Chapter 2
The AK Model

April 2, 2008

1 Introduction

The neoclassical model presented in the previous chapter takes the rate of technological change as being determined exogenously, by non-economic forces. There is good reason, however, to believe that technological change depends on economic decisions, because it comes from industrial innovations made by profit-seeking firms, and depends on the funding of science, the accumulation of human capital and other such economic activities. Technology is thus an endogenous variable, determined within the economic system. Growth theories should take this endogeneity into account, especially since the rate of technological progress is what determines the long-run growth rate.

Incorporating endogenous technology into growth theory forces us to deal with the difficult phenomenon of increasing returns to scale. More specifically, people must be given an incentive to improve technology. But because the aggregate production function F exhibits constant returns in K and L alone, Euler's theorem tells us that it will take all of the economy's output to pay capital and labor their marginal products in producing final output, leaving nothing over to pay for the resources used in improving technology.[1] Thus a theory

[1] Euler's Theorem states that if F is homogeneous of degree 1 in K and L (the definition of constant returns) then

$$F_1(K,L)K + F_2(K,L)L = F(K,L) \tag{E}$$

where F_i is the partial derivative with respect to the i^{th} argument. The marginal products of K and L are F_1 and F_2 respectively. So if K and L are paid their marginal products then the left-hand side is the total payment to capital (price F_1 times quantity K) plus the total payment to labor, and the equation states that these payments add up to total output.

To verify Euler's Theorem take the equation

$$F(\lambda K, \lambda L) = \lambda F(K,L) \tag{C}$$

that defines homogeneity of degree one, and differentiate both sides with respect to λ at the point $\lambda = 1$. Since (C) must hold for all $\lambda > 0$ therefore the two derivatives must be equal, which implies (E).

of endogenous technology cannot be based on the usual theory of competitive equilibrium, which requires that all factors be paid their marginal products.

Arrow's (1962) solution to this problem was to suppose that technological progress is an unintended consequence of producing new capital goods, a phenomenon dubbed "learning by doing." Learning by doing was assumed to be purely external to the firms responsible for it. That is, if technological progress depends on the aggregate production of capital, and firms are all very small, then they can be assumed all to take the rate of technological progress as being given independently of their own production of capital goods. So each firm maximizes profit by paying K and L their marginal products, without offering any additional payment for their contribution to technological progress.

Learning by doing formed the basis of the first model of endogenous growth theory, which is know as the AK model. The AK model assumes that when people accumulate capital, learning by doing generates technological progress that tends to raise the marginal product of capital, thus offsetting the tendency for the marginal product to diminish when technology is unchanged. The model results in a production function of the form $Y = AK$, in which the marginal product of capital is equal to the constant A.

The AK model predicts that a country's long-run growth rate will depend on economic factors such as thrift and the efficiency of resource allocation. In subsequent chapters we will develop alternative models of endogenous growth that emphasize not thrift and efficiency but creativity and innovation, which see as the main driving forces behind economic growth. But given its historical place as the first endogenous growth model, the AK paradigm is an important part of any economist's toolkit. Accordingly we devote this chapter to developing the AK model and to summarizing the empirical debate that took place in the 1990s between its proponents and proponents of the neoclassical model of Solow and Swan.

1.1 The Harrod-Domar model

An early precursor of the AK model was the Harrod-Domar model,[2] which assumes that the aggregate production function has fixed technological coefficients:

$$Y = F(K, L) = \min\{AK, BL\},$$

where A and B are the fixed coefficients. Under this technology, producing a unit of output requires $1/A$ units of capital and $1/B$ units of labor; if either input falls short of this minimum requirement there is no way to compensate by substituting the other input.

[2]See Harrod (1939) and Domar (1946).

With a fixed-coefficient technology, there will either be surplus capital or surplus labor in the economy, depending on whether the historically given supply of capital is more or less than (B/A) times the exogenous supply of labor. When $AK < BL$, which is the case that Harrod and Domar emphasize, capital is the limiting factor. Firms will produce the amount

$$Y = AK,$$

and hire the amount $(1/B)\,Y = (1/B)\,AK < L$ of labor.

Now, with a fixed saving rate, we know that the capital stock will grow according to the same equation as in the neoclassical model:

$$\dot{K} = sY - \delta K. \tag{1}$$

These last two equations imply:

$$\dot{K} = sAK - \delta K,$$

so that the growth rate of capital will be:

$$g = \dot{K}/K = sA - \delta.$$

Because output is strictly proportional to capital, g will also be the rate of growth of output. It follows immediately that the growth rate of output is increasing in the saving rate s.

The problem with the Harrod-Domar model is that it cannot account for the sustained growth in output per person that has taken place in the world economy since the Industrial Revolution. To see this, let ν be the rate of population growth. Then the growth rate of output per person is $g - \nu$. But if this is positive, then so is the growth rate of capital per person K/L, since K also grows at the rate g. Eventually a point will be reached where capital is no longer the limiting factor in the production function. That is, K/L will eventually exceed the limit B/A above which labor becomes the limiting factor. From then on we will instead have $Y = BL$, implying that Y will grow at the same rate as L; that is, output per person Y/L will cease to grow.

2 The Frankel model

2.1 Basic setup

The first AK model that could account for sustained growth in per-capita output was that of Frankel (1962), who was motivated by the challenge of constructing a model that would combine the virtues of the Solow-Swan and Harrod-Domar models. As in Solow-Swan, this model would display perfect competition, substitutable factors (with Cobb-Douglas production technologies) and full employment. As in Harrod-Domar, the model would generate a long-run growth rate that depends on the saving rate.

Frankel built his model on the foundation of learning by doing. He recognized that because individual firms contribute to the accumulation of technological knowledge when they accumulate capital,[3] the AK structure of the Harrod-Domar model does not require fixed coefficients. Instead, he assumed that each firm $j \in \{1, 2, .., N\}$ has a production function of the form

$$y_j = \overline{A} k_j^{\alpha} L_j^{1-\alpha},$$

where k_j and L_j are the firm's own employment of capital and labor, and $\overline{A}$ is (aggregate) productivity. Aggregate productivity in turn depends upon the total amount of capital that has been accumulated by all firms, namely:

$$\overline{A} = A_0 \cdot \left(\sum_{j=1}^{N} k_j \right)^{\eta},$$

where η is a positive exponent that reflects the extent of the knowledge externalities generated among firms.

For simplicity assume that

$$L_j = 1$$

for all j, let

$$K = \sum_{j=1}^{N} k_j$$

denote the aggregate capital stock, and let

$$Y = \sum_{j=1}^{N} y_j$$

denote the aggregate output flow.

Since all firms face the same technology and the same factor prices, they will hire factors in the same proportions, so that

$$k_j = K/N \text{ for all } j.$$

[3] He called it "development" rather than "knowledge."

This in turn implies that in equilibrium

$$\overline{A} = A_0 K^{\eta};$$

hence individual outputs are all equal to

$$y_j = A_0 K^{\eta} (K/N)^{\alpha}$$

and therefore aggregate output is

$$Y = N A_0 K^{\eta} (K/N)^{\alpha}$$

which can be written as

$$Y = A K^{\alpha+\eta}. \tag{2}$$

where $A = A_0 N^{1-\alpha}$.

The model is then closed by assuming a constant saving rate, which generates the same capital accumulation equation (1) as in Solow-Swan and Harrod-Domar. Using the output equation (2) to substitute for Y in this equation we have

$$\dot{K} = s A K^{\alpha+\eta} - \delta K$$

so the growth rate of the capital stock is

$$g_K = \dot{K}/K = s A K^{\alpha+\eta-1} - \delta \tag{3}$$

2.2 Three cases

We now analyze the dynamic path of the economy defined by equation (3). Three cases must be considered.

1. $\alpha + \eta < 1$

   In this case the extent of knowledge spillovers η is not sufficiently strong to counteract the effect $1 - \alpha$ of decreasing returns to individual capital accumulation, and the long-run growth rate is zero. The case produces the same aggregate dynamics as the Solow-Swan model with no technological progress and no population growth, which we analyzed in the previous chapter. That is, according to (3) there is a steady state capital stock at which the growth rate g_K of capital is zero, namely

$$K^* = (sA/\delta)^{1/(1-\alpha-\eta)} \tag{4}$$

If K were to rise above K^* the growth rate would turn negative, since in this case (3) makes g_K a decreasing function of K. Thus K would fall back to its steady state, at which the growth rate of capital is zero and therefore the growth rate of output (2) is zero.

2. $\alpha + \eta > 1$

   In this case learning externalities are so strong that the aggregate economy experiences an ever-increasing growth rate. That is, again (4) defines a unique steady-state capital stock but it is no longer stable, because g_K is now an increasing function of K, so that if K were to rise above K^* it would keep on rising, at an ever-increasing rate. This is known as the "explosive growth" case.

3. $\alpha + \eta = 1$

   In this knife-edge case, learning externalities exactly compensate decreasing returns to individual capital accumulation, so that the aggregate production function becomes an AK function, namely:

   $$Y = AK.$$

   Thus the aggregate growth rate becomes:

   $$g = \dot{K}/K = sA - \delta$$

   which is nothing but the Harrod-Domar growth rate, now obtained as the long-run growth rate in a model with substitutable factors and full market clearing. In other words, as capital increases, output increases in proportion, even though there is continual full employment of labor and even though there is substitutability in the aggregate production function, because knowledge automatically increases by just the right amount. Unlike in the Harrod-Domar model, here an increase in the saving propensity s will increase the growth rate permanently even though output per person is growing at a positive rate (namely $g - 0 = g$).

3 An AK model with intertemporal utility maximization

Romer (1986)[4] developed a Ramsey version of the AK model, in which the constant saving rate is replaced by intertemporal utility maximization by a representative individual, again with the assumption that individuals do not internalize the externalities associated with the growth of knowledge.

3.1 The setup

Romer assumed a production function with externalities of the same sort as considered by Frankel, and focused on the case in which the labor supply per firm was equal to unity and the rate of capital depreciation was zero ($\delta = 0$). Saving is determined by the owner of the representative one-worker firm, whose dynamic optimization problem is to

$$\max \int_0^\infty u(c_t) e^{-\rho t} dt$$

$$\text{subject to: } \dot{k} = \overline{A} k^\alpha - c$$

where k is the capital stock of the individual firm, $y = \overline{A} k^\alpha$ is its output, $c = c_t$ is the current consumption of its owner-worker, and $\overline{A}$ denotes aggregate productivity which is taken as given by each individual firm.[5]

As in the previous section, aggregate productivity depends upon the aggregate capital stock, namely:

$$\overline{A} = A_0 K^\eta,$$

where

$$K = \sum\nolimits_1^N k_j.$$

Assuming a constant intertemporal elasticity of substitution as in the previous chapter,

[4] Romer actually laid out more than an AK model, in as much as his approach allowed for general production and utility functions and assumed that there were strictly *increasing* social returns to capital. What we present here is the limiting special case that many followers have extracted from Romer's analysis, in which there are constant social returns to capital and an isoelastic utility function.

[5] This maximization problem is the same as that of the Cass-Koopmans-Ramsey model analyzed in the previous chapter, but in the limiting case where the length of the time period has shrunk to zero, so time has become continuous.

namely $u(c) = \frac{c^{1-\varepsilon}-1}{1-\varepsilon}$, one obtains the Euler condition[6]

$$-\varepsilon\dot{c}/c = \rho - \alpha\overline{A}k^{\alpha-1}.$$

Having rational expectations, individuals correctly anticipate that all firms will employ the same capital in equilibrium (given that these firms are all identical), so

$$K = Nk$$

and therefore the above Euler condition can be written as

$$-\varepsilon\dot{c}/c = \rho - \alpha A_0 N^{\eta} k^{\alpha+\eta-1}. \tag{5}$$

3.2 Long-run growth

Aggregate output Y is given by the same equation (2) as in the Frankel model, because:

$$Y = Ny = NA_0K^{\eta}\left(K/N\right)^{\alpha} = AK^{\alpha+\eta},$$

and again there are three cases to consider depending on the exponent $\alpha + \eta$:

1. $\alpha + \eta < 1$

   In the case of decreasing returns, again growth will vanish asymptotically as in the neoclassical model without technological progress. To see this, assume, on the contrary, that the growth rate is bounded above zero. The following argument shows that this assumption leads to a contradiction. Positive growth implies that the capital stock k will converge to infinity over time, which implies that the right hand side of (5) must converge to ρ, since the exponent $\alpha + \eta - 1$ is negative, and this in turn implies that the growth rate $\dot{c}/c$ will become negative, which contradicts our assumption of positive growth.

2. $\alpha + \eta > 1$

   In the case of increasing returns to capital, then as in the Frankel model there will be explosive growth. This can be seen using the Euler equation (5). Specifically, if growth is positive in the long run, then the capital stock k converges to infinity over time. This, together with the fact that $\alpha + \eta > 1$, implies that the right hand side of

[6] This Euler condition follows from our analysis of the Ramsey model in the previous chapter, but where in this case the net private marginal product of capital is: $F_1(k, A) - \delta = \alpha\overline{A}k^{\alpha-1} - 0 = \alpha\overline{A}k^{\alpha-1}$.

(5) converges to negative infinity, which in turn implies that the growth rate $\dot{c}/c$ must converge to infinity.

3. $\alpha + \eta = 1$

   In the AK case where there are constant social returns to capital, then as in the Frankel model the economy will sustain a strictly positive but finite growth rate g, in which diminishing private returns to capital are just offset by the external improvements in technology $\overline{A}$ that they bring about. More precisely, in a steady state consumption and output will grow at the same rate, so this case (5) implies

$$g = \dot{c}/c = \left(\alpha A_0 N^\eta - \rho\right)/\varepsilon.$$

   In particular, we see that the higher the discount rate ρ (that is the lower the propensity to save), or the lower the intertemporal elasticity of substitution measured by $1/\varepsilon$, the lower will be the steady-state growth rate g.

3.3 Welfare

So far we have just reproduced the results already generated by the model with a constant saving rate. However, moving to a Ramsey model where savings behavior results from explicit intertemporal utility maximization allows us to also conduct a welfare analysis. In particular one can show that, because individuals and individual firms do not internalize the effect of individual capital accumulation on knowledge $\overline{A}$ when optimizing on c and k, the equilibrium growth rate $g = \left(\alpha A_0 N^\eta - \rho\right)/\varepsilon$ is less than the socially optimal rate of growth. More formally, the social planner who internalizes the knowledge externalities induced by individual capital accumulation, would solve the dynamic program

$$\max \int_0^\infty e^{-\rho t} u(c_t) dt$$

$$\text{s.t.} \quad \dot{k} = A_0 (Nk)^\eta k^\alpha - c,$$

that is, he would internalize the fact that $\overline{A} = A_0(Nk)^\eta$ when choosing k. When $u(c) = \frac{c^{1-\varepsilon}-1}{1-\varepsilon}$, we obtain the Euler equation

$$-\varepsilon \dot{c}/c = \rho - (\alpha+\eta) A_0 N^\eta k^{\alpha+\eta-1}$$

With constant social returns to capital ($\alpha + \eta = 1$), this yields the socially optimal rate of growth

$$g^* = (N^\eta A_0 - \rho)/\varepsilon > g = (\alpha N^\eta A_0 - \rho)/\varepsilon.$$

3.4 Concluding remarks

First, although growth has been endogenized, it relies entirely on *external* (and therefore unremunerated) accumulation of knowledge. Introducing rewards to technological progress as we shall do in the following chapters, adds a new dimension of complexity, because it moves us away from a world of perfect competition into a world of imperfect competition among large individual firms.

Second, in the case where $\alpha + \eta = 1$, cross-country variations in parameters such as α and ρ will result in permanent differences in rates of economic growth. Thus, the simple AK approach does *not* predict conditional convergence in income per capita; the cross-section distribution of income should instead exhibit both absolute and conditional divergence. We shall return to this issue in the next sections of the chapter.

4 The debate between neoclassical and AK advocates in a nutshell

In this section, we briefly reflect on a now closed debate between advocates of the neoclassical approach and those of the AK model. A first argument in favor of the AK approach is that it can account for the persistently positive growth rates of per-capita GDP which we observe in most countries worldwide, whereas the neoclassical model cannot explain it.

However, advocates of the neoclassical model can argue that the AK model cannot explain cross-country or cross-regional convergence. Two main types of convergence appear in the discussions about growth across regions or countries. *Absolute convergence* takes place when poorer areas grow faster than richer ones whatever their respective characteristics. On the other hand, as we already saw in the previous chapter, there is *conditional convergence* when a country (or a region) grows faster the farther it is below its own steady state; or equivalently, if we take two countries or regions with identical savings rates, depreciation rates and aggregate production technologies, the country that begins with lower output per capita has a higher growth rate than the country that begins with higher output per capita. This latter form of convergence is definitely the weaker.

Now consider two regions within a country (for example the US) or two countries that share the same underlying characteristics (in particular the same saving rate and the same depreciation rate of capital). Under constant returns in the aggregate production function, as in AK models, per-capita GDP in a country or region with lower initial capital stock will never converge to that of countries with higher initial capital stock, even in the weak sense of conditional convergence, simply because these two countries will always grow at the same rate independently of their amounts of accumulated capital. On the other hand, with diminishing returns to capital, the level of income per capita should converge toward its steady-state value, with the speed of convergence increasing in the distance to the steady state. In other words, lower initial values of income per capita should generate higher transitional growth rates, once the determinants of the steady state are controlled for. The following figure, drawn from Barro and Sala-i-Martin (1995) shows a clear conditional convergence pattern among US states. This, in turn, questions the AK approach.

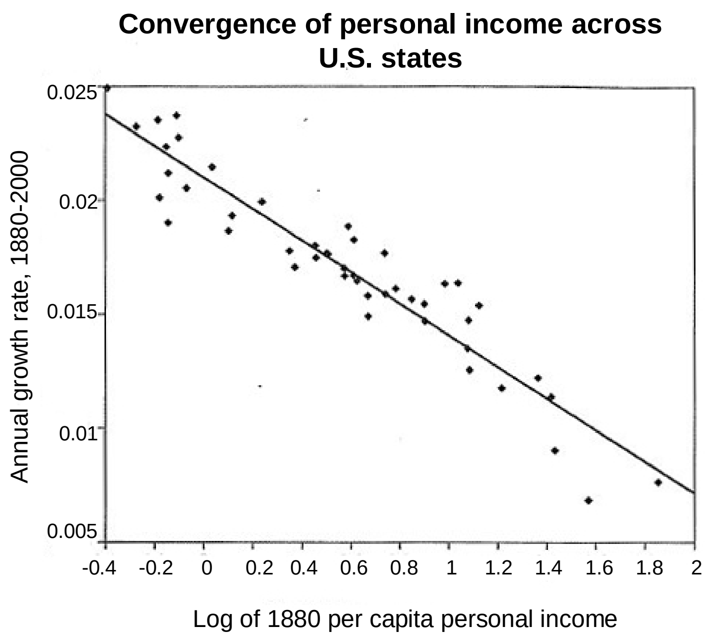

Figure 2.1

Turning to the evidence on convergence of countries, we find that in some countries (in particular China and the Asian tigers, namely Singapore, Hong Kong, Taiwan and Korea) per-capita GDP manages to converge towards per-capita GDP levels in industrialized countries. This again may indicate that the Solow-Swan model, with its emphasis on diminishing returns to capital and transitional dynamics, is closer to the truth than the AK model. However, what the neoclassical framework cannot account for is the fact that while some countries appear to converge towards the world technology frontier, other countries (for example in Africa) diverge from it. In Chapter 10 below we will how this phenomenon, commonly referred to as *club convergence,* can be explained using an innovation-based model of endogenous growth, the Schumpeterian growth model. But before we present this new framework and compare it to the AK and neoclassical models, let us briefly see how the AK advocates have tried to argue their case against the proponents of the neoclassical model.

A first counterargument by AK advocates involves the issue of returns to capital and the estimation of the elasticity of output with respect to physical capital. Early empirical work on endogenous-growth models centered on this question. In particular, Romer (1987) carries out the following test. Suppose first that the consumption good is produced according to a Cobb-Douglas production function as in the simplest version of the Solow-Swan model:

$$Y = K^{\alpha}(AL)^{1-\alpha}, \quad 0 < \alpha < 1.$$

Under perfect competition in the market for final goods, and given the assumption of constant returns to scale implicit in this Cobb-Douglas function, the coefficients α and $(1-\alpha)$ should be equal to the shares of capital and labor in national income respectively, that is approximately 1/3 and 2/3 in the U.S. case. To see why, notice that under perfect competition for final goods, capital and labor are each paid their marginal products. The share of labor in national income is thus equal to

$$\frac{\partial Y}{\partial L}L = (1-\alpha)K^{\alpha}A^{1-\alpha}L^{-\alpha}.L = (1-\alpha)Y$$

and similarly

$$\frac{\partial Y}{\partial K}K = \alpha K^{\alpha-1}A^{1-\alpha}L^{1-\alpha}.K = \alpha Y.$$

Now, using both time series and cross-section data, Romer estimated the true elasticity of final goods output with respect to physical capital to be higher than the value 1/3 predicted by the Solow-Swan model, and perhaps lying in the range between 0.7 and 1.0. This result in turn appeared to be consistent with the existence of externalities to capital accumulation, as captured by the formalization $A \approx K^{\eta}$ in the AK model. Such externalities imply that the elasticity of final output with respect to physical capital will be larger than the share of capital income in value added.

A second counterargument involves the speed of convergence predicted by the neoclassical model. First, it is intuitive that the lower α, that is, the more decreasing returns to capital are, the faster convergence will be over time, since decreasing returns are the source of convergence. And indeed, in the limit case where $\alpha = 1$, that is in the constant returns case, convergence never happens! This negative correlation between α and the speed of convergence can be formally established (see problem set XX). Now, performing a cross-country regression as specified in Chapter 1, section 2.3, Mankiw, Romer, and Weil (1992) find that the rate at which countries converge to their steady states is slower than that predicted by a Solow-Swan model with a capital share of one-third. The empirically observed speed of convergence suggests a share of broad capital in output of around 0.7–0.8. One explanation may again be the existence of externalities in capital accumulation of the kind emphasized by AK models. However, Mankiw-Romer-Weil propose an alternative explanation, which boils down to augmenting the Solow-Swan model by including human capital on top of physical capital. They specify the following constant-returns production function:

$$Y = K^{\alpha} H^{\beta} (AL)^{1-\alpha-\beta}.$$

Using a simple proxy for the rate of investment in human capital, they argue that this technology is consistent with the cross-country data. Their cross-section regressions indicate that both α and β are about 1/3, suggesting that the AK model is wrong in assuming constant returns to broad capital.

Interestingly, the elasticity of output with respect to the investment ratio becomes equal to $\frac{\alpha}{1-\alpha-\beta}$ in the augmented model, instead of $\frac{\alpha}{1-\alpha}$. In other words, the presence of human capital accumulation increases the impact of physical investment on the steady state level of output. Moreover, the Solow-Swan model augmented with human capital can account for a very low rate of convergence to steady states. It is also consistent with evidence on international capital flows; see Barro, Mankiw and Sala-i-Martín (1995) and Manzocchi and Martin (1996).Yet, the constant-returns specification of Mankiw-Romer-Weil delivers the same long-run growth predictions as the basic Solow-Swan model.

Overall, empirical evidence regarding returns to capital tends to discriminate in favor of decreasing returns, and hence in favor of the neoclassical growth model. Mankiw, Romer, and Weil claim that the neoclassical growth model is correct not only in assuming diminishing returns, but also in suggesting that efficiency grows at the same rate across countries.

True, their augmented model suffers from the same basic problem as the original Solow model, namely that it cannot explain sustained long-run growth. But here the line of defense is to say that the model can still explain growth on the transition path to the steady-state, and that such transition may last for many years so that the data would not allow us to tell apart transitional growth from steady-state growth.

However, another criticism to the augmented neoclassical model came from Benhabib and Spiegel (1994). Their point is that the neoclassical and MRW models all predict that growth can increase only as a result of a higher rate of factor accumulation. However, based on cross-country panel data, they argue that the countries that accumulated human capital most quickly between 1965 and 1985 have not grown accordingly, and more importantly, they show that long-run growth appears to be related to the initial *level* of human capital. This in turn suggests that, when trying to explain the historical experience of developing countries, one should turn to models in which technology differs across countries, and where human capital stocks promotes technological catch up and/or innovation. We will return to this in Chapter 17.

Overall, if the AK model appears to be dominated by the neoclassical model, one must still come to grip with the fact that growth appears to be sustained over time, and also positively correlated with variables such as human capital stocks. Building new endogenous growth models that account for those facts, and also convergence, will be a main challenge of the theories developed in the next chapters.

5 An open economy AK model with convergence

In this section we present a recent attempt at saving the AK model from the criticism that it cannot explain convergence. This attempt, developed by Acemoglu and Ventura (2002), henceforth AV, links to international trade and the notion of terms of trade. The idea is to show that, if we take into account that countries are tied together by international trade, even AK models can exhibit convergence in growth rates. This is because in an open economy A will depend on the country's terms of trade, and if it keeps growing faster than the rest of the world the supply of its goods on world markets will keep growing relative to the supply of foreign goods, which will drive down their relative price - the terms of trade will fall, lowering A and therefore lowering the country's growth rate until it converges to the growth rate of the rest of the world. In such a world each country will look like an AK model for given terms of trade and yet it will converge, not because of a diminishing physical marginal product of capital but because of a diminishing value of the marginal product of capital. The argument is clever and logically tight, even though one may object to it on empirical grounds.

Here we present in stages a simplified version of the AV model, with constant savings rates. First we study a closed economy with two sectors - a final sector and an intermediate sector. Next we assume that production in the final sector requires as input not just the domestically produced intermediate product but also a foreign intermediate product, under the assumption that the relative price of that foreign product (the inverse of this country's terms of trade) is given. This produces a similar version of the AK model, one in which A depends positively on the terms of trade. Finally we close the model off by supposing that the foreign country is just like the domestic country but possibly with a different saving rate. In this last model the terms of trade will depend inversely on this country's capital stock relative to the foreign country's.

5.1 A two-sector closed economy

The final good Y and the intermediate good X are both produced under perfect competition. The final good is produced with capital K and intermediates according to:

$$Y = K^{\alpha} X^{1-\alpha} \tag{6}$$

and the intermediate is produced with the final good one for one.

Let the final good Y be the numeraire. Then the unit price of good Y is equal to one, and this is also the unit cost of producing the intermediate good. Since markets are perfectly competitive, the price of intermediate good X is equal to its unit cost, thus it is also equal to one. Given this, the demand for the intermediate good X is determined by profit-maximization in the final sector. That is, the optimal X maximizes final-sector profits:

$$\Pi = K^{\alpha} X^{1-\alpha} - X.$$

The first-order condition for this problem is:

$$(1-\alpha) K^{\alpha} X^{-\alpha} = 1$$

or equivalently

$$X = (1-\alpha)^{1/\alpha} K.$$

Substituting back into (6) we obtain:

$$Y = (1-\alpha)^{\frac{1-\alpha}{\alpha}} K$$

So even though the production function (6) has a diminishing marginal product of capital we still have an AK model, with $Y = AK$, where the marginal product of capital A is a constant given by:

$$A = (1-\alpha)^{\frac{1-\alpha}{\alpha}}$$

The reason for this is that the production technology for the final good has constant returns with respect to K and X , both of which are produced with K.

Now, let us assume a constant saving rate so that we can close the model with the same capital accumulation equation as in the Solow model, namely:[7]

$$\dot{K} = sY - \delta K.$$

Then, as in the Frankel analyzed in Section 2.3, the country's growth rate depends positively on its saving rate according to:

$$g = \dot{K}/K = sA - \delta = s\left(1-\alpha\right)^{\frac{1-\alpha}{\alpha}} - \delta.$$

5.2 Opening up the economy with fixed terms of trade

Now suppose that producing the Y good requires not just X but also a foreign-produced intermediate product X_f , according to the production function

$$Y = K^{\alpha} X^{\frac{1-\alpha}{2}} \left(X_f\right)^{\frac{1-\alpha}{2}}$$

Both X and X_f are tradable goods however capital is not tradable.

As before, since it takes one unit of final good to produce one unit of X, and since the market for X is perfectly competitive, the price of X is unity. Suppose the price p_f of the foreign good is given. Then domestic producers of Y will choose X and X_f to solve the problem:

$$\max_{X,X_f} \left\{ K^{\alpha} X^{\frac{1-\alpha}{2}} \left(X_f\right)^{\frac{1-\alpha}{2}} - X - p_f X_f \right\}$$

which yields:

$$X = \left(\frac{1-\alpha}{2}\right) Y \quad (7)$$

$$p_f X_f = \left(\frac{1-\alpha}{2}\right) Y \quad (8)$$

[7] This is not quite the same as in the neoclassical model because Y is not the country's GDP, just its production of the final good, some of which is used as an input to the intermediate sector. However, Y is proportional to GDP because:

$$\begin{aligned} \text{GDP} &= \text{value added in final sector} + \text{value added in intermediate sector} \\ &= (Y - X) + 0 \\ &= (1-\alpha)^{\frac{1-\alpha}{\alpha}} K - (1-\alpha)^{\frac{1}{\alpha}} K \\ &= \alpha\left(1-\alpha\right)^{\frac{1-\alpha}{\alpha}} K = \alpha Y \end{aligned}$$

so the parameter s is actually the country's saving rate, as conventionally defined, multiplied by the constant α.

Substituting back into the production function yields:

$$Y = \left(\frac{1-\alpha}{2}\right)^{\frac{1-\alpha}{\alpha}} (p_f)^{-\frac{1-\alpha}{2\alpha}} K \tag{9}$$

So again we have an AK model, with $Y = AK$, where now the constant marginal product of capital A depends negatively on the relative price of foreign goods (that is,. positively on the country's terms of trade which is just $1/p_f$):

$$A = \left(\frac{1-\alpha}{2}\right)^{\frac{1-\alpha}{\alpha}} (p_f)^{-\frac{1-\alpha}{2\alpha}}$$

The reason is that, as p_f goes up, the domestic country needs to spend more on the imported inputs that are combined with capital, thus lowering the amount of income it can generate from an extra unit of capital.

It follows from this that the growth rate depends not just on saving but also on the terms of trade:

$$g = \dot{K}/K = sA - \delta = s\left(\frac{1-\alpha}{2}\right)^{\frac{1-\alpha}{\alpha}} (p_f)^{-\frac{1-\alpha}{2\alpha}} - \delta. \tag{10}$$

Now assume that the domestic country can only export good X in exchange for good X_f, and suppose that "initially" the domestic growth rate exceeds the world growth rate. Then the foreign demand for the country's exported good X will not grow as fast as the country's demand for X_f, which implies that the relative price of the foreign good, p_f, must increase so as to preserve trade balance. This in turn will tend to bring the domestic country's growth rate down to the world level. To see how this might work in more detail, in the next subsection we will suppose that the rest of the world consists of a single country which behaves just like the domestic country.

5.3 Closing the model with a 2-country analysis

Suppose that the rest of the world consists of a single country, just like the domestic country, except possibly with a different saving rate s_f. This foreign country will produce its final good using the same technology as used in the domestic country, so that its output of final product and its use of its own intermediate product and this country's will be determined as above except that from the foreign country's point of view the price of imported intermediate products is $1/p_f$ instead of p_f.

Proceeding as above we see that the foreign country will import the amount F_X of the domestic country's intermediate good, where F_X is given by:

$$(1/p_f)\, F_X = \left(\frac{1-\alpha}{2}\right) Y_f$$

which is the same as (8) except that Y has been replaced by the foreign production Y_f and p_f by $1/p_f$.

By analogy to (9) we have:

$$Y_f = \left(\frac{1-\alpha}{2}\right)^{\frac{1-\alpha}{\alpha}} (p_f)^{\frac{1-\alpha}{2\alpha}}\, K_f$$

where K_f is the foreign capital stock.[8] From these last two equations:

$$F_X = \left(\frac{1-\alpha}{2}\right)^{\frac{1}{\alpha}} (p_f)^{\frac{1+\alpha}{2\alpha}}\, K_f$$

But F_X is not just the foreign country's imports; it is also the domestic country's exports. And trade balance imposes that this in turn be equal to the value (in domestic goods) of the domestic country's imports, namely: $p_f X_f$, because exports are what we use to buy imports. So from equations (8) and (9) above we have:

$$F_X = \left(\frac{1-\alpha}{2}\right)^{\frac{1}{\alpha}} (p_f)^{-\frac{1-\alpha}{2\alpha}}\, K$$

By equating the right-hand sides of these last two equations we can solve for the equilibrium relative price of foreign goods:

$$p_f = k_R^{\alpha}$$

where k_R is the relative capital stock:

$$k_R = \frac{K}{K_f}$$

If the domestic capital stock grows faster than the foreign stock, k_R will rise, so will the relative price p_f of foreign intermediates, and hence the domestic growth rate will fall. This

[8] Note that the exponent of p_f in this equation is the negative of the exponent of p_f in the analogous domestic equation (9). This is because the price of the other country's intermediate product is p_f for the domestic country but $1/p_f$ for the foreign country.

will stabilize k_R. More formally, we have from the domestic growth equation (10):

$$\dot{K}/K = s\left(\frac{1-\alpha}{2}\right)^{\frac{1-\alpha}{\alpha}}(p_f)^{-\frac{1-\alpha}{2\alpha}} - \delta = s\left(\frac{1-\alpha}{2}\right)^{\frac{1-\alpha}{\alpha}} k_R^{-\frac{1-\alpha}{2}} - \delta$$

and from the analogous foreign growth equation:[9]

$$\dot{K}_f/K_f = s_f\left(\frac{1-\alpha}{2}\right)^{\frac{1-\alpha}{\alpha}}(p_f)^{\frac{1-\alpha}{2\alpha}} - \delta = s_f\left(\frac{1-\alpha}{2}\right)^{\frac{1-\alpha}{\alpha}} k_R^{\frac{1-\alpha}{2}} - \delta$$

Since the growth rate of the relative stock k_R is just the differential growth rate $\dot{K}/K - \dot{K}_f/K_f$, these last two equations imply:

$$\dot{k}_R/k_R = \left(\frac{1-\alpha}{2}\right)^{\frac{1-\alpha}{\alpha}}\left[s k_R^{-\frac{1-\alpha}{2}} - s_f k_R^{\frac{1-\alpha}{2}}\right]$$

This is a stable ordinary differential equation with the unique steady state:

$$k_R^* = \left(\frac{s}{s_f}\right)^{\frac{1}{1-\alpha}}$$

The steady state is asymptotically stable because the RHS of the differential equation is decreasing in k_R. So the growth rate of k_R will approach zero, implying the growth rates of K and K_f will approach each other - convergence.

5.4 Concluding comment

We have now seen how the AV model delivers convergence through international trade and its effects on capital accumulation. Faster growth in the domestic economy increases the price of the imported intermediate good, thus resulting in a deterioration of the country's terms of trade, which in turn reduces the rate of capital accumulation. Unfortunately, the model is not fully consistent with empirical evidence. In particular, the prediction that growth reduces a country's terms of trade is counterfactual. So although the AV model is an instructive extension of AK theory to the case of an open economy, in the end it too cannot account for the evidence on cross-country convergence.

[9] Again, the coefficient of p_f in one equation is the negative of the coefficient in the other, because the price of the other country's intermediate is p_f for the domestic country but $1/p_f$ for the foreign country. Hence the coefficient of k_R in one equation is also the negative of the coefficient in the other.

6 Conclusion

In the previous chapter we saw that the neoclassical model provides the standard for parsimoniously modelling growth and convergence. However, the model leaves the rate of technological change exogenous and hence unexplained, which means that it cannot explain sustained long-run growth. In this chapter we have shown that the AK model can explain long-run growth using the same basic assumptions as the neoclassical model but adding knowledge externalities among firms that accumulate physical capital. However, the AK model does not provide a convincing explanation for convergence.

In our view the underlying source of the difficulties faced by the AK model is that it does not make an explicit distinction between capital accumulation and technological progress. In effect it just lumps together the physical and human capital whose accumulation is studied by neoclassical theory with the intellectual capital that is accumulated when technological progress is made. So starting with the next chapter we will focus mainly on innovation-based models that make explicit the distinction between capital accumulation and technological progress. Innovation-based models do a better job of fitting the data with respect to long-run growth and convergence, and they also generate a rich set of predictions on the determinants of growth across firms and industries, in contrast with the high level of aggregation in neoclassical and AK models.

7 Literature Notes

The first AK models go back to Harrod (1939) and Domar (1946) who assume an aggregate production function with fixed coefficients. Frankel (1962) develops the first AK model with substitutable factors and knowledge externalities, with the purpose of reconciling the positive long-run growth result of Harrod-Domar with the factor substitutability and market clearing features of the neoclassical model. The Frankel model has a constant saving rate as in Solow (1956), whereas Romer (1986) develops an AK model with intertemporal consumer maximization. The idea that productivity could increase as the result of learning-by-doing externalities was most forcefully pushed forward by Arrow (1962).

Lucas (1988) developed an AK model where the creation and transmission of knowledge occurs through human capital accumulation. Rebelo (1991) uses AK models to explain how heterogeneity in growth experiences can be the result of cross-country differences in government policy. King and Rebelo (1991) use the AK model to analyze the effect of fiscal policy on growth. Jones, Manuelli and Stachetti (1999) use again the AK framework to analyze the effect of macroeconomic volatility on growth. And Acemogu and Ventura (2003) use the AK model to analyze the effects of terms of trade on growth.

For a comprehensive account of the AK growth literature, we again refer the readers to the Handbook survey by Jones and Manuelli (2005).