

# Fusion of heterogeneous incomplete hesitant preference relations in group decision making

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## Abstract

In this paper, we focus on the fusion of heterogeneous incomplete hesitant preference relations (including hesitant fuzzy preference relations and hesitant multiplicative preference relations) under group decision making settings. First, some simple formulae are developed to derive a priority weight vector from an incomplete hesitant fuzzy preference relation or an incomplete hesitant multiplicative preference relation based on the logarithmic least squares method. Based on the priority weight vector, an induced fuzzy or multiplicative preference relation can be derived for an incomplete hesitant preference relation. Moreover, the consistency indices of hesitant fuzzy preference relations and hesitant multiplicative preference relations are defined. Afterwards, an approach to group decision making based on incomplete hesitant fuzzy preference relations and incomplete hesitant multiplicative preference relations is developed to deal with group decision making problems with multiple decision organizations. Finally, three examples are used to illustrate the proposed approach.

*Keywords:* preference relation; hesitant fuzzy set; group decision making; logarithmic least squares

## 1. Introduction

Group decision making (GDM) is a common activity occurring in human beings' life, which usually needs a group of decision makers to achieve the final decision for a specific decision making problem<sup>1</sup>. For actual decision making problems, preference relation is a useful tool for decision makers to express their preferences over alternatives through pairwise comparisons<sup>2,3</sup>. In recent years, a number of methods have been proposed to deal with GDM problems with preference relations<sup>4,5,6,7,8</sup>.

The fuzzy preference relation is one of the most widely used preference relations for actual decision making problems. Given a set of alternatives  $X =$

$\{x_1, x_2, \dots, x_n\}$ , the degree to which the alternative  $x_i$  is preferred to  $x_j$  is characterized by a membership function  $\mu_P(x_i, x_j) \in [0, 1]$ , then the fuzzy preference relation over the alternatives is represented by a matrix  $P = (p_{ij})_{n \times n}$ , where  $p_{ij} = \mu_P(x_i, x_j)$ . To avoid misleading solutions, two important issues are usually considered for GDM with fuzzy preference relations: (1) Individual consistency which is used to measure the agreement degree among the preference values provided by an individual decision maker<sup>9</sup>; (2) Group consensus which is used to measure the agreement degree among different decision makers' judgments<sup>10</sup>. Up to now, different models for checking and improving the consistency and consensus for fuzzy preference relations have

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been developed. For instance, Herrera-Viedma et al.<sup>11</sup> put forward a new characterization of the additive consistency property of fuzzy preference relations and utilized the property to construct consistent fuzzy preference relations from a set of  $n - 1$  preference values. Ma et al.<sup>12</sup> proposed an analysis method to check and improve the consistency of a fuzzy preference relation based on graph theory. Herrera-Viedma et al.<sup>13</sup> developed a consensus model for GDM with incomplete fuzzy preference relations by considering both the consensus and consistency measures. Chiclana et al.<sup>9</sup> defined the cardinal consistency for a fuzzy preference relation and showed that cardinal consistency with the conjunctive representable cross ratio uninorm is equivalent to Tanino's multiplicative transitivity property<sup>14</sup>. Moreover, they concluded that multiplicative transitivity is the most appropriate property for modeling cardinal consistency of a fuzzy preference relation. Wu and Xu<sup>15</sup> developed a concise consensus support model for GDM based on deviation measures of fuzzy preference relations. Xia et al.<sup>16</sup> proposed some consistency and consensus models for GDM with fuzzy preference relations based on multiplicative consistency.

Due to the lack of knowledge and decision makers' limited expertise, preference relations provided by a decision maker are sometimes incomplete. Thus, how to manage incomplete information for GDM with preference relations is also a quite important issue<sup>13,17,18,19,20,21</sup>. As surveyed by Ureña et al.<sup>22</sup>, the technique for dealing with incomplete fuzzy preference relations can be divided into two categories. One is iterative approaches which estimate the missing elements based on the consistency property of fuzzy preference relations<sup>23,24</sup>, and the other is to estimate the missing preference values or to directly derive the priority weight vector of alternatives without previously completing the preference relations through establishing some optimization models<sup>17,18,25,26,27,28</sup>. For a detailed review about these techniques, readers can refer to Ref. 22.

To overcome the limitations of fuzzy sets theory when dealing with imprecise and vague information with different sources of vagueness simultaneously, some extensions of fuzzy sets were pro-

posed by scholars, such as intuitionistic fuzzy sets<sup>29</sup>, type-2 fuzzy sets<sup>30</sup>, interval-valued fuzzy sets<sup>31</sup> and fuzzy multisets<sup>32</sup>. Considering the common difficulty that often appears when the membership degree of an element must be established and the difficulty is not because of an error margin (as in intuitionistic fuzzy set) or due to some possibility distribution (as in type-2 fuzzy sets)<sup>33,34</sup>, Torra and Narukawa<sup>35,36</sup> defined the hesitant fuzzy set, which allows the membership degree of an object to have a set of possible values. The hesitant fuzzy set may be helpful in the following two cases: (1) When providing assessments over alternatives, a decision maker may have some hesitancy due to the uncertainty of the environment and lack of knowledge. In this case, the hesitant fuzzy set can be used to represent the possible assessment values of the decision maker. (2) For a GDM problem conducted within organizations, each member of an organization may be required to provide assessments about the alternatives, then the hesitant fuzzy set can be used to represent the evaluation information of the organization, especially when anonymity is required to protect decision makers' privacy or avoid influencing each other<sup>37</sup>.

Since the appearance of hesitant fuzzy sets, much work has been conducted on the aggregation of hesitant fuzzy elements<sup>37</sup>, distance and similarity measures for hesitant fuzzy sets<sup>38,39</sup> and decision making methods based on hesitant fuzzy sets<sup>40</sup>. In recent years, research on hesitant preference relations has also received more and more attentions<sup>33</sup>. For instance, Zhu and Xu<sup>41</sup> proposed the concept of hesitant fuzzy preference relation (HFPR) and developed a regression method to transform HFPRs to fuzzy preference relations with the highest consistency level. Moreover, Liao et al.<sup>42</sup> investigated the multiplicative consistency of an HFPR and developed some algorithms to improve the group consensus. Based on hesitant fuzzy linguistic term sets<sup>43</sup>, Zhu and Xu<sup>44</sup> presented the concept of hesitant fuzzy linguistic preference relation and provided two optimization methods to improve the consistency of a hesitant fuzzy linguistic preference relation. Chen et al.<sup>45</sup> extended the hesitant fuzzy set to interval-valued hesitant fuzzy set and defined

the interval-valued HFPR, based on which they developed an approach to GDM with interval-valued HFPRs. Zhu et al.<sup>46</sup> proposed some approaches to deriving the ranking of alternatives based on HFPRs using the  $\alpha$ -normalization and the  $\beta$ -normalization. Recently, Xia and Xu<sup>47</sup> defined the concept of hesitant multiplicative elements and hesitant multiplicative preference relation (HMPR) and developed some approaches to managing hesitant information in GDM problems with fuzzy and multiplicative preference relations. For GDM problems with HMPRs, Zhang and Wu<sup>48</sup> defined the individual consistency index and group consensus index, and then developed a consistency and consensus-based decision support model. Based on the concept of HMPR, Zhu and Xu<sup>49</sup> proposed the analytic hierarchy process-hesitant GDM approach to deal with decision making problem with hierarchal structures.

The previous research has significantly advanced the research of GDM based on hesitant preference relations. However, there are still some challenges which need to be tackled: (1) For some GDM problems with high uncertainty, decision makers may have limited knowledge about the problems to be solved, and incomplete hesitant preference relations may be provided. However, there is little research focusing on managing incomplete hesitant preference relations. (2) Most traditional approaches derive a ranking of alternatives from a hesitant preference relation based on the aggregation operators of hesitant fuzzy elements and hesitant multiplicative elements, which sometimes are quite complex, as the dimension of the derived hesitant fuzzy/multiplicative elements may increase significantly. (3) Due to the differences of culture and education backgrounds, heterogeneous hesitant preference relations may be provided by different decision makers or organizations. However, there is no study that is devoted to the fusion of heterogeneous (incomplete) hesitant preference relations.<sup>†</sup>

The main contribution of this paper is to tackle the three challenges mentioned above, which is summarized as below. First, motivated by the logarithmic

least squares method<sup>50,51,52</sup>, this paper develops some formulae to derive priority weights from an incomplete HFPR or an incomplete HMPR, and then an approach to GDM based on heterogeneous incomplete hesitant preference relations is proposed, which can be used to deal with GDM problems with multiple decision organizations. The proposed GDM approach avoids the complex operations of hesitant fuzzy elements or hesitant multiplicative elements, and allows decision makers to express their preference information flexibly.

The rest of this paper is organized as follows. In section 2, some preliminaries related to hesitant preference relations are presented. Afterwards, section 3 develops some simple formulae to derive priority weights from an incomplete hesitant preference relation. In section 4, an approach to GDM with incomplete HFPRs and incomplete HMPRs is proposed. Three examples are provided to illustrate the proposed approach in section 5, and section 6 discusses the characteristics of the proposed approach. Finally, some conclusions are drawn in section 7.

## 2. Preliminaries

In this section, some preliminaries related to HFPRs and HMPRs are presented.

**Definition 1.** (Saaty 1977)<sup>2</sup> Let  $X = \{x_1, x_2, \dots, x_n\}$  be a fixed set. A multiplicative preference relation over  $X$  is represented by a pairwise matrix  $C = (c_{ij})_{n \times n}$  with  $c_{ij}c_{ji} = 1$ ,  $1/9 \leq c_{ij} \leq 9$ , where  $c_{ij}$  denotes the ratio degree to which the alternative  $x_i$  is preferred to  $x_j$ ,  $i, j \in I = \{1, 2, \dots, n\}$ .

**Definition 2.** (Orlovsky 1978)<sup>3</sup> Let  $X = \{x_1, x_2, \dots, x_n\}$  be a fixed set. A fuzzy preference relation over  $X$  is represented by a pairwise matrix  $D = (d_{ij})_{n \times n}$  with  $d_{ij} + d_{ji} = 1$ ,  $0 \leq d_{ij} \leq 1$ , where  $d_{ij}$  denotes the degree to which the alternative  $x_i$  is preferred to  $x_j$ ,  $i, j \in I$ .

**Definition 3.** (Tanino 1984)<sup>14</sup> Let  $D = (d_{ij})_{n \times n}$  be a fuzzy preference relation, then  $D$  is called a mul-

<sup>†</sup> In this paper, when referring to an incomplete hesitant preference relation, it means an incomplete HFPR or an incomplete HMPR, while heterogeneous incomplete hesitant preference relations mean different types of incomplete hesitant preference relations, i.e. HFPRs and HMPRs.

tiplicatively consistent fuzzy preference relation if  $d_{ij} > 0$ ,  $d_{ij}d_{jk}d_{ki} = d_{ji}d_{kj}d_{ik}$  for all  $i, j, k \in I$ , and such a fuzzy preference relation can also be given by  $d_{ij} = \frac{w_i}{w_i + w_j}$ ,  $i, j \in I$ , where  $w = (w_1, w_2, \dots, w_n)^T$  is the priority weight vector of  $D$ , and  $w_i > 0$ ,  $i \in I$ ,  $\sum_{i=1}^n w_i = 1$ .

**Remark 1.** Usually, there are two different scales for a fuzzy preference relation  $D$ , including the 0-1 scale (i.e.  $d_{ij} \in [0, 1]$ ,  $i, j \in I$ ) and the 0.1-0.9 scale (i.e.  $d_{ij} \in [0.1, 0.9]$ ,  $i, j \in I$ ). According to Definition 3,  $d_{ij} > 0$  is usually assumed when the multiplicative consistency of fuzzy preference relations is used. To facilitate the transformation between a fuzzy preference relation and a multiplicative preference relation, the 0.1-0.9 scale is applied in this paper when the multiplicative consistency is used for a fuzzy preference relation.

To allow the membership degree of an object to have a set of possible values, Torra and Narukawa<sup>35,36</sup> defined the hesitant fuzzy set as below.

**Definition 4.** Let  $X$  be a fixed set, a hesitant fuzzy set on  $X$  in terms of a function that when applied to  $X$  returns a subset of  $[0, 1]$ <sup>36</sup>. To be easily understood, Xia and Xu<sup>37</sup> expressed the hesitant fuzzy set by a mathematical symbol  $E = \{\langle x, h_E(x) \rangle | x \in X\}$ , where  $h_E(x)$  is a set of some values in  $[0, 1]$ , denoting all the possible membership degrees of the element  $x \in X$  to the set  $E$ . For convenience,  $h = h_E(x)$  is called a hesitant fuzzy element.

**Definition 5.** (Zhu & Xu 2013)<sup>41</sup> Let  $X = \{x_1, x_2, \dots, x_n\}$  be a fixed set, where  $x_i$  denotes the  $i$ th alternative,  $i \in I$ , then an HFPR  $A$  over  $X$  is denoted by a matrix  $A = (a_{ij})_{n \times n}$ , where  $a_{ij} = \{a_{ij}^s | s = 1, 2, \dots, l_{ij}\}$  is a hesitant fuzzy element indicating all the possible degrees to which the alternative  $x_i$  is preferred to  $x_j$ . Moreover,  $a_{ij}$  should satisfy the following conditions:

$$a_{ij}^{\sigma(s)} + a_{ji}^{\sigma(l_{ij}-s+1)} = 1, a_{ij}^s \in [0, 1],$$

$$a_{ii} = \{0.5\}, l_{ij} = l_{ji}, s \in S_{ij} = \{1, 2, \dots, l_{ij}\}, i, j \in I,$$
(1)

where  $\{\sigma(1), \sigma(2), \dots, \sigma(l_{ij})\}$  is a permutation of  $\{1, 2, \dots, l_{ij}\}$  such that  $a_{ij}^{\sigma(s-1)} < a_{ij}^{\sigma(s)}$ ,  $s =$

$2, 3, \dots, l_{ij}$ , i.e.  $a_{ij}^{\sigma(s)}$  is the  $s$ th smallest element in  $a_{ij}$ ,  $i, j \in I$ .

Based on the multiplicative consistency of fuzzy preference relations (i.e. Definition 3), a multiplicative consistent HFPR is defined as follows.

**Definition 6.** (Zhu, Xu & Xu 2014)<sup>46</sup> Let  $A = (a_{ij})_{n \times n}$  be an HFPR, where  $a_{ij} = \{a_{ij}^s | s = 1, 2, \dots, l_{ij}\}$  is a hesitant fuzzy element,  $i, j \in I$ . If the following convex feasible region

$$\Theta = \left\{ w = (w_1, w_2, \dots, w_n)^T \mid \frac{w_i}{w_i + w_j} \in a_{ij}, \right.$$

$$\left. w_i > 0, i, j \in I, \sum_{i=1}^n w_i = 1 \right\}$$
(2)

is non-empty, then  $A$  is a multiplicative consistent HFPR.

**Remark 2.** According to Remark 1, it is assumed that  $a_{ij}^s \in [0.1, 0.9]$  when the multiplicative consistency is used for an HFPR.

**Definition 7.** (Xia & Xu 2013)<sup>47</sup> Let  $X$  be a fixed set, a hesitant multiplicative set on  $X$  is defined as  $F = \{\langle x, t_F(x) \rangle | x \in X\}$ , where  $t_F(x)$  denotes the set of possible multiplicative preference information of the element  $x \in X$  to the set  $F$  and  $t_F(x) \subset [1/9, 9]$ ,  $\forall x \in X$ . For convenience,  $t = t_F(x)$  is called a hesitant multiplicative element.

**Definition 8.** (Xia & Xu 2013)<sup>47</sup> Let  $X = \{x_1, x_2, \dots, x_n\}$  be a fixed set, where  $x_i$  denotes the  $i$ th alternative,  $i \in I$ , then an HMPR  $B$  over  $X$  is denoted by a matrix  $B = (b_{ij})_{n \times n}$ , where  $b_{ij} = \{b_{ij}^s | s = 1, 2, \dots, l_{ij}\}$  is a hesitant multiplicative element indicating all the possible ratio degrees to which the alternative  $x_i$  is preferred to  $x_j$ . Moreover,  $b_{ij}$  should satisfy the following conditions:

$$b_{ij}^{\delta(s)} b_{ji}^{\delta(l_{ij}-s+1)} = 1, b_{ij}^s \in [1/9, 9],$$

$$b_{ii} = \{1\}, l_{ij} = l_{ji}, s \in S_{ij} = \{1, 2, \dots, l_{ij}\}, i, j \in I,$$
(3)

where  $\{\delta(1), \delta(2), \dots, \delta(l_{ij})\}$  is a permutation of  $\{1, 2, \dots, l_{ij}\}$  such that  $b_{ij}^{\delta(s-1)} < b_{ij}^{\delta(s)}$ ,  $s = 2, 3, \dots, l_{ij}$ , i.e.  $b_{ij}^{\delta(s)}$  is the  $s$ th smallest element in  $b_{ij}$ ,  $i, j \in I$ .

As a multiplicative preference relation  $C = (c_{ij})_{n \times n}$  can be transformed into a fuzzy preference relation  $D = (d_{ij})_{n \times n}$  by<sup>53,54</sup>

$$d_{ij} = \frac{c_{ij}}{1 + c_{ij}}, i, j \in I, \quad (4)$$

the following theorems are introduced to achieve the transformation between an HFPR and an HMPR.

**Theorem 1.** Let  $B = (b_{ij})_{n \times n}$  be an HMPR, then it can be transformed into an HFPR  $A = (a_{ij})_{n \times n}$ , where  $a_{ij} = \{a_{ij}^s | s = 1, 2, \dots, l_{ij}\}$  and

$$a_{ij}^s = \frac{b_{ij}^s}{1 + b_{ij}^s}, s \in S_{ij}, i, j \in I. \quad (5)$$

**Theorem 2.** Let  $A = (a_{ij})_{n \times n}$  be an HFPR, then it can be transformed into an HMPR  $B = (b_{ij})_{n \times n}$ , where  $b_{ij} = \{b_{ij}^s | s = 1, 2, \dots, l_{ij}\}$  and

$$b_{ij}^s = \frac{a_{ij}^s}{1 - a_{ij}^s}, s \in S_{ij}, i, j \in I. \quad (6)$$

**Theorem 3.** Let  $A = (a_{ij})_{n \times n}$  be an HFPR. If we first transform it into an HMPR  $B = (b_{ij})_{n \times n}$  by Eq. (6) and then transform  $B$  into an HFPR  $A' = (a'_{ij})_{n \times n}$  by Eq. (5), where  $a'_{ij} = \{a'_{ij}^s | s = 1, 2, \dots, l_{ij}\}$ ,  $i, j \in I$ , then  $A = A'$ .

The proofs of Theorems 1-3 are provided in Appendix A.

**Theorem 4.** Let  $B = (b_{ij})_{n \times n}$  be an HMPR. If we first transform it into an HFPR  $A = (a_{ij})_{n \times n}$  by Eq. (5) and then transform  $A$  into an HMPR  $B' = (b'_{ij})_{n \times n}$  by Eq. (6), where  $b'_{ij} = \{b'_{ij}^s | s = 1, 2, \dots, l_{ij}\}$ ,  $i, j \in I$ , then  $B = B'$ .

**Proof.** The proof is similar to that of Theorem 3.  $\square$

**Remark 3.** Theorems 1 and 2 show that an HFPR and an HMPR can be transformed into each other, and Theorems 3 and 4 demonstrate that the transformation is invertible. Note that there are some other approaches to conducting the transformation processes, and the reason why Eqs. (5) and (6) are utilized is that such transformations coincide with

the consistency definition of a multiplicative preference relation and a fuzzy preference relation from the priority weight vector perspective, which can result in the priority weight derivation model easily (see the model (15)).

### 3. Deriving priority weights and consistent preference relations from an incomplete hesitant preference relation

Due to the lack of time and knowledge, decision maker's limited expertise, or incapacity to quantify the preference degree, incomplete preference relations are usually provided by a decision maker. Motivated by the definition of acceptable incomplete preference relations<sup>55</sup>, we give the definitions of acceptable incomplete HFPRs and acceptable incomplete HMPRs, respectively.

**Definition 9.** Let  $A = (a_{ij})_{n \times n}$  be an HFPR. If some elements of  $A$  are unknown,  $A$  is called an incomplete HFPR. If there exists at least one known element (except diagonal elements) in each row or each column of  $A$ , then  $A$  is called an acceptable incomplete HFPR.

**Definition 10.** Let  $B = (b_{ij})_{n \times n}$  be an HMPR. If some elements of  $B$  are unknown, we call  $B$  an incomplete HMPR. If there exists at least one known element (except diagonal elements) in each row or each column of  $B$ , then  $B$  is called an acceptable incomplete HMPR.

Up to now, most previous approaches focus on deriving priority weights from a complete HFPR or a complete HMPR, and little work has been conducted on deriving priority weights from incomplete hesitant preference relations. Motivated by the logarithmic least squares method<sup>50,51,52</sup>, in this section we develop some formulae to derive priority weights from an incomplete HFPR or an incomplete HMPR. Note that all the incomplete hesitant preference relations considered in this paper are acceptable ones.

First of all, we focus on how to derive priority weights from an incomplete HFPR. Let  $A = (a_{ij})_{n \times n}$  be an incomplete HFPR. If the element  $a_{ij}$  is known, we denote it as  $a_{ij} = \{a_{ij}^s | s = 1, 2, \dots, l_{ij}\}$ . For the missing elements, let  $a_{ij} = \varphi$ . For convenience, we introduce the indication matrix  $\Delta = (\delta_{ij})_{n \times n}$  for  $A$  to

indicate whether the elements of  $A$  are known or not, where

$$\delta_{ij} = \begin{cases} 1 & \text{for } a_{ij} \neq \varphi \\ 0 & \text{for } a_{ij} = \varphi \end{cases}, \quad i, j \in I. \quad (7)$$

Motivated by Ref. 46 and Definition 6, if  $A$  is consistent, we have

$$\frac{w_i}{w_i + w_j} \in a_{ij}, \text{ for all } \delta_{ij} = 1, i, j \in I,$$

i.e.

$$\delta_{ij} \frac{w_i}{w_i + w_j} = \delta_{ij} a_{ij}^1 \text{ or } \delta_{ij} a_{ij}^2 \text{ or } \dots \text{ or } \delta_{ij} a_{ij}^{l_{ij}}, i, j \in I,$$

which can be written as

$$\delta_{ij} \frac{w_i}{w_j} = \delta_{ij} \frac{a_{ij}^1}{a_{ji}^1} \text{ or } \delta_{ij} \frac{a_{ij}^2}{a_{ji}^2} \text{ or } \dots \text{ or } \delta_{ij} \frac{a_{ij}^{l_{ij}}}{a_{ji}^{l_{ij}}}, i, j \in I.$$

As we assume that  $w_i > 0, i \in I$ , we have

$$\begin{aligned} & \delta_{ij} (\ln w_i - \ln w_j) \\ &= \delta_{ij} \ln \frac{a_{ij}^1}{a_{ji}^1} \text{ or } \delta_{ij} \ln \frac{a_{ij}^2}{a_{ji}^2} \text{ or } \dots \text{ or } \delta_{ij} \ln \frac{a_{ij}^{l_{ij}}}{a_{ji}^{l_{ij}}}, i, j \in I. \end{aligned} \quad (8)$$

However, Eq. (8) may usually not hold, i.e. there are some deviations which are calculated as

$$\varepsilon_{ij}^s = \delta_{ij} (\ln w_i - \ln w_j - \ln \frac{a_{ij}^s}{a_{ji}^{l_{ij}-s+1}}), s \in S_{ij}, i, j \in I.$$

It is obvious that the values of all the  $\varepsilon_{ij}^s$  should be kept as small as possible. Therefore, we establish the following optimization model:

$$\begin{aligned} \min \quad & z_1 = \sum_{i=1}^n \sum_{j=1}^n \sum_{s=1}^{l_{ij}} (\varepsilon_{ij}^s)^2 \\ &= \sum_{i=1}^n \sum_{j=1}^n \sum_{s=1}^{l_{ij}} \delta_{ij} (\ln w_i - \ln w_j - \ln \frac{a_{ij}^s}{a_{ji}^{l_{ij}-s+1}})^2 \\ \text{s.t.} \quad & \sum_{i=1}^n w_i = 1, \\ & w_i > 0, i \in I. \end{aligned} \quad (9)$$

**Theorem 5.** Let

$$D = \begin{pmatrix} \sum_{j=2}^n l_{1j} \delta_{1j} & -l_{12} \delta_{12} & \dots & -l_{1(n-1)} \delta_{1(n-1)} \\ -l_{21} \delta_{21} & \sum_{\substack{j=1 \\ j \neq 2}}^n l_{2j} \delta_{2j} & \dots & -l_{2(n-1)} \delta_{2(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ -l_{(n-1)1} \delta_{(n-1)1} & -l_{(n-1)2} \delta_{(n-1)2} & \dots & \sum_{\substack{j=1 \\ j \neq n-1}}^n l_{(n-1)j} \delta_{(n-1)j} \end{pmatrix}, \quad (10)$$

and

$$Y = \begin{pmatrix} \sum_{j=1}^n \sum_{s=1}^{l_{1j}} \delta_{1j} \ln \frac{a_{1j}^s}{a_{j1}^{l_{1j}-s+1}} \\ \sum_{j=1}^n \sum_{s=1}^{l_{2j}} \delta_{2j} \ln \frac{a_{2j}^s}{a_{j2}^{l_{2j}-s+1}} \\ \vdots \\ \sum_{j=1}^n \sum_{s=1}^{l_{(n-1)j}} \delta_{(n-1)j} \ln \frac{a_{(n-1)j}^s}{a_{j(n-1)}^{l_{(n-1)j}-s+1}} \end{pmatrix}. \quad (11)$$

then the solution to the model (9) is  $w = (w_1, w_2, \dots, w_n)^T$ , where

$$w_i = \begin{cases} \frac{\exp(p_i)}{\sum_{j=1}^{n-1} \exp(p_j) + 1}, & i = 1, 2, \dots, n-1, \\ \frac{1}{\sum_{j=1}^{n-1} \exp(p_j) + 1}, & i = n. \end{cases} \quad (12)$$

with  $P = (p_1, p_2, \dots, p_{n-1}) = D^{-1}Y$ .

The proof of Theorem 5 is provided in Appendix A.

Based on Eq. (12), we can derive a consistent fuzzy preference relation  $\bar{A} = (\bar{a}_{ij})_{n \times n}$ , where  $\bar{a}_{ij} = \frac{w_i}{w_i + w_j}, i, j \in I$ . Here, we call  $\bar{A}$  the induced fuzzy preference relation of the HFPR  $A$ . The objective function value of the model (9) reflects the consistency level of the HFPR  $A$ , i.e. the smaller the

value of  $z_1$ , the more consistent the HFPR  $A$ . Considering the numbers of the known elements, we define the consistency index of  $A$  as

$$CI(A) = 1 / \left( \frac{z_1}{\sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n l_{ij} \delta_{ij}} \right) = \frac{\sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n l_{ij} \delta_{ij}}{\sum_{i=1}^n \sum_{j=1}^n \sum_{s=1}^{l_{ij}} \delta_{ij} (\ln w_i - \ln w_j - \ln \frac{a_{ij}^s}{a_{ji}^{l_{ij}-s+1}})^2} \quad (13)$$

**Remark 4.** Some authors have already proposed distinct consistency measures for HFPRs, For instance, Liao et al. <sup>42</sup> and Zhu et al. <sup>46</sup> first constructed some consistent HFPRs based on the defined consistency property, and then measured the consistency by calculating the distance between the initial preference relation and the consistent preference relation. When calculating the distances, the numbers of the elements in the two preference relations need to be equal and some normalization operations are usually needed, which may distort the initial preference relations.

Unlike their consistency measures, Eq. (13) utilizes the priority weights derived from the HFPR to measure the consistency, which avoids the normalization operations used in previous consistency measures and makes fully use of the original preference information provided by the decision maker. Moreover, it should be emphasized that Eq. (13) can be used to measure the consistency of an incomplete HFPR, while the consistency indices in Refs. 42 and 46 only work well for complete HFPRs.

In what follows, we consider how to derive a priority weight vector from an HMPR. Let  $B = (b_{ij})_{n \times n}$  be an incomplete HMPR. If the element  $b_{ij}$  is known, we denote it as  $b_{ij} = \{b_{ij}^s | s = 1, 2, \dots, l_{ij}\}$ . For the missing elements, let  $b_{ij} = \varphi$ . Similarly, we also introduce the indication matrix  $\Delta = (\delta_{ij})_{n \times n}$  for  $B$  to indicate whether the elements of  $B$  are known or not, where

$$\delta_{ij} = \begin{cases} 1 & \text{for } b_{ij} \neq \varphi \\ 0 & \text{for } b_{ij} = \varphi \end{cases}, \quad i, j \in I. \quad (14)$$

For an HMPR  $B = (b_{ij})_{n \times n}$ , we rewrite the objective function of the model (9) based on Theorem 1 as

$$z_2 = \sum_{i=1}^n \sum_{j=1}^n \sum_{s=1}^{l_{ij}} \delta_{ij} (\ln w_i - \ln w_j - \ln \frac{b_{ij}^s / (1 + b_{ij}^s)}{b_{ji}^{l_{ij}-s+1} / (1 + b_{ji}^{l_{ij}-s+1})})^2 = \sum_{i=1}^n \sum_{j=1}^n \sum_{s=1}^{l_{ij}} \delta_{ij} (\ln w_i - \ln w_j - \ln \frac{b_{ij}^s / (1 + b_{ij}^s)}{1 / b_{ij}^s})^2 = \sum_{i=1}^n \sum_{j=1}^n \sum_{s=1}^{l_{ij}} \delta_{ij} (\ln w_i - \ln w_j - \ln b_{ij}^s)^2.$$

Therefore, we can construct an optimization model for an HMPR  $B = (b_{ij})_{n \times n}$  as

$$\begin{aligned} \min \quad z_2 &= \sum_{i=1}^n \sum_{j=1}^n \sum_{s=1}^{l_{ij}} \delta_{ij} (\ln w_i - \ln w_j - \ln b_{ij}^s)^2 \\ \text{s.t.} \quad &\sum_{i=1}^n w_i = 1, \\ &w_i > 0, i \in I. \end{aligned} \quad (15)$$

Let  $Q = (q_1, q_2, \dots, q_{n-1})^T = F^{-1}Y'$ , where  $F = D$  and

$$Y' = \begin{pmatrix} \sum_{j=1}^n \sum_{s=1}^{l_{1j}} \delta_{1j} \ln b_{1j}^s \\ \sum_{j=1}^n \sum_{s=1}^{l_{2j}} \delta_{2j} \ln b_{2j}^s \\ \dots \\ \sum_{j=1}^n \sum_{s=1}^{l_{(n-1)j}} \delta_{(n-1)j} \ln b_{(n-1)j}^s \end{pmatrix}, \quad (16)$$

in a similar manner, the solution to the model (15) is derived as

$$w_i = \begin{cases} \frac{\exp(q_i)}{\sum_{j=1}^{n-1} \exp(q_j) + 1}, & i = 1, 2, \dots, n-1 \\ \frac{1}{\sum_{j=1}^{n-1} \exp(q_j) + 1}, & i = n, \end{cases} \quad (17)$$

Based on Eq. (17), we can derive a consistent multiplicative preference relation  $\bar{B} = (\bar{b}_{ij})_{n \times n}$ ,

where  $\bar{b}_{ij} = \frac{w_i}{w_j}$ ,  $i, j \in I$ . We call  $\bar{B}$  the induced multiplicative preference relation of the HMPR  $B$ . Similarly, we define the consistency index for the HMPR  $B = (b_{ij})_{n \times n}$  as

$$CI(B) = 1 / \left( \frac{\sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n l_{ij} \delta_{ij}}{\sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n l_{ij} \delta_{ij}} \right) = \frac{\sum_{i=1}^n \sum_{j=1}^n \sum_{s=1}^{l_{ij}} \delta_{ij} (\ln w_i - \ln w_j - \ln b_{ij}^s)^2}{\sum_{i=1}^n \sum_{j=1}^n \sum_{s=1}^{l_{ij}} \delta_{ij}} \quad (18)$$

Based on the consistency index, the consistency level of an HMPR can also be measured.

**Remark 5.** It is worth noting that an induced fuzzy preference relation can also be derived from an HMPR and an induced multiplicative preference relation can be derived from an HFPR. Whether to derive a fuzzy preference relation or a multiplicative preference relation should be determined according to actual situations.

#### 4. GDM based on heterogeneous incomplete hesitant preference relations

In section 3, we have developed some formulae to derive a priority weight vector and an induced fuzzy or multiplicative preference relation from an incomplete hesitant preference relation. For actual GDM problems, different decision makers may prefer to utilize different types of preference relations due to the difference in culture, experience and educational backgrounds. As a result, how to fuse heterogeneous preference information from different sources becomes an important issue in GDM<sup>56,57,58,59</sup>. In this paper, we particularly focus on the fusion of heterogeneous incomplete hesitant preference relations, including incomplete HFPRs and incomplete HMPRs.

<sup>‡</sup> In the proposed approach, induced fuzzy preference relations are derived.

#### 4.1. The proposed approach

We consider the following GDM problems. Let  $X = \{x_1, x_2, \dots, x_n\}$  be the set of alternatives,  $O = \{o_1, o_2, \dots, o_m\}$  be the set of  $m$  organizations and  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)^T$  be the weight vector of the organizations which needs to be determined, where  $\sum_{k=1}^m \lambda_k = 1$  and  $\lambda_k \geq 0$ ,  $k \in K = \{1, 2, \dots, m\}$ . Decision makers in each organization provide their preference information using fuzzy preference relations or multiplicative preference relations. Here, we assume that decision makes in the same organization use the same type of preference relations. For convenience, let  $o_k$  ( $k \in K_1 = \{1, 2, \dots, m_1\}$ ) be the decision organizations whose decision makers use fuzzy preference relations and  $o_k$  ( $k \in K_2 = \{m_1 + 1, m_1 + 2, \dots, m\}$ ) be the decision organizations whose decision makers use multiplicative preference relations. To protect the privacy of the decision makers, the preference information of each organization is denoted as an HFPR or an HMPR. Let the HFPR of the  $k$  th organization be  $A^k = (a_{ij,k})_{n \times n}$ ,  $k \in K_1$ , and the HMPR of the  $k$  th organization be  $B^k = (b_{ij,k})_{n \times n}$ ,  $k \in K_2$ . If the alternatives  $x_i$  and  $x_j$  are compared by the  $k$  th organization ( $k \in K_1$ ), then  $a_{ij,k} = \{a_{ij,k}^s | s = 1, 2, \dots, l_{ij}^k\}$  is a hesitant fuzzy element; if the alternatives  $x_i$  and  $x_j$  are compared by the  $k$  th organization ( $k \in K_2$ ),  $b_{ij,k} = \{b_{ij,k}^s | s = 1, 2, \dots, l_{ij}^k\}$  is a hesitant multiplicative element. If no comparison between the alternatives  $x_i$  and  $x_j$  is provided by the  $k$  th organization, let  $a_{ij,k} = \varphi$  or  $b_{ij,k} = \varphi$ ,  $k \in K$ .

As the hesitant preference relations are heterogeneous and some elements of the hesitant preference relations are missing, we cannot aggregate them into a collective one directly. To deal with this problem, we propose a novel procedure to fuse heterogeneous incomplete hesitant preference relations. The basic ideas are as follows.

By Eqs. (12) and (17), a priority weight vector which reflects the hesitant preference information of a decision organization can be derived from each HFPR or HMPR. Based on these priority weight vectors, some induced consistent fuzzy preference relations (or multiplicative preference relations) can be obtained,<sup>‡</sup> which are homogeneous and complete.



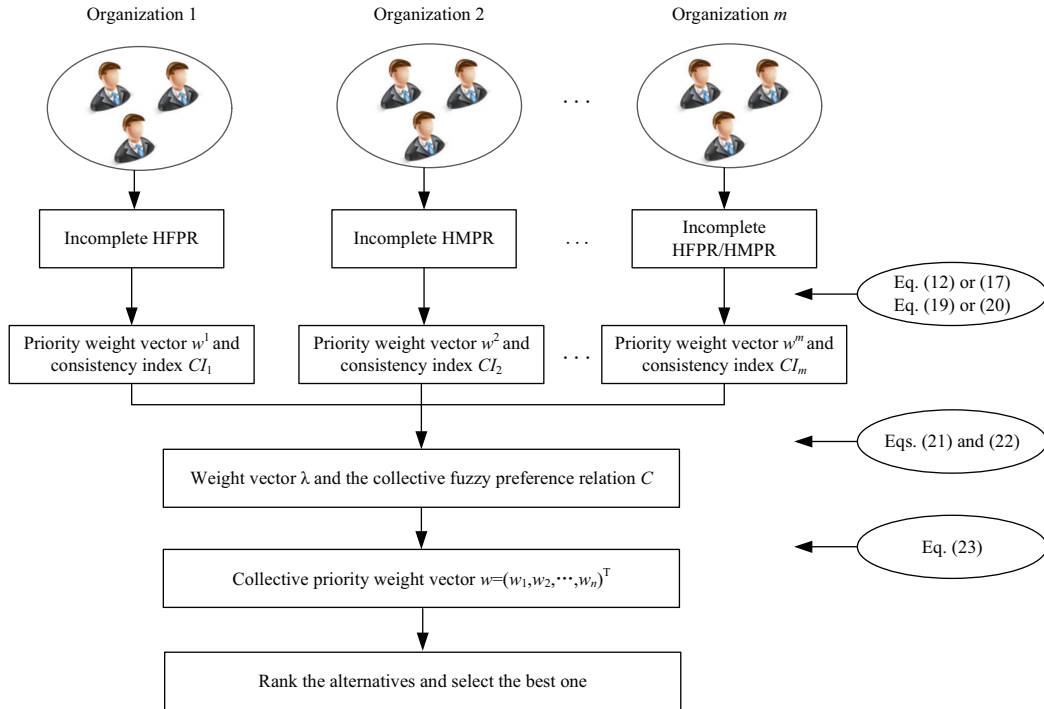


Figure 1: GDM with heterogeneous incomplete hesitant preference relations

By fusing these induced fuzzy preference relations (or multiplicative preference relations), the collective preference relation is derived. Finally, a collective priority weight vector can be calculated based on the collective preference relation, and then the alternatives can be ranked.

Based on the above analysis, the following GDM approach (Approach I) is developed, as illustrated by Figure 1.

**Approach I**

**Step 1:** Calculate the indication matrices  $\Delta^k = (\delta_{ij}^k)_{n \times n}$  for each organization, where

$$\delta_{ij}^k = \begin{cases} 1 & \text{for } a_{ij,k} \neq \varphi \\ 0 & \text{for } a_{ij,k} = \varphi \end{cases}, \quad i, j \in I, k \in K_1,$$

and

$$\delta_{ij}^k = \begin{cases} 1 & \text{for } b_{ij,k} \neq \varphi \\ 0 & \text{for } b_{ij,k} = \varphi \end{cases}, \quad i, j \in I, k \in K_2.$$

**Step 2:** Derive the priority weight vector of each individual hesitant preference relation as  $w^k = (w_1^k, w_2^k, \dots, w_n^k)$  by Eq. (12) for all  $k \in K_1$  and Eq. (17) for all  $k \in K_2$ , respectively, and then derive the induced fuzzy preference relation for each organization as  $\bar{A}^k = (\bar{a}_{ij,k})_{n \times n}$ , where  $\bar{a}_{ij,k} = \frac{w_i^k}{w_i^k + w_j^k}$ ,  $i, j \in I, k \in K$ .

**Step 3:** Calculate the consistency index for each HFPR by Eq. (13) as

$$CI_k = CI(A^k)$$

$$= \frac{\sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n l_{ij}^k \delta_{ij}^k}{\sum_{i=1}^n \sum_{s=1}^n \sum_{j=1}^{l_{ij}^k} \delta_{ij}^k (\ln w_i^k - \ln w_j^k - \ln \frac{a_{ij,k}^s}{a_{ji,k}^{l_{ij}^k - s + 1}})^2}, k \in K_1, \tag{19}$$

and calculate the consistency index for

each HMPR by Eq. (18) as

$$CI_k = CI(B^k) = \frac{\sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n l_{ij}^k \delta_{ij}^k}{\sum_{i=1}^n \sum_{s=1}^n \sum_{j=1}^n \delta_{ij}^k (\ln w_i^k - \ln w_j^k - \ln b_{ij,k}^s)^2}, k \in K_2. \quad (20)$$

**Step 4:** Derive the weight vector of the organizations as  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)^T$ , where

$$\lambda_k = \frac{CI_k}{\sum_{h=1}^m CI_h}, k \in K. \quad (21)$$

**Step 5:** Aggregate the individual induced fuzzy preference relations of each organization to obtain the collective fuzzy preference relation  $C = (c_{ij})_{n \times n}$ <sup>60</sup>, where

$$c_{ij} = \frac{\prod_{k=1}^m (\bar{a}_{ij,k})^{\lambda_k}}{\prod_{k=1}^m (\bar{a}_{ij,k})^{\lambda_k} + \prod_{k=1}^m (\bar{a}_{ji,k})^{\lambda_k}}, i, j \in I. \quad (22)$$

**Step 6:** Derive the priority weight vector  $w = (w_1, w_2, \dots, w_n)^T$  by<sup>61</sup>

$$w_i = \frac{\prod_{j=1}^n (c_{ij})^{1/n}}{\sum_{i=1}^n (\prod_{j=1}^n (c_{ij})^{1/n})}, i \in I. \quad (23)$$

**Step 7:** Rank the alternatives based on  $w$  and select the best alternative.

### 5. Illustrative examples

In this section, we present three examples to illustrate the proposed GDM approach.

**Example 1.** First, we consider the supplier selection problems in high-tech companies. Suppose that a high-tech company which manufactures electronic products intends to select a supplier of USB connectors (adapted from Ref. 64). There are four suppliers  $x_1, x_2, x_3$  and  $x_4$  to be selected. Three departments of the company including finance department ( $o_1$ ), engineering department ( $o_2$ ) and quality control department ( $o_3$ ) are invited to evaluate the four suppliers. The experts in each department provide their preference information over the alternatives using fuzzy preference relations or multiplicative preference relations, and experts in the same department use the same type of preference relation. After collecting the preference information of the experts in each department, three heterogeneous incomplete hesitant preference relations are obtained, as listed in Tables 1-3.

Table 1: HFPR ( $A^1$ ) of the organization  $o_1$

$o_1$	$x_1$	$x_2$	$x_3$	$x_4$
$x_1$	{0.5}	{0.3,0.5}	$\varnothing$	{0.7}
$x_2$	{0.5,0.7}	{0.5}	{0.25,0.5}	{0.6,0.7}
$x_3$	$\varnothing$	{0.5,0.75}	{0.5}	{0.5,0.7}
$x_4$	{0.3}	{0.3,0.4}	{0.3,0.5}	{0.5}

In what follows, we use the proposed approach to deal with the decision making problem. First of

all, we derive the indication matrices as follows:

$$\Delta^1 = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \Delta^2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix},$$

Table 2: HFPR ( $A^2$ ) of the organization  $o_2$

$o_2$	$x_1$	$x_2$	$x_3$	$x_4$
$x_1$	{0.5}	{0.3}	{0.3,0.6}	{0.5,0.6,0.7}
$x_2$	{0.7}	{0.5}	{0.3,0.6}	$\varnothing$
$x_3$	{0.4,0.7}	{0.4,0.7}	{0.5}	{0.4,0.6}
$x_4$	{0.3,0.4,0.5}	$\varnothing$	{0.4,0.6}	{0.5}

Table 3: HMPR ( $B^3$ ) of the organization  $o_3$

$o_3$	$x_1$	$x_2$	$x_3$	$x_4$
$x_1$	{1}	{1/2,1}	{1/3, 1/2}	{1}
$x_2$	{1,2}	{1}	{1,2,3}	{2}
$x_3$	{2,3}	{1/3,1/2,1}	{1}	$\varnothing$
$x_4$	{1}	{1/2}	$\varnothing$	{1}

$$\Delta^3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

$$\bar{A}^2 = \begin{pmatrix} 0.5 & 0.4384 & 0.4579 & 0.5456 \\ 0.5616 & 0.5 & 0.5197 & 0.6060 \\ 0.5421 & 0.4803 & 0.5 & 0.5871 \\ 0.4544 & 0.3940 & 0.4129 & 0.5 \end{pmatrix},$$

Afterwards, we calculate the priority weight vector for each hesitant preference relation. Let us take the HFPR  $A^1$  as an example. By Eq. (10), we calculate  $D$  and  $Y$  for  $A^1$  as

$$\bar{A}^3 = \begin{pmatrix} 0.5 & 0.3175 & 0.3872 & 0.4910 \\ 0.6825 & 0.5 & 0.5759 & 0.6746 \\ 0.6128 & 0.4241 & 0.5 & 0.6042 \\ 0.5090 & 0.3254 & 0.3958 & 0.5 \end{pmatrix}.$$

$$D = \begin{pmatrix} 3 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 4 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 \\ 1.0014 \\ 1.9459 \end{pmatrix}.$$

Therefore, the priority weight vector of HFPR  $A^1$  is derived as  $w^1 = (0.2282, 0.2731, 0.3393, 0.1594)^T$ , and the induced fuzzy preference relation  $\bar{A}^{-1}$  can be calculated as

The consistency indices of the hesitant preference relations are calculated by Eqs. (19) and (20) as  $CI(A^1) = 4.0301, CI(A^2) = 3.1107, CI(B^3) = 4.6665$ . Therefore, the weight vector of the organizations is derived as  $\lambda = (0.3413, 0.2635, 0.3952)^T$ .

Based on the induced fuzzy preference relations and the weight vector of the organizations, the collective fuzzy preference relation is calculated as

$$\bar{A}^{-1} = \begin{pmatrix} 0.5 & 0.4552 & 0.4021 & 0.5888 \\ 0.5448 & 0.5 & 0.4459 & 0.6315 \\ 0.5979 & 0.5541 & 0.5 & 0.6804 \\ 0.4112 & 0.3685 & 0.3196 & 0.5 \end{pmatrix}.$$

$$C = \begin{pmatrix} 0.5 & 0.3944 & 0.4106 & 0.5390 \\ 0.6056 & 0.5 & 0.5169 & 0.6423 \\ 0.5894 & 0.4831 & 0.5 & 0.6266 \\ 0.4610 & 0.3577 & 0.3734 & 0.5 \end{pmatrix}.$$

Similarly, we also derive the priority weight vectors and the induced fuzzy preference relations of  $A^2$  and  $B^3$  as follows:

$$w^2 = (0.2327, 0.2981, 0.2754, 0.1938)^T,$$

$$w^3 = (0.1733, 0.3726, 0.2744, 0.1797)^T,$$

By Eq. (23), the priority weight vector of the four alternatives is obtained as  $w = (0.2072, 0.3182, 0.2974, 0.1772)^T$ . Therefore, the best supplier is  $x_2$ .

To further illustrate the effectiveness of the proposed approach, we make some comparisons with the approaches developed by Xia & Xu<sup>47</sup>.

**Example 2.**<sup>47</sup> Consider a problem of selecting a loading-hauling system for a hypothetical iron ore open pit mine. Three potential transportation system

alternatives are evaluated, i.e.  $x_1$ : shovel-truck system,  $x_2$ : shovel-truck-in-pit crusher-belt conveyor system and  $x_3$ : loader truck system.

Three decision organizations provides their HFPRs for the three alternatives, as shown in Tables 4-6.

Table 4: HFPR ( $A^1$ ) of the organization  $o_1$

$o_1$	$x_1$	$x_2$	$x_3$
$x_1$	{0.5}	{0.1,0.3,0.4}	{0.4,0.6,0.7,0.8}
$x_2$	{0.6,0.7,0.9}	{0.5}	{0.5,0.6,0.9}
$x_3$	{0.2,0.3,0.4,0.6}	{0.1,0.4,0.5}	{0.5}

Table 5: HFPR ( $A^2$ ) of the organization  $o_2$

$o_2$	$x_1$	$x_2$	$x_3$
$x_1$	{0.5}	{0.2,0.4}	{0.5,0.6,0.8}
$x_2$	{0.6,0.8}	{0.5}	{0.5,0.6,0.8,0.9}
$x_3$	{0.2,0.4,0.5}	{0.1,0.2,0.4,0.5}	{0.5}

Table 6: HFPR ( $A^3$ ) of the organization  $o_3$

$o_3$	$x_1$	$x_2$	$x_3$
$x_1$	{0.5}	{0.3,0.4,0.5,0.7}	{0.5,0.8}
$x_2$	{0.3,0.5,0.6,0.7}	{0.5}	{0.5,0.8,0.9}
$x_3$	{0.2,0.5}	{0.1,0.2,0.5}	{0.5}

Next, we utilize the proposed approach to deal with this GDM problem. By Eq. (12), the induced fuzzy preference relations of the three HFPRs are derived as

$$\bar{A}^1 = \begin{pmatrix} 0.5 & 0.3006 & 0.5815 \\ 0.6994 & 0.5 & 0.7637 \\ 0.4185 & 0.2363 & 0.5 \end{pmatrix},$$

$$\bar{A}^2 = \begin{pmatrix} 0.5 & 0.3392 & 0.6094 \\ 0.6608 & 0.5 & 0.7524 \\ 0.3906 & 0.2476 & 0.5 \end{pmatrix},$$

$$\bar{A}^3 = \begin{pmatrix} 0.5 & 0.4517 & 0.7064 \\ 0.5483 & 0.5 & 0.7449 \\ 0.2936 & 0.2551 & 0.5 \end{pmatrix}.$$

By Eq. (21), the weight vector of the three organization is derived as  $\lambda = (0.2803, 0.3773, 0.3423)^T$ . Thus, the collective fuzzy preference relation is calculated as

$$C = \begin{pmatrix} 0.5 & 0.3648 & 0.6366 \\ 0.6352 & 0.5 & 0.7531 \\ 0.3634 & 0.2469 & 0.5 \end{pmatrix}.$$

By Eq. (23), the priority weights of the three

Table 7: HMPR ( $B^1$ ) of the organization  $o_1$

$o_1$	$x_1$	$x_2$	$x_3$
$x_1$	{1}	{1/8,1/5,1/2}	{3,5,7,9}
$x_2$	{2,5,8}	{1}	{4,6,9}
$x_3$	{1/9,1/7,1/5,1/3}	{1/9,1/6,1/4}	{1}

Table 8: HMPR ( $B^2$ ) of the organization  $o_2$

$o_2$	$x_1$	$x_2$	$x_3$
$x_1$	{1}	{1/5,1/2}	{5,6,8}
$x_2$	{2,5}	{1}	{2,4,7,9}
$x_3$	{1/8,1/6,1/5}	{1/9,1/7,1/4,1/2}	{1}

Table 9: HMPR ( $B^3$ ) of the organization  $o_3$

$o_3$	$x_1$	$x_2$	$x_3$
$x_1$	{1}	{1/7,1/5,1/4,1/3}	{2,4}
$x_2$	{3,4,5,7}	{1}	{3,8,9}
$x_3$	{1/4,1/2}	{1/9,1/8,1/3}	{1}

alternatives can be derived as  $w_1 = 0.3019$ ,  $w_2 = 0.5257$ ,  $w_3 = 0.1724$ . Therefore, the ranking of the alternatives is  $x_2 \succ x_1 \succ x_3$ , which demonstrates that  $x_2$  is the best alternative. The ranking is also consistent with that derived by Xia & Xu’s approach<sup>47</sup>. However, by contrast, we find that the approach in Ref. 47 is based on the operations of hesitant fuzzy elements, which is more complex than the proposed approach, as the dimension of the derived hesitant fuzzy elements by the operations in Ref. 47 may increase significantly. Moreover, the approach in Ref. 47 can only deal with GDM problems based on complete HFPRs.

**Example 3.** Reconsider Example 2. If the preference information of the three decision organizations are provided in the form of HMPR, as demonstrated in Tables 7-9<sup>47</sup>, the following results are obtained.

By Eq. (17), the induced fuzzy preference relations of the three HMPRs are derived as

$$\bar{A}^1 = \begin{pmatrix} 0.5 & 0.2772 & 0.7918 \\ 0.7228 & 0.5 & 0.9084 \\ 0.2082 & 0.0916 & 0.5 \end{pmatrix},$$

$$\bar{A}^2 = \begin{pmatrix} 0.5 & 0.3788 & 0.8005 \\ 0.6212 & 0.5 & 0.8681 \\ 0.1995 & 0.1319 & 0.5 \end{pmatrix},$$

$$\bar{A}^3 = \begin{pmatrix} 0.5 & 0.2083 & 0.6659 \\ 0.7917 & 0.5 & 0.8834 \\ 0.3341 & 0.1166 & 0.5 \end{pmatrix}.$$

The weight vector of the three organization is derived as  $\lambda = (0.2541, 0.2499, 0.496)^T$ . Therefore, the collective fuzzy preference relation can be calculated as

$$C = \begin{pmatrix} 0.5 & 0.2632 & 0.7367 \\ 0.7368 & 0.5 & 0.8868 \\ 0.2633 & 0.1132 & 0.5 \end{pmatrix}.$$

By Eq. (23), the priority weights of the three alternatives are derived as  $w_1 = 0.2406$ ,  $w_2 = 0.6734$ ,

$w_3 = 0.086$ , which results in a ranking  $x_2 \succ x_1 \succ x_3$ . Therefore,  $x_2$  is the best alternative, which is also consistent with that shown in Ref. 47.

## 6. Discussions on the proposed approach

In this section, we make some discussions to show the characteristics of the proposed approach. Compared with previous approaches, the proposed approach has the following advantages.

(1) The proposed approach is a first attempt to fuse heterogeneous incomplete hesitant preference relations. By literature, we can see that previous work only concentrates on the fusion of complete HFPRs or HMPRs (see Refs. 46,47,48,49). In case of incompleteness and heterogeneity, their approaches will not work well. However, the proposed approach can deal with such cases effectively, which allows decision makers or decision organizations to express their preference information more flexibly. Moreover, when the hesitant preference relations are complete and homogeneous, i.e. only HFPRs or HMPRs, the proposed approach can also be used to derive priority weights (see Examples 2 and 3). Therefore, the proposed approach has more extensive applications in GDM based on hesitant preference relations.

(2) The proposed approach first uses the logarithmic least squares method to derive an individual priority weight vector and an induced fuzzy (or multiplicative) preference relation from each incomplete hesitant preference relation, and then a collective preference relation is derived by fusing the induced preference relations to output the collective priority weight vector. While the aggregation-based approaches (see Refs. 46,47) utilize hesitant fuzzy aggregation operators or hesitant multiplicative aggregation operators to aggregate the hesitant preference information, which is more complex than the proposed approach, since the dimension of the collective values may increase significantly. Here, we again emphasize that the aggregation-based approaches cannot be used to deal with GDM problems with incomplete hesitant preference relations.

(3) The priority weights derived by the proposed approach lie in the unit interval. Therefore, the pro-

posed approach can be integrated with the Analytic Hierarchy Process<sup>62</sup> to deal with multi-criteria decision making problems under GDM settings, especially when the preference information of the decision makers are hesitant, incomplete and heterogeneous.

(4) The proposed approach also develops a simple formula to determine the weights of decision makers based on the proposed consistency indices, which is helpful when the weight vector of decision makers is unknown.

However, the proposed approach also has some limitations. First, the proposed approach only considers the situations that incomplete hesitant preference relations are acceptable ones, which cannot deal with the ignorance situations that sometimes may appear in actual GDM problems<sup>63</sup>. Second, the proposed approach can only fuse two types of incomplete hesitant preference relations, which cannot fuse hesitant fuzzy linguistic preference relations<sup>44</sup>. All the issues will be studied in the future.

## 7. Conclusions

In this paper, based on the logarithmic least squares method, some simple formulae are developed to derive priority weights from an incomplete HFPR or an incomplete HMPR. Based on the individual priority weight vectors, the consistency indices of an HFPR and an HMPR are also defined. To deal with GDM problems with heterogeneous incomplete hesitant preference relations, a novel approach is developed. In the proposed approach, the individual priority weight vector derived from each incomplete hesitant preference relation is utilized to obtain the corresponding induced fuzzy preference relation, and the consistency indices of the incomplete hesitant preference relations are used to determine the weights of decision makers. The proposed approach can allow decision makers or decision organizations to express their preference information more flexibly.

In terms of future research, we intend to integrate hesitant fuzzy linguistic preference relations<sup>44,65</sup> into the GDM approach. Moreover, we will apply the proposed approach to deal with other prac-

tical decision making problems, such as company performance appraisal and quality function deployment.

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**Appendix A Proofs of the theorems**

**Proof of Theorem 1**

**Proof.** Let  $b_{ij}^{\delta(s)}$  be the  $s$  th smallest element of  $b_{ij}$  and  $a_{ij}^{\sigma(s)}$  be the  $s$  th smallest element of  $a_{ij}$ . As  $f(x) = \frac{x}{1+x}$  is a monotone increasing function of

$x$  for all  $x > 0$ , we know that  $\frac{b_{ij}^{\delta(s)}}{1+b_{ij}^{\delta(s)}}$  is also the

$s$  th smallest element of  $a_{ij}$ , i.e.  $a_{ij}^{\sigma(s)} = \frac{b_{ij}^{\delta(s)}}{1+b_{ij}^{\delta(s)}}$ .

Moreover, we have

$$\begin{aligned} a_{ij}^{\sigma(s)} + a_{ji}^{\sigma(l_{ij}-s+1)} &= \frac{b_{ij}^{\delta(s)}}{1+b_{ij}^{\delta(s)}} + \frac{b_{ji}^{\delta(l_{ij}-s+1)}}{1+b_{ji}^{\delta(l_{ij}-s+1)}} \\ &= \frac{b_{ij}^{\delta(s)}}{1+b_{ij}^{\delta(s)}} + \frac{1/b_{ij}^{\delta(s)}}{1+1/b_{ij}^{\delta(s)}} = 1, \end{aligned}$$

and  $a_{ij}^s \in [0.1, 0.9]$ ,  $a_{ii} = \{0.5\}$ ,  $s \in S_{ij}$ ,  $i, j \in I$ . Therefore,  $A = (a_{ij})_{n \times n}$  is an HFPR.  $\square$

**Proof of Theorem 2**

**Proof.** Let  $a_{ij}^{\sigma(s)}$  be the  $s$  th smallest element of  $a_{ij}$  and  $b_{ij}^{\delta(s)}$  be the  $s$  th smallest element of  $b_{ij}$ . As

$f(x) = \frac{x}{1-x}$  is a monotone increasing function of

$x$  for all  $x > 0$ , we know that  $\frac{a_{ij}^{\sigma(s)}}{1-a_{ij}^{\sigma(s)}}$  is also the

$s$  th smallest element of  $b_{ij}$ , i.e.  $b_{ij}^{\delta(s)} = \frac{a_{ij}^{\sigma(s)}}{1-a_{ij}^{\sigma(s)}}$ .

Moreover, we have

$$\begin{aligned} b_{ij}^{\delta(s)} \times b_{ji}^{\delta(l_{ij}-s+1)} &= \frac{a_{ij}^{\sigma(s)}}{1-a_{ij}^{\sigma(s)}} \times \frac{a_{ji}^{\sigma(l_{ij}-s+1)}}{1-a_{ji}^{\sigma(l_{ij}-s+1)}} \\ &= \frac{a_{ij}^{\sigma(s)}}{1-a_{ij}^{\sigma(s)}} \times \frac{1-a_{ij}^{\sigma(s)}}{a_{ij}^{\sigma(s)}} = 1. \end{aligned}$$

As  $a_{ij}^s \in [0.1, 0.9]$ , we have  $b_{ij}^s \in [1/9, 9]$ ,  $b_{ii} = \{1\}$ ,  $s \in S_{ij}$ ,  $i, j \in I$ . Therefore,  $B = (b_{ij})_{n \times n}$  is an HMPR.  $\square$

**Proof of Theorem 3**

**Proof.** By Eq. (6), an HMPR  $B = (b_{ij})_{n \times n}$  is derived, where  $b_{ij} = \{b_{ij}^s | s = 1, 2, \dots, l_{ij}\} = \{\frac{a_{ij}^s}{1-a_{ij}^s} | s = 1, 2, \dots, l_{ij}\}$ ,  $i, j \in I$ . By Eq. (5), we have

$$\begin{aligned} a_{ij}'^s &= \frac{b_{ij}^s}{1+b_{ij}^s} = \frac{a_{ij}^s/(1-a_{ij}^s)}{1+a_{ij}^s/(1-a_{ij}^s)} \\ &= \frac{a_{ij}^s/(1-a_{ij}^s)}{(1-a_{ij}^s+a_{ij}^s)/(1-a_{ij}^s)} = a_{ij}^s, s \in S_{ij}, i, j \in I. \end{aligned}$$

Therefore,  $A = A'$ .  $\square$

**Proof of Theorem 5**

**Proof.** To solve the model (9), we construct the following Lagrange function as

$$\begin{aligned} L(w, \theta) &= \sum_{i=1}^n \sum_{j=1}^n \sum_{s=1}^{l_{ij}} \delta_{ij} (\ln w_i - \ln w_j - \ln \frac{a_{ij}^s}{a_{ji}^{l_{ij}-s+1}})^2 \\ &\quad + 4\theta (\sum_{i=1}^n w_i - 1), \end{aligned}$$

where  $\theta$  is the Lagrange multiplier.

Let  $\frac{\partial L}{\partial w_i} = 0$ , then we have

$$\sum_{j=1}^n \sum_{s=1}^{l_{ij}} 4\delta_{ij}(\ln w_i - \ln w_j - \ln \frac{a_{ij}^s}{a_{ji}^{l_{ij}-s+1}}) \cdot \frac{1}{w_i} + 4\theta = 0, i \in I,$$

i.e.

$$\sum_{j=1}^n \sum_{s=1}^{l_{ij}} \delta_{ij}(\ln w_i - \ln w_j - \ln \frac{a_{ij}^s}{a_{ji}^{l_{ij}-s+1}}) + \theta w_i = 0, i \in I. \tag{A.1}$$

Summing both sides of Eq. (A.1) for all  $i \in I$ , we have

$$\sum_{i=1}^n \sum_{j=1}^n \sum_{s=1}^{l_{ij}} \delta_{ij}(\ln w_i - \ln w_j - \ln \frac{a_{ij}^s}{a_{ji}^{l_{ij}-s+1}}) + \theta = 0,$$

which is equivalent to

$$\sum_{i=1}^n \sum_{j=1}^n \sum_{s=1}^{l_{ij}} \delta_{ij}(\ln w_i - \ln w_j) - \sum_{i=1}^n \sum_{j=1}^n \sum_{s=1}^{l_{ij}} \ln(\frac{a_{ij}^s}{a_{ji}^{l_{ij}-s+1}}) \delta_{ij} + \theta = 0,$$

i.e.

$$\sum_{i=1}^n \sum_{j=1}^n l_{ij} \delta_{ij}(\ln w_i - \ln w_j) - \ln(\prod_{i=1}^n \prod_{j=1}^n \prod_{s=1}^{l_{ij}} (\frac{a_{ij}^s}{a_{ji}^{l_{ij}-s+1}}) \delta_{ij}) + \theta = 0, \tag{A.2}$$

As  $\sum_{i=1}^n \sum_{j=1}^n l_{ij} \delta_{ij}(\ln w_i - \ln w_j) = \ln \prod_{i=1}^n \prod_{j=1}^n (\frac{w_i}{w_j})^{l_{ij} \delta_{ij}} =$

$\ln 1 = 0$  and  $\ln(\prod_{i=1}^n \prod_{j=1}^n \prod_{s=1}^{l_{ij}} (\frac{a_{ij}^s}{a_{ji}^{l_{ij}-s+1}}) \delta_{ij}) = \ln 1 = 0$ , by

Eq. (A.2), we have  $\theta = 0$ .

Therefore, Eq. (A.1) can be written as

$$\sum_{j=1}^n l_{ij} \delta_{ij}(\ln w_i - \ln w_j) = \sum_{j=1}^n \sum_{s=1}^{l_{ij}} \delta_{ij} \ln \frac{a_{ij}^s}{a_{ji}^{l_{ij}-s+1}}, i \in I. \tag{A.3}$$

Let

$$\bar{D} = \begin{pmatrix} \sum_{j=2}^n l_{1j} \delta_{1j} & -l_{12} \delta_{12} & \cdots & -l_{1n} \delta_{1n} \\ -l_{21} \delta_{21} & \sum_{j=1, j \neq 2}^n l_{2j} \delta_{2j} & \cdots & -l_{2n} \delta_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ -l_{n1} \delta_{n1} & -l_{n2} \delta_{n2} & \cdots & \sum_{j=1, j \neq n}^n l_{nj} \delta_{nj} \end{pmatrix} \tag{A.4}$$

and

$$\bar{Y} = \begin{pmatrix} \sum_{j=1}^n \sum_{s=1}^{l_{1j}} \delta_{1j} \ln \frac{a_{1j}^s}{a_{j1}^{l_{1j}-s+1}} \\ \sum_{j=1}^n \sum_{s=1}^{l_{2j}} \delta_{2j} \ln \frac{a_{2j}^s}{a_{j2}^{l_{2j}-s+1}} \\ \vdots \\ \sum_{j=1}^n \sum_{s=1}^{l_{nj}} \delta_{nj} \ln \frac{a_{nj}^s}{a_{jn}^{l_{nj}-s+1}} \end{pmatrix}, \tag{A.5}$$

then Eq. (A.3) is equivalent to

$$\bar{D} \bar{P} = \bar{Y}, \tag{A.6}$$

where  $\bar{P} = (\ln w_1, \ln w_2, \dots, \ln w_n)^T$ .

As the sum of each row or column of  $\bar{D}$  is zero,  $\bar{D}$  is a singular matrix. Let  $\ln w_n = 0$ , then we can solve the first  $n - 1$  equations of Eq. (A.6). The solution is  $P = (p_1, p_2, \dots, p_{n-1})^T = (\ln w_1, \ln w_2, \dots, \ln w_{n-1})^T = D^{-1}Y$ .

Considering the normalization constraint, the solution to the model (9) is derived as Eq. (12). This completes the proof of Theorem 5.  $\square$

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