordered weighted geometric aggregation operator (*HFDOWG*) operators. We have utilized these operators in step-2 of the proposed algorithm. The final score values of the alternatives and the ranking order are summarized in a tabular form, numbered by 4. It is very much translucent from Table 4 that despite the appearance of slight difference occurs to the respective ranking order of the alternatives; the best i.e., most desirable alternative is absolutely same as found in the existing methods [43,55,56,59,62].

Alongside the above comparative study (Table 4 and Figure 2), we discuss also some characteristic comparison of our proposed *MADM* approach and the decision-making methods suggested by Xia and Xu [43], Yu [55], Tan *et al.* [56], Qin *et al.* [59] and He [62], which are summarized in Table 5.

Table 4 Comparative study with existing approaches.

Operators	Score Values						Ranking Order
	A_1	A_2	A_3	A_4	A_5	A_6	
HFDAWAA (proposed)	0.0181	0.0218	0.0193	0.0178	0.0207	**0.0248**	$A_6 \succ A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4$
HFDAWGA (proposed)	0.0236	0.0281	0.0253	0.0209	0.0259	**0.0350**	$A_6 \succ A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4$
HFDAOWAA (proposed)	0.0208	0.0221	0.0212	0.0187	0.0212	**0.0250**	$A_6 \succ A_2 \succ A_5 = A_3 \succ A_1 \succ A_4$
HFDAOWGA (proposed)	0.0264	0.0292	0.0276	0.0215	0.0276	**0.0364**	$A_6 \succ A_2 \succ A_5 = A_3 \succ A_1 \succ A_4$
HFHWA ($\gamma = 2$) [56]	0.0246	0.0319	0.0266	0.0232	0.0274	**0.0377**	$A_6 \succ A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4$
HFHWG ($\gamma = 2$) [56]	0.0232	0.0291	0.0252	0.0199	0.0254	**0.0359**	$A_6 \succ A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4$
HFHOWA ($\gamma = 2$) [56]	0.0272	0.0327	0.0285	0.0247	0.0291	**0.0389**	$A_6 \succ A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4$
HFHOWG ($\gamma = 2$) [56]	0.0261	0.0305	0.0276	0.0210	0.0276	**0.0373**	$A_6 \succ A_2 \succ A_5 = A_3 \succ A_1 \succ A_4$
HFDWA ($k = 2$) [62]	0.0284	0.0350	0.0296	0.0304	0.0304	**0.0421**	$A_6 \succ A_2 \succ A_4 \succ A_5 \succ A_3 \succ A_1$
HFDWG ($k = 2$) [62]	0.0195	0.0217	0.0211	0.0141	0.0185	**0.0316**	$A_6 \succ A_2 \succ A_3 \succ A_1 \succ A_5 \succ A_4$
HFDOWA ($k = 2$) [62]	0.0301	0.0353	0.0305	0.0308	0.0311	**0.0425**	$A_6 \succ A_2 \succ A_5 \succ A_4 \succ A_3 \succ A_1$
HFDOWG ($k = 2$) [62]	0.0218	0.0235	0.0230	0.0146	0.0201	**0.0328**	$A_6 \succ A_2 \succ A_3 \succ A_1 \succ A_5 \succ A_4$
HFEWA [55]	0.0246	0.0319	0.0266	0.0232	0.0274	**0.0377**	$A_6 \succ A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4$
HFEWG [55]	0.0232	0.0291	0.0252	0.0199	0.0254	**0.0359**	$A_6 \succ A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4$
HFEOWA [55]	0.0272	0.0327	0.0285	0.0247	0.0291	**0.0389**	$A_6 \succ A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4$
HFEOWG [55]	0.0261	0.0305	0.0276	0.0210	0.0276	**0.0373**	$A_6 \succ A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4$
HFWA [43]	0.0249	0.0320	0.0268	0.0238	0.0277	**0.0380**	$A_6 \succ A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4$
HFWG [43]	0.0229	0.0285	0.0249	0.0193	0.0249	**0.0355**	$A_6 \succ A_2 \succ A_5 = A_3 \succ A_1 \succ A_4$
HFFWA [59]	0.0248	0.0321	0.0267	0.0235	0.0275	**0.0378**	$A_6 \succ A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4$
HFFWG [59]	0.0231	0.0288	0.0251	0.0196	0.0251	**0.0357**	$A_6 \succ A_2 \succ A_5 = A_3 \succ A_1 \succ A_4$

HFDAWAA, Hesitant fuzzy Dombi–Archimedean weighted arithmetic aggregation operator; *HFDAWGA*, Hesitant fuzzy Dombi–Archimedean weighted geometric aggregation operator; *HFDAOWAA*, Hesitant fuzzy Dombi–Archimedean ordered weighted arithmetic aggregation operator; *HFDAOWGA*, Hesitant fuzzy Dombi–Archimedean ordered weighted geometric aggregation operator; *HFHWA*, Hesitant fuzzy Hamachar weighted arithmetic aggregation operator; *HFHWG*, Hesitant fuzzy Hamachar weighted geometric aggregation operator; *HFHOWA*, Hesitant fuzzy Hamachar ordered weighted arithmetic aggregation operator; *HFHOWG*, Hesitant fuzzy Hamachar ordered weighted geometric aggregation operator; *HFDWA*, Hesitant fuzzy Dombi weighted arithmetic aggregation operator; *HFDWG*, Hesitant fuzzy Dombi weighted geometric aggregation operator; *HFDOWA*, Hesitant fuzzy Dombi ordered weighted arithmetic aggregation operator; *HFDOWG*, Hesitant fuzzy Dombi ordered weighted geometric aggregation operator; *HFEWA*, Hesitant fuzzy Einstein weighted arithmetic aggregation operator; *HFEWG*, Hesitant fuzzy Einstein weighted geometric aggregation operator; *HFEOWA*, Hesitant fuzzy Einstein ordered weighted arithmetic aggregation operator; *HFEOWG*, Hesitant fuzzy Einstein ordered weighted geometric aggregation operator; *HFWA*, Hesitant fuzzy weighted arithmetic aggregation operator; *HFWG*, Hesitant fuzzy weighted geometric aggregation operator; *HFFWA*, Hesitant fuzzy Frank weighted averaging operator; *HFFWG*, Hesitant fuzzy Frank weighted geometric operator.

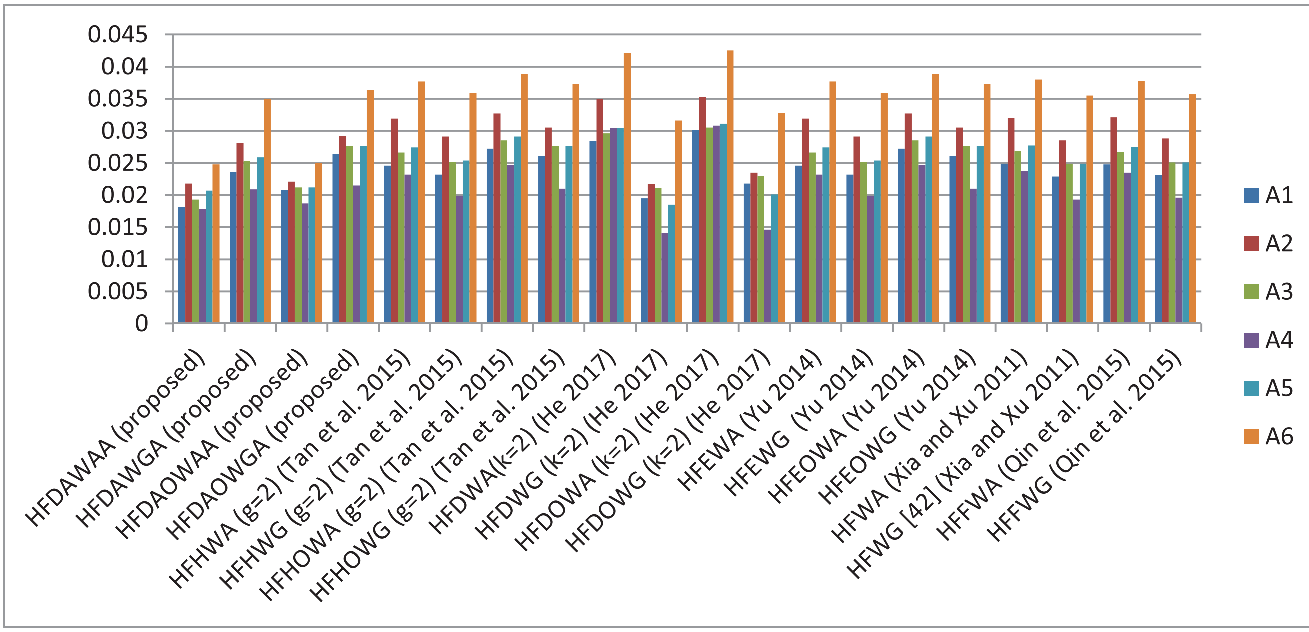

Figure 2 Comparative study.

Table 5 Characteristic comparisons.

Methods	Whether Handle *MADM* Problems?	Whether Aggregation Operators Are in Generalized Form?	Whether the Aggregation Operators Are Flexible in Nature?	Whether the Method Reduces Computational Complexity?	Whether Compatible with Risk Preferences?
Xia and Xu [43]	Yes	No	No	Yes	No
Yu [55]	Yes	No	No	Yes	No
Tan *et al.* [56]	Yes	Yes	No	No	Yes
He [62]	Yes	No	No	Yes	Yes
Proposed	**Yes**	**Yes**	**Yes**	**Yes**	**Yes**

Thus, our proposed approach leads to the following advantageous facts:

1. The proposed *AO*s are obtained through the coalescence of the Dombi and Archimedean operations under *HF* environment and hence the raised *MADM* approach can be considered as one such blooming works because it develops a new flexible measure for decision-makers to choose the appropriate functions and parameters in accordance with the risk preferences whereas the existing methods [43,55,56,59,62] envisage the *HF* information in nonappearance of the simultaneous act of flexibility of functions and parameters.
2. Our proposed *AO*s include a parameter "*k*" which can try on the aggregate value based on the real decision needs. Thus, our proposed operators reveal themselves with higher generality and flexibility.

8. CONCLUSION

Keeping in mind the sensitive issue of growing perplexity and dubiousness of real-world decision-making problems, at the time of the formation of *MADM*, the attribute value is represented suitably as a *HFE*. The existing many information fusion methods developed so far for aggregating *HF* information, are restricted to algebraic t-norm and t-conorm, Einstein t-norm and t-conorm, Hammacher t-norm and t-conorm, and even we observe the inflexibility in the process of aggregation. Motivated by the idea of Dombi and Archimedean operations, in this paper, we have introduced some new operations between *HFE*. The prominent characteristics of these proposed operators are studied. Furthermore, we have developed some *HF AO*s based on the proposed operations, such as *HFDAWAA, HFDAOWAA, HFDAHAA, HFDAWGA, HFDAOWGA*, and *HFDAHGA* operators. Some essential properties such as idempotency, boundedness, shift invariance, monotonicity etc., of the proposed *AO*s are discussed in detail. Next, a procedure of *MADM* based on the proposed operators is presented under a *HF* environment. At the end, we fetch a practical example of a human resource selection to be comfortable with the decision steps in the proposed method. The result demonstrates the practicality and effectiveness of the new method.

CONFLICTS OF INTEREST

The authors declare that they don't have any conflict of interest.

AUTHORS' CONTRIBUTIONS

Abhijit Saha: Conceptualization, methodology, validation, writing original draft; **Peide Liu, Samarjit Kar, Debjit Dutta:** Formal analysis, writing original draft, review and editing.

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