CARF Working Paper

CARF-F-407

Estimating the Hurst parameter from short term volatility swaps: a Malliavin calculus approach

Elisa Alòs
Dpt. d'Economia i Empresa
Universitat Pompeu Fabra

Kenichiro Shiraya
Graduate School of Economics
The University of Tokyo

First version : March 29, 2017
This version : November 7, 2018

CARF is presently supported by Bank of Tokyo-Mitsubishi UFJ, Ltd., The Dai-ichi Life Insurance Company, Limited, Nomura Holdings, Inc. and Sumitomo Mitsui Banking Corporation (in alphabetical order). This financial support enables us to issue CARF Working Papers.

Estimating the Hurst parameter from short term volatility swaps: a Malliavin calculus approach

Elisa Alòs*
Dpt. d'Economia i Empresa
Universitat Pompeu Fabra
c/Ramon Trias Fargas, 25-27
08005 Barcelona, Spain

Kenichiro Shiraya†
Graduate School of Economics
The University of Tokyo
7-3-1 Hongo, Bunkyo-ku
Tokyo 113-0033, Japan

Abstract

This paper is devoted to studying the difference between the fair strike of a volatility swap and the at-the-money implied volatility (ATMI) of a European call option. It is well-known that the difference between these two quantities converges to zero as the time to maturity decreases. In this paper, we make use of a Malliavin calculus approach to derive an exact expression for this difference. This representation allows us to establish that the order of the convergence is different in the correlated and in the uncorrelated case, and that it depends on the behavior of the Malliavin derivative of the volatility process. In particular, we will see that for volatilities driven by a fractional Brownian motion, this order depends on the corresponding Hurst parameter H. Moreover, in the case $H \geq 1/2$, we develop a model-free approximation formula for the volatility swap, in terms of the ATMI and its skew.

Keywords: Malliavin calculus, fractional volatility models, volatility swaps.

AMS subject classification: 91G99

1 Introduction

A volatility swap is a forward contract whose underlying is the future realized volatility asset price. It is well known (see for example Feinstein (1989), Friz and Gatheral (2005), Carr and Lee (2008, 2009)), that the difference between the ATMI of a vanilla option and the corresponding volatility swap price tends to zero as the time to maturity decreases. Moreover, the sign of the difference between the above two quantities is related to the skew of the implied volatility (see for example Demeterfi, Derman, Kamal and Zou (1999) and Carr and Lee (2008)).

This paper is devoted to contributing to the study of the link between volatility derivatives and the ATMI of vanilla options, in the context of stochastic volatility models. Our analysis does not require a specific model and can be applied to the case of fractional volatility models, introduced by Comte and Renault (1998) (with Hurst parameter $H > 1/2$), to explain the long-time behavior of the implied volatility. Alòs, León and Vives (2007), proposed to consider volatility models with $H < 1/2$ to explain

*Supported by grants ECO2014-59885-P and MTM2013-40782-P

†Supported by CARF (Center for Advanced Research in Finance)

the empirical short-time skew of the ATMI. Recently, fractional models with $H < 1/2$ have been further studied by Fukasawa (2011) and have been proved to be interesting as a tool to describe real market data (see for example Gatheral, Jaisson and Rosenbaum (2014)).

Even when the classical literature focuses on volatility models that are diffusion processes, several recent works include the case of fractional volatilities. Among them, we can quote the paper by Bergomi and Guyon (2011), where the authors presented a vol-of-vol expansion of the ATMI around the variance swap. More recently, El Euch, Fukasawa, Gatheral and Rosenbaum (2018) proved a small-time Edgeworth expansion of the density of the asset price, from where they deduced an expansion (again around the variance swap) of the ATMI, for models with Hurst parameter $H \in (0, \frac{1}{2}]$.

Our approach uses Malliavin calculus techniques that allow us to find an explicit expression for the difference between the ATMI and the fair strike of the volatility swap in terms of the Malliavin derivative of the volatility process, both in the uncorrelated case (see Proposition 2) and the correlated case (see Proposition 7). As an application of these explicit decompositions, we compute the rate of convergence of this difference and see that this rate depends on the regularity properties of the Malliavin derivative. In particular, for models based on the fractional Brownian motion, we prove that this difference is of the order $O\left(T^{1+2H}\right)$ in the uncorrelated case, where T denotes the time to maturity. In the correlated case, this difference is of the order $O\left(T^{2H}\right)$ if $H \leq 1/2$,and of the order $O\left(T^{H+\frac{1}{2}}\right)$ if $H > 1/2$. These results give us a tool to estimate the Hurst parameter of fractional volatilities, as we see in the numerical examples in Section 5.

The paper is organized as follows. Section 2 is devoted to introducing the main concepts and notations. In Section 3 we prove a representation of the difference between the ATMI and the volatility swap, in the case when the volatility process and the asset price are uncorrelated processes. This representation allows us to deduce the order of convergence of this difference, in terms of the Hurst parameter of the volatility process. Moreover, we prove a limit relationship between the implied volatility, its curvature, and the volatility swap. In Section 4 we extend the results in Section 3 to the correlated case. Finally, some numerical examples of a fractional volatility model are presented in Section 5.

2 The main problem and notations

In this paper, we consider the following model for the log-price of a stock under a risk-neutral probability measure P.

$$X_t = X_0 - \frac{1}{2}\int_0^t \sigma_s^2 ds + \int_0^t \sigma_s \left(\rho dW_s + \sqrt{1-\rho^2} dB_s\right), \quad t \in [0,T]. \tag{1}$$

Here, X_0 is the current log-price, W and B are standard Brownian motions defined on a complete probability space $(\Omega, \mathcal{G}, P)$, and σ is a square-integrable and right-continuous stochastic process adapted to the filtration generated by W. In the following, we denote by $\mathcal{F}^W$ and $\mathcal{F}^B$ the filtrations generated by W and B. Moreover we define $\mathcal{F} := \mathcal{F}^W \vee \mathcal{F}^B$. We assume the interest rate r to be zero for the sake of simplicity. The same arguments in this paper hold even if there exists a deterministic drift term in Equation (**??**).

The price of a European call with strike price K is given by the equality

$$V_t = E_t[(e^{X_T} - K)_+],$$

where E_t is the $\mathcal{F}_t-$conditional expectation with respect to P (i.e., $E_t[Z] = E[Z|\mathcal{F}_t]$). In the sequel, we make use of the following notation:

- $v(t, Y_t) = \sqrt{\frac{Y_t}{T-t}}$, where $Y_t = \int_t^T \sigma_u^2 du$, and it is abbreviated as $v_t = v(t, Y_t)$.

  That is, v represents the future average volatility, and it is not an adapted process. Notice that $E_t[v_t]$ is the fair strike of a volatility swap with maturity time T.

- $BS(t, T, x, k, \sigma)$ denotes the price of a European call option under the classical Black-Scholes model with constant volatility σ, current log stock price x, time to maturity $T - t$, and strike price $K = \exp(k)$. Remember that in this case

$$BS(t, T, x, k, \sigma) = e^x N(d_+(k, \sigma)) - e^k N(d_-(k, \sigma)),$$

  where N denotes the cumulative probability function of the standard normal law and

$$d_\pm(k, \sigma) := \frac{k_t^* - k}{\sigma\sqrt{T-t}} \pm \frac{\sigma}{2}\sqrt{T-t},$$

  where k_t^* denotes the at-the-money strike, that coincides with x when the interest rate is zero.

- For any fixed t, T, X_t, k we define the implied volatility $I(t, T, X_t, k)$ as the quantity such that

$$BS(t, T, X_t, k, I(t, T, X_t, k)) = V_t,$$

  and the inverse function of the Black-Scholes formula with respect to the volatility parameter is defined as

$$BS^{-1}(t, T, X_t, k, V_t) = I(t, T, X_t, k).$$

  We also define a simplified notation of the inverse function $BS^{-1}(k, \lambda) := BS^{-1}(t, T, X_t, k, \lambda)$.

- $H(t, T, x, k, \sigma) := \left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2}\right) BS(t, T, x, k, \sigma)$.

We assume that the reader is familiar with the elementary results of the Malliavin calculus, as given for instance in Nualart (2006). In the remaining of this paper $\mathbb{D}_W^{1,2}$ denotes the domain of the Malliavin derivative operator D^W with respect to the Brownian motion W. It is well-known that $\mathbb{D}_W^{1,2}$ is a dense subset of $L^2(\Omega)$ and that D^W is a closed and unbounded operator from $\mathbb{D}_W^{1,2}$ to $L^2([0,T] \times \Omega)$. We also consider the iterated derivatives $D^{n,W}$, for $n > 1$, whose domains will be denoted by $\mathbb{D}_W^{n,2}$. We will use the notation $\mathbb{L}_W^{n,2} = L^2([0,T]; \mathbb{D}_W^{n,2})$.

We will make use of the following anticipating Itô's formula (see for example Alòs (2006)).

Proposition 1 *Assume model (??) and $\sigma^2 \in \mathbb{L}_W^{1,2}$. Let $F : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be a function in $C^{1,2}([0,T] \times \mathbb{R}^2)$ such that there exists a positive constant C such that, for all $t \in [0,T]$, F and its partial derivatives evaluated in (t, X_t, Y_t) are bounded by C. Then it follows that*

$$\begin{aligned} F(t, X_t, Y_t) &= F(0, X_0, Y_0) + \int_0^t \partial_s F(s, X_s, Y_s) ds \\ &- \int_0^t \partial_x F(s, X_s, Y_s) \frac{\sigma_s^2}{2} ds \\ &+ \int_0^t \partial_x F(s, X_s, Y_s) \sigma_s (\rho dW_s + \sqrt{1-\rho^2} dB_s) \end{aligned}$$

$$
\begin{aligned}
&-\int_0^t \partial_y F(s,X_s,Y_s)\sigma_s^2 ds + \rho \int_0^t \partial_{xy}^2 F(s,X_s,Y_s)\Theta_s ds \\
&+\frac{1}{2}\int_0^t \partial_{xx}^2 F(s,X_s,Y_s)\sigma_s^2 ds, \qquad (2)
\end{aligned}
$$

where $\Theta_s := (\int_s^t D_s^W \sigma_r^2 dr)\sigma_s$.

3 The uncorrelated case

Let us consider the following hypotheses:

(H1) There exist two positive constants a, b such that $a \leq \sigma_t \leq b$, for all $t \in [0,T]$.

(H2) $\sigma^2 \in \mathbb{L}^{1,2}$.

The key tool in our analysis will be the following relationship between the ATMI and the volatility swap fair strike.

Proposition 2 *Consider the model (??) with $\rho = 0$ and assume that hypotheses (H1) and (H2) hold. Then the at-the-money implied volatility admits the representation*

$$
\begin{aligned}
&I(t,T,X_t,k_t^*) \\
&= E_t[v_t] - \frac{1}{32(T-t)} \\
&\times E_t\left[\int_t^T \frac{BS^{-1}(k_t^*,\Lambda_r)}{\left(N'\left(d_+\left(k_t^*, BS^{-1}(k_t^*,\Lambda_r)\right)\right)\right)^2}\left(E_r\left[N'(d_+(k_t^*,v_t))\frac{\int_r^T D_r^W \sigma_s^2 ds}{v_t}\right]\right)^2 dr\right],
\end{aligned}
$$

where

$$
\Lambda_r := E_r\left[BS(t,T,X_t,k_t^*,v_t)\right]
$$

Proof. Notice that, in the uncorrelated case, the Hull and White formula gives us that the option price can be writen as

$$
V_t = E_t\left[BS(t,T,X_t,k_t^*,v_t)\right].
$$

Then the implied volatility satisfies that

$$
\begin{aligned}
&I(t,T,X_t,k_t^*) \\
&= BS^{-1}(k_t^*,V_t) \\
&= E_t\left[BS^{-1}(k_t^*, E_t[BS(t,T,X_t,k_t^*,v_t)])\right] \\
&= E_t\left[BS^{-1}(k_t^*, E_t[BS(t,T,X_t,k_t^*,v_t)])\right. \\
&\quad - BS^{-1}(k_t^*, BS(t,T,X_t,k_t^*,v_t)) \\
&\quad \left.+ BS^{-1}(k_t^*, BS(t,T,X_t,k_t^*,v_t))\right] \\
&= E_t\left[BS^{-1}(k_t^*, E_t[BS(t,T,X_t,k_t^*,v_t)]) - BS^{-1}(k_t^*, BS(t,T,X_t,k_t^*,v_t))\right] \\
&\quad + E_t[v_t].
\end{aligned}
$$

Now, as in Alòs and León (2017), we can write

$$BS\left(t,T,X_t,k_t^*,v_t\right)=E_t\left[BS\left(t,T,X_t,k_t^*,v_t\right)\right]+\int_t^T U_s dW_s,$$

where U_s can be computed by Clark-Ocone formula and W is the Brownian motion that drives the volatility process. Then

$$\begin{aligned}
&E_t\left[BS^{-1}(k,E_t\left[BS\left(t,T,X_t,k_t^*,v_t\right)\right])-BS^{-1}(k,BS\left(t,T,X_t,k_t^*,v_t\right))\right]\\
&= E_t\left[BS^{-1}(k_t^*,\Lambda_t)-BS^{-1}(k_t^*,\Lambda_T)\right]\\
&= E_t\left[-\int_t^T \left(BS^{-1}\right)'\left(k_t^*,\Lambda_r\right)U_r dW_r-\frac{1}{2}\int_t^T\left(BS^{-1}\right)''\left(k_t^*,\Lambda_r\right)U_r^2 dr\right], \qquad (3)
\end{aligned}$$

where $(BS^{\ -1})'$ and $(BS^{\ -1})''$ denote, respectively, the first and second derivatives of BS^{-1} with respect to Λ. Notice that

$$\begin{aligned}
U_r &= E_r\left[D_r^W\left(BS\left(t,T,X_t,k_t^*,v_t\right)\right)\right]\\
&= E_r\left[\exp(X_t)N'(d_+\left(k_t^*,v_t\right))\frac{\int_r^T D_r^W\sigma_s^2 ds}{2\sqrt{T-t}v_t}\right],
\end{aligned}$$

which, jointly with (H1), implies that

$$\begin{aligned}
E_t\left[\int_t^T\left[(BS^{-1})'(k_t^*,\Lambda_r)U_r\right]^2 dr\right] &\leq \frac{\exp(2X_t)}{4(T-t)}E_t\left[\int_t^T\left[E_r\int_r^T D_r^W\sigma_s^2 ds\right]^2 dr\right]\\
&\leq C(T,t).
\end{aligned}$$

This gives us that the expectation of the stochastic integral in (**??**) is zero. Then,

$$\begin{aligned}
&E_t\left[BS^{-1}(X_t,E_t\left[BS\left(t,T,X_t,k_t^*,v_t\right)\right])-BS^{-1}(X_t,BS\left(t,T,X_t,k_t^*,v_t\right))\right]\\
&=-\frac{1}{2}E_t\left[\int_t^T\left(BS^{-1}\right)''\left(k_t^*,\Lambda_r\right)U_r^2 dr\right]. \qquad (4)
\end{aligned}$$

Now, as

$$\left(BS^{-1}\right)''\left(k_t^*,\Lambda_r\right)=\frac{BS^{-1}(k_t^*,\Lambda_r)}{4\left(\exp(X_t)N'(d_+\left(k_t^*,BS^{-1}(k_t^*,\Lambda_r)\right))\right)^2}, \qquad (5)$$

the proof is complete. ■

In order to prove our limit results, we will need the following hypotheses.

(H3) Hypothesis (H2) holds, and there exist two constants $\delta\in\left(-\frac{1}{2},\frac{1}{2}\right)$ and $C>0$ such that, for any $0<r<s<T$,

$$E_r\left[D_r^W\sigma_s^2\right]\leq C\left(s-r\right)^{\delta}.$$

(H4) Hypotheses (H2) and (H3) hold and the term

$$\frac{1}{(T-t)^{3+2\delta}} E_t \left[\int_t^T \left(E_r \left[\int_r^T D_r^W \sigma_s^2 ds \right] \right)^2 dr \right],$$

has a finite limit as $T \to t$.

The following theorem gives us that, with reasonable parameters, the difference between the volatility swap and the ATM implied volatility is very small, according to the previous results by Carr and Lee (2009).

Theorem 3 *Consider the model (??) with $\rho = 0$ and assume that hypotheses (H1), (H2), (H3) and (H4) hold. Then*

$$\lim_{T\to t} \frac{I(t,T,X_t,k_t^*) - E_t[v_t]}{(T-t)^{2+2\delta}} = -\frac{1}{32\sigma_t} \lim_{T\to t} \frac{1}{(T-t)^{3+2\delta}} E_t \left[\int_t^T \left(E_r \left[\int_r^T D_r^W \sigma_s^2 ds \right] \right)^2 dr \right].$$

Proof. Proposition 2 gives us that

$$\begin{aligned} & I(t,T,X_t,k_t^*) \\ = & \; E_t[v_t] - \frac{1}{32(T-t)} \\ & \times E_t \left[\int_t^T \frac{BS^{-1}(k_t^*, \Lambda_r)}{\left(N'\left(d_+\left(k_t^*, BS^{-1}(k_t^*, \Lambda_r)\right)\right)\right)^2} \left(E_r \left[N'(d_+(k_t^*, v_t)) \frac{\int_r^T D_r^W \sigma_s^2 ds}{v_t} \right] \right)^2 dr \right]. \end{aligned}$$

Here, using similar arguments as in the proof of Proposition ?? we see that $BS^{-1}(k_t^*, \Lambda_r)$ might be expanded as

$$\begin{aligned} & BS^{-1}(k_t^*, \Lambda_r) \\ = & \; E_r \left[BS^{-1}(k_t^*, \Lambda_r) \right] \\ = & \; E_r \left[BS^{-1} \left(X_t, \Lambda_T - \int_r^T U_s dW_s \right) \right] \\ = & \; E_r \left[BS^{-1}(k_t^*, \Lambda_T) \right] - \frac{1}{2} E_r \left[\int_r^T \left(BS^{-1}\right)''(k_t^*, \Lambda_\theta) U_\theta^2 d\theta \right] \\ = & \; E_r[v_t] - \frac{1}{2} E_r \left[\int_r^T \left(BS^{-1}\right)''(k_t^*, \Lambda_\theta) U_\theta^2 d\theta \right]. \end{aligned}$$

Notice that $\left(BS^{-1}\right)''(k_t^*, \cdot)$ is bounded. This comes from the hypothesis (H1) and (??). Moreover, (H1) and (H3) imply that

$$U_r = E_r \left[\exp(X_t) N'(d_+(k_t^*, v_t)) \frac{\int_r^T D_r^W \sigma_s^2 ds}{2\sqrt{T-t} v_t} \right] \leq C_t (T-t)^{\frac{1}{2}+\delta},$$

where C_t is a positive constant.Then,

$$BS^{-1}(k_t^*, \Lambda_r) = E_r\left[v_t\right] + O((T-r)^{2+2\delta}),$$

which implies that

$$\begin{aligned}
&\lim_{T\to t} \frac{I\left(t,T,X_t,k_t^*\right) - E_t\left[v_t\right]}{(T-t)^{2+2\delta}} \\
&= -\frac{1}{32}\lim_{T\to t}\frac{1}{(T-t)^{3+2\delta}}E_t\left[\int_t^T BS^{-1}(k_t^*,\Lambda_r)\left(E_r\left[\exp\left(\frac{1}{8}\left(BS^{-1}(k_t^*,\Lambda_r)^2 - v_t^2\right)(T-t)\right)\frac{\int_r^T D_r^W \sigma_s^2 ds}{v_t}\right]\right)^2 dr\right] \\
&= -\frac{1}{32\sigma_t}\lim_{T\to t}\frac{1}{(T-t)^{3+2\delta}}E_t\left[\int_t^T \left(E_r\left[\int_r^T D_r^W \sigma_s^2 ds\right]\right)^2 dr\right], \qquad (6)
\end{aligned}$$

and now the proof is complete. ■

Corollary 4 *Assume a fractional volatility model of the form $\sigma_t = f(B_t^H)$, where $f \in \mathcal{C}_b^1$ is a function with range in a compact set of $\mathbb{R}^+$ and B_t^H is a fractional Brownian motion with Hurst parameter H (see for example Alòs, León and Vives (2007)). Then, (H1)-(H4) hold with $\delta = H - 1/2$, and then $I\left(t,T,X_t,k_t^*\right) - E_t\left[v_t\right] = O\left((T-t)^{1+2H}\right)$.*

Remark 5 *Corollary 3.10 of Alòs and León (2017) shows that, under some regularity conditions*

$$\lim_{T\to t}\frac{1}{(T-t)^{2\delta}}\frac{\partial^2 I}{\partial k^2}\left(t,T,X_t,k_t^*\right) = \frac{1}{4\sigma_t^5}\lim_{T\to t}\frac{1}{(T-t)^{3+2\delta}}E_t\left[\int_t^T\left(E_r\left[\int_r^T D_r^W\sigma_s^2 ds\right]\right)^2 dr\right].$$

Then, the above result gives us, that, in the uncorrelated case

$$\lim_{T\to t}\frac{I\left(t,T,X_t,k_t^*\right) - E_t\left[v_t\right]}{(T-t)^{2+2\delta}} = -\frac{\sigma_t^4}{8}\lim_{T\to t}\frac{\frac{\partial^2 I}{\partial k^2}\left(t,T,X_t,k_t^*\right)}{(T-t)^{2\delta}}.$$

That is,

$$\lim_{T\to t}\frac{I\left(t,T,X_t,k_t^*\right) - E_t\left[v_t\right]}{(T-t)^{2+2\delta}} = -\frac{1}{8}\lim_{T\to t}\frac{I\left(t,T,X_t,k_t^*\right)^4\frac{\partial^2 I}{\partial k^2}\left(t,T,X_t,k_t^*\right)}{(T-t)^{2\delta}}.$$

Remark 6 *Assume that (H3) holds for $\delta = 0$ and that, for every $t \in [0,T]$, there exists a random variable $D_t^+\sigma_t^2$ such that*

$$\lim_{T\to t}\sup_{r\in[t,T]}\left|E_r\left[D_r^W\sigma_s^2\right] - D_t^+\sigma_t^2\right| = 0. \qquad (7)$$

Then, Theorem ?? gives us that

$$\begin{aligned}
&\lim_{T\to t}\frac{I\left(t,T,X_t,k_t^*\right) - E_t\left[v_t\right]}{(T-t)^2} \\
&= -\frac{\left(D_t^+\sigma_t^2\right)^2}{96\sigma_t} \\
&\quad -\frac{1}{32\sigma_t}\lim_{T\to t}\frac{1}{(T-t)^3}E_t\left[\int_t^T\left(\left(E_r\left[\int_r^T D_r^W\sigma_s^2 ds\right]\right)^2 - \left(\int_r^T D_t^+\sigma_t^2 ds\right)^2\right)dr\right]. \qquad (8)
\end{aligned}$$

Now, notice that

$$E_t\left[\int_t^T\left(\left(E_r\left[\int_r^T D_r^W\sigma_s^2ds\right]\right)^2-\left(\int_r^T D_t^+\sigma_t^2ds\right)^2\right)dr\right]$$
$$= E_t\left[\int_t^T E_r\left[\left(\int_r^T D_r^W\sigma_s^2ds-\int_r^T D_t^+\sigma_t^2ds\right)\left(\int_r^T D_r^W\sigma_s^2ds+\int_r^T D_t^+\sigma_t^2ds\right)\right]dr\right],$$

*which gives us, jointly with (**??**) and (**??**), that*

$$\lim_{T\to t}\frac{I\left(t,T,X_t,k_t^*\right)-E_t\left[v_t\right]}{(T-t)^2}=-\frac{\left(D_t^+\sigma_t^2\right)^2}{96\sigma_t}.$$

4 The correlated case

We will consider the following hypothesis

(H2') $\sigma\in\mathbb{L}_W^{3,2}$.

(H3') Hypothesis (H2') holds and there exist two constants $\delta\in\left(-\frac{1}{2},\frac{1}{2}\right)$ and $C>0$ such that, for any $0<r<s<T$,

$$E_r\left[D_r^W\sigma_s^2\right]\leq C\left(s-r\right)^{\delta},$$
$$E_r\left[D_\theta^W D_r^W\sigma_s^2\right]\leq C\left(s-r\right)^{\delta}\left(s-\theta\right)^{\delta},$$

and

$$E_r\left[D_u^W D_\theta^W D_r^W\sigma_s^2\right]\leq C\left(s-r\right)^{\delta}\left(s-\theta\right)^{\delta}\left(s-u\right)^{\delta}.$$

(H5) Hypotheses (H1), (H2'), (H3') and (H4) hold and the terms

$$\frac{1}{(T-t)^{2+\delta}}E_t\left[\int_t^T\int_s^T D_s^W\sigma_r^2drds\right],$$
$$\frac{1}{(T-t)^{4+2\delta}}E_t\left[\left(\int_t^T\int_s^T D_s^W\sigma_r^2drds\right)^2\right],$$
$$\frac{1}{(T-t)^{3+2\delta}}E_t\left[\int_t^T\left(\int_s^T D_s^W\sigma_rdr\right)^2ds\right],$$

and

$$\frac{1}{(T-t)^{3+2\delta}}E_t\left[\int_t^T\int_s^T\int_r^T D_s^W D_r^W\sigma_u^2dudrds\right],$$

have a finite limit as $T\to t$.

The following result gives us an exact decomposition for the at-the-money implied volatility that will be the main tool in this Section.

Proposition 7 *Consider the model (??) and assume that hypotheses (H1), (H2') and (H3') hold for some $\delta \in \left(-\frac{1}{2}, \frac{1}{2}\right)$. Then*

$$\begin{aligned} I\left(t, T, X_t, k_t^*\right) &= I^0\left(t, T, X_t, k_t^*\right) \\ &+\frac{\rho}{2} E_t\left[\int_t^T \left(BS^{-1}\right)'\left(k_t^*, \Gamma_s\right) H(s, T, X_s, k_t^*, v_s)\Phi_s ds\right], \end{aligned} \tag{9}$$

where $I^0\left(t, T, X_t, k_t^*\right)$ denotes the implied volatility in the uncorrelated case $\rho = 0$,

$$\Gamma_s := E_t\left[BS(t, T, X_t, k_t^*, v_t)\right] + \frac{\rho}{2} E_t\left[\int_t^s H(r, T, X_r, k_t^*, v_r)\Phi_r dr\right],$$

and $\Phi_t := \sigma_t \int_t^T D_t^W \sigma_r^2 dr$.

Proof. We can write (see Alòs, León and Vives (2007)):

$$V_t = E_t\left[BS(t, T, X_t, k_t^*, v_t)\right] + A_t^T,$$

where

$$A_t^T = \frac{\rho}{2} E_t\left[\int_t^T H(s, T, X_s, k_t^*, v_s)\Phi_s ds\right].$$

Thus,

$$I\left(t, T, X_t, k_t^*\right) = BS^{-1}\left(k_t^*, V_t\right) = E_t\left[BS^{-1}(k_t^*, V_t^0 + A_t^T)\right], \tag{10}$$

where $V_t^0 := E_t\left[BS\left(t, T, X_t, k_t^*, v_t\right)\right]$ denotes the option price in the uncorrelated case $\rho = 0$. Then, it follows that

$$\begin{aligned} & E_t\left[BS^{-1}(k_t^*, V_t^0 + A_t^T) - BS^{-1}(k_t^*, V_t^0)\right] \\ &= \frac{\rho}{2} E_t\left[\int_t^T \left(BS^{-1}\right)'\left(k_t^*, \Gamma_s\right) H(s, T, X_s, k_t^*, v_s)\Phi_s ds\right], \end{aligned}$$

which proves (??). ■

Theorem ?? and Proposition ?? allow us to prove the following result.

Theorem 8 *Consider the model (??) and assume that hypotheses (H1), (H2'), (H3'), (H4) and (H5) hold for some $\delta \in \left(-\frac{1}{2}, \frac{1}{2}\right)$. Then*

- *If $\delta < 0$*

$$\begin{aligned} & \lim_{T \to t} \frac{I\left(t, T, X_t, k_t^*\right) - E_t\left[v_t\right]}{(T-t)^{1+2\delta}} \\ &= \lim_{T \to t} \frac{3\rho^2}{8\sigma_t^3 (T-t)^{4+2\delta}} E_t\left[\left(\int_t^T \int_s^T D_s^W \sigma_r^2 dr ds\right)^2\right] \\ &- \lim_{T \to t} \frac{\rho^2}{2\sigma_t (T-t)^{3+2\delta}} E_t\left[\int_t^T \left(\int_s^T D_s^W \sigma_r dr\right)^2 ds\right] \\ &- \lim_{T \to t} \frac{\rho^2}{2\sigma_t (T-t)^{3+2\delta}} E_t\left[\int_t^T \int_s^T \int_r^T D_s^W D_r^W \sigma_u^2 du dr ds\right]. \end{aligned} \tag{11}$$

- *If $\delta > 0$*

$$\lim_{T\to t}\frac{I\left(t,T,X_t,k_t^*\right)-E_t\left[v_t\right]}{(T-t)^{1+\delta}}=\lim_{T\to t}\frac{\rho}{4(T-t)^{2+\delta}}E_t\left[\int_t^T\int_s^T D_s^W\sigma_r^2drds\right]. \tag{12}$$

- *If $\delta = 0$*

$$\begin{aligned}
&\lim_{T\to t}\frac{I\left(t,T,X_t,k_t^*\right)-E_t\left[v_t\right]}{(T-t)}\\
&= \lim_{T\to t}\frac{3\rho^2}{8\sigma_t^3(T-t)^4}E_t\left[\left(\int_t^T\int_s^T D_s^W\sigma_r^2drds\right)^2\right]\\
&\quad-\lim_{T\to t}\frac{\rho^2}{2\sigma_t(T-t)^3}E_t\left[\int_t^T\left(\int_s^T D_s^W\sigma_r dr\right)^2 ds\right]\\
&\quad-\lim_{T\to t}\frac{\rho^2}{2\sigma_t(T-t)^3}E_t\left[\int_t^T\int_s^T\int_r^T D_s^W D_r^W\sigma_u^2dudrds\right]\\
&\quad+\lim_{T\to t}\frac{\rho}{4(T-t)^2}E_t\left[\int_t^T\int_s^T D_s^W\sigma_r^2drds\right].
\end{aligned} \tag{13}$$

Proof. The main idea of the proof is to see that $I\left(t,T,X_t,k_t^*\right)-E[v_t]$ is a sum of terms of the order $O\left((T-t)^{1+\delta}\right)$, terms of order $O\left((T-t)^{1+2\delta}\right)$ and higher-order terms. Then if $\delta<0$, $1+\delta>1+2\delta$ and the leading terms will be those of order $O\left((T-t)^{1+2\delta}\right)$, while if $\delta>0$ the leading terms will be those of order $O\left((T-t)^{1+\delta}\right)$. Notice that Proposition **??** gives us that

$$I\left(t,T,X_t,k_t^*\right)-E[v_t]=T_1+T_2,$$

where

$$\begin{aligned}
T_1 &= I^0\left(t,T,X_t,k_t^*\right)-E[v_t],\\
T_2 &= \frac{\rho}{2}E_t\left[\int_t^T\left(BS^{-1}\right)'\left(k_t^*,\Gamma_s\right)H(s,T,X_s,k_t^*,v_s)\Phi_s ds\right].
\end{aligned}$$

We know, from the previous section, that $T_1=O\left((T-t)^{2+2\delta}\right)$. Then, the proof reduces to see that T_2 is a sum of terms of the orders $O\left((T-t)^{1+\delta}\right)$ and $O\left((T-t)^{1+2\delta}\right)$. Towards this end, we apply the anticipating Itô's formula (**??**) to the process

$$H(s,T,X_s,k_t^*,v_s)J_s,$$

where $J_s=\int_s^T\left(BS^{-1}\right)'\left(k_t^*,\Gamma_u\right)\Phi_u du$. Then, taking conditional expectations we get

$$\begin{aligned}
0 &= E_t\Bigg[H(t,T,X_t,k_t^*,v_t)J_t\\
&\quad+\int_t^T H(s,T,X_s,k_t^*,v_s)dJ_s
\end{aligned}$$

$$
\begin{aligned}
&+\int_t^T \frac{\partial^2}{\partial x \partial \sigma} H(s,T,X_s,k_t^*,v_s) J_s \frac{\partial v}{\partial y}(D_s^W Y_s)\sigma_s ds \\
&+\int_t^T \frac{\partial}{\partial x} H(s,T,X_s,k_t^*,v_s)(D_s^W J_s)\sigma_s ds \\
&+\int_t^T \frac{\partial}{\partial t} H(s,T,X_s,k_t^*,v_s) J_s ds \\
&+\int_t^T \frac{\partial}{\partial \sigma} H(s,T,X_s,k_t^*,v_s)\frac{\partial v}{\partial t} J_s ds \\
&+\int_t^T \frac{\partial}{\partial \sigma} H(s,T,X_s,k_t^*,v_s)\frac{\partial v}{\partial y} J_s dY_s \\
&+\int_t^T \frac{\partial}{\partial x} H(s,T,X_s,k_t^*,v_s) J_s dX_s \\
&+\frac{1}{2}\int_t^T \frac{\partial^2}{\partial x^2} H(s,T,X_s,k_t^*,v_s) J_s d\langle X\rangle_s \Bigg].
\end{aligned}
$$

Now, using the relationships

$$
\begin{aligned}
&\frac{1}{\sigma(T-t)}\frac{\partial}{\partial \sigma} BS(t,T,x,k,\sigma) = \left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right) BS(t,T,x,k,\sigma), \\
&\left(\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} - \frac{1}{2}\sigma^2\frac{\partial}{\partial x}\right) BS(t,T,x,k,\sigma) = 0, \\
&D_s^W J_s = \rho \int_s^T \left(BS^{-1}\right)'(k_t^*,\Gamma_r)\, D_s^W \Phi_r dr, \\
&D_s^W Y_s = \rho \int_s^T D_s^W \sigma_r^2 dr,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
0 \quad = \quad & E_t\Bigg[H(t,T,X_t,k_t^*,v_t) J_t \\
& -\int_t^T H(s,T,X_s,k_t^*,v_s)\left(BS^{-1}\right)'(X_t,\Gamma_s)\,\Phi_s ds \\
& +\frac{\rho}{2}\int_t^T \left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2}\right) H(s,T,X_s,k_t^*,v_s) J_s \Phi_s ds \\
& +\rho\int_t^T \frac{\partial}{\partial x} H(s,T,X_s,k_t^*,v_s)\left(\int_s^T \left(BS^{-1}\right)'(k_t^*,\Gamma_r)\left(D_s^W\Phi_r\right) dr\right)\sigma_s ds\Bigg],
\end{aligned}
$$

which implies that

$$
T_2 \quad = \quad E_t\Bigg[\frac{\rho}{2} H(t,T,X_t,k_t^*,v_t) J_t
$$

$$
\begin{aligned}
&+\frac{\rho^2}{4}\int_t^T \left(\frac{\partial^3}{\partial x^3}-\frac{\partial^2}{\partial x^2}\right) H(s,T,X_s,k_t^*,v_s)J_s\Phi_s ds \\
&+\frac{\rho^2}{2}\int_t^T \frac{\partial}{\partial x}H(s,T,X_s,k_t^*,v_s)\left(\int_s^T \left(BS^{-1}\right)'(k_t^*,\Gamma_r)\left(D_s^W\Phi_r\right)dr\right)\sigma_s ds\Bigg] \\
= \;& T_2^1+T_2^2+T_2^3.
\end{aligned}
$$

Now, the proof will be decomposed into three steps.

Step 1 Here we claim that T_2^1 is of the order $O\left((T-t)^{1+\delta}\right)$. As

$$
\left(BS^{-1}\right)'(k_t^*,\Gamma_s)=\frac{1}{e^{X_t}N'(d_+(k_t^*,BS^{-1}(k_t^*,\Gamma_s)))\sqrt{T-t}},
$$

and

$$
H(t,T,X_t,k_t^*,v_t)=\frac{e^{X_t}N'(d_+(k_t^*,v_t))}{v_t\sqrt{T-t}}\left(1-\frac{d_+(k_t^*,v_t)}{v_t\sqrt{T-t}}\right),
$$

we have that

$$
\begin{aligned}
&\lim_{T\to t}\frac{T_2^1}{(T-t)^{1+\delta}} \\
&= \lim_{T\to t}\frac{1}{(T-t)^{1+\delta}}E_t\left[\frac{\rho}{2}H(t,T,X_t,k_t^*,v_t)J_t\right] \\
&= \lim_{T\to t}\frac{\rho}{2(T-t)^{1+\delta}}E_t\left[\frac{e^{X_t}N'(d_+(k_t^*,v_t))}{v_t\sqrt{T-t}}\left(1-\frac{d_+(k_t^*,v_t)}{v_t\sqrt{T-t}}\right)\right. \\
&\quad\left.\times\int_t^T\frac{1}{e^{X_t}N'(d_+(k_t^*,BS^{-1}(k_t^*,\Gamma_s)))\sqrt{T-t}}\Phi_s ds\right] \\
&= \lim_{T\to t}\frac{\rho}{4(T-t)^{2+\delta}}E_t\left[\int_t^T\frac{\Phi_s}{v_s}ds\right] \\
&= \lim_{T\to t}\frac{\rho}{4(T-t)^{2+\delta}}E_t\left[\int_t^T\frac{\sigma_s}{v_s}\int_s^T D_s^W\sigma_r^2 dr ds\right] \\
&= \lim_{T\to t}\frac{\rho}{4(T-t)^{2+\delta}}E_t\left[\int_t^T\int_s^T D_s^W\sigma_r^2 dr ds\right].
\end{aligned}
\tag{14}
$$

Step 2. Here we see that T_2^2 and T_2^3 are the sum of terms of the order $O\left((T-t)^{1+2\delta}\right)$ plus terms of the order $O\left((T-t)^{\frac{3}{2}+3\delta}\right)$. Notice that $\frac{3}{2}+3\delta>\max(1+\delta,1+2\delta)$. Applying again the anticipating Itô's formula to the processes

$$
\left(\frac{\partial^3}{\partial x^3}-\frac{\partial^2}{\partial x^2}\right)H(s,T,X_s,k_t^*,v_s)Z_s,
$$

and

$$
\frac{\partial H}{\partial x}(s,T,X_s,k_t^*,v_s)R_s,
$$

where

$$
\begin{aligned}
Z_s &:= \int_s^T \Phi_u J_u du, \\
R_s &:= \int_s^T \left(\int_u^T \left(BS^{-1}\right)' (k_t^*, \Gamma_r) \left(D_s^W \Phi_r\right) dr \right) \sigma_u du,
\end{aligned}
$$

we get

$$
\begin{aligned}
T_2^2 &= \frac{\rho^2}{4} E_t \Bigg[\left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2} \right) H(t, T, X_t, k_t^*, v_t) Z_t \\
&+ \frac{\rho}{2} \int_t^T \left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2} \right)^2 H(s, T, X_s, k_t^*, v_s) Z_s \Phi_s ds \\
&+ \rho \int_t^T \frac{\partial}{\partial x} \left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2} \right) H(s, T, X_s, k_t^*, v_s) \left(D_s^W Z_s\right) \sigma_s ds \Bigg], \quad (15)
\end{aligned}
$$

and

$$
\begin{aligned}
T_2^3 &= \frac{\rho^2}{2} E_t \Bigg[\frac{\partial H}{\partial x}(t, T, X_t, k_t^*, v_t) R_t \\
&+ \frac{\rho}{2} \int_t^T \left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2} \right) \frac{\partial H}{\partial x}(s, T, X_s, k_t^*, v_s) R_s \Phi_s ds \\
&+ \rho \int_t^T \frac{\partial^2 H}{\partial x^2}(s, T, X_s, k_t^*, v_s) \left(\int_s^T \int_r^T \left(BS^{-1}\right)' (k_t^*, \Gamma_u) \left(D_s^W D_r^W \Phi_u\right) dudr \right) \sigma_s ds \Bigg]. \quad (16)
\end{aligned}
$$

Lemma 4.1 in Alòs, León and Vives gives us that the last two terms in (**??**) and (**??**) are $O\left((T-t)^{\frac{3}{2}+3\delta}\right)$. Now, as

$$
\left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2} \right) H(t, T, X_t, k_t^*, v_t) = -\frac{1}{16} \frac{e^{X_t} N'\left(d_+(k_t^*, v_t)\right)}{\left(v_t \sqrt{T-t}\right)^5} \left(v_t^4 (T-t)^2 - 48\right),
$$

and

$$
\frac{\partial H}{\partial x}(t, T, X_t, k_t^*, v_t) = \frac{1}{4} \frac{e^{X_t} N'(d_+(k_t^*, v_t))}{\left(v_t \sqrt{T-t}\right)^3} \left(v_t^2 (T-t) - 4\right).
$$

It follows that

$$
\begin{aligned}
&\lim_{T \to t} \frac{T_2^2}{(T-t)^{1+2\delta}} \\
&= \frac{\rho^2}{4(T-t)^{1+2\delta}} E_t \left[\left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2} \right) H(t, T, X_t, k_t^*, v_t) Z_t \right] \\
&= \frac{\rho^2}{4(T-t)^{1+2\delta}} E_t \Bigg[-\frac{1}{16} \frac{e^{X_t} N'\left(d_+(k_t^*, v_t)\right)}{\left(v_t \sqrt{T-t}\right)^5} \left(v_t^4 (T-t)^2 - 48\right)
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_t^T \sigma_s \left(\int_t^T D_s^W \sigma_r^2 dr \right) \left(\int_s^T \frac{\Phi_r}{e^{X_t} N'(d_+(k_t^*, BS^{-1}(k_t^*, \Gamma_r)))\sqrt{T-t}} dr \right) ds \Bigg] \\
= &\lim_{T\to t} \frac{3\rho^2}{4\sigma_t^5 (T-t)^{4+2\delta}} E_t \left[\int_t^T \left(\int_s^T D_s^W \sigma_r^2 dr \right) \left(\int_s^T \Phi_r dr \right) \sigma_s ds \right] \\
= &\lim_{T\to t} \frac{3\rho^2}{4\sigma_t^5 (T-t)^{4+2\delta}} E_t \left[\int_t^T \left(\int_s^T D_s^W \sigma_r^2 dr \right) \left(\int_s^T \sigma_r \int_r^T D_r^W \sigma_\theta^2 d\theta dr \right) \sigma_s ds \right] \\
= &\lim_{T\to t} \frac{3\rho^2}{4\sigma_t^3 (T-t)^{4+2\delta}} E_t \left[\int_t^T \left(\int_s^T D_s^W \sigma_r^2 dr \right) \left(\int_s^T \int_r^T D_r^W \sigma_\theta^2 d\theta dr \right) ds \right] \\
= &\lim_{T\to t} \frac{3\rho^2}{8\sigma_t^3 (T-t)^{4+2\delta}} E_t \left[\left(\int_t^T \int_s^T D_s^W \sigma_r^2 dr ds \right)^2 \right], \qquad (17)
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{T\to t} \frac{T_2^3}{(T-t)^{1+2\delta}} \\
= &\lim_{T\to t} \frac{\rho^2}{2(T-t)^{1+2\delta}} E_t \left[\frac{\partial H}{\partial x}(t, T, X_t, k_t^*, v_t) R_t \right] \\
= &\lim_{T\to t} \frac{\rho^2}{2(T-t)^{1+2\delta}} E_t \Bigg[\frac{1}{4} \frac{e^{X_t} N'(d_+(k_t^*, v_t))}{\left(v_t\sqrt{T-t}\right)^3} \left(v_t^2 (T-t) - 4 \right) \\
&\times \int_t^T \int_s^T \frac{1}{e^{X_r} N'(d_+(k_t^*, BS^{-1}(k_t^*, \Gamma_r)))\sqrt{T-t}} \left(D_s^W \left(\sigma_r \int_r^T D_s^W \sigma_u^2 du \right) \right) dr \sigma_s ds \Bigg] \\
= &-\lim_{T\to t} \frac{\rho^2}{2\sigma_t^2 (T-t)^{3+2\delta}} E_t \Bigg[\int_t^T \int_s^T D_s^W \sigma_r \int_r^T D_r^W \sigma_u^2 du dr ds \\
&+ \int_t^T \int_s^T \sigma_r \int_r^T D_s^W D_r^W \sigma_u^2 du dr ds \Bigg] \\
= &-\lim_{T\to t} \frac{\rho^2}{2\sigma_t (T-t)^{3+2\delta}} E_t \left[\int_t^T \left(\int_s^T D_s^W \sigma_r dr \right)^2 ds \right] \\
&-\lim_{T\to t} \frac{\rho^2}{2\sigma_t (T-t)^{3+2\delta}} E_t \left[\int_t^T \int_s^T \int_r^T D_s^W D_r^W \sigma_u^2 du dr ds \right]. \qquad (18)
\end{aligned}
$$

Step 3 From the results in the last steps, we deduce that $I(t, T, X_t, k_t^*) - E[v_t]$ is the sum of terms of the orders $O\left((T-t)^{1+\delta}\right)$, $O\left((T-t)^{1+2\delta}\right)$ and higher-order terms. Then we conclude that, for $\delta < 0$, $I(t, T, X_t, k_t^*) - E[v_t]$ is of the order $O\left((T-t)^{1+2\delta}\right)$, and that if $\delta > 0$, this difference is of the order $O\left((T-t)^{1+\delta}\right)$. Now, taking into account (**??**), (**??**) and (**??**), the result follows. ■

Corollary 9 *Assume that $\sigma_t = f(B_t^H)$, where $f \in \mathcal{C}_b^3$ with range in a compact set of $\mathbb{R}^+$ and B_t^H is a fBm with Hurst parameter H. Then $\delta = H - 1/2$ and the above result proves that, in the correlated case*

- *If* $H \leq 1/2$*, then* $I(t,T,X_t,k_t^*) - E_t[v_t] = O\left((T-t)^{2H}\right)$.
- *If* $H \geq 1/2$*, then* $I(t,T,X_t,k_t^*) - E_t[v_t] = O\left((T-t)^{H+1/2}\right)$.

Notice that, if we compare with Corollary 4, we see that the order of the convergence is not affected only by the Hurst parameter but also by the correlation. This result is in line with the results by Fukasawa (2014), where it was established that the leverage effect (the negative correlation observed between the asset price and its volatility) plays a crucial role in the ATM short-time behaviour of the implied volatility.

Remark 10 *The above theorem gives us that, if* $\delta \geq 0$*,*

$$\lim_{T\to t} \frac{I(t,T,X_t,k_t^*) - E_t[v_t]}{(T-t)^{1+\delta}} = \frac{\rho}{4} \lim_{T\to t} \frac{1}{(T-t)^{2+\delta}} \int_t^T \int_s^T D_s^W \sigma_r^2 dr ds + O(\rho^2).$$

Now, taking into account the representation of the short-time limit skew in term of the Malliavin derivative of the volatility process (see Alòs, León and Vives (2007)) we get

$$\lim_{T\to t} \frac{I(t,T,X_t,k_t^*) - E_t[v_t]}{(T-t)^{1+\delta}} = \frac{\sigma_t^2}{2} \lim_{T\to t} \frac{\frac{\partial I}{\partial k}(t,T,X_t,k_t^*)}{(T-t)^{\delta}} + O(\rho^2). \tag{19}$$

*This equality is in line with the previous results in Section 6.5 of Carr and Lee (2008) on the impact of correlation on volatility swap prices. Moreover (**??**) gives us, in the case* $H \geq 1/2$*, the following model-free approximation formula:*

$$E_t[v_t] \approx I(t,T,X_t,k_t^*) - \frac{I(t,T,X_t,k_t^*)^2}{2} \frac{\partial I}{\partial k}(t,T,X_t,k_t^*)(T-t), \tag{20}$$

that is similar to the model-free first-order vol-of-vol expansion around the variance swap by Bergomi and Guyon (2011). In the case $\delta < 0$*, the obtained limit expressions are more complex. Even when they would allow us to construct an approximation for the volatility swap fair strike, it is not easy to establish a model-free relationship between the fair strike of the volatility swap and the implied volatility skew.*

Remark 11 *Hypotheses (H1)-(H5) have been chosen for the sake of simplicity. The same results can be extended to other stochastic volatility models (see Section 5).*

5 Numerical examples

Consider the model (**??**) whose volatility process is given by the following form

$$\sigma_s = \sigma_0 \exp\left(\nu W_s^H - \frac{\nu^2 s^{2H}}{4H}\right), \; s \in [0,T], \tag{21}$$

for some positive constants ν and σ_0 and where

$$W_s^H := \int_0^s \frac{dW_r}{(s-r)^{\frac{1}{2}-H}},$$

for some Hurst parameter $H \in (0,1)$. Also, we assume $t < T < \infty$. This model is similar to the fractional SABR model (see e.g. Gatheral and Jaisson (2014)). We can prove that this model satisfies (H2'), (H4) and (H5). Nevertheless, it does not satisfy (H1) nor (H3). In order to see that the results in Theorem **??** still hold, we can make use of an approximation argument. Let us define $\phi(x) := \sigma_0 \exp(x)$. For every $n > 1$, consider a function $\phi_n \in \mathcal{C}_b^2$ satisfying that $\phi_n(x) = \phi(x)$ for any $x \in [-n, n]$, $\phi_n(x) \in [\phi(-2n) \vee \phi(x), \phi(-n)]$ for $x \leq -n$, and $\phi_n(x) \in [\phi(n), \phi(x) \wedge \phi(2n)]$ for $x \geq n$. We will define

$$\sigma_s^n := \phi_n \left(\nu W_s^H - \frac{\nu^2 s^{2H}}{4H} \right).$$

It is easy to see that σ_s^n satisfies (H1) and (H3') with $\delta = H - \frac{1}{2}$. Then, we can write (we consider $t = 0$ for the sake of simplicity)

$$\begin{aligned} & I(0, T, X_0, k_0^*) - E[v_0] \\ &= I(0, T, X_0, k_0^*) - I^n(0, T, X_0, k_0^*) \\ &\quad + I^n(0, T, X_0, k_0^*) - E[v_0^n] \\ &\quad + E[v_0^n] - E[v_0] \\ &=: T_1 + T_2 + T_3, \end{aligned} \tag{22}$$

where I^n and $E_t[v_t^n]$ denote, respectively, the implied volatility and the fair price of the volatility swap under the volatility process σ^n. Now, Theorem **??** gives us that (in the correlated case) $T_2 = O\left(T^{2H}\right)$ if $H \leq \frac{1}{2}$ and $T_2 = O\left(T^{H+\frac{1}{2}}\right)$ if $H \geq \frac{1}{2}$. On the other hand,

$$v_0^n = \sqrt{\frac{1}{T} \int_0^T (\sigma_s^n)^2 ds} = \sqrt{\frac{1}{T} \int_0^T (\phi_n(a_s))^2 ds}$$

where $a_s := \nu W_s^H - \frac{\nu^2 s^{2H}}{4H}$. Now, as $\phi_n(x) \leq \max(\phi(x), \phi(-n)) < \phi(x) + \phi(-n)$, $2(a^2 + b^2) \geq (a + b)^2$, and $\sqrt{a} + \sqrt{b} \geq \sqrt{a + b}$ for $a, b > 0$, we get that

$$\begin{aligned} v_0^n &\leq \sqrt{\frac{1}{T} \int_0^T (\phi(a_s) + \phi(-n))^2 ds} \\ &\leq \sqrt{2} \left(\sqrt{\frac{1}{T} \int_0^T (\phi(a_s))^2 ds} + \sqrt{\frac{1}{T} \int_0^T (\phi(-n))^2 ds} \right) \\ &= \sqrt{2} \left(v_0 + \phi(-n) \right), \end{aligned} \tag{23}$$

Then $|v_0^n - v_0| \leq (1 + \sqrt{2}) v_0 + \sqrt{2} \phi(-n)$, which implies that

$$\begin{aligned} T_3 &\leq E\left[|v_0^n - v_0|\right] \\ &\leq (1 + \sqrt{2}) E\left[|v_0 + \sigma_0 \exp(-n)| \mathbf{1}_{\sup_{s \in [0,T]} |\ln(\sigma_s/\sigma_0)| > n} \right] \\ &\leq (1 + \sqrt{2}) \left(E\left[(v_0 + \sigma_0 \exp(-n))^2 \right] \right)^{\frac{1}{2}} \left(P\left(\sup_{s \in [0,T]} \left| \nu W_s^H - \frac{\nu^2 s^{2H}}{4H} \right| > n \right) \right)^{\frac{1}{2}} \end{aligned} \tag{24}$$

Then, if $T<1$, taking $n>\frac{\nu^2}{2H}$ it follows that

$$T_3 \leq (1+\sqrt{2})\left(E\left[(v_0+\sigma_0\exp(-n))^2\right]\right)^{\frac{1}{2}}\left(P\left(\sup_{s\in[0,T]}\left|W_s^H\right|>\frac{n}{2\nu}\right)\right)^{\frac{1}{2}}.$$

Now, Markov's inequality gives us that

$$P\left(\sup_{s\in[0,T]}\left|W_s^H\right|>\frac{n}{2\nu}\right)\leq\left(\frac{2\nu}{n}\right)^p E\left[\left(\sup_{s\in[0,T]}\left|W_s^H\right|\right)^p\right]. \tag{25}$$

Moreover, by Lemma 7.4 in Nualart and Răşcanu (2002) we can easily deduce that, for any $p>4$, there exists a positive constant C_1 such that

$$E\left[\left(\sup_{s\in[0,T]}\left|W_s^H\right|\right)^p\right]\leq C_1T^{pH}. \tag{26}$$

This, jointly with (**??**) and (**??**) give us that $T_3=O\left(T^{\frac{pH}{2}}\right)$. Then, taking $p>4$ it follows that the order of this term is higher than the order of T_2. Next, by the mean value theorem, there exists a point $\xi\in(V_0^n,V_0)$ such that

$$\begin{aligned} T_1 &= I(0,T,X_0,k_0^*)-I^n(0,T,X_0,k_0^*) \\ &= (BS^{-1})'(k_0^*,\xi)(V_0-V_0^n), \end{aligned}$$

where V_0^n is the option premium with the approximated volatility (σ^n). Then,

$$\lim_{T\to0}\frac{I(0,T,X_0,k_0^*)-I^n(0,T,X_0,k_0^*)}{T^\alpha}=C_2\lim_{T\to0}\frac{V_0-V_0^n}{T^{\alpha+\frac{1}{2}}}, \tag{27}$$

for any α, and for some C_2. Now, let us consider the following extension of the Hull and White formula (see Willard (1997) and Romano and Touzi (1997)):

$$\begin{aligned} V_t &= E_t\left[BS\left(t,T,\hat{X}_t,k_t^*,\sqrt{1-\rho^2}v_t\right)\right], \\ V_t^n &= E_t\left[BS\left(t,T,\hat{X}_t^n,k_t^*,\sqrt{1-\rho^2}v_t^n\right)\right], \end{aligned}$$

where

$$\hat{X}_t=X_t+\rho\int_t^T\sigma_s dW_s-\frac{1}{2}\int_t^T(\sigma_s)^2ds$$

and

$$\hat{X}_t^n=X_t+\rho\int_t^T\sigma_s^n dW_s-\frac{1}{2}\int_t^T(\sigma_s^n)^2ds$$

Then, similar arguments as for T_3 give us that, if $T<1$ and $n>\frac{\nu^2}{2H}$

$$|V_0-V_0^n| \leq E\left[\left|e^{\hat{X}_0}-e^{\hat{X}_0^n}\right|\right]+E\left[e^{\hat{X}_0}\sqrt{T(1-\rho^2)}\,|v_0-v_0^n|\right]$$

$$
\begin{aligned}
&\le \left(E\left[\left(e^{\hat{X}_0}+e^{\hat{X}_0^n}\right)^2\right]\right)^{\frac{1}{2}}\left(E\left[\mathbf{1}_{\sup_{s\in[0,T]}|\ln(\sigma_s/\sigma_0)|>n}\right]\right)^{\frac{1}{2}} \\
&\quad +\sqrt{1-\rho^2}\left(E\left[e^{2\hat{X}_0}\right]\right)^{\frac{1}{2}}\left(E\left[|v_0-v_0^n|^2\,\mathbf{1}_{\sup_{s\in[0,T]}|\ln(\sigma_s/\sigma_0)|>n}\right]\right)^{\frac{1}{2}} \\
&\le \left(E\left[\left(e^{\hat{X}_0}+e^{\hat{X}_0^n}\right)^2\right]\right)^{\frac{1}{2}}\left(E\left[\mathbf{1}_{\sup_{s\in[0,T]}|\ln(\sigma_s/\sigma_0)|>n}\right]\right)^{\frac{1}{2}} \\
&\quad +\sqrt{1-\rho^2}\left(E\left[e^{2\hat{X}_0}\right]\right)^{\frac{1}{2}}\left(E\left[(v_0+\sigma_0\exp(-n))^2\mathbf{1}_{\sup_{s\in[0,T]}|\ln(\sigma_s/\sigma_0)|>n}\right]\right)^{\frac{1}{2}} \\
&\le C_3\left(P\left(\sup_{s\in[0,T]}\left(\nu W_s^H-\frac{\nu^2 s^{2H}}{4H}\right)>n\right)\right)^{\frac{1}{4}} \\
&\le C_3\left(P\left(\sup_{s\in[0,T]}\left|W_s^H\right|>\frac{n}{2\nu}\right)\right)^{\frac{1}{4}},
\end{aligned}
$$

for some $C_3>0$. Then, Markov's inequality gives us that

$$
|V_0-V_0^n|\le C_3\left(\frac{2\nu}{n}\right)^{\frac{p}{4}}E\left[\left(\sup_{s\in[0,T]}\left|W_s^H\right|\right)^p\right]^{\frac{1}{4}},
$$

which implies that

$$
T_1=O\left(T^{\frac{pH}{4}-\frac{1}{2}}\right),
$$

and taking $p>\frac{8}{H}$, the order of T_1 is also higher than that of T_2.

5.1 Estimating the Hurst parameters

Hereafter, we use the parameters $\sigma_0=10\%$, $\nu=0.2$ and the correlation between the asset price and its volatility is $\rho=-0.8$.

Let us consider a linear regression analysis with dependent variable $\ln|I(0,T,X_0,k_0^*)-E[v_0]|$ and independent variable $\ln T$. According to our previous results, the corresponding slope will be approximately $2H$ for $H\le 1/2$ and $1/2+H$ for $H>1/2$. This gives us a tool to estimate the Hurst parameter of the fractional volatility model. In fact, if the obtained slope a is lesser than 1, then we will estimate H as $a/2$, while if $a\ge 1$, the Hurst parameter will be estimated by $a-1/2$. In order to check the goodness of this methodology, we have checked it numerically for different Hurst parameters. The results have been compared with the estimate obtained from the fact that the skew is of the order $H-1/2$ (which implies that a linear regression analysis with dependent variable $\frac{\partial I}{\partial k}(0,T,X_0,k_0^*)$ and independent variable $\ln T$ will have a slope equal to $H-1/2$).

Firstly we obtain the ATM option premiums whose maturities are from 0.0001 to 0.5 by using Monte Carlo simulation with 500 time steps for one year (the least number of partition is 100) and one billion trials. Then, the ATM implied volatilities are calculated by the bisection method. The ATM skews ($\frac{\partial I}{\partial k}$) are obtained by the difference method of the implied volatilities. The variance swaps are also calculated by Monte Carlo method. We apply the Black-Scholes model as the control variate to the Monte Carlo simulations for pricing option premiums. The Hurst parameters are set as 0.1, 0.3, 0.5, 0.6 and 0.9. In order to estimate the Hurst parameters, we calculate the ATM implied volatilities, volatility swaps, and ATM skews with these steps, and the results are shown in Table **??**.

H index	Maturity	0.5	0.4	0.3	0.2
0.1	$I(0,T,X_0,k_0^*)$	9.9590%	9.9504%	9.9389%	9.9233%
	$E[v_0]$	10.1292%	10.1077%	10.0816%	10.0477%
	$\partial I/\partial k$	-0.21216	-0.23179	-0.25938	-0.30334
0.3	$I(0,T,X_0,k_0^*)$	9.9825%	9.9858%	9.9889%	9.9922%
	$E[v_0]$	10.0548%	10.0475%	10.0395%	10.0303%
	$\partial I/\partial k$	-0.12601	-0.13183	-0.13941	-0.15073
0.5	$I(0,T,X_0,k_0^*)$	9.9808%	9.9848%	9.9886%	9.9925%
	$E[v_0]$	10.0166%	10.0133%	10.0100%	10.0067%
	$\partial I/\partial k$	-0.07937	-0.07940	-0.07930	-0.07906
0.7	$I(0,T,X_0,k_0^*)$	9.9870%	9.9901%	9.9930%	9.9957%
	$E[v_0]$	10.0071%	10.0052%	10.0035%	10.0020%
	$\partial I/\partial k$	-0.05240	-0.05013	-0.04727	-0.04342
0.9	$I(0,T,X_0,k_0^*)$	9.9914%	9.9937%	9.9958%	9.9976%
	$E[v_0]$	10.0037%	10.0025%	10.0015%	10.0007%
	$\partial I/\partial k$	-0.03585	-0.03280	-0.02919	-0.02470
H index	Maturity	0.1	0.01	0.001	0.0001
0.1	$I(0,T,X_0,k_0^*)$	9.93962763%	9.96946549%	9.98260328%	9.98947785%
	$E[v_0]$	10.04169028%	10.02661269%	10.01691433%	10.01072121%
	$\partial I/\partial k$	-0.40128	-1.01318	-2.55196	-6.42015
0.3	$I(0,T,X_0,k_0^*)$	9.99612543%	9.99983575%	10.00008748%	10.00004253%
	$E[v_0]$	10.02000722%	10.00502817%	10.00126319%	10.00031732%
	$\partial I/\partial k$	-0.17339	-0.27530	-0.43630	-0.69192
0.5	$I(0,T,X_0,k_0^*)$	9.99623704%	9.99962458%	9.99996267%	9.99999633%
	$E[v_0]$	10.00333211%	10.00033331%	10.00003334%	10.00000334%
	$\partial I/\partial k$	-0.07914	-0.07914	-0.07909	-0.07912
0.7	$I(0,T,X_0,k_0^*)$	9.99813219%	9.99988195%	9.99999258%	9.99999954%
	$E[v_0]$	10.00074288%	10.00002959%	10.00000119%	10.00000006%
	$\partial I/\partial k$	-0.03782	-0.02390	-0.01509	-0.00950
0.9	$I(0,T,X_0,k_0^*)$	9.99908619%	9.99996321%	9.99999854%	9.99999994%
	$E[v_0]$	10.00020248%	10.00000322%	10.00000006%	10.00000001%
	$\partial I/\partial k$	-0.01873	-0.00746	-0.00297	-0.00118

Table 1: ATM implied volatilities, volatility swaps, and ATM skews

The linear regression with dependent variable $\ln|I(0,T,X_0,k_0^*) - E[v_0]|$ and independent variable $\ln T$ give us the slopes which are used for calculating the Hurst parameters. Moreover, we also estimate H from the obtained skew. The results are summarized in Table **??**.

H index		Maturities	$T \leq 0.5$	$T \leq 0.4$	$T \leq 0.3$	$T \leq 0.2$	$T \leq 0.1$	$T \leq 0.01$
0.1	(A)	Slopes	0.244	0.241	0.237	0.232	0.227	0.215
		estimated H	0.122	0.120	0.118	0.116	0.113	0.107
	(B)	Slopes	-0.401	-0.401	-0.401	-0.402	-0.401	-0.401
		estimated H	0.099	0.099	0.099	0.098	0.099	0.099
0.3	(A)	Slopes	0.655	0.654	0.652	0.649	0.646	0.638
		estimated H	0.328	0.327	0.326	0.325	0.323	0.319
	(B)	Slopes	-0.200	-0.200	-0.200	-0.200	-0.200	-0.200
		estimated H	0.300	0.300	0.300	0.300	0.300	0.300
0.5	(A)	Slopes	1.002	1.002	1.002	1.001	1.002	1.002
		estimated H	0.501	0.501	0.501	0.501	0.501	0.501
	(B)	Slopes	0.000	0.000	0.000	-0.000	0.000	0.000
		estimated H	0.500	0.500	0.500	0.500	0.500	0.500
0.7	(A)	Slopes	1.242	1.241	1.240	1.238	1.235	1.229
		estimated H	0.742	0.741	0.740	0.738	0.735	0.729
	(B)	Slopes	0.200	0.200	0.200	0.200	0.200	0.200
		estimated H	0.700	0.700	0.700	0.700	0.700	0.700
0.9	(A)	Slopes	1.427	1.424	1.421	1.416	1.408	1.388
		estimated H	0.927	0.924	0.921	0.916	0.908	0.888
	(B)	Slopes	0.400	0.400	0.400	0.400	0.400	0.400
		estimated H	0.900	0.900	0.900	0.900	0.900	0.900

Table 2: Hurst parameters obtained from linear regressions.

$T \leq x$ means that the maturities of the data, which are used for the linear regression, are less than or equal to x years. For example, in the case of $T \leq 0.2$, the linear regression analysis uses the data whose maturities are 0.2, 0.1, 0.01, 0.001, 0.0001. (A) shows the results calculated by the implied volatilities and volatility swaps, and (B) shows those calculated by the skews. In the cases of using the ATM implied volatility and the volatility swap, the Hurst parameters (the values in estimated H rows) are calculated by "slope / 2" for $H \leq 0.5$ and are calculated by "slope - 0.5" for $H > 0.5$. In the cases of using the ATM skew, the Hurst parameters are calculated by "slope + 0.5".

These results show that most of the Hurst parameters are obtained accurately. In particular, the Hurst parameters obtained by skews are more precise than those obtained by implied volatilities and volatility swaps. Nevertheless, this latest methodology does not need to compute the implied volatility skew from real market data. We also notice that the estimates obtained from the implied volatilities and the volatility swaps tend to be more accurate when we only use very short times to maturity. The techniques presented in this paper could have a potential interest in FX markets, where volatility swaps are more popular than variance swaps and where maturities can be very short.

5.2 Approximation of volatility swaps

Even when formula (**??**) is only valid in the case $H \geq \frac{1}{2}$, Theorem **??** gives us that, in the uncorrelated case $\rho = 0$, the ATM implied volatility (that coincides in this case with (**??**)) must be an accurate approximation for the volatility swap fair price. In this subsection we compare the values of our formula (**??**) with those of the ATM implied volatility as the approximated values of volatility swaps. Table **??** and Table **??** show the approximated volatility swaps using the ATM implied volatility (ATMI) and our correction (formula (??)) for $\rho = -0.8$ and $\rho = 0$, respectively. The rows of "volatility swap" are the original volatility swap values obtained by the Monte Carlo simulation.

H index	Maturities	0.5	0.4	0.3	0.2
0.5	vol swap	10.0166%	10.0133%	10.0100%	10.0067%
	ATMI	9.9808%	9.9848%	9.9886%	9.9925%
	formula (??)	10.0006%	10.0006%	10.0005%	10.0004%
	error (ATMI)	-0.358%	-0.285%	-0.214%	-0.142%
	error (formula (??))	-0.160%	-0.127%	-0.095%	-0.063%
0.7	vol swap	10.0071%	10.0052%	10.0035%	10.0020%
	ATMI	9.9870%	9.9901%	9.9930%	9.9957%
	formula (??)	10.0000%	10.0001%	10.0001%	10.0001%
	error (ATMI)	-0.201%	-0.151%	-0.105%	-0.062%
	error (formula (??))	-0.070%	-0.051%	-0.034%	-0.019%
0.9	vol swap	10.0037%	10.0025%	10.0015%	10.0007%
	ATMI	9.9914%	9.9937%	9.9958%	9.9976%
	formula (??)	10.0003%	10.0003%	10.0002%	10.0001%
	error (ATMI)	-0.123%	-0.087%	-0.057%	-0.031%
	error (formula (??))	-0.033%	-0.022%	-0.013%	-0.006%
H index	Maturities	0.1	0.01	0.001	0.0001
0.5	vol swap	10.0033321%	10.0003333%	10.0000333%	10.0000033%
	ATMI	9.9962370%	9.9996246%	9.9999627%	9.9999963%
	formula (??)	10.0001909%	10.0000203%	10.0000022%	10.0000003%
	error (ATMI)	-0.070927%	-0.007087%	-0.000707%	-0.000070%
	error (formula (??))	-0.031401%	-0.003130%	-0.000311%	-0.000031%
0.7	vol swap	10.0007429%	10.0000296%	10.0000012%	10.0000001%
	ATMI	9.9981322%	9.9998820%	9.9999926%	9.9999995%
	formula (??)	10.0000225%	10.0000014%	10.0000001%	10.0000000%
	error (ATMI)	-0.026105%	-0.001476%	-0.000086%	-0.000005%
	error (formula (??))	-0.007204%	-0.000282%	-0.000011%	0.000000%
0.9	vol swap	10.0002025%	10.0000032%	10.0000001%	10.0000000%
	ATMI	9.9990862%	9.9999632%	9.9999985%	9.9999999%
	formula (??)	10.0000223%	10.0000005%	10.0000000%	10.0000000%
	error (ATMI)	-0.011163%	-0.000400%	-0.000015%	-0.000001%
	error (formula (??))	-0.001802%	-0.000027%	0.000000%	0.000000%

Table 3: approximated volatility swaps ($\rho = -0.8$)

H index	Maturities	0.5	0.4	0.3	0.2
0.1	vol swap	10.1292%	10.1077%	10.0816%	10.0477%
	ATMI = formul (??)	10.1286%	10.1075%	10.0814%	10.0476%
	error	-0.005%	-0.002%	-0.002%	0.000%
0.3	vol swap	10.0548%	10.0475%	10.0395%	10.0303%
	ATMI = formul (??)	10.0546%	10.0475%	10.0395%	10.0303%
	error	-0.002%	0.000%	-0.001%	0.000%
0.5	vol swap	10.0166%	10.0133%	10.0100%	10.0067%
	ATMI = formul (??)	10.0165%	10.0133%	10.0100%	10.0067%
	error	-0.001%	0.000%	0.000%	0.000%
0.7	vol swap	10.0071%	10.0052%	10.0035%	10.0020%
	ATMI = formul (??)	10.0070%	10.0052%	10.0035%	10.0020%
	error	-0.001%	0.000%	0.000%	0.000%
0.9	vol swap	10.0037%	10.0025%	10.0015%	10.0007%
	ATMI = formul (??)	10.0036%	10.0025%	10.0015%	10.0007%
	error	0.000%	0.000%	0.000%	0.000%
H index	Maturities	0.1	0.01	0.001	0.0001
0.1	vol swap	10.0416903%	10.0266127%	10.0169143%	10.0107212%
	ATMI = formul (??)	10.0417006%	10.0266294%	10.0169207%	10.0107220%
	error	0.000103%	0.000166%	0.000064%	0.000008%
0.3	vol swap	10.0200072%	10.0050282%	10.0012632%	10.0003173%
	ATMI = formul (??)	10.0200206%	10.0050341%	10.0012656%	10.0003186%
	error	0.000134%	0.000059%	0.000024%	0.000013%
0.5	vol swap	10.0033321%	10.0003333%	10.0000333%	10.0000033%
	ATMI = formul (??)	10.0033428%	10.0003366%	10.0000344%	10.0000037%
	error	0.000106%	0.000033%	0.000011%	0.000003%
0.7	vol swap	10.0007429%	10.0000296%	10.0000012%	10.0000001%
	ATMI = formul (??)	10.0007510%	10.0000311%	10.0000015%	10.0000001%
	error	0.000081%	0.000015%	0.000003%	0.000001%
0.9	vol swap	10.0002025%	10.0000032%	10.0000001%	10.0000000%
	ATMI = formul (??)	10.0002079%	10.0000038%	10.0000001%	10.0000000%
	error	0.000054%	0.000006%	0.000001%	0.000000%

Table 4: approximated volatility swaps ($\rho = 0$)

The rows named “error” are calculated as

$$(approximated\ value\ -\ volatility\ swap\ value)/(volatility\ swap\ value),$$

and are expressed as a percent.

In the correlated case (i.e. $\rho = -0.8$), we can see that all of the errors of the new approximation are lower than those of obtained by ATM implied volatility. In the uncorrelated case, as predicted and according to Carr and Lee (2009), the differences between the volatility swap and the ATMI are much smaller than those of the correlated case.

References

[1] Alòs, E. "A generalization of the Hull and White formula with applications to option pricing approximation," Finance and Stochastics 10 (3), (2006): 353-365.

[2] Alòs, E. and León, J. A. "On the curvature of the smile in stochastic volatility models," SIAM J. Financial Math. 8, (2017): 373-399.

[3] Alòs, E., León, J. A., and Vives, J. "On the short-time behavior of the implied volatility for jump-diffusion models with stochastic volatility," Finance and Stochastics 11 (4), (2007): 571-589.

[4] Bergomi, L. and Guyon, J.. The Smile in Stochastic Volatility Models (December 2, 2011). Available at SSRN: https://ssrn.com/abstract=1967470

[5] Carr, P. and Lee, R. "Robust replication of volatility derivatives," PRMIA award for Best Paper in Derivatives, MFA 2008 Annual Meeting, (2008).

[6] Carr, P. and Lee, R. "Volatility derivatives," Annu. Rev. Financ. Econ. 1 (1), (2009): 319-339.

[7] Comte, F. and Renault, E.: "Long memory in continuous-time stochastic volatility models," Math. Financ. 8, (1998): 291-323.

[8] Demeterfi, K., Derman, E., Kamal, M., and Zou, J. "More Than You Ever Wanted To Know About Volatility Swaps," Goldman Sachs Quantitative Strategies Research Notes, (1999).

[9] El Euch, O. Fukasawa, M., Gatheral, J. and Rosenbaum, M. Short-Term At-the-Money Asymptotics Under Stochastic Volatility Models (January 27, 2018). Available at SSRN: https://ssrn.com/abstract=3111471

[10] Feinstein S. "The black-scholes formula is nearly linear in sigma for at-the-money options: Therefore implied volatilities from at-the-money options are virtually unbiased," Work. Pap., Fed. Reserve Bank Atlanta, (1989).

[11] Fukasawa, M. "Asymptotic analysis for stochastic volatility: martingale expansion," Finance and Stochastics 15 (4), (2011): 635-654.

[12] Fukasawa, M. "Volatility derivatives and model-free implied leverage," International Journal of Theoretical and Applied Finance 17 (1), (2014).

[13] Friz, P., and Gatheral, J. "Valuation of volatility derivatives as an inverse problem," Quantitative Finance 5 (6), (2005): 531-542.

[14] Gatheral, J. and Jaisson, T. "Fractional volatility models. Presentation at Bloomberg seminar," http://mfe.baruch.cuny.edu/wp-content/uploads/2012/09/FractionalVolatility2014BBQ.pdf, (2014).

[15] Gatheral, J., Jaisson, T. and Rosenbaum, M. "Volatility is rough." Preprint, 2014. arXiv:1410.3394.

[16] Nualart, D. "The Malliavin Calculus and Related Topics. Second Edition," Probability and its Applications. Springer-Verlag, (2006).

[17] Nualart, D. and Răşcanu, A. "Differential equations driven by fractional Brownian motion," Collect. Math. 53 (1), (2002): 55-81.

[18] Romano, M., and Touzi, N. "Contingent claims and market completeness in a stochastic volatility model," Math. Financ. 7, (1997): 399-410.

[19] Willard, G.A. "Calculating prices and sensitivities for path-independent securities in multifactor models," J. Deriv. 5, (1997): 45-61.