Prediction Intervals for *k*-records in Terms of Current Records

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In this paper, we discuss the prediction of *k*-records from the future sequence based on observed current records of either kind upper or lower, record coverage and *k*-records from the same distribution. It is shown that the coverage probability of all proposed prediction intervals are distribution-free. Exact and explicit expressions for the prediction coefficient of these intervals are obtained. The existence and optimality of these intervals are discussed. At the end, a numerical example is given for illustrating and comparing the proposed procedure.

Keywords: Coverage probability, Current records, *K*-records, Prediction intervals.

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1. Introduction

Let $\{X_i, \ i \geqslant 1\}$ be a sequence of independent and identically distributed (iid) random variables with an absolutely continuous cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. An observation X_j is called an *upper record* if its value exceeds all previous observations, i.e., X_j is an *upper record* if $X_j > X_i$ for every $i < j$. An analogous definition can be given for *lower records*. Interested readers may refer to the book by Arnold *et al.* [5] and the references contained therein. Upper *k*-record process is defined in terms of the *k*-th largest X yet seen. For a formal definition, in the continuous case, we follow Arnold *et al.* [5](p. 43). Let $T_{0,k} = k, R_{0,k} = X_{1:k}$ and for $n \geqslant 1$, let $T_{n,k} = \min\{j : j > T_{n-1,k}, X_j > X_{T_{n-1,k}-k+1:T_{n-1,k}}\}$, where $X_{i:m}$ denotes the *i*-th order statistic in a sample of size *m*. The sequence of *upper k-records* is then defined by $R_{n,k} = X_{T_{n,k}-k+1:T_{n,k}}$ for $n \geqslant 0$. An analogous definition can be given for *lower k-records* as well. Let us denote the pdf of $R_{i,k}$ by $f_{i,k}(t)$, then for $i \geqslant 0$,

$$f_{i,k}(t) = \frac{k^{i+1}[-\log \bar{F}(t)]^i}{i!}[\bar{F}(t)]^{k-1}f(t), \tag{1.1}$$

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where $\bar{F}(t) = 1 - F(t)$ is the survival function of X. Several applications of k-records can be found in the literature, for instance, see the examples cited in Kamps [10].

Current records are another type of record data which was introduced by Houchens [9]. Consider the largest and the smallest observations, respectively, at the time when a new record of either kind (upper or lower) occurs. In the literature, they are called the n-th *upper current record* and the n-th *lower current record*, respectively, of the X_n sequence when the n-th record of any kind (either an upper or lower) is observed. In this paper, we denote them by R_n^l and R_n^s, respectively ($R_0^s = R_0^l = X_1$). For $n \geqslant 1$, the interval (R_n^s, R_n^l) will be called the *record coverage*. The marginal density of R_i^l is given by (see Arnold *et al.* [5], p. 276)

$$f_{R_i^l}(x) = 2^i f(x) \left(1 - \bar{F}(x) \sum_{j=0}^{i-1} \frac{\{-\log \bar{F}(x)\}^j}{j!} \right); \tag{1.2}$$

for convenience in notation, we suppose the empty sum to be 0, i.e., $\sum_{j=0}^{-1} a_j = 0$. By replacing $\bar{F}$ in (1.2) by F, the pdf of R_i^s is derived. For fixing these concepts, let us consider the following sequence of observations:

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3.0, 2.0, 2.5, 2.6, 1.7, 3.7, 2.2, 1.5, 2.7, 2.3, 1.2,
4.0, 2.5, 4.3, 4.1, ...
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The current records and upper k-records (for $k = 1, 2, 3$) extracted from the above sequence are then as follows:

m	0	1	2	3	4	5	6	7
R_m^l	3.0	3.0	3.0	3.7	3.7	3.7	4.0	4.3
R_m^s	3.0	2.0	1.7	1.7	1.5	1.2	1.2	1.2
$R_{m,1}$	3.0	3.7	4.0	4.3	-	-	-	-
$R_{m,2}$	2.0	2.5	2.6	3.0	3.7	4.0	4.1	-
$R_{m,3}$	2.0	2.5	2.6	2.7	3.0	3.7	4.0	-

The problem of prediction of future records based on observed record data has been extensively studied by several statisticians in parametric and nonparametric setup. Among others are Ahsanullah [3], Dunsmore [8], Awad and Raqab [7], AL-Hussaini *et al.* [4], Malinowska and Szynal [11], Raqab [12] and Asgharzadeh and Valiollahi [6]. Recently, Ahmadi and Balakrishnan [1] considered the problem of predicting order statistics (record values) based on record values (order statistics) and obtained several nonparametric prediction intervals. Ahmadi and Balakrishnan [2] obtained prediction intervals for order statistics of future sample based on current records. Here, we consider two sample prediction intervals and intend to construct prediction intervals for k-records (upper and lower) from the future sequence based on observed record coverage, current records of either kind upper or lower from the same distribution. There are presented in Section 2. Also, prediction intervals based on k-records are given in Section 3. Finally, a numerical example is given in Section 4 for illustrating and comparing the proposed procedure.

2. Prediction based on current records

In this section, we consider three cases and discuss how one can construct prediction intervals for k-record based on current records. In what follows, let $\{X_i, i \geqslant 1\}$ be a sequence of iid continuous random variables with cdf $F(x)$ and pdf $f(x)$, also denote the corresponding i-th and j-th lower

and upper current records by R_i^s and R_j^l, respectively. Independently of the X_i-sequence, let $\{Y_i,\ i \geqslant 1\}$ be a sequence of iid continuous random variables from the same distribution and denote the corresponding m-th k-record by $R_{m,k}^Y$ ($m \geqslant 0$, $k \geqslant 1$).

2.1. *PIs based on record coverage*

Here, we show that (R_i^s, R_i^l), is a two-sided prediction interval for $R_{m,k}^Y$, whose coverage probability is free of F and is given by

$$\alpha_1(i;m,k) = 2^i \sum_{s=0}^{i-1} \frac{1}{2^{s+1}} F_{NB}(m, s+1, \frac{2}{k+2}) - 2^i \sum_{s=0}^{i-1} \sum_{\ell=s}^{\infty} \frac{C_\ell(s)}{s!}$$
$$\times \left[\frac{1}{\ell+1} \left(1 - \left(\frac{k}{k+\ell+1} \right)^{m+1} \right) - \frac{1}{\ell+2} \left(1 - \left(\frac{k}{k+\ell+2} \right)^{m+1} \right) \right], \qquad (2.1)$$

where

$$F_{NB}(m,n,p) = \sum_{x=0}^{m} \binom{n+x-1}{x} p^n q^x; \quad x = 0,1,2,\ldots, \qquad (2.2)$$

in fact $F_{NB}(m,n,p)$ is the cdf of Negative binomial with parameters n and p. Also, $C_\ell(s)$ is the coefficient of u^ℓ in $\left(\sum_{i=1}^{+\infty} u^i / i \right)^s$.

Proof for (2.1): From the definition of current records, R_i^s is smaller than R_i^l for $i \geqslant 1$ with probability one, then we have

$$\alpha_1(i;m,k) = P(R_i^s \leqslant R_{m,k}^Y \leqslant R_i^\ell)$$
$$= P(R_i^s \leqslant R_{m,k}^Y) - P(R_i^l \leqslant R_{m,k}^Y)$$
$$= \varphi_1(i,m,k) - \varphi_2(i,m,k), \quad \text{say.} \qquad (2.3)$$

By using the expressions in (1.1) and (1.2), and apply the same argument given in the proof of Lemma 1 of Ahmadi and Balakrishnan [2], we obtain an exact expression for $\varphi_1(i,m,k)$ which is stated in the following.

$$\varphi_1(i,m,k) = P(R_{m,k}^Y \geqslant R_i^s)$$
$$= \int_{-\infty}^{\infty} P(R_{m,k}^Y \geqslant y) dF_{R_i^s}(y) \quad \text{(by independence of } R_i^s \text{ and } R_{m,k}^Y)$$
$$= \int_{-\infty}^{\infty} [\bar{F}(y)]^k \sum_{j=0}^{m} \frac{(-k \log \bar{F}(y))^j}{j!} \left(2^i \left[1 - F(y) \sum_{s=0}^{i-1} \frac{(-\log F(y))^s}{s!} \right] \right) f(y) dy$$
$$= 2^i \left\{ 1 - \left(\frac{k}{k+1} \right)^{m+1} - \sum_{j=0}^{m} \sum_{s=0}^{i-1} \frac{k^j}{s! j!} [I(j,k,s) - I(j,k+1,s)] \right\}, \qquad (2.4)$$

where

$$I(j,k,s) = \int_0^1 y^k (-\log(1-y))^s (-\log y)^j dy$$
$$= \int_0^\infty u^j e^{-(k+1)u} (-\log(1-e^{-u}))^s du,$$

which, upon writing $-\log(1-e^{-u}) = \sum_{i=1}^{\infty} e^{-iu}/i$, immediately gives

$$I(j,s,k) = \sum_{\ell=s}^{\infty} C_{\ell}(s) j!/(k+\ell+1)^{j+1}, \tag{2.5}$$

where $C_{\ell}(s)$ is the coefficient of u^{ℓ} in $(\sum_{i=1}^{+\infty} u^i/i)^s$. It may be noted that the coefficient $C_{\ell}(s)$ in (2.5) can be obtained by recursive manner, see Arnold *et al.* [5] (pp. 70–71), $C_{\ell}(1) = \frac{1}{\ell}$, $\ell \geqslant 1$ and for $s \geqslant 2$,

$$C_{\ell}(s) = \sum_{i=s-1}^{\ell-1} C_i(s-1)/(\ell-i).$$

Also an explicit form of $C_{\ell}(s)$ is given by

$$C_{\ell}(s) = \sum_{r_{s-1}=s-1}^{\ell-1} \sum_{r_{s-2}=s-2}^{r_{s-1}-1} \cdots \sum_{r_2=2}^{r_3-1} \sum_{r_1=1}^{r_2-1} \prod_{i=1}^{s} \frac{1}{r_i - r_{i-1}}, \quad \ell = s, s+1, \ldots,$$

where $r_0 = 0$ and $r_s = \ell$.

Similarly for the second term in (2.3), we have

$$\begin{aligned}
\varphi_2(i,m,k) &= P(R_{m,k}^Y \geqslant R_i^l) \\
&= \int_{-\infty}^{\infty} P(R_{m,k}^Y \geqslant y) dF_{R_i^l}(y) \\
&= \int_{-\infty}^{\infty} [\bar{F}(y)]^k \sum_{j=0}^{m} \frac{(-k\log\bar{F}(y))^j}{j!} \left\{ 2^i \left[1 - \bar{F}(y) \sum_{s=0}^{i-1} \frac{(-\log\bar{F}(y))^s}{s!} \right] \right\} f(y) dy \\
&= 2^i \left\{ \sum_{j=0}^{m} \frac{k^j}{j!} \int_0^1 (-\log y)^j y^k dy - \sum_{j=0}^{m} \sum_{s=0}^{i-1} \frac{k^j}{j!s!} \int_0^1 (-\log y)^{j+s} y^{k+1} dy \right\} \\
&= 2^i \left\{ \sum_{j=0}^{m} \left(\frac{k}{k+1} \right)^j \frac{1}{k+1} - \sum_{j=0}^{m} \sum_{s=0}^{i-1} \frac{(j+s)!}{j!s!} \frac{k^j}{(k+2)^{j+s+1}} \right\} \\
&= 2^i \left\{ 1 - \left(\frac{k}{k+1} \right)^{m+1} - \sum_{j=0}^{m} \sum_{s=0}^{i-1} \binom{j+s}{s} \left(\frac{k}{k+2} \right)^j \left(\frac{1}{k+2} \right)^{s+1} \right\}. \tag{2.6}
\end{aligned}$$

So, first, using (2.5) and (2.4) we find an explicit expression for $\varphi_1(i,m,k)$, then upon substituting the resulting expression (2.6) and (2.4) into (2.3), the required result follows. $\square$

Thus, the i-th record coverage (R_i^s, R_i^l), $i \geqslant 1$, is a two-sided prediction interval for $R_{m,k}^Y$, $m \geqslant 0$, the m-th upper k-record from the future Y-sequence, whose coverage probability is free of F and is given by (2.1). Table 1 presents some numerical values of $\alpha_1(i;m,k)$ for $i = 1$ up to 10, $k = 1, 2, 3, 5$, and some selected values of m. From this table, we observe that for fixed m and k, the prediction coefficient $\alpha_1(i;m,k)$ increases as i increases and for fixed i and k, decreases as m increases.

2.1.1. *Optimal prediction interval*

For a given α_0 the two-sided prediction interval (R_i^s, R_i^l), $i \geqslant 1$, exists if and only if $\alpha_1(i, m, k) \geqslant \alpha_0$. For choosing the optimal interval, if k and m and the desired prediction level α_0 are specified, we have to start with i from 1 and gradually increase it until $\alpha_1(i, m, k)$ exceeds α_0, resulting in an optimal $i_{opt} = g_1(k, m)$. With this in mind, we obtain i_{opt} for $m = 2, ..., 10$, $k = 1, ..., 5$ and $\alpha_0 = 0.90$. These are presented in Table 2. From Table 2, we observe that $i_{opt} = g_1(k, m)$ is a decreasing function with respect to k and increasing function with respect to m.

2.2. *PIs based on upper current records*

Let R_n^l be the n-th upper current record from the X-sequence. In this section, we show that (R_i^l, R_j^l) is a prediction interval for the m-th upper k-record from the future Y-sequence, $R_{m,k}^Y$. The problem is to obtain the probability of the event $(R_i^l \leqslant R_{m,k}^Y \leqslant R_j^l)$. It may be noted that R_n^l (for $n \geqslant 0$) is a continuous random variable, and for $i < j$, $R_i^l \leqslant R_j^l$ with probability one, and moreover $R_{m,k}^Y$ and (R_i^l, R_j^l) are independent. Then, using (2.6) we find that

$$
\begin{aligned}
\alpha_2(i, j; m, k) &= P(R_i^l \leqslant R_{m,k}^Y \leqslant R_j^l) \\
&= 2^i \left\{ 1 - (\frac{k}{k+1})^{m+1} - \sum_{j=0}^{m} \sum_{s=0}^{i-1} \binom{j+s}{s} \left(\frac{k}{k+2}\right)^j \left(\frac{1}{k+2}\right)^{s+1} \right\} \\
&\quad - 2^j \left\{ 1 - (\frac{k}{k+1})^{m+1} - \sum_{j=0}^{m} \sum_{s=0}^{j-1} \binom{j+s}{s} \left(\frac{k}{k+2}\right)^j \left(\frac{1}{k+2}\right)^{s+1} \right\},
\end{aligned} \tag{2.7}
$$

Thus, we have a prediction interval (R_i^l, R_j^l), $j > i \geqslant 0$, for $R_{m,k}$, $k \geqslant 1$, whose prediction coefficient given by (2.7), is free of F. From (2.7), it is obvious that for fixed m and k, the prediction coefficient $\alpha_2(i, j; m, k)$ is decreasing in i and increasing in j, as we would expect.

If we are interested in finding a one-sided prediction interval of the form $(R_i^l, +\infty)$ or $(-\infty, R_j^l)$ for future upper k-records, the coverage probabilities of these intervals can be easily obtained from (2.7). For example, the prediction coefficient of the interval $(R_i^l, +\infty)$ is

$$
2^i \left\{ 1 - (\frac{k}{k+1})^{m+1} - \sum_{s=0}^{i-1} \frac{1}{2^{s+1}} F_{NB}(m, s+1, \frac{2}{k+2}) \right\},
$$

where $F_{NB}(m, n, p)$ is defined in (2.2).

2.2.1. *Existence*

For a given α_0, m and k, the two-sided prediction interval exists if and only if, for a large n,

$$
P(R_0^l \leqslant R_{m,k}^Y \leqslant R_n^l) \geqslant \alpha_0. \tag{2.8}
$$

But, since the prediction coefficient $\alpha_2(i, j; m, k)$ is decreasing in i and increasing in j we have

$$
\max_{i,j} \alpha_2(i, j; m, k) = 1 - \left(\frac{k}{k+1} \right)^{m+1}.
$$

Thus, for a given α_0, m and k, the two sided prediction interval exists if and only if $\max_{i,j} \alpha_2(i, j; m, k) \geqslant \alpha_0$, which means that we can construct a prediction interval for $R_{m,k}$ with a

coverage probability of at least α_0 by using upper record values if

$$1 - \left(\frac{k}{k+1}\right)^{m+1} \geqslant \alpha_0. \tag{2.9}$$

Ahmadi and Balakrishnan [1], found the same result on the basis of ordinary upper records.

Table 3 contains the numerical values of $\alpha_2(i, j; m, k)$ for some choice of m and k. This table shows that if m and k and desired α_0 are specified, we can choose i and j so that $\alpha_2(i, j; m, k)$ exceeds α_0. For example if we want to construct a prediction interval for $R_{5,2}^Y$ with coverage probability at least $\alpha_0 = 0.80$, it suffices to select $i = 0$ and $j = 10$. Thus (R_i^l, R_j^l) is a $100\alpha_2\%$ prediction interval for $R_{m,k}^Y$, the m-th k-record from a future sample, where $\alpha_2(i, j; m, k)$ given by (2.7).

2.2.2. *Optimal prediction interval*

In the previous subsection we proved that, for a given α_0, m and k the prediction interval, (R_i^ℓ, R_j^ℓ), for $R_{m,k}^Y$ exists if and only if $1 - \left(\frac{k}{k+1}\right)^{m+1} \geqslant \alpha_0$. Now, we obtain an optimal prediction interval for $R_{m,k}^Y$, based on the observed current records from X-sequence. Let α_0, m and k be fixed, we have to find i_{opt} and j_{opt} such that the expected width of the prediction interval, $(R_{i_{opt}}^\ell, R_{j_{opt}}^\ell)$, be less than any prediction interval, (R_i^ℓ, R_j^ℓ). It should be noted that the expected width and the coefficient of the prediction interval are decreasing in i and increasing in j. With this in mind, the steps of the algorithm for finding the optimal prediction interval can be expressed as:

(1) For a given α_0, m and k, set $i = i_0 = 0$ and $j = i_0 + 1$.
(2) Gradually increase j until $\alpha_2(i_0, j; m, k)$ becomes greater than or equal to α_0.
(3) Calculate the expected width of the prediction interval resulting from step 2.
(4) Set $i = i_0 + 1$, and start with $j = i_0 + 2$ and follow the above procedure.

By the above procedure, we can find all the pairs of (i, j) such that $\alpha_2(i, j; m, k)$ is at least α_0. With comparing the expected width of these intervals, we can find the optimal prediction interval with prediction coefficient at least α_0. Since the prediction coefficient, $\alpha_2(i, j; m, k)$, is distribution free, we calculate the expected width of (R_i^ℓ, R_j^ℓ) for uniform distribution. From (1.2), we readily obtain

$$E_{i,j} = E(U_j^\ell - U_i^\ell) = \frac{1}{2}\left[\left(\frac{2}{3}\right)^i - \left(\frac{2}{3}\right)^j\right],$$

where U_i^ℓ stands for the i-th upper current record from $U(0, 1)$ distribution.

For some selected values of α_0, m and k, by employing the above algorithm, we obtained (i_{opt}, j_{opt}), $E_{i,j}$ and $\alpha_2(i_{opt}, j_{opt}; m, k)$. These results are presented in Table 4. From Table 4, we observe that for $k \geqslant 2$, in the most cases we have to start with $i = 0$ and gradually increase it until $\alpha_2(i_{opt}, j_{opt}; m, k)$ exceeds α_0, resulting in optimal i_{opt} and j_{opt}.

2.3. *PIs based on lower current records*

In this Section we construct a prediction interval for $R_{m,k}^Y$ based on R_i^s and R_j^s, $j > i \geqslant 0$, the *i*-th and the *j*-th lower records, respectively. We suppose that the assumptions of the previous subSection hold. We recall that $R_j^s \leqslant R_i^s$ with probability one, and moreover $R_{m,k}^Y$ and (R_j^s, R_i^s) are independent. Then we readily find that

$$
\begin{aligned}
\alpha_3(i,j;m,k) &= P(R_j^s \leqslant R_{m,k}^Y \leqslant R_i^s) \\
&= 2^j \left\{ 1 - \left(\frac{k}{k+1} \right)^{m+1} \right. \\
&\quad - \sum_{s=0}^{j-1} \sum_{t=0}^{m} \sum_{\ell=s}^{\infty} \frac{C_\ell(s)k^t}{s!} \left[\frac{1}{(k+\ell+1)^{t+1}} - \frac{1}{(k+\ell+2)^{t+1}} \right] \Bigg\} \\
&\quad - 2^i \left\{ 1 - \left(\frac{k}{k+1} \right)^{m+1} \right. \\
&\quad - \sum_{s=0}^{i-1} \sum_{t=0}^{m} \sum_{\ell=s}^{\infty} \frac{C_\ell(s)k^t}{s!} \left[\frac{1}{(k+\ell+1)^{t+1}} - \frac{1}{(k+\ell+2)^{t+1}} \right] \Bigg\}.
\end{aligned} \tag{2.10}
$$

It is obvious that the prediction coefficient $\alpha_3(i,j;m,k)$ is increasing in *j* and decreasing in *i*. For a given α_0, *m* and *k*, the two-sided prediction interval exists if and only if , for the large values of *n*, the inequality $P(R_n^s \leqslant R_{m,k}^Y \leqslant R_0^s) \geqslant \alpha_0$ holds. But in other hand for a large *n*, $\varphi(i,j;m,k) = 1$, thus we have

$$
\max_{i,j} \alpha_3(i,j;m,n) = \left(\frac{k}{k+1} \right)^{m+1} \geqslant \alpha_0
$$

which means we can construct a prediction interval for $R_{m,k}^Y$ with coverage probability of at least α_0 based on current lower records values if $k \geqslant \sqrt[m+1]{\alpha_0} / (1 - \sqrt[m+1]{\alpha_0})$. Table 5 contains the numerical values of $\alpha_3(i,j;m,k)$ for some choice of *m* and *k*. From Table 5, we observe that lower current records are not suitable for constructing the prediction intervals for upper *k*−records, as we expected. It seems more applicable which we use lower current records for predicting lower *k*−records.

3. PIs based on upper *r*-records

In this section, we construct prediction intervals for upper *k*-record of future *Y*-sequence based on observed *k*-records from *X*-sequence. With this in mind for convenience of notations, let us denote the observed *k*-records for *X*-sequence by $R_{n,r}$, $n \geqslant 0$ (use subscript *r* instead of *k*) and consider the interval $(R_{i,r}, R_{j,r})$, $i < j$, as a prediction interval for the *m*-th upper *k*-record from future *Y*-sequence. Since $R_{i,r}$ and $R_{j,r}$ are continuous random variables, and for $i < j$, $R_{i,r} \leqslant R_{j,r}$, with probability one, and moreover $R_{m,k}^Y$ and $(R_{i,r}, R_{j,r})$ are independent, we have

$$
P(R_{i,r} \leqslant R_{m,k}^Y \leqslant R_{j,r}) = P(R_{m,k}^Y \geqslant R_{i,r}) - P(R_{m,k}^Y \geqslant R_{j,r}). \tag{3.1}
$$

By (1.1) and using conditional argument, we have

$$
\begin{aligned}
P(R_{m,k}^Y \geqslant R_{n,r}) &= \sum_{t=0}^{m} \frac{k^t r^{n+1}}{t! n!} \int_0^1 u^{k+r-1}(-\log u)^{t+n} du \\
&= \sum_{t=0}^{m} \frac{k^t r^{n+1}}{t! n!} \int_0^\infty e^{-(k+r)y} y^{t+n} dy \\
&= \sum_{t=0}^{m} \binom{n+t}{t} \left(\frac{k}{k+r}\right)^t \left(\frac{r}{k+r}\right)^{n+1}.
\end{aligned}
\tag{3.2}
$$

Hence by substituting (3.2) in (3.1), we find the prediction coefficient of the interval $(R_{i,r}, R_{j,r})$, which is given by

$$
\alpha_4(i,j;m,r,k) = F_{NB}\left(m, i+1, \frac{r}{k+r}\right) - F_{NB}\left(m, j+1, \frac{r}{k+r}\right),
\tag{3.3}
$$

where $F_{NB}(.,.,.)$ is given by (2.2). It may noted that if $k = r$ then $\alpha_4(i,j;m,\ell,k)$, does not depend on k.

Notice that, for a given α_0, m, r and k, the two sided prediction interval exists if and only if , for a large n,

$$
P(R_{0,r} \leqslant R_{m,k}^Y \leqslant R_{n,r}) \geqslant \alpha_0.
$$

From (3.3), this condition is equivalent to

$$
\max_{i,j} \alpha_4(i,j;m,r,k) = 1 - \left(\frac{k}{k+r}\right)^{m+1} \geqslant \alpha_0.
\tag{3.4}
$$

Thus, we can construct a prediction interval for $R_{m,k}^Y$ based on observed r-records from X-sequence with a coverage probability of at least α_0 if the inequality (3.4) holds. We have obtained the numerical values of $\alpha_4(i,j;m,r,k)$ for some selected choices of i,j,m,r,k. These are presented in Table 6. From Table 6, we observe that, for fixed r,m and k the prediction coefficient $\alpha_4(i,j;m,r,k)$ is increasing in j and decreasing in i.

With following the procedure as described in Section 2, we can find the optimal prediction interval for $R_{m,k}^Y$ based on the event $(R_{i,r}, R_{j,r})$. For some selected values of m,k and r, we obtain the optimal prediction intervals and present them in Table 7. In Table 7,

$$
E'_{i,j} = E(U_{j,r} - U_{i,r}) = \left(\frac{k}{k+1}\right)^i - \left(\frac{k}{k+1}\right)^j,
$$

where $U_{i,r}$ stands for the i-th upper r-record from Uniform $(0, 1)$ distribution.

4. Numerical example

For illustrating the proposed procedure in this paper, we use the data set in Arnold *et al.* [5] (pp. 49-50) which represent the record values of the average July temperatures (in degrees centigrade) of Neurenburg, Switzerland, during the period 1864-1993, the extracted current records and r-records, $r = 1, 2, 3, 4$, are given in Table 8.

Based on the data in Table 8, we obtained prediction intervals for the future k-records with shortest length based on three record data set, namely, record coverage, upper current records and r-records.

These are presented in Table 9. Using Table 9, it is observed that when $r = k$ and the three mentioned data set are available, the length of prediction intervals in terms of r-records are smaller than others. It is also observed that for a given α_0, the optimal prediction interval for $R_{m,k}$, based on record coverage has more width than the optimal prediction interval based on upper current records or upper r-records, which support our intuition.

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Appendix

Table 1. The values of $\alpha_1(i;m,k)$ for $i = 1,\ldots,10$ and some selected choices of m and k.

k	m	i 1	2	3	4	5	6	7	8	9	10
1	1	0.2777	0.4669	0.6051	0.7083	0.7857	0.8435	0.8864	0.9180	0.9411	0.9578
	2	0.1759	0.3025	0.4154	0.5183	0.6098	0.6888	0.7552	0.8097	0.8536	0.8884
	3	0.1003	0.1770	0.2591	0.3463	0.4344	0.5193	0.5980	0.6685	0.7302	0.7829
	4	0.0542	0.0982	0.1524	0.2172	0.2896	0.3662	0.4435	0.5185	0.5890	0.6536
	5	0.0285	0.0527	0.0861	0.1298	0.1831	0.2441	0.3106	0.3799	0.4496	0.5176
2	1	0.3888	0.6429	0.7970	0.8867	0.9376	0.9660	0.9816	0.9901	0.9947	0.9971
	2	0.3425	0.5713	0.7245	0.8260	0.8921	0.9342	0.9605	0.9766	0.9863	0.9920
	3	0.2700	0.4578	0.6051	0.7209	0.8086	0.8722	0.9166	0.9467	0.9665	0.9793
	4	0.2008	0.3471	0.4812	0.6025	0.7060	0.7892	0.8530	0.8999	0.9332	0.9563
	5	0.1443	0.2544	0.3696	0.4865	0.5964	0.6928	0.7729	0.8364	0.8848	0.9205
	6	0.1014	0.1821	0.2766	0.3817	0.4890	0.5909	0.6819	0.7592	0.8220	0.8712
	8	0.0481	0.0892	0.1460	0.2191	0.3049	0.3978	0.4921	0.5824	0.6649	0.7371
	10	0.0221	0.0420	0.0729	0.1172	0.1752	0.2453	0.3246	0.4088	0.4939	0.5760
3	1	0.4050	0.6676	0.8214	0.9064	0.9518	0.9754	0.9876	0.9937	0.9968	0.9984
	2	0.4117	0.6780	0.8318	0.9150	0.9581	0.9797	0.9903	0.9954	0.9978	0.9990
	3	0.3736	0.6201	0.7765	0.8725	0.9294	0.9619	0.9799	0.9896	0.9947	0.9973
	4	0.3190	0.5364	0.6937	0.8056	0.8815	0.9303	0.9602	0.9779	0.9880	0.9936
	5	0.2626	0.4482	0.6016	0.7257	0.8196	0.8862	0.9306	0.9590	0.9764	0.9867
	7	0.1666	0.2934	0.4248	0.5543	0.6706	0.7668	0.8412	0.8955	0.9333	0.9586
	9	0.1005	0.1822	0.2823	0.3962	0.5132	0.6232	0.7193	0.7981	0.8594	0.9049
	10	0.0772	0.1417	0.2262	0.3282	0.4392	0.5496	0.6514	0.7394	0.8114	0.8675
4	1	0.3911	0.6464	0.8010	0.8905	0.9408	0.9684	0.9833	0.9913	0.9954	0.9976
	2	0.4314	0.7076	0.8594	0.9356	0.9715	0.9877	0.9948	0.9978	0.9990	0.9996
	3	0.4241	0.6968	0.8498	0.9288	0.9674	0.9855	0.9936	0.9973	0.9988	0.9995
	4	0.3919	0.6486	0.8056	0.8968	0.9473	0.9740	0.9876	0.9942	0.9973	0.9988
	5	0.3487	0.5830	0.7434	0.8496	0.9160	0.9550	0.9768	0.9884	0.9944	0.9973
	7	0.2575	0.4416	0.5991	0.7287	0.8266	0.8946	0.9387	0.9657	0.9815	0.9902
	9	0.1800	0.3170	0.4578	0.5939	0.7124	0.8066	0.8758	0.9234	0.9545	0.9739
	10	0.1486	0.2650	0.3941	0.5274	0.6508	0.7546	0.8354	0.8941	0.9344	0.9607
5	1	0.3684	0.6121	0.7681	0.8652	0.9236	0.9576	0.9769	0.9876	0.9934	0.9965
	2	0.4285	0.7033	0.8555	0.9326	0.9696	0.9866	0.9942	0.9975	0.9989	0.9995
	3	0.4438	0.7262	0.8762	0.9474	0.9786	0.9916	0.9968	0.9988	0.9995	0.9998
	4	0.4318	0.7085	0.8607	0.9369	0.9726	0.9885	0.9953	0.9981	0.9992	0.9997
	5	0.4041	0.6673	0.8240	0.9115	0.9575	0.9804	0.9913	0.9962	0.9984	0.9993
	7	0.3296	0.5547	0.7183	0.8325	0.9062	0.9503	0.9749	0.9878	0.9943	0.9974
	9	0.2538	0.4369	0.5973	0.7308	0.8313	0.9002	0.9439	0.9699	0.9844	0.9922
	10	0.2197	0.3825	0.5373	0.6757	0.7869	0.8679	0.9222	0.9563	0.9764	0.9877

Table 2. The minimum values of i which satisfy the inequality $\alpha_1(i,m;k) \geqslant 0.90$.

k $\quad$ m	2	3	4	5	6	7	8	9	10
1	11	14	17	-	-	-	-	-	-
2	6	7	9	10	11	13	14	15	16
3	4	5	6	7	8	9	9	10	11
4	4	4	5	5	6	7	7	8	9
5	4	4	4	4	5	5	6	6	7

Table 3. The values of $\alpha_2(i,j;m,k)$ for $k=1,2,3$ and some selected choices of i,j and m.

k	m	i	j=4	6	8	10	12	14	16	18	20
1	1	0	0.4701	0.5963	0.6687	0.7080	0.7288	0.7394	0.7447	0.7474	0.7487
		1	0.3312	0.4574	0.5298	0.5692	0.5899	0.6005	0.6059	0.6085	0.6098
		2	0.2016	0.3278	0.4001	0.4395	0.4602	0.4709	0.4762	0.4789	0.4802
	3	0	0.2840	0.4568	0.6061	0.7205	0.8012	0.8548	0.8887	0.9093	0.9216
		1	0.2339	0.4067	0.5559	0.6703	0.7510	0.8046	0.8385	0.8592	0.8714
		2	0.1665	0.3393	0.4885	0.6029	0.6837	0.7372	0.7711	0.7918	0.8040
	5	0	0.1142	0.2285	0.3643	0.5020	0.6266	0.7300	0.8099	0.8685	0.9095
		1	0.0999	0.2142	0.3500	0.4878	0.6124	0.7157	0.7957	0.8543	0.8953
		2	0.0769	0.1912	0.3270	0.4647	0.5894	0.6927	0.7727	0.8312	0.8723
	7	0	0.0378	0.0910	0.1725	0.2770	0.3942	0.5122	0.6216	0.7162	0.7934
		1	0.0341	0.0873	0.1688	0.2733	0.3904	0.5085	0.6179	0.7124	0.7896
		2	0.0274	0.0806	0.1621	0.2666	0.3837	0.5018	0.6112	0.7057	0.7829
		3	0.0164	0.0696	0.1511	0.2556	0.3728	0.4908	0.6002	0.6948	0.7720
2	1	0	0.4791	0.5312	0.5481	0.5533	0.5549	0.5553	0.5555	0.5555	0.5555
		1	0.2847	0.3368	0.3537	0.3589	0.3604	0.3609	0.3610	0.3610	0.3611
		2	0.1458	0.1979	0.2148	0.2200	0.2215	0.2220	0.2221	0.2222	0.2222
	3	0	0.5254	0.6749	0.7492	0.7818	0.7948	0.7998	0.8015	0.8021	0.8023
		1	0.3904	0.5398	0.6142	0.6468	0.6598	0.6647	0.6665	0.6671	0.6673
		2	0.2453	0.3948	0.4691	0.5017	0.5148	0.5197	0.5214	0.5220	0.5222
	5	0	0.3989	0.6050	0.7486	0.8327	0.8763	0.8969	0.9060	0.9098	0.9113
		1	0.3267	0.5329	0.6764	0.7606	0.8042	0.8248	0.8338	0.8376	0.8391
		2	0.2292	0.4354	0.5790	0.6631	0.7067	0.7273	0.7364	0.7401	0.7416
	7	0	0.2532	0.4518	0.6337	0.7703	0.8587	0.9098	0.9368	0.9501	0.9563
		1	0.2181	0.4167	0.5986	0.7352	0.8236	0.8747	0.9017	0.9150	0.9212
		2	0.1635	0.3621	0.5440	0.6806	0.7690	0.8201	0.8471	0.8604	0.8666
		3	0.0894	0.2880	0.4700	0.6065	0.6949	0.7461	0.7731	0.7864	0.7925
3	1	0	0.4109	0.4320	0.4364	0.4372	0.4374	0.4374	0.4374	0.4374	0.4374
		1	0.2084	0.2295	0.2339	0.2347	0.2349	0.2349	0.2349	0.2349	0.2349
		2	0.0914	0.1125	0.1169	0.1177	0.1179	0.1179	0.1179	0.1179	0.1179
	3	0	0.5626	0.6463	0.6733	0.6810	0.6829	0.6834	0.6835	0.6835	0.6835
		1	0.3757	0.4595	0.4865	0.4941	0.4961	0.4966	0.4967	0.4967	0.4967
		2	0.2095	0.2932	0.3203	0.3279	0.3299	0.3303	0.3305	0.3305	0.3305
	5	0	0.5485	0.7082	0.7811	0.8088	0.8181	0.8209	0.8217	0.8219	0.8220
		1	0.4171	0.5769	0.6497	0.6775	0.6867	0.6896	0.6904	0.6906	0.6906
		2	0.2665	0.4262	0.4991	0.5268	0.5361	0.5389	0.5397	0.5399	0.5400
	7	0	0.4542	0.6667	0.7954	0.8585	0.8850	0.8949	0.8983	0.8994	0.8997
		1	0.3709	0.5834	0.7121	0.7752	0.8017	0.8116	0.8150	0.8161	0.8164
		2	0.2580	0.4705	0.5992	0.6623	0.6888	0.6987	0.7021	0.7032	0.7035
		3	0.1290	0.3414	0.4702	0.5333	0.5598	0.5697	0.5731	0.5742	0.5745

Table 4. The optimal prediction intervals for future upper k-record based on upper current records.

(m,k,α_0)	(i_{opt}, j_{opt})	$E_{i,j}$	$\alpha_2(i_{opt}, j_{opt}; m, k)$
(1, 1, 0.70)	(0, 10)	0.4913	0.7080
(2, 1, 0.80)	(0, 12)	0.4661	0.8117
(3, 1, 0.80)	(2, 20)	0.2220	0.8040
(3, 1, 0.90)	(0, 17)	0.4994	0.9003
(4, 1, 0.80)	(3, 20)	0.1479	0.8097
(5, 1, 0.80)	(5, 27)	0.0658	0.8033
(5, 1, 0.90)	(3, 27)	0.1481	0.9002
(2, 2, 0.70)	(0, 12)	0.4961	0.7011
(3, 2, 0.75)	(0, 9)	0.4869	0.7690
(4, 2, 0.80)	(0, 9)	0.4869	0.8015
(5, 2, 0.80)	(1, 12)	0.3294	0.8042
(5, 2, 0.90)	(0, 15)	0.4988	0.9024
(7, 2, 0.95)	(0, 18)	0.2217	0.9501
(3, 3, 0.65)	(0, 7)	0.4707	0.6638
(4, 3, 0.75)	(0, 9)	0.4869	0.7507
(5, 3, 0.80)	(0, 10)	0.4913	0.8088
(7, 3, 0.85)	(0, 10)	0.4913	0.8585

Table 5. The values of $\alpha_3(i,j;m,k)$ for $k=1,2,3$ and some selected choices of i,j and m.

k	m	i	j 4	5	6	7	8	9	10	11	12
1	1	0	0.2382	0.2442	0.2471	0.2486	0.2493	0.2496	0.2497	0.2497	0.2497
		1	0.0993	0.1053	0.1083	0.1097	0.1104	0.1107	0.1108	0.1109	0.1108
		2	0.0397	0.0457	0.0487	0.0501	0.0508	0.0511	0.0512	0.0513	0.0512
	2	0	0.1234	0.1244	0.1247	0.1249	0.1249	0.1249	0.1249	0.1248	0.1247
		1	0.0354	0.0364	0.0368	0.0369	0.0369	0.0369	0.0369	0.0368	0.0367
		2	0.0106	0.0116	0.0120	0.0121	0.0121	0.0121	0.0121	0.0120	0.0119
2	1	0	0.4075	0.4254	0.4347	0.4395	0.4419	0.44317	0.4437	0.4440	0.4441
		1	0.2131	0.2310	0.2403	0.2450	0.2475	0.2487	0.2493	0.2496	0.2497
		2	0.0979	0.1158	0.1251	0.1299	0.1323	0.1335	0.1341	0.1344	0.1345
	2	0	0.2878	0.2929	0.2949	0.2957	0.2960	0.2961	0.2962	0.2962	0.2961
		1	0.1165	0.1216	0.1236	0.1244	0.1247	0.1248	0.1249	0.1249	0.1248
		2	0.0429	0.0479	0.0499	0.0507	0.0511	0.0512	0.0512	0.0512	0.0512
3	1	0	0.4954	0.5264	0.5434	0.5525	0.5573	0.5598	0.5611	0.5617	0.5620
		1	0.2929	0.3239	0.3409	0.3500	0.3548	0.3573	0.3586	0.3592	0.3595
		2	0.1473	0.1783	0.1953	0.2044	0.2092	0.2117	0.2129	0.2136	0.2139
	3	0	0.3099	0.3141	0.3156	0.3161	0.3163	0.3163	0.3163	0.3163	0.3162
		1	0.1231	0.1273	0.1288	0.1293	0.1295	0.1295	0.1295	0.1295	0.1294
		2	0.0428	0.0470	0.0485	0.0490	0.0492	0.0492	0.0492	0.0492	0.0491
4	1	0	0.5415	0.5850	0.6100	0.6239	0.6315	0.6355	0.6376	0.6387	0.6392
		1	0.3459	0.3894	0.4144	0.4284	0.4359	0.4400	0.4421	0.4432	0.4437
		2	0.1854	0.2289	0.2539	0.2678	0.2754	0.2794	0.2815	0.2826	0.2831
	3	0	0.3961	0.4045	0.4077	0.4089	0.4093	0.4095	0.4095	0.4095	0.4094
		1	0.1841	0.1924	0.1956	0.1968	0.1972	0.1974	0.1974	0.1974	0.1974
		2	0.0721	0.0805	0.0837	0.0849	0.0853	0.0854	0.0855	0.0855	0.0854

Table 6. The values of $\alpha_4(i,j;m,r,k)$ for $i = 0,1,2,3$ and $j = 3,4,\ldots,10$ and some selected choices of m, r and k.

		j	3	4	5	6	7	8	9	10
m	i									
2	0		0.5312	0.6484	0.7304	0.7851	0.8203	0.8422	0.8557	0.8637
	1		0.3437	0.4609	0.5429	0.5976	0.6328	0.6547	0.6682	0.6762
	2		0.1562	0.2734	0.3554	0.4101	0.4453	0.4672	0.4807	0.4887
3	0		0.4375	0.5742	0.6835	0.7656	0.8242	0.8645	0.8913	0.9088
	1		0.3125	0.4492	0.5585	0.6406	0.6992	0.7395	0.7663	0.7838
	2		0.1562	0.2929	0.4023	0.4843	0.5429	0.5832	0.6101	0.6275
4	0		0.3320	0.4687	0.5917	0.6943	0.7749	0.8353	0.8789	0.9095
	1		0.2539	0.3906	0.5136	0.6162	0.6967	0.7572	0.8008	0.8313
	2		0.1367	0.2734	0.3964	0.4990	0.5795	0.6400	0.6836	0.7142

$k = 2$, $r = 3$

		j	3	4	5	6	7	8	9	10
1	0		0.5030	0.6067	0.6813	0.7336	0.7694	0.7936	0.8097	0.8204
	1		0.3110	0.4147	0.4893	0.5416	0.5774	0.6016	0.6177	0.6284
	2		0.1382	0.2419	0.3165	0.3688	0.4046	0.4288	0.4449	0.4556
2	0		0.3916	0.5160	0.6206	0.7042	0.7687	0.8170	0.8525	0.8780
	1		0.2764	0.4008	0.5054	0.5890	0.6535	0.7018	0.7373	0.7628
	2		0.1382	0.2626	0.3671	0.4507	0.5152	0.5636	0.5991	0.6246
3	0		0.2641	0.3803	0.4917	0.5921	0.6781	0.7490	0.8058	0.8500
	1		0.2027	0.3188	0.4303	0.5306	0.6166	0.6876	0.7443	0.7886
	2		0.1105	0.2267	0.3381	0.4385	0.5245	0.5954	0.6522	0.6964

$k = 3$, $r = 2$

		j	3	4	5	6	7	8	9	10
2	0		0.6048	0.6877	0.7341	0.7589	0.7717	0.7780	0.7811	0.7826
	1		0.3456	0.4285	0.4749	0.4997	0.5125	0.5188	0.5219	0.5234
	2		0.1382	0.2211	0.2676	0.2924	0.3051	0.3115	0.3146	0.3161
3	0		0.5806	0.6967	0.7710	0.8156	0.8411	0.8551	0.8626	0.8664
	1		0.3732	0.4893	0.5636	0.6082	0.6337	0.6477	0.6552	0.6591
	2		0.1658	0.2820	0.3563	0.4009	0.4263	0.4404	0.4478	0.4517
4	0		0.5163	0.6556	0.7560	0.8228	0.8649	0.8901	0.9047	0.9128
	1		0.3608	0.5001	0.6004	0.6673	0.7094	0.7346	0.7492	0.7573
	2		0.1741	0.3135	0.4138	0.4807	0.5227	0.5480	0.5625	0.5707

Table 7. The optimal prediction intervals for future upper *k*-record based on *r*-records.

(m, k, r, α_0)	(i_{opt}, j_{opt})	$E'_{i,j}$	$\alpha_4(i_{opt}, j_{opt}; m, r, k)$
(2, 1, 1, 0.80)	(0, 7)	0.9921	0.8203
(2, 1, 1, 0.85)	(0, 9)	0.9980	0.8557
(3, 2, 2, 0.85)	(0, 8)	0.9609	0.8645
(3, 2, 2, 0.90)	(0, 10)	0.9826	0.9088
(4, 2, 2, 0.70)	(2, 10)	0.4271	0.7142
(4, 2, 2, 0.85)	(1, 11)	0.6551	0.8522
(4, 2, 2, 0.95)	(0, 13)	0.9948	0.9533
(1, 2, 3, 0.80)	(0, 9)	0.9249	0.8097
(2, 2, 3, 0.80)	(1, 13)	0.7262	0.8024
(4, 2, 3, 0.80)	(3, 18)	0.4162	0.8073
(2, 3, 2, 0.70)	(0, 5)	0.8683	0.7341
(3, 3, 2, 0.80)	(0, 6)	0.9122	0.8156
(4, 3, 2, 0.70)	(1, 7)	0.6081	0.7094
(4, 3, 2, 0.90)	(0, 9)	0.9739	0.9047
(7, 3, 2, 0.95)	(0, 12)	0.9922	0.9602
(5, 4, 3, 0.80)	(1, 9)	0.6749	0.8204
(5, 4, 3, 0.95)	(0, 12)	0.9683	0.9537
(7, 4, 3, 0.95)	(0, 12)	0.9683	0.9500

Table 8. The current records and *r*-records, $r = 1, 2, 3, 4$ extracted from Arnold *et al.* (1998, pp. 49–50).

m	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
R^s_m	19.0	19.0	18.4	17.4	17.4	17.4	17.2	15.6	15.6	15.6	15.6	15.3	15.3	15.3	-
R^l_m	19.0	20.1	20.1	20.1	21.0	21.4	21.4	21.4	21.7	22.0	22.1	22.1	22.6	23.4	-
$R_{m,1}$	19.0	20.1	21.0	21.4	21.7	22.0	22.1	22.6	23.4	-	-	-	-	-	-
$R_{m,2}$	19.0	19.7	20.1	21.0	21.4	21.7	22.0	22.1	22.3	22.6	-	-	-	-	-
$R_{m,3}$	18.4	19.0	19.7	20.1	20.4	20.9	21.0	21.4	21.7	22.0	22.1	22.3	-	-	-
$R_{m,3}$	17.4	18.4	19.0	19.7	19.9	20.1	20.4	20.9	21.0	21.4	21.5	21.6	21.7	22.0	22.1

Table 9. Prediction intervals based on the data set in Table 8.

(m, k, r, α_0)	(R^s_i, R^l_i)	$\alpha_1(i; m, k)$	(R^l_i, R^l_j)	$\alpha_2(i, j; m, k)$	$(R_{i,r}, R_{j,r})$	$\alpha_4(i, j, r; m, k)$
(2, 2, 1, 0.70)	(17.4, 20.1)	0.7245	(19.0, 22.6)	0.7011	(19.0, 22.1)	0.7002
(2, 2, 2, 0.70)	(17.4, 20.1)	0.7245	(19.0, 22.6)	0.7011	(19.0, 21.7)	0.7304
(2, 2, 3, 0.70)	(17.4, 20.1)	0.7245	(19.0, 22.6)	0.7011	(19.0, 22.3)	0.7018
(3, 2, 1, 0.75)	(17.4, 21.4)	0.8086	(19.0, 22.0)	0.7690	(19.0, 21.7)	0.7600
(3, 2, 2, 0.75)	(17.4, 21.4)	0.8086	(19.0, 22.0)	0.7690	(19.7, 22.6)	0.7663
(4, 3, 1, 0.70)	(17.4, 21.0)	0.8056	(19.0, 21.4)	0.7230	(18.4, 20.4)	0.7137
(4, 3, 2, 0.70)	(17.4, 21.0)	0.8056	(19.0, 21.4)	0.7230	(19.0, 21.4)	0.7094
(4, 3, 3, 0.70)	(17.4, 21.0)	0.8056	(19.0, 21.4)	0.7230	(19.7, 22.1)	0.7142
(5, 3, 1, 0.75)	(17.4, 21.4)	0.8196	(19.0, 21.4)	0.7527	(18.4, 20.9)	0.7876
(5, 3, 2, 0.75)	(17.4, 21.4)	0.8196	(19.0, 21.4)	0.7527	(18.4, 21.0)	0.7951
(5, 3, 3, 0.75)	(17.4, 21.4)	0.8196	(19.0, 21.4)	0.7527	(19.7, 22.1)	0.7504