

ISSN PRINT 2055-3706
ISSN ONLINE 2055-3714

Journal of Applied Logics

The IfCoLog Journal of Logics and their Applications

Volume 5 ● Issue 8 ● November 2018

Available online at
www.collegepublications.co.uk/journals/ifcolog/
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JOURNAL OF APPLIED LOGICS - IFCoLOG
JOURNAL OF LOGICS AND THEIR APPLICATIONS

Volume 5, Number 8

November 2018

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ISBN 978-1-84890-291-6
ISSN (E) 2055-3714
ISSN (P) 2055-3706

College Publications
Scientific Director: Dov Gabbay
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<http://www.collegepublications.co.uk>

Printed by Lightning Source, Milton Keynes, UK

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A PROPOSITIONAL LOGIC WITH BINARY METRIC OPERATORS

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Abstract

The aim of this paper is to combine distance functions and Boolean propositions by developing a formalism suitable for speaking about distances between Boolean formulas. We introduce and investigate a formal language that is an extension of classical propositional language obtained by adding new binary (modal-like) operators of the form $D_{\leq s}$ and $D_{\geq s}$, $s \in \mathbb{Q}_0^+$. Our language allows making formulas such as $D_{\leq s}(\alpha, \beta)$ with the intended meaning ‘distance between formulas α and β is less than or equal to s ’. The semantics of the proposed language consists of possible worlds with a distance function defined between sets of worlds. Our main concern is a complete axiomatization that is sound and strongly complete with respect to the given semantics.

Keywords: metric operators, soundness, completeness

We thank the anonymous reviewers for their careful reading of our manuscript and their many insightful comments and suggestions.

The work presented here was partially supported by the Serbian Ministry of Education and Science (projects 174026)

1 Introduction

Formalisms for representing uncertain, incomplete or vague data, information and knowledge, as well as reasoning about them are the subject of increasing interest in many scientific fields closely related to many applications (such as technology development). Besides many mathematical concepts that are useful in these fields, we emphasize two of them: *distance functions* and *Boolean propositions*.

In general, distance functions are fundamental for many areas of mathematics and computer science. Roughly speaking, distance functions express the degree of similarity (or dissimilarity) between two objects: matrices (in algebra), graphs (in discrete mathematics, combinatorics), strategies (in game theory), probability distributions (in probability theory), knowledge (in artificial intelligence), messages (in coding theory), strings (in information theory, linguistics), etc.

Boolean propositions (Boolean functions or propositional formulas, what the reader prefers) are important in many of the above mentioned areas. The language of Boolean propositions is very suited for a representation of different discrete systems. Some recent applications include circuit design, social choice theory, learning theory etc.

The aim of this paper is to combine distance functions and Boolean propositions by developing a formalism suitable for speaking about distances between Boolean formulas. More precisely, we introduce and investigate a formal language that is an extension of classical propositional language obtained by adding new binary (modal-like) operators of the form $D_{\leq s}$ and $D_{\geq s}$, $s \in \mathbb{Q}_0^+$ (where \mathbb{Q}_0^+ is the set of non negative rational numbers). The language allows making formulas such as $D_{\leq s}(\alpha, \beta)$ with the intended meaning ‘distance between formulas α and β is less than or equal to s ’. Thus, our formalism is substantially related to distance functions between Boolean propositions, and it enables us to infer consistent conclusions from propositional and metric statements. In the next section, Example 3 gives an illustrative sketch of possible applications.

The idea of constructing logical formalisms that include the notion of distance is not new ([2], [3], [6], [9], [10], [11], [13], [20], [22] and [25]). More attention has been devoted to metric (or quantitative) temporal logics (see [1], [7] and [14]), which reflects the fact that temporal logic in general is more developed than spatial logic.

In this paper, we adopt an approach similar to the development of probabilistic propositional logic (see [5], [15], [16], [17], [18] and [19]). The semantics of our language consists of possible worlds with a distance function defined between sets of words. Our main concern is a complete axiomatization that is sound and strongly complete with respect to the given semantics (‘Every consistent set of formulas has a model’ in contrast to the weak completeness ‘every consistent formula has a

model'). Finitary axiomatizations could not be expected, because of the inherent non-compactness: in the proposed language it is possible to define an inconsistent set of formulas such that all its finite subsets are consistent ($T = \{\neg D_{=0}(\alpha, \beta)\} \cup \{D_{<\frac{1}{n}}(\alpha, \beta) \mid n \text{ is a positive integer}\}$). The lack of compactness forces us to consider an infinitary axiomatization.

The rest of the paper is organized as follows. Section 2 presents some preliminaries and gives several examples that motivate our investigation and indicate possible applications. Syntax and semantics of our logic are introduced in Section 3. In Section 4 we examine a sound and (strongly) complete axiomatization, and discuss several modifications of the proposed logic. This Section contains the main results of this paper. Concluding remarks are given in Section 5.

2 Preliminaries

The term *distance* generally refers to a function satisfying some properties of the (most common) distance between two points in Euclidean space. In this paper, we focus on metrics and pseudometrics.

Recall that a metric space is a pair (X, d) , where X is a nonempty set and $d : X \times X \rightarrow [0, +\infty)$ is a metric, i.e., a function satisfying the following constraints:

(D1) $d(x, y) = 0$ iff $x = y$ (identity of indiscernibles),

(D2) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality),

(D3) $d(x, y) = d(y, x)$ (symmetry),

for all $x, y, z \in X$. The value $d(x, y)$ is called the distance from x to y . Although acceptable in many cases, the requirements (D1), (D2) and (D3) all together are too strong in many real contexts. This is especially true for the condition: $d(x, y) = 0$ implies $x = y$. In a pseudometric space, the distance between two distinct points can be zero; $d : X \times X \rightarrow [0, +\infty)$ is a pseudometric if it satisfies (D2), (D3) and the following constraint less strong than (D1):

(D1-) $d(x, x) = 0$, for all $x \in X$.

There are many ways of relaxing the constraints on metrics. For instance, it is reasonable to omit the symmetry of distance (e.g., $d(x, y)$ = 'work required to get from x to y in a mountainous region'.) A quasimetric is defined as a function that satisfies (D1) and (D2). Any quasimetric traditionally can be symmetrized, e.g. by one of the procedures: $(d(x, y) + d(y, x))/2$ or $\max\{d(x, y), d(y, x)\}$. A pseudoquasimetric (also called hemimetric) satisfies (D1-) and (D2).

In many naturally arising examples, distance functions are bounded. If it is not bounded, there are well-known procedures that normalize a metric space (X, d) into an 1-bounded topologically-equivalent metric space such as $(X, d/(1 + d))$ or $(X, \min\{1, d\})$. Especially, if d is bounded by M , $(X, d/M)$ is a straightforward conversion.

The rest of this section brings some motivating examples that highlight possible applications of our logic.

EXAMPLE 1. Given a bounded pseudometric space (X, d) , one can define a pseudo-quasimetric on the subsets of X , called the one-sided Hausdorff distance:

$$d_H(A, B) = \sup_{a \in A} \inf_{b \in B} d(a, b), \quad A, B \subseteq X.$$

The (bidirectional) Hausdorff distance is defined as:

$$D_H(A, B) = \max\{d_H(A, B), d_H(B, A)\}, \quad A, B \subseteq X.$$

D_H is a pseudometric on $\mathcal{P}(X)$ (the power set of X) that is of a great importance in many applications (e.g. [21]).

Besides the Hausdorff metric, many other distance functions between sets are important for applications (see for instance [4] and the references given there). Any such distance function is closely related to our semantics (specified in the next section) that is based on a distance function defined on a Boolean algebra (more precisely, on the Lindenbaum-Tarski algebra of a classical propositional theory).

EXAMPLE 2. Given a probability space (W, \mathcal{F}, P) , a natural example of distance between two sets (events) is the probability of their symmetric difference, $d_P(A, B) = P(A \Delta B)$. It is well-known that d_P is a pseudometric. This example could inspire development of logics that extend probabilistic propositional logics by enriching their languages with distance operators. Some interesting ideas in that direction are given in [12].

EXAMPLE 3. Let For_n be the set of classical propositional formulas over the propositional variables p_1, \dots, p_n . Let a_1, \dots, a_N , where $N = 2^n$, run through the 2^n conjunctive clauses of the form $p_1^{e_1} \wedge \dots \wedge p_n^{e_n}$, where $e_1, \dots, e_n \in \{0, 1\}$ ($p^1 = p$ and $p^0 = \neg p$). We call a_i 's atoms, and denote the set of these atoms by A . It is obvious that for a given atom a there is a unique valuation $v_a : \{p_1, \dots, p_n\} \rightarrow \{0, 1\}$ such that $v_a(a) = 1$, and vice versa. Moreover, each formula can be regarded as a set of atoms. For each α there is $S_\alpha \subseteq A$ such that α is classically equivalent to $\bigvee S_\alpha$:

$$S_\alpha = \{a \in A \mid a \models \alpha\} = \{a \in A \mid v_a(\alpha) = 1\}.$$

Identifying the atoms from A with the binary strings of the length n , any distance function between strings can be transferred into the logical context. For instance,

the Hamming distance could be very useful (the Hamming distance between two strings of equal length is the number of positions at which the corresponding symbols are different). Note that this distance can be derived from the notion of logical consequence, or more precisely from the mapping:

$$v(a, \alpha) = \begin{cases} 1, & a \models \alpha, \\ 0, & a \not\models \alpha. \end{cases}$$

Any distance between atoms can be lifted to a distance between formulas in a manner analogous to the way in which one obtains the Hausdorff metrics (see Example 1). The Hamming distance between atoms defines $\overline{D} : \text{For} \times \text{For} \rightarrow [0, +\infty)$,

$$\overline{D}(\alpha, \beta) = \sum_{i=1}^N |v(a_i, \alpha) - v(a_i, \beta)|,$$

which is a metric. Normalizing the metric \overline{D} , we obtain another metric

$$D(\alpha, \beta) = \left(\sum_{i=1}^N |v(a_i, \alpha) - v(a_i, \beta)| \right) / N.$$

Using the metric D , we sketch out an idea for more serious applications. Suppose that the patient can use four different types of medicines A, B, C, D for medical treatment (taking only one type or mixing two or more types simultaneously). We use a, b, c, d to denote the (propositional) statements: the patient takes A, B, C, D , respectively. Let p denote ‘the patient is cured’. Three experimentally approved facts can be expressed by the following formulas: $a \wedge b \rightarrow p, a \vee b \vee c \rightarrow p, (a \wedge d) \vee (b \wedge c) \rightarrow p$. In order to identify the most efficient medicine, i.e. to prescribe only one medicine to a new patient, the doctor could consider the distances from $a \rightarrow p, b \rightarrow p, c \rightarrow p$ or $d \rightarrow p$ to each of the approved facts. The following table is obtained by easy calculations. For instance, $D(a \wedge b \rightarrow p, a \rightarrow p) = 4/32 = 0.125$. The table shows that $a \rightarrow p$ or $b \rightarrow p$ are the closest to one of the approved statements.

D	$a \rightarrow p$	$b \rightarrow p$	$c \rightarrow p$	$d \rightarrow p$
$(a \wedge b) \rightarrow p$	0.125	0.125	0.25	0.25
$(a \vee c \vee d) \rightarrow p$	0.1875	0.25	0.1875	0.1875
$((a \wedge d) \vee (b \wedge c)) \rightarrow p$	0.15625	0.15625	0.15625	0.15625

Note that D has very interesting properties:

1. $D(\alpha, \neg\alpha) = 1$,

2. $D(\alpha, \neg\beta) = 1 - D(\alpha, \beta)$,
3. $D(\alpha, \beta \vee \gamma) = D(\alpha, \beta) + D(\alpha, \gamma) - D(\alpha, \beta \wedge \gamma)$,

for all formulas α , β and γ . Since the metric D shares some properties with conditional probabilities, it would be interesting to investigate some deeper connections between our logic and the appropriate probabilistic logics (see [5], [8], [15], [16], [17], [18] and [19]).

3 Syntax and Semantics

Syntax. The language of distance logics consists of a countable set $P = \{p_1, p_2, \dots\}$ of propositional letters, classical connectives \wedge and \neg , and a list of binary metric operators $D_{\leq s}$ and $D_{\geq s}$ for every $s \in \mathbb{Q}_0^+$. The set For_C of all classical propositional formulas over the set P is defined as usual. The formulas from the set For_C will be denoted by $\alpha, \beta, \gamma, \dots$. If $\alpha, \beta \in \text{For}_C$ and $s \in \mathbb{Q}_0^+$, then $D_{\leq s}(\alpha, \beta)$ and $D_{\geq s}(\alpha, \beta)$ are basic metric formulas. The set For_M of all metric formulas is the smallest set:

- containing all basic metric formulas, and
- closed under formation rules: if $A, B \in \text{For}_M$, then $\neg A, A \wedge B \in \text{For}_M$.

The formulas from the set For_M will be denoted by A, B, C, \dots . Let $\text{For} = \text{For}_C \cup \text{For}_M$. The formulas from For will be denoted by $\Phi, \Psi \dots$. For example, the following is a formula: $D_{\leq 0.4}(\alpha, \beta) \wedge \neg D_{\geq 0.1}(\alpha \wedge \gamma, \neg\beta)$.

We use the usual abbreviations for the other classical connectives \vee , \rightarrow and \leftrightarrow , and the standard conventions for the omission of parentheses. We also abbreviate:

- $\neg D_{\leq s}(\alpha, \beta)$ to $D_{> s}(\alpha, \beta)$,
- $\neg D_{\geq s}(\alpha, \beta)$ to $D_{< s}(\alpha, \beta)$,
- $D_{\leq s}(\alpha, \beta) \wedge D_{\geq s}(\alpha, \beta)$ to $D_{=s}(\alpha, \beta)$, and
- $\neg D_{=s}(\alpha, \beta)$ to $D_{\neq s}(\alpha, \beta)$.

Both $\alpha \wedge \neg\alpha$ and $A \wedge \neg A$ are denoted by \perp , for arbitrary formulas $\alpha \in \text{For}_C$ and $A \in \text{For}_M$. Note that neither mixing of pure propositional formulas and metric formulas, nor nested metric operators are allowed. Thus, $\alpha \vee D_{\geq 0.2}(\gamma, \beta)$ and $D_{\geq 0.4}(D_{< 0.9}(\alpha, \beta), D_{=0.6}(\gamma, \beta))$ do not belong to the set For .

Semantics. The semantics of our logic is essentially based on a distance function $D : \text{For}_C \times \text{For}_C \rightarrow [0, +\infty)$ satisfying the corresponding constraints for every $\alpha, \beta, \gamma \in \text{For}_C$:

(D1-) If $\alpha \leftrightarrow \beta$ is a tautology, then $D(\alpha, \beta) = 0$;

(D2) $D(\alpha, \beta) \leq D(\alpha, \gamma) + D(\gamma, \beta)$;

(D3) $D(\alpha, \beta) = D(\beta, \alpha)$;

possibly with the stronger version of (D1-):

(D1) $\alpha \leftrightarrow \beta$ is a tautology iff $D(\alpha, \beta) = 0$.

In the other words, the semantics is based on a (pseudo)metric on the Lindenbaum–Tarski algebra of a classical propositional theory, i.e., on the quotient algebra obtained by factoring For_C by the equivalence relation which identifies any two formulas provably equivalent in the theory. However, we propose a slightly more general semantics based on the possible-world approach. A possible-world interpretation of propositional language is specified by a nonempty set W of worlds and the truth values of all propositional letters at each world, $v : W \times P \rightarrow \{0, 1\}$. Given a world w , the truth values of all propositional formulas is defined in the standard recursive way, and we write $v(w, \alpha)$ for the truth value of α determined by the valuation assigned to w . Given a possible-world interpretation (W, v) , each formula $\alpha \in \text{For}_C$ defines a set of worlds $[\alpha] = \{w : v(w, \alpha) = 1\}$. Let $\mathcal{F} = \{[\alpha] \mid \alpha \in \text{For}_C\}$.

Definition 3.1. An *LPM*-model is a structure $\mathbf{M} = \langle W, v, d \rangle$ where:

- W is a nonempty set of objects called worlds and $v : W \times P \rightarrow \{0, 1\}$ provides for each world $w \in W$ a two-valued evaluation of the propositional letters;
- $d : \mathcal{F} \times \mathcal{F} \rightarrow [0, +\infty)$ is a pseudometric.

An *LPM*-model $\mathbf{M} = \langle W, v, d \rangle$ is an *LM*-model if d is a metric.

Note that our semantics is completely analogous to the semantics for some probabilistic propositional logics ([5]).

Definition 3.2. The satisfiability relation fulfills the following conditions for every *LPM*-model or *LM*-model $\mathbf{M} = \langle W, v, d \rangle$:

- if $\alpha \in \text{For}_C$, $\mathbf{M} \models \alpha$ iff for every $w \in W$, $v(w, \alpha) = \text{true}$,
- if $\alpha, \beta \in \text{For}_C$, $\mathbf{M} \models D_{\leq s}(\alpha, \beta)$ iff $d([\alpha], [\beta]) \leq s$,
- if $\alpha, \beta \in \text{For}_C$, $\mathbf{M} \models D_{\geq s}(\alpha, \beta)$ iff $d([\alpha], [\beta]) \geq s$,
- if $A \in \text{For}_M$, $\mathbf{M} \models \neg A$ iff $\mathbf{M} \not\models A$,

- if $A, B \in \text{For}_M$, $\mathbf{M} \models A \wedge B$ iff $\mathbf{M} \models A$ and $\mathbf{M} \models B$.

A formula $\Phi \in \text{For}$ is (*LPM-satisfiable*) *LM-satisfiable* if there is an (*LPM-model*) *LM-model* \mathbf{M} such that $\mathbf{M} \models \Phi$. A set T of formulas is (*LPM-satisfiable*) *LM-satisfiable* if there is an (*LPM-model*) *LM-model* \mathbf{M} such that $\mathbf{M} \models \Phi$ for every $\Phi \in T$.

Φ is (*LPM-valid*) *LM-valid* if for every (*LPM-model*) *LM-model* \mathbf{M} , $\mathbf{M} \models \Phi$.

We could further restrict the class of *LM-models* to those models whose distance function fulfills some additional conditions. For instance, the following conditions are motivated by the examples 2 and 3:

$$(D4) \quad d([\alpha], [\neg\alpha]) = 1;$$

$$(D5) \quad \text{if } [\beta] \cap [\gamma] = \emptyset \text{ then } d([\alpha], [\beta]) + d([\alpha], [\gamma]) \leq 1.$$

Such model will be called *LM⁺-model*. The equality $d([\alpha], [\beta]) + d([\alpha], [\neg\beta]) = 1$ is an easy consequence of (D2), (D3), (D4) and (D5): $1 = d([\beta], [\neg\beta]) \leq d([\alpha], [\beta]) + d([\alpha], [\neg\beta]) \leq 1$. Note that in general $D_{\leq s}$ and $D_{\geq s}$ are not interdefinable. But if we consider *LM⁺-models*, then one type of our operators can be defined by the other: e.g., $D_{\leq s}(\alpha, \beta) = D_{\geq 1-s}(\alpha, \neg\beta)$. At the end of the next section, we will briefly discuss a complete axiomatization with respect to *LM⁺-models*.

4 Sound and complete axiomatization

The set of all *LPM-valid* formulas can be characterized by the following set of axiom schemata:

(A1) all For_C -instances of classical propositional tautologies,

(A2) all For_M -instances of classical propositional tautologies,

$$(A3) \quad D_{\geq 0}(\alpha, \beta),$$

$$(A4) \quad D_{\leq s}(\alpha, \beta) \rightarrow D_{< r}(\alpha, \beta), \quad r > s,$$

$$(A5) \quad D_{< s}(\alpha, \beta) \rightarrow D_{\leq s}(\alpha, \beta),$$

$$(A6) \quad D_{\leq s}(\alpha, \beta) \wedge D_{\leq r}(\beta, \gamma) \rightarrow D_{\leq s+r}(\alpha, \gamma),$$

$$(A7) \quad D_{\leq s}(\alpha, \beta) \rightarrow D_{\leq s}(\beta, \alpha);$$

and inference rules:

(R1) From Φ and $\Phi \rightarrow \Psi$ infer Ψ ,

(R2) From $\alpha \leftrightarrow \beta$ infer $D_{=0}(\alpha, \beta)$,

(R3) From $A \rightarrow D_{<s+\frac{1}{k}}(\alpha, \beta)$, for every positive integer k , infer $A \rightarrow D_{\leq s}(\alpha, \beta)$,

(R4) From $A \rightarrow D_{>s-\frac{1}{k}}(\alpha, \beta)$, for every positive integer $k > \frac{1}{s}$, infer $A \rightarrow D_{\geq s}(\alpha, \beta)$
($s \neq 0$).

We denote this axiomatic system by Ax .

Let us shortly discuss the above axioms and rules. The classical propositional logic is a sublogic of our logics, because of the axioms (A1), (A2) and the rule (R1).

The axioms (A3), (A4), (A5) and the rules (R3), (R4) force the range of (pseudo)-metrics to be the set of non negative reals, $[0, +\infty)$. The rules (R3) and (R4) are the infinitary inference rules. Each of them has a countable set of assumptions and one conclusion. The rules correspond to the Archimedean axiom for real numbers.

The axioms (A6), (A7) and the rule (R2) describe the conditions (D1-), (D2) and (D3). The rule (R2) can be considered as the rule of necessitation in modal logics, but it can be applied on the classical propositional formulas only.

Definition 4.1. A formula Φ is deducible from a set T of formulas ($T \vdash \Phi$) if there is an at most countable sequence of formulas $\Phi_0, \Phi_1, \dots, \Phi$ such that every Φ_i is an axiom or a formula from the set T , or it is derived from the preceding formulas by an inference rule.

A formula Φ is a theorem ($\vdash \Phi$) if it is deducible from the empty set, and a proof for Φ is the corresponding sequence of formulas.

A set T of formulas is consistent if there is at least one formula from For_C , and at least one formula from For_M that are not deducible from T , otherwise T is inconsistent.

A consistent set T of formulas is said to be maximal consistent if for every $A \in \text{For}_M$, either $A \in T$ or $\neg A \in T$.

A set T is deductively closed if for every $\Phi \in \text{For}$, if $T \vdash \Phi$, then $\Phi \in T$.

Lemma 4.1. *Let T^* be a maximal consistent set of formulas, and $\alpha, \beta \in \text{For}_C$. If $T \vdash \alpha \leftrightarrow \beta$, then $D_{=0}(\alpha, \beta) \in T$.*

PROOF. If $T \vdash \alpha \leftrightarrow \beta$, then $T \vdash D_{=0}(\alpha, \beta)$ by the rule (R2). If $D_{=0}(\alpha, \beta) \notin T$, then $\neg D_{=0}(\alpha, \beta) \in T$ (since T is maximal) which contradicts the consistency of T . \square

Theorem 4.1. (*Deduction theorem*). *If T is a set of formulas, Φ is a formula, and $T \cup \{\Phi\} \vdash \Psi$, then $T \vdash \Phi \rightarrow \Psi$, where Φ and Ψ are either both classical or both metric formulas.*

PROOF. We use the transfinite induction on the length of the proof of Ψ from $T \cup \{\Phi\}$. The classical cases follow as usual.

Suppose that $\Psi = D_{=0}(\alpha, \beta)$ is obtained from $T \cup \{\Phi\}$ by an application of the inference rule (R2) and $\Phi \in \text{For}_M$. In that case:

$$\begin{aligned} T \cup \{\Phi\} &\vdash \alpha \leftrightarrow \beta \\ T \cup \{\Phi\} &\vdash D_{=0}(\alpha, \beta), \text{ by (R2)}. \end{aligned}$$

However, since $\alpha \leftrightarrow \beta \in \text{For}_C$, and $\Phi \in \text{For}_M$, Φ does not affect the proof of $\alpha \leftrightarrow \beta$ from $T \cup \{\Phi\}$. Note that a classical propositional formula can be inferred only by the rule (R1) applied on classical formulas. Thus, we have:

$$\begin{aligned} T &\vdash \alpha \leftrightarrow \beta \\ T &\vdash D_{=0}(\alpha, \beta), \text{ by (R2)} \\ T &\vdash D_{=0}(\alpha, \beta) \rightarrow (\Phi \rightarrow D_{=0}(\alpha, \beta)), \text{ by (A2), since } p \rightarrow (q \rightarrow p) \text{ is a tautology,} \\ T &\vdash \Phi \rightarrow D_{=0}(\alpha, \beta), \text{ by (R1)} \\ T &\vdash \Phi \rightarrow \Psi \end{aligned}$$

Next, let us consider the case where $\Psi = A \rightarrow D_{\leq s}(\alpha, \beta)$ is obtained from $T \cup \{\Phi\}$ by an application of (R3), and $\Phi \in \text{For}_M$. Then:

$$\begin{aligned} T \cup \{\Phi\} &\vdash A \rightarrow D_{< s + \frac{1}{k}}(\alpha, \beta), \text{ for every positive integer } k \\ T \vdash \Phi &\rightarrow (A \rightarrow D_{< s + \frac{1}{k}}(\alpha, \beta)), \text{ every positive integer } k, \text{ by the induction hypothesis} \\ T &\vdash (\Phi \wedge A) \rightarrow D_{< s + \frac{1}{k}}(\alpha, \beta), \text{ every positive integer } k \\ T &\vdash (\Phi \wedge A) \rightarrow D_{\leq s}(\alpha, \beta), \text{ by (R3)} \\ T &\vdash \Phi \rightarrow (A \rightarrow D_{\leq s}(\alpha, \beta)) \\ T &\vdash \Phi \rightarrow \Psi \end{aligned}$$

The case concerning formulas obtained by (R4) can be proved in the same way. □

The perceptive reader might think that it is a bit strange having a deduction theorem in the presence of an analogue of the necessary rule. However, we assure the reader that this is a common situation in probabilistic logics. Please see [18], and the references therein.

Theorem 4.2. (*Soundness theorem*). *The axiomatic system Ax is sound with respect to the LPM-models (and therefore to the LM-models).*

PROOF. Soundness of the axiomatic system Ax follows from the soundness of propositional classical logics and from the properties of pseudometrics. We can show that

every instance of an axiom schemata holds in every *LPM*-model, while the inference rules preserve validity.

It is easy to see that if α is an instance of a classical propositional tautologies, then for every model $\mathbf{M} = \langle W, v, d \rangle$, $\mathbf{M} \models \alpha$. The axioms (A3-7) concern the properties of the ordering on \mathbb{Q}_0^+ and the conditions (D2) and (D3), and these axioms obviously hold in every LPM-model.

The rule (R1) is validity-preserving for the same reason as in classical logic. Consider Rule (R2) and suppose that a formula $\alpha \leftrightarrow \beta$ is valid. Then $[\alpha] = [\beta]$ holds in any LPM-model and hence $D_{=0}(\alpha, \beta)$ must be true in that LPM-model. The rules (R3) and (R4) preserves validity because of the Archimedean property. \square

Theorem 4.3. *Every consistent set of formulas can be extended to a maximal consistent set.*

PROOF. Let T be a consistent set of formulas and let A_0, A_1, A_2, \dots be an enumeration of all formulas from For_M . We define a sequence of sets T_i , $i = 0, 1, 2, \dots$ as follows:

- (1) $T_0 = T \cup \text{Con}_C(T) \cup \{D_{=0}(\alpha, \beta) : \alpha \leftrightarrow \beta \in \text{Con}_C(T)\}$, where $\text{Con}_C(T)$ is the set of all classical consequences of T ($\text{Con}_C(T) \subset \text{For}_C$);

For every $i \geq 0$,

- (2) if $T_i \cup \{A_i\}$ is consistent, then $T_{i+1} = T_i \cup \{A_i\}$;
- (3) otherwise, if $T_i \cup \{A_i\}$ is inconsistent, we have:
 - (a) if A_i is of the form $B \rightarrow D_{\leq s}(\alpha, \beta)$, then $T_{i+1} = T_i \cup \{\neg A_i, B \rightarrow D_{\geq s + \frac{1}{k}}(\alpha, \beta)\}$, where k is a positive integer chosen so that T_{n+1} is consistent;
 - (b) if A_i is of the form $B \rightarrow D_{\geq s}(\alpha, \beta)$, then $T_{i+1} = T_i \cup \{\neg A_i, B \rightarrow D_{\leq s - \frac{1}{k}}(\alpha, \beta)\}$, where k is a positive integer chosen so that T_{n+1} is consistent;
 - (c) otherwise, $T_{i+1} = T_i \cup \{\neg A_i\}$.

Note that at each stage we extend the previous set of formulas by finitely many formulas.

Let $T^* = \cup_{i=0}^{\infty} T_i$. The rest of the proof is divided into tree parts.

Claim 1. T_i is consistent for each $i \geq 0$.

Proof of Claim 1. The sets obtained by the steps (1) and (2) are obviously consistent. The sets obtained by the step (3c) are consistent by classical arguments:

if $T_i \cup \{A_i\} \vdash \perp$, by the deduction theorem we have $T_i \vdash \neg A_i$, and since T_i is consistent, so it is $T_i \cup \{\neg A_i\}$.

Let us consider the step (3a).

If $T_i \cup \{B \rightarrow D_{\leq s}(\alpha, \beta)\}$ is not consistent, then the set T_i can be consistently extended as it is described above. Suppose that it is not the case. Then

$T_i, \neg(B \rightarrow D_{\leq s}(\alpha, \beta)), B \rightarrow \neg D_{< s + \frac{1}{k}}(\alpha, \beta) \vdash \perp$, for every positive integer k

$T_i, \neg(B \rightarrow D_{\leq s}(\alpha, \beta)) \vdash \neg(B \rightarrow \neg D_{< s + \frac{1}{k}}(\alpha, \beta))$, for every positive integer k , by the deduction theorem

$T_i, \neg(B \rightarrow D_{\leq s}(\alpha, \beta)) \vdash B \rightarrow D_{< s + \frac{1}{k}}(\alpha, \beta)$, for every positive integer k , by the classical tautology $\neg(p \rightarrow q) \rightarrow (p \rightarrow \neg q)$

$T_i, \neg(B \rightarrow D_{\leq s}(\alpha, \beta)) \vdash B \rightarrow D_{\leq s}(\alpha, \beta)$, by (R3)

The last line contradicts the consistency of T_i . In the same manner we prove that the step (3b) produces consistent sets. Thus, the proof of the Claim 1 is completed.

Claim 2. T^* is deductively closed.

Proof of Claim 2. We can show that T^* is a deductively closed set.

Let Φ be a formula from For. It can be proved by induction on the length of the inference that if $T^* \vdash \Phi$, then $\Phi \in T^*$. Note that if $T_i \vdash A$ and $A = A_n$, it must be $A \in T^*$ because $T_{\max\{n, i\}+1}$ is consistent.

Suppose that the sequence $\Phi_1, \Phi_2, \dots, \Phi$ is a formal inference of φ from T^* .

If the sequence is finite, there must be a set T_i such that $T_i \vdash \Phi$, and $\Phi \in T^*$. Thus, suppose that the sequence is countable infinite. We can show that for every i , if Φ_i is obtained by an application of an inference rule, and all the premises belong to T^* , then it must be $\Phi_i \in T^*$.

If the rule is a finitary one (either (R1) or (R2)), then we conclude $\Phi_i \in T^*$ by reasoning as above. Next we consider the infinitary rule (R3). Let $\Phi_i = B \rightarrow D_{\leq s}(\alpha, \beta)$ be obtained by (R3) from the premises $\Phi_i^j = B \rightarrow D_{< s + \frac{1}{j}}(\alpha, \beta) \in T^*$, for every positive integer j . Assume $\Phi_i \notin T^*$. The step (3a) of the construction of T^* provides a positive integer k , such that $B \rightarrow \neg D_{< s + \frac{1}{k}}(\alpha, \beta) \in T^*$. Thus, there is m , such that T_m contains both $B \rightarrow D_{< s + \frac{1}{k}}(\alpha, \beta)$ and $B \rightarrow \neg D_{< s + \frac{1}{k}}(\alpha, \beta)$. It follows that $T_m \cup \{B\}$ is not consistent. $T_m \vdash B \rightarrow \perp$ implies $T_m \vdash B \rightarrow D_{\leq s}(\alpha, \beta)$, and hence $B \rightarrow D_{\leq s}(\alpha, \beta) \in T^*$, i.e. $\Phi_i \in T^*$ which contradicts the assumption $\Phi_i \notin T^*$. The case when $\Phi_i = B \rightarrow D_{\geq s}(\alpha, \beta)$ is obtained by (R4) follows similarly.

Henceforth, the set T^* is deductively closed.

Claim 3. T^* is maximal consistent.

Proof of Claim 3. It is easy to see that T^* does not contain all formulas. If $\alpha \in \text{For}_C$, by the definition of T_0 , α and $\neg\alpha$ cannot be simultaneously in T_0 . If for some A , both A and $\neg A$ belong to T^* , then there is a set T_i such that $A, \neg A \in T_i$, contrary to the consistency of T_i . In summary, for a formula Φ , either $\Phi \in T^*$

or $\neg\Phi \in T^*$, and the set T^* does not contain both. Thus, T^* is consistent. The construction guarantees that it is maximal. Note that T^* could not be complete for classical formulas, in the sense that T^* may contain neither α nor $\neg\alpha$. \square

The next lemma gives some auxiliary statements which will be needed for the proof of the completeness theorem.

Lemma 4.2. *Let T^* be a maximal consistent set of formulas as in the proof of the previous theorem. Then the following hold:*

- (i) T^* is deductively closed, and consequently contains all valid formulas.
- (ii) T^* contains either A or $\neg A$ (and certainly not both), for each $A \in \text{For}_M$.
- (iii) $A, B \in T^*$ iff $A \wedge B \in T^*$, for every $A, B \in \text{For}_M$.
- (iv) $\inf\{s \in \mathbb{Q}_0^+ \mid D_{\leq s}(\alpha, \beta) \in T^*\} \leq r$ iff $D_{\leq r}(\alpha, \beta) \in T^*$, for every nonnegative rational number r .
- (v) $\inf\{s \in \mathbb{Q}_0^+ \mid D_{\leq s}(\alpha, \beta) \in T^*\} < r$ iff $D_{< r}(\alpha, \beta) \in T^*$, for every positive rational number r .

PROOF. The statements (i) and (ii) were already proved. The proof of the statement (iii) is standard. Assume $\inf\{s \mid D_{\leq s}(\alpha, \beta) \in T^*\} \leq r$ in order to prove the nontrivial part of the statement (iv). If $D_{\leq r}(\alpha, \beta) \notin T^*$, then $\neg D_{\leq r}(\alpha, \beta) \in T^*$, and by the step (3a) there is a positive integer k such that $D_{\geq r + \frac{1}{k}}(\alpha, \beta) \in T^*$. Because of the consistency of T^* , there is no rational $s < r + \frac{1}{k}$ such that $D_{\leq s}(\alpha, \beta) \in T^*$, but that is in contradiction with the assumption. Finally, let us prove the nontrivial part of (v). If $D_{< r}(\alpha, \beta) \in T^*$, then $D_{\leq r}(\alpha, \beta) \in T^*$, by (A5) and (i), and hence $\inf\{s \mid D_{\leq s}(\alpha, \beta) \in T^*\} \leq r$, by (iv). The equality $\inf\{s \mid D_{\leq s}(\alpha, \beta) \in T^*\} = r$ implies $D_{\leq r - \frac{1}{n}}(\alpha, \beta) \notin T^*$, and therefore $D_{> r - \frac{1}{n}}(\alpha, \beta) \in T^*$, for every integer $n > \frac{1}{r}$. By the rule (R4), we obtain $D_{\geq r}(\alpha, \beta) \in T^*$, a contradiction. We thus get $\inf\{s \mid D_{\leq s}(\alpha, \beta) \in T^*\} < r$. \square

Theorem 4.4. *(Completeness theorem for LPM-models) Every consistent set T of formulas has an LPM-model.*

PROOF. Let T be a consistent set of formulas, and T^* its maximal consistent extension as in the proof of Theorem 4.3. Using T^* , we define a tuple $\mathbf{M} = \langle W, v, d \rangle$, where:

- W contains all classical propositional interpretations (valuations of propositional letters) that satisfy the set $\text{Con}_C(T)$ of all classical consequences of T ;

- $v : W \times P \rightarrow \{0, 1\}$ is an assignment such that for every world $w \in W$ and every propositional letter $p \in P$, $v(w, p) = 1$ iff $w \models p$,
- $d : \mathcal{F} \times \mathcal{F} \rightarrow [0, +\infty)$, such that $d([\alpha], [\beta]) = \inf\{s : D_{\leq s}(\alpha, \beta) \in T^*\}$.

Remember $[\alpha] = \{w \in W : w \models \alpha\}$ and $\mathcal{F} = \{[\alpha] : \alpha \in \text{For}_C\}$.

Claim 1. \mathbf{M} is an LPM-model.

Proof of Claim 1. For every formulas $\alpha, \beta \in \text{For}_C$,

$$d([\alpha], [\beta]) = \inf\{s : D_{\leq s}(\alpha, \beta) \in T^*\} \geq 0,$$

because $D_{\geq 0}(\alpha, \beta)$ is an axiom, and $D_{\geq 0}(\alpha, \beta) \in T^*$ by the statement (i) in the previous lemma. Therefore, d fulfills the non-negativity constraint.

(D1-) Assume $[\alpha] = [\beta]$. Then, for every $w \in W$, $w \models \alpha$ iff $w \models \beta$, and consequently $\text{Con}_C(T) \vdash \alpha \leftrightarrow \beta$, by the Completeness theorem for the classical propositional logic. Thus, $\alpha \leftrightarrow \beta \in \text{Con}_C(T)$ and $D_{=0}(\alpha, \beta) \in T_0 \subseteq T^*$. It follows that

$$d([\alpha], [\beta]) = \inf\{s : D_{\leq s}(\alpha, \beta) \in T^*\} = 0.$$

(D2) Let

$$d([\alpha], [\gamma]) = \inf\{s : D_{\leq s}(\alpha, \gamma) \in T^*\} = s_1$$

and

$$d([\gamma], [\beta]) = \inf\{s : D_{\leq s}(\gamma, \beta) \in T^*\} = s_2.$$

According to the statement (iv) of the previous lemma, for every rationals $r \geq s_1$ and $t \geq s_2$, $D_{\leq r}(\alpha, \gamma) \in T^*$ and $D_{\leq t}(\gamma, \beta) \in T^*$. The axiom (A6) and (i) in the previous lemma imply that $D_{\leq r+t}(\alpha, \beta) \in T^*$, i.e.

$$d([\alpha], [\beta]) = \inf\{s : D_{\leq s}(\alpha, \beta) \in T^*\} \leq r + t, \text{ for all rationals } r \geq s_1, t \geq s_2.$$

Therefore, $d([\alpha], [\beta]) \leq s_1 + s_2 = d([\alpha], [\gamma]) + d([\gamma], [\beta])$.

(D3) In this case we omit details which are the same as above. If $d([\alpha], [\beta]) = \inf\{s : D_{\leq s}(\alpha, \beta) \in T^*\} = s_0$, then for any rational $r \geq s_0$, $D_{\leq r}(\alpha, \beta) \in T^*$, and so $D_{\leq r}(\beta, \alpha) \in T^*$, by the axiom (A7). It follows that $d([\beta], [\alpha]) = \inf\{s : D_{\leq s}(\beta, \alpha) \in T^*\} \leq s_0$. If there were a rational $t < s_0$, such that $D_{\leq t}(\beta, \alpha) \in T^*$, we would have $D_{\leq t}(\alpha, \beta) \in T^*$, a contradiction. This gives $d([\beta], [\alpha]) = s_0 = d([\alpha], [\beta])$.

Claim 2. For every formula Φ , $\mathbf{M} \models \Phi$ iff $\Phi \in T^*$.

Proof of Claim 2. For every $\alpha \in \text{For}_C$:

$$\begin{aligned} \mathbf{M} \models \alpha &\text{ iff } w \models \alpha, \text{ for every } w \in W \\ &\text{ iff } \text{Con}_C(T) \vdash \alpha, \text{ by the definition of } W \\ &\text{ iff } \alpha \in T_0 \\ &\text{ iff } \alpha \in T^*. \end{aligned}$$

For every $\alpha, \beta \in \text{For}_C$, and $r \in \mathbb{Q}_0^+$:

$$\begin{aligned}
 \mathbf{M} \models D_{\leq r}(\alpha, \beta) & \text{ iff } d([\alpha], [\beta]) \leq r \\
 & \text{ iff } \inf\{s : D_{\leq s}(\alpha, \beta) \in T^*\} \leq r, \text{ by the definition of } d, \\
 & \text{ iff } D_{\leq r}(\alpha, \beta) \in T^*, \text{ by the statement (iv) of Lemma 4.2;} \\
 \mathbf{M} \models D_{\geq r}(\alpha, \beta) & \text{ iff } d([\alpha], [\beta]) \geq r \\
 & \text{ iff } d([\alpha], [\beta]) < r \text{ does not hold} \\
 & \text{ iff } \inf\{s : D_{\leq s}(\alpha, \beta) \in T^*\} < r \text{ does not hold} \\
 & \text{ iff } D_{< r}(\alpha, \beta) \notin T^*, \text{ by the statement (v) of Lemma 4.2} \\
 & \text{ iff } D_{\geq r}(\alpha, \beta) \in T^*
 \end{aligned}$$

For every $A, B \in \text{For}_M$:

$$\begin{aligned}
 \mathbf{M} \models A \wedge B & \text{ iff } \mathbf{M} \models A \text{ and } \mathbf{M} \models B \\
 & \text{ iff } A \in T^* \text{ and } B \in T^*, \text{ by the induction hypothesis} \\
 & \text{ iff } A \wedge B \in T^*, \text{ by the statement (iii) of Lemma 4.2;}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{M} \models \neg A & \text{ iff } \mathbf{M} \not\models A \\
 & \text{ iff } A \notin T^*, \text{ by the induction hypothesis} \\
 & \text{ iff } \neg A \in T^*, \text{ by the statement (ii) of Lemma 4.2.}
 \end{aligned}$$

□

Theorem 4.5. *(Completeness theorem for LM-models) Every consistent set T of formulas has an LM-model.*

PROOF. The main points in this proof are the same as in the proof of Theorem 4.4. We first extend T to a maximal consistent set. But, the extension given in the proof of Theorem 4.3 will be slightly changed. The sequence of sets T_i , $i = 0, 1, 2, \dots$ is now defined as follows:

- (1) $T_0 = T \cup \text{Con}_C(T) \cup \{D_{=0}(\alpha, \beta) : \alpha \leftrightarrow \beta \in \text{Con}_C(T)\}$, where $\text{Con}_C(T)$ is the set of all classical consequences of T ($\text{Con}_C(T) \subset \text{For}_C$);

For every $i \geq 0$,

- (2) if $T_i \cup \{A_i\}$ is consistent, then $T_{i+1} = T_i \cup \{A_i\}$;
- (3) otherwise, if $T_i \cup \{A_i\}$ is inconsistent, we have:

- (a) if A_i is of the form $B \rightarrow D_{\leq s}(\alpha, \beta)$, then $T_{i+1} = T_i \cup \{\neg A_i, B \rightarrow D_{\geq s + \frac{1}{k}}(\alpha, \beta)\}$, where k is a positive integer chosen so that T_{n+1} is consistent;
- (b) if A_i is of the form $B \rightarrow D_{\geq s}(\alpha, \beta)$, then $T_{i+1} = T_i \cup \{\neg A_i, B \rightarrow D_{\leq s - \frac{1}{k}}(\alpha, \beta)\}$, where k is a positive integer chosen so that T_{n+1} is consistent;
- (c) if A_i is of the form $D_{=0}(\alpha, \beta)$, then $T_{i+1} = T_i \cup \{\neg A_i, \neg(\alpha \leftrightarrow \beta)\}$;
- (d) otherwise, $T_{i+1} = T_i \cup \{\neg A_i\}$.

We show that the step (3c) produces consistent sets.

Suppose $T_i \cup \{\neg A_i, \neg(\alpha \leftrightarrow \beta)\} \vdash \perp$, i.e., $T_i \cup \{\neg A_i\} \vdash \alpha \leftrightarrow \beta$. Since $\alpha \leftrightarrow \beta \in \text{For}_C$, $\alpha \leftrightarrow \beta$ belongs to Con_C , and consequently $D_{=0}(\alpha, \beta) \in T_0$, which contradicts the consistency of T_i .

The rest of the proof is the same as for Theorem 4.4. □

The fact that the axiomatic system Ax is sound and complete with respect to two different classes of models is quite similar to the one for probabilistic logics (see [18]), or from the modal framework where, for instance, the modal system K is characterized by the class of all models, but also by the class of all irreflexive models. Consequently, our syntax cannot express differences between the mentioned classes of distance models, LPM-models and LM-models.

Note that with the *LPM*-semantics, as well as *LM*-semantics the set formulas $\{D_{\neq s}(\alpha, \beta) : s \in \mathbb{Q}_0^+\}$ is satisfiable (in a model where $d([\alpha], [\beta])$ is an irrational number). Although there are no formal reasons why this would be problematic, it is possible to determine, at syntax level, a countable range of distance functions. If we want the range to be \mathbb{Q}_0^+ , the rule (R3) should be replaced with the following rule:

(R) From $A \rightarrow D_{\neq s}(\alpha, \beta)$, for every $s \in \mathbb{Q}_0^+$, infer $\neg A$.

Following the ideas given in the previous theorems, one could prove the completeness theorem for *LPM*-models (*LM*-models) with \mathbb{Q}_0^+ -valued pseudometrics (metrics).

If we extended Ax with the following axioms:

(A8) $D_{\leq 1}(\alpha, \beta)$,

(A9) $D_{=1}(\alpha, \neg\alpha)$,

(A10) $D_{=0}(\beta \wedge \gamma, \perp) \wedge D_{\geq s}(\alpha, \beta) \rightarrow D_{\leq 1-s}(\alpha, \gamma)$, $s \leq 1$,

we would be able to prove the completeness theorem with respect to the class of LM^+ -models.

5 Conclusion

In this paper we have introduced propositional metric logics with binary metric operators and provided strongly complete axiomatizations. One of interesting problems for further investigation might be to find axiomatization of a logic that allows the iterations of metric operators and mixing of classical and metric formulas. Namely, allowing iterations of the metric operators can help us formalize many things. Another direction for research might be extending our logic to corresponding first order logics. All these formalizations could be useful tool in modelling and understanding real-world problems.

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FORMALIZATION OF LERCH'S THEOREM USING HOL LIGHT

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Abstract

The Laplace transform is an algebraic method that is widely used for analyzing physical systems by either solving the differential equations modeling their dynamics or by evaluating their transfer function. The dynamics of the given system are firstly modeled using differential equations and then Laplace transform is applied to convert these differential equations to their equivalent algebraic equations. These equations can further be simplified to either obtain the transfer function of the system or to find out the solution of the differential equations in frequency domain. Next, the uniqueness of the Laplace transform provides the solution of these differential equations in the time domain. The traditional Laplace transform based analysis techniques, i.e., paper-and-pencil proofs and computer simulation methods are error-prone due to their inherent limitations and thus are not suitable for the analysis of the systems. Higher-order-logic theorem proving can overcome these limitations of these techniques and can ascertain accurate analysis of the systems. In this paper, we extend our higher-order logic formalization of the Laplace transform, which includes the formal definition of the Laplace transform and verification of its various classical properties. One of the main contributions of the paper is the formalization of Lerch's theorem, which describes the uniqueness of the Laplace transform and thus plays a vital role in solving linear differential equations in the frequency domain. For illustration, we present the formal analysis of a $4\text{-}\pi$ soft error crosstalk model, which is widely used in nanometer technologies, such as, Integrated Circuits (ICs).

1 Introduction

The engineering and physical systems exhibiting the continuous-time dynamical behaviour are mathematically modeled using differential equations, which need to be solved to judge system characteristics. Laplace transform method allows us to solve these differential equations or evaluate the transfer function of the signals in these systems using algebraic techniques and thus is very commonly used in system analysis. Taking the Laplace transform of differential equations allows us to convert the time-varying functions involved in these differential equations to their corresponding s -domain representations, i.e., the integral and differential operators in time domain are converted to their equivalent multiplication and division operators in the s -domain, where s represents the angular frequency. These algebraic equations can then be further simplified to either obtain the transfer function of the system or solution of the differential equations in frequency domain. In the last step, the uniqueness of the Laplace transform is used to obtain the solution of these differential equations in time domain.

Traditionally, the Laplace transform is used for analyzing the engineering and physical systems using paper-and-pencil proofs, numerical methods and symbolic techniques. However, these analysis techniques cannot ascertain accuracy due to their inherent limitations, like human-error proneness, discretization and numerical errors. For example, the Laplace transform based analysis provided by the computer algebra systems, like Mathematica and Maple, and Symbolic Math Toolbox of Matlab use the algorithms that consider the improper integral involved in the definition of the Laplace transform as the continuous analog of the power series, i.e., the integral is discretized to summation and the complex exponentials are sampled [38]. Given the wide-spread usage of these systems in many safety-critical domains, such as medicine, military and transportation, accurate transform method based analysis has become a dire need. With the same motivation, the Laplace transform has been formalized in the HOL Light theorem prover and it has been successfully used for formally analyzing the Linear Transfer Converter (LTC) circuit [38], Sallen key low-pass filters [39], Unmanned Free-swimming Submersible (UFSS) vehicle [27] and platoon of the automated vehicles [31]. Similarly, the Fourier transform [10] has also been formalized in the same theorem prover and has been used for formally analyzing an Automobile Suspension System (ASS) [26], audio equalizer [29] and MEMs accelerometer [29]. However, both of these formalizations can only be used for the frequency domain analysis. In order to relate this frequency-domain analysis to the corresponding linear differential equations in the time domain, we need the uniqueness of the Laplace and Fourier transforms and Lerch's theorem fulfills this requirement for the former. However, to the best of our knowledge, Lerch's theo-

rem has not been formally verified in a theorem prover so far. We overcome this limitation in this paper with the motivation that the verification of Lerch's theorem along with the existing formalization of the Laplace transform [38, 27] would facilitate the formal reasoning about the time domain solutions of differential equations in the sound core of a theorem prover and thus the formal analysis of many engineering systems [40]. For this purpose, we extend our formalization of the Laplace transform in higher-order logic [38, 27], which includes the formal definition of the Laplace transform and verification of its various classical properties. We present a new definition of the Laplace transform [27, 28], which is based on the notion of sets. Moreover, we formally verify its various properties, which include time scaling, time shifting, modulation, Laplace transform of n -order differential equation and transfer function of a generic n -order system [27], in addition to the properties, which were verified using the older definition namely linearity, frequency shifting, and differentiation and integration in time domain [38].

Lerch's theorem [12, 22] provides the uniqueness for the Laplace transform and thus allows to evaluate the solution of differential equations using the Laplace transform in the frequency domain [24]. Mathematically, if

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st} dt, \quad \text{Re } s \geq \gamma \tag{1}$$

is satisfied by a continuous function f , then there is no other continuous function other than f that satisfies Equation (1). The complex term $\mathcal{L}[f(t)] = F(s)$ in the above equation represents the Laplace transform of the time varying function f . The above statement can alternatively be interpreted by assuming that there is another continuous function g , which satisfies the following condition:

$$\mathcal{L}[g(t)] = G(s) = \int_0^\infty g(t)e^{-st} dt, \quad \text{Re } s \geq \gamma \tag{2}$$

and if $\mathcal{L}[f(t)] = \mathcal{L}[g(t)]$, then both of the functions f and g are the same, i.e., $f(t) = g(t)$ in $0 \leq t$ [12, 9].

We found a couple of paper-and-pencil proofs of Lerch's theorem [12, 22] in literature and both are mainly based on the following lemma.

Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function on $[0, 1]$ and

$$\int_0^1 x^n \phi(x) dx = 0, \quad \text{for } n = 0, 1, 2, \dots \tag{3}$$

Then

$$\phi(x) = 0, \quad \text{in } 0 \leq x \leq 1 \tag{4}$$

In both the cases, the authors adopt different strategies for the proof of the above lemma. Cohen [12] considers splitting the region of integration, i.e., interval $[0, 1]$ into three regions, namely, $[0, a]$, $[a, b]$ and $[b, 1]$, and uses approximation of the function $\phi(x)$ with the corresponding polynomials in each of the regions. However, the author does not provide a way to handle the singularity problem of the logarithm function at the value 0 in the interval $[0, 1]$. We propose to cater for this singularity problem by considering the notion of right-hand limit and continuity at one-sided open interval, and the notion of the improper integrals. On the other hand, Orloff [22] provides the proof of the lemma by approximating the function $\phi(x)$ with a polynomial $p(x)$ in the interval $[0, 1]$. This can be achieved by either using the Stone-Weierstrass theorem [19] or by using the approximation of the function $\phi(x)$ with a polynomial $p(x)$ with respect to L^2 norm and is based on L^p spaces. In this paper, we adopt the strategy based on L^p spaces [7] because of the availability of a rich formalization of L^p spaces in HOL Light [5]. Whereas, in the case of Cohen's proof, we need to verify the properties of the improper integrals. The formal proof based on this strategy is more efficient, i.e., it requires less effort in the form of lines-of-code and man-hours as will be elaborated in Sections 4 and 5. Moreover, it is more generic than the other two methods, i.e., it considers an arbitrary interval $[a, b]$ as the region of integration and thus can be directly used for the formal verification of the uniqueness property of the Fourier transform, which is our next goal.

The formalization presented in this paper is developed in higher-order logic (HOL) using the HOL Light theorem prover. The main motivation behind this choice is the availability of the multivariate calculus [18] (differentiation [3], integration [4] and L^p spaces [5]) and Laplace transform theories [38, 27]. The proposed formalization is presented using a mix Math/HOL Light notation to make the paper easy to read for non-experts of HOL Light. The complete HOL Light script is available at [25] for readers interested in viewing the HOL Light code. In order to demonstrate the practical effectiveness of the Laplace transform theory in reasoning about the system analysis problems, we use it to conduct the formal analysis of a $4\text{-}\pi$ soft error crosstalk model, which is widely used in integrated circuits (ICs).

The rest of the paper is organized as follows: Section 2 provides a brief introduction about the HOL Light theorem prover and the multivariable calculus theories of HOL Light, which act as preliminaries for the reported formalization. Section 3 provides the formalization of the Laplace transform. We describe the formalization of the lemma, given in Equations (3) and (4), for Lerch's theorem in Section 4. Section 5 presents the formalization of Lerch's theorem. Section 6 presents our formal analysis of the soft error crosstalk model. Finally, Section 7 concludes the paper.

2 Preliminaries

In this section, we present an introduction to the HOL Light theorem prover and an overview about the multivariable calculus theories of HOL Light, which provide the foundational support for the proposed formalization.

2.1 HOL Light Theorem Prover

HOL Light [14, 17] is an interactive theorem proving environment for conducting proofs in higher-order logic. The logic in the HOL Light system is represented in the strongly-typed functional programming language ML [23]. Various mathematical foundations have been formalized and saved as HOL Light theories. A HOL Light theory is a collection of valid HOL Light types, constants, axioms, definitions and theorems. A theorem is a formalized statement that may be an axiom or could be deduced from already verified theorems by an inference rule. It consists of a finite set Ω of Boolean terms, called the assumptions, and a Boolean term S , called the conclusion. Soundness is assured as every new theorem must be verified by applying the basic axioms and primitive inference rules or any other previously verified theorems/inference rules. The HOL Light theorem prover provides an extensive support of theorems regarding, boolean, arithmetics, real analysis and multivariate analysis in the form of theories, which are extensively used in our formalization. In fact, one of the primary reasons to chose the HOL Light theorem prover for the proposed formalization was the presence of an extensive support of multivariable calculus theories [1].

2.2 Multivariable Calculus Theories in HOL Light

A N -dimensional vector is represented as a \mathbb{R}^N column matrix with each of its element as a real number in HOL Light [18, 15]. All of the vector operations can thus be performed using matrix manipulations and all the multivariable calculus theorems are verified for functions with an arbitrary data-type $\mathbb{R}^N \rightarrow \mathbb{R}^M$. For example, a complex number is defined as a 2-dimensional vector, i.e., a \mathbb{R}^2 column matrix.

Some of the frequently used HOL Light functions in our work are explained below:

Definition 2.1. Cx and ii
 $\vdash \forall a. \text{Cx } a = \text{complex } (a, \&0)$
 $\vdash \text{ii} = \text{complex } (\&0, \&1)$

`Cx` is a type casting function from real (\mathbb{R}) to complex (\mathbb{R}^2), whereas the `&` operator type casts a natural number (\mathbb{N}) to its corresponding real number (\mathbb{R}). Similarly, `ii` (iota) represents a complex number having the real part equal to zero and the magnitude of the imaginary part equal to 1 [16].

Definition 2.2. Re, Im, lift and drop

```

⊢ ∀ z. Re z = z$1
⊢ ∀ z. Im z = z$2
⊢ ∀ x. lift x = (lambda i. x)
⊢ ∀ x. drop x = x$1

```

The functions `Re` and `Im` accept a complex number and return its real and imaginary part, respectively. Here, the notation `z$i` represents the i^{th} component of vector `z`. Similarly, the functions `lift` : $\mathbb{R} \rightarrow \mathbb{R}^1$ and `drop` : $\mathbb{R}^1 \rightarrow \mathbb{R}$ map a real number to a 1-dimensional vector and a 1-dimensional vector to a real number, respectively [16]. Here, the function `lambda` is used to construct a vector componentwise [18].

Definition 2.3. Exponential, Complex Cosine and Sine

```

⊢ ∀ x. exp x = Re (cexp (Cx x))
⊢ ∀ z. ccos z = (cexp (ii * z) + cexp (--ii * z)) / Cx (&2)
⊢ ∀ z. csin z = (cexp (ii * z) - cexp (--ii * z)) / (Cx (&2) * ii)

```

The complex exponential, real exponential, complex cosine and complex sine are represented as `cexp` : $\mathbb{R}^2 \rightarrow \mathbb{R}^2$, `exp` : $\mathbb{R} \rightarrow \mathbb{R}$, `ccos` : $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ and `csin` : $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ in HOL Light, respectively [2].

Definition 2.4. Vector Integral and Real Integral

```

⊢ ∀ f i. integral i f = (@y. (f has_integral y) i)
⊢ ∀ f i. real_integral i f = (@y. (f has_real_integral y) i)

```

The function `integral` represents the vector integral and is defined using the Hilbert choice operator `@` in the functional form. It takes the integrand function `f` : $\mathbb{R}^N \rightarrow \mathbb{R}^M$, and a vector-space `i` : $\mathbb{R}^N \rightarrow \mathbb{B}$, which defines the region of integration, and returns a vector \mathbb{R}^M , which is the integral of `f` on `i`. The function `has_integral` represents the same relationship in the relational form. Similarly, the function `real_integral` accepts an integrand function `f` : $\mathbb{R} \rightarrow \mathbb{R}$ and a set of real numbers `i` : $\mathbb{R} \rightarrow \mathbb{B}$ and returns the real-valued integral of the function `f` over `i`.

Definition 2.5. Vector Derivative and Real Derivative

$\vdash \forall f \text{ net. vector_derivative } f \text{ net} =$
 $\quad (\text{@f'}. (f \text{ has_vector_derivative } f')) \text{ net}$
 $\vdash \forall f x. \text{real_derivative } f x =$
 $\quad (\text{@f'}. (f \text{ has_real_derivative } f')) (\text{atreal } x)$

The function `vector_derivative` takes a function $f : \mathbb{R}^1 \rightarrow \mathbb{R}^M$ and a `net` : $\mathbb{R}^1 \rightarrow \mathbb{B}$, which defines the point at which f has to be differentiated, and returns a vector of data-type \mathbb{R}^M , which represents the differential of f at `net`. The function `has_vector_derivative` defines the same relationship in the relational form. Similarly, the function `real_derivative` accepts a function $f : \mathbb{R} \rightarrow \mathbb{R}$ and a real number x , which represents the point where f has to be differentiated, and returns the real-valued differential of f at x .

Definition 2.6. Limit of a Vector and a Real function

$\vdash \forall f \text{ net. lim } \text{net } f = (\text{@l}. (f \rightarrow l)) \text{ net}$
 $\vdash \forall f \text{ net. reallim } \text{net } f = (\text{@l}. (f \rightarrow l)) \text{ net}$

The function `lim` accepts a `net` with elements of an arbitrary data-type \mathbb{A} and a function $f : \mathbb{A} \rightarrow \mathbb{R}^M$ and returns $l : \mathbb{R}^M$, i.e., the value to which f converges at the given `net`. Similarly, the function `reallim` accepts a `net` with elements of data-type \mathbb{R} and a function $f : \mathbb{R} \rightarrow \mathbb{R}$ and returns $l : \mathbb{R}$, i.e., the value to which f converges at the given `net`.

In order to facilitate the understanding of the paper, we present the formalization of the Laplace transform, Lerch's theorem and the associated lemma using a mix Math/HOL Light notation. Some of the terms used, listed in Table 1, correlate with the traditional conventions, whereas the others are considered only to facilitate the understanding of this paper.

Table 1: Conventions used for HOL Light Functions

HOL Light Functions	Mathematical Conventions	Description
<code>lift x</code>	\bar{x}	Conversion of a real number to 1-dimensional vector
<code>drop x</code>	\underline{x}	Conversion of a 1-dimensional vector to a real number
<code>Cx a</code>	\vec{a}^2	Type casting from real (\mathbb{R}) to complex (\mathbb{R}^2)

<code>exp x</code>	e^x	Real exponential function
<code>cexp x</code>	\vec{e}^x	Complex exponential function
<code>integral</code>	$\vec{\int}$	Integral of a vector-valued function
<code>has_integral</code>	$\vec{\int}$	Integral of a vector-valued function (Relational form)
<code>real_integral</code>	\int	Integral of a real-valued function
<code>has_real_integral</code>	\int	Integral of a real-valued function (Relational form)
<code>lim</code>	$\vec{\lim}$	Limit of a vector-valued function
<code>real_lim</code>	\lim	Limit of a real-valued function
<code>abs x</code>	$ x $	Absolute value of a variable x
<code>norm \vec{x}</code>	$\ \vec{x}\ $	Norm of a vector x
<code>vsum</code>	$\vec{\sum}$	Summation of a vector-valued function
<code>sum</code>	\sum	Summation of a real-valued function
<code>vector_derivative f (at t)</code>	$\frac{d\vec{f}}{dt}$	Derivative of a vector-valued function f w.r.t t
<code>real_derivative f (at t)</code>	$\frac{df}{dt}$	Derivative of a real-valued function f w.r.t t
<code>higher_vector_derivative n f t</code>	$\frac{d^n \vec{f}}{dt^n}$	n^{th} order derivative of a vector-valued function f w.r.t t
<code>higher_real_derivative n f t</code>	$\frac{d^n f}{dt^n}$	n^{th} order derivative of a real-valued function f w.r.t t

We build upon the above-mentioned fundamental functions of multivariable calculus in HOL Light to formalize the Laplace transform theory in the next sections.

3 Formalization of the Laplace Transform

Mathematically, the Laplace transform is defined for a function $f : \mathbb{R}^1 \rightarrow \mathbb{R}^2$ as [9]:

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}dt, \quad s \in \mathbb{C} \tag{5}$$

We formalize Equation (5) in HOL Light as follows [27]:

Definition 3.1. Laplace Transform

$$\vdash \forall s f. \text{laplace_transform } f \ s = \int_{\gamma} \overrightarrow{e}^{-s(\underline{t})} f(\underline{t}) dt, \quad \gamma = \{\underline{t} \mid 0 \leq \underline{t}\}$$

The function `laplace_transform` accepts a complex-valued function $f : \mathbb{R}^1 \rightarrow \mathbb{R}^2$ and a complex number s and returns the Laplace transform of f as represented by Equation (5). In the above definition, we used the complex exponential function $\overrightarrow{e} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ because the return data-type of the function f is \mathbb{R}^2 . Here, the data-type of \underline{t} is \mathbb{R}^1 and to multiply it with the complex number s , it is first converted into a real number \underline{t} by using `drop` and then it is converted to data-type \mathbb{R}^2 using `Cx`. Next, we use the vector function `integral` (Definition 2.4), i.e., $\overrightarrow{\int}$ to integrate the expression $f(t)e^{-i\omega t}$ over the positive real line since the data-type of this expression is \mathbb{R}^2 . The region of integration is γ , which represents the positive real line or the set $\{\underline{t} \mid 0 \leq \underline{t}\}$. The Laplace transform was earlier formalized using a limiting process as [38]:

$$\vdash \forall s f. \text{laplace_transform } f \ s = \lim_{b \rightarrow \infty} \int_{\underline{0}}^{\underline{b}} \overrightarrow{e}^{-s(\underline{t})} f(\underline{t}) dt$$

However, the HOL Light definition of the integral function implicitly encompasses infinite limits of integration. So, our definition covers the region of integration, i.e., $[0, \infty)$, as $\{\underline{t} \mid 0 \leq \underline{t}\}$ and is equivalent to the definition given in [38]. However, our definition considerably simplifies the reasoning process in the verification of the Laplace transform properties since it does not involve the notion of limit.

The Laplace transform of a function f exists, if f is piecewise smooth and is of exponential order on the positive real line [38, 9]. A function is said to be piecewise smooth on an interval if it is piecewise differentiable on that interval.

Definition 3.2. Laplace Existence

$$\vdash \forall s f. \text{laplace_exists } f \ s = \left(\forall b. f \text{ piecewise_differentiable_on } [\underline{0}, \underline{b}] \right) \wedge \left(\exists M a. \text{Re}(s) > \underline{a} \wedge \text{exp_order_cond } f \ M \ a \right)$$

The function `exp_order_cond` in the above definition represents the exponential order condition necessary for the existence of the Laplace transform [38, 9]:

Definition 3.3. Exponential Order Condition

$$\vdash \forall f M a. \text{exp_order_cond } f M a \Leftrightarrow 0 < M \wedge \left(\forall t. 0 \leq t \Rightarrow \|f(\bar{t})\| \leq M e^{at} \right)$$

We used Definitions 3.1, 3.2 and 3.3 to formally verify some of the classical properties of the Laplace transform, given in Table 2. The properties namely linearity, frequency shifting, differentiation and integration were already verified using the formal definition of the Laplace transform [38]. We formally verified these using our new definition of the Laplace transform [27]. Moreover, we formally verified some new properties, such as, time shifting, time scaling, cosine and sine-based modulations and the Laplace transform of a n -order differential equation [27]. The assumptions of these theorems describe the existence of the corresponding Laplace transforms. For example, the predicate `laplace_exists_higher_deriv` in the theorem corresponding to the n -order differential equation ensures that the Laplace of all the derivatives up to the n^{th} order of the function `f` exist. The function `diff_eq_n_order` models the n -order differential equation itself. Similarly, the predicate `differentiable_higher_derivative` provides the differentiability of the function `f` and its higher derivatives up to the n^{th} order. Moreover, the HOL Light function `EL k lst` returns the k^{th} element of a list `lst`. The verification of these properties not only ensures the correctness of our definitions but also plays a vital role in minimizing the user effort in reasoning about the Laplace transform based analysis of systems, as will be depicted in Section 6 of this paper.

Table 2: Properties of the Laplace Transform

Property	Formalized Form
Integrability	
$e^{-st} f(t)$ integrable on $[0, \infty)$	$\vdash \forall f s. \text{laplace_exists } f s$ $\Rightarrow \overrightarrow{e^{-s}}(\overline{t})^2 f(t) \text{ integrable_on } \{t \mid 0 \leq t\}$
Linearity	
$\mathcal{L}[\alpha f(t) + \beta g(t)] = \alpha F(s) + \beta G(s)$	$\vdash \forall f g s a b.$ $\text{laplace_exists } f s \wedge \text{laplace_exists } g s$ $\Rightarrow \text{laplace_transform } (a * f(t) + b * g(t)) s =$ $a * \text{laplace_transform } f s +$ $b * \text{laplace_transform } g s$
Frequency Shifting	
$\mathcal{L}[e^{s_0 t} f(t)] = F(s - s_0)$	$\vdash \forall f s s_0. \text{laplace_exists } f s$ $\Rightarrow \text{laplace_transform } \left(\overrightarrow{e^{s_0}}(\overline{t})^2 f(t) \right) s =$ $\text{laplace_transform } f (s - s_0)$

First-order Differentiation in Time Domain	
$\mathcal{L}\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0)$	$\begin{aligned} &\vdash \forall f s. \text{laplace_exists } f s \wedge \\ &\quad (\forall t. f \text{ differentiable at } t) \wedge \\ &\quad \text{laplace_exists } \left(\frac{\overrightarrow{df}}{dt}\right) s \\ &\Rightarrow \text{laplace_transform } \left(\frac{\overrightarrow{df}}{dt}\right) s = \\ &\quad s * \text{laplace_transform } f s - f(\overline{0}) \end{aligned}$
Higher-order Differentiation in Time Domain	
$\begin{aligned} \mathcal{L}\left[\frac{d^n}{dt^n}f(t)\right] &= s^n F(s) \\ - \sum_{k=1}^n s^{k-1} \frac{d^{n-k}f(0)}{dx^{n-k}} \end{aligned}$	$\begin{aligned} &\vdash \forall f s n. \\ &\quad \text{laplace_exists_higher_deriv } n f s \wedge \\ &\quad (\forall t. \text{differentiable_higher_derivative } n f t) \\ &\Rightarrow \text{laplace_transform } \left(\frac{\overrightarrow{d^n f}}{dt^n}\right) s = \\ &\quad s^n * \text{laplace_transform } f s - \\ &\quad \overrightarrow{\sum_{k=1}^n} \left(s^{k-1} \frac{\overrightarrow{d^{n-k}f(\overline{0})}}{dt^{n-k}} \right) \end{aligned}$
Integration in Time Domain	
$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{1}{s}F(s)$	$\begin{aligned} &\vdash \forall f s. 0 < \text{Re } s \wedge \text{laplace_exists } f s \wedge \\ &\quad \text{laplace_exists } \left(\int_0^t f(\tau)d\tau\right) s \wedge \\ &\quad (\forall x. f \text{ continuous_on interval } [\overline{0}, x]) \\ &\Rightarrow \text{laplace_transform } \left(\int_0^t f(\tau)d\tau\right) s = \\ &\quad \frac{\overrightarrow{1}}{s} * \text{laplace_transform } f s \end{aligned}$
Time Shifting	
$\mathcal{L}[f(t - t_0)u(t - t_0)] = e^{-t_0 s}F(s)$	$\begin{aligned} &\vdash \forall f s t_0. 0 < t_0 \wedge \text{laplace_exists } f s \\ &\Rightarrow \text{laplace_transform } (\text{shifted_fun } f t_0) s = \\ &\quad \overrightarrow{e^{-s}}(\overline{t_0}) * \text{laplace_transform } f s \end{aligned}$
Time Scaling	
$\mathcal{L}[f(ct)] = \frac{1}{c}F\left(\frac{s}{c}\right), \quad 0 < c$	$\begin{aligned} &\vdash \forall f s c. 0 < c \wedge \text{laplace_exists } f s \wedge \\ &\quad \text{laplace_exists } f \left(\frac{s}{c^2}\right) \\ &\Rightarrow \text{laplace_transform } (f(c \% t)) s = \\ &\quad \frac{\overrightarrow{1}}{c^2} * \text{laplace_transform } f \left(\frac{s}{c^2}\right) \end{aligned}$
Modulation (Cosine and Sine-based)	

$\mathcal{L}[f(t)\cos(\omega_0 t)] = \frac{F(s - i\omega_0)}{2} + \frac{F(s + i\omega_0)}{2}$	$\begin{aligned} &\vdash \forall f s \omega_0. \text{laplace_exists } f s \\ &\Rightarrow \text{laplace_transform} \\ &\quad \left(\text{ccos } (\overrightarrow{\omega_0}^2 * \overrightarrow{t})^2 \right) f (t) \Big) s = \\ &\quad \frac{\text{laplace_transform } f (s - ii * \overrightarrow{\omega_0}^2)}{\overrightarrow{2}^2} + \\ &\quad \frac{\text{laplace_transform } f (s + ii * \overrightarrow{\omega_0}^2)}{\overrightarrow{2}^2} \end{aligned}$
$\mathcal{L}[f(t)\sin(\omega_0 t)] = \frac{F(s - i\omega_0)}{2i} - \frac{F(s + i\omega_0)}{2i}$	$\begin{aligned} &\vdash \forall f s \omega_0. \text{laplace_exists } f s \\ &\Rightarrow \text{laplace_transform} \\ &\quad \left(\text{csin } (\overrightarrow{\omega_0}^2 * \overrightarrow{t})^2 \right) f (t) \Big) s = \\ &\quad \frac{\text{laplace_transform } f (s - ii * \overrightarrow{\omega_0}^2)}{\overrightarrow{2}^2 * ii} - \\ &\quad \frac{\text{laplace_transform } f (s + ii * \overrightarrow{\omega_0}^2)}{\overrightarrow{2}^2 * ii} \end{aligned}$
<i>n</i>-order Differential Equation	
$\mathcal{L}\left(\sum_{k=0}^n \alpha_k \frac{d^k y}{dt^k}\right) = \frac{F(s) \sum_{k=0}^n \alpha_k s^k}{s^{i-1} \frac{d^{k-i} f(0)}{dt^{k-i}}}$	$\begin{aligned} &\vdash \forall f \text{lst } s n. \\ &\quad \text{laplace_exists_higher_deriv } n f s \wedge \\ &\quad (\forall t. \text{differentiable_higher_derivative } n f t) \\ &\Rightarrow \text{laplace_transform} \\ &\quad \left(\text{diff_eq_n_order } n \text{lst } f t \right) s = \\ &\quad \text{laplace_transform } f s * \overrightarrow{\sum_{k=0}^n} \left(\text{EL } k \text{lst} * s^k \right) \\ &\quad - \overrightarrow{\sum_{k=0}^n} \left(\text{EL } k \text{lst} * \overrightarrow{\sum_{i=1}^k} \left(s^{i-1} \left(\frac{d^{k-i} f(\vec{0})}{dt^{k-i}} \right) \right) \right) \end{aligned}$

The generalized linear differential equation describes the input-output relationship for a generic n -order system [6]:

$$\sum_{k=0}^n \alpha_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^m \beta_k \frac{d^k}{dt^k} x(t), \quad m \leq n \tag{6}$$

where $y(t)$ is the output and $x(t)$ is the input to the system. The constants α_k and β_k are the coefficients of the output and input differentials with order k , respectively. The greatest index n of the non-zero coefficient α_n determines the order of the underlying system. The corresponding transfer function is obtained by setting the initial conditions equal to zero [20]:

$$\frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^m \beta_k s^k}{\sum_{k=0}^n \alpha_k s^k} \tag{7}$$

We verified the transfer function, given in Equation (7), for the generic n -order system as the following HOL Light theorem [27].

Theorem 3.1. Transfer Function of a Generic n -order System

$$\begin{aligned}
 & \vdash \forall y \ x \ m \ n \ \text{inlst} \ \text{outlst} \ s. \\
 & (\forall t. \text{differentiable_higher_deriv} \ m \ n \ x \ y \ t) \wedge \\
 & \text{laplace_exists_of_higher_deriv} \ m \ n \ x \ y \ s \wedge \\
 & \text{zero_init_conditions} \ m \ n \ x \ y \wedge \\
 & \text{diff_eq_n_order_sys} \ m \ n \ \text{inlst} \ \text{outlst} \ y \ x \wedge \\
 & \text{laplace_transform} \ x \ s \neq \vec{0}^2 \wedge \sum_{k=0}^n \left(\text{EL} \ k \ \text{outlst} \ * \ s^k \right) \neq \vec{0}^2 \\
 & \Rightarrow \frac{\text{laplace_transform} \ y \ s}{\text{laplace_transform} \ x \ s} = \frac{\sum_{k=0}^m \left(\text{EL} \ k \ \text{inlst} \ * \ s^k \right)}{\sum_{k=0}^n \left(\text{EL} \ k \ \text{outlst} \ * \ s^k \right)}
 \end{aligned}$$

The first assumption ensures that the functions y and x are differentiable up to the n^{th} and m^{th} order, respectively. The next assumption represents the Laplace transform existence condition up to the n^{th} order derivative of function y and m^{th} order derivative of the function x . The next assumption models the zero initial conditions for both of the functions y and x , respectively. The next assumption represents the formalization of Equation (6) and the last two assumptions provide the conditions for the design of a reliable system. Finally, the conclusion of the above theorem represents the transfer function given by Equation (7). The verification of this theorem is mainly based on *n-order differential equation* property of the Laplace transform and is very useful as it allows to automate the verification of the transfer function of any system as will be seen in Section 6 of the paper. The formalization, described in this section, took around 2000 lines of HOL Light code [25] and around 110 man-hours.

4 Lemma for Lerch's Theorem

We formally verify the lemma (Equation (4)) involved in verifying Lerch's theorem for a function f as the following HOL Light theorem:

Theorem 4.1. Lemma for a Vector-valued Function

$$\begin{aligned}
 & \vdash \forall f \ s. \\
 & \text{bounded} \ s \wedge \\
 & \overline{\|f(x)\|^2} \ \text{integrable_on} \ s \wedge \\
 & \left(\forall n. \int_s \underline{x}^n f(x) = \vec{0}^2 \right) \\
 & \Rightarrow \text{negligible} \ \{x \mid x \text{ IN } s \wedge f(x) \neq \vec{0}^2\}
 \end{aligned}$$

The above theorem is the general version of the lemma (Equations (3) and (4)) and it is verified for a vector-valued function $\mathbf{f} : \mathbb{R}^1 \rightarrow \mathbb{R}^2$ and an arbitrary interval, i.e., set \mathbf{s} . The first assumption of Theorem 4.1 ensures that the set \mathbf{s} is bounded. The next assumption models the integrability condition for $\overline{\|\mathbf{f}(\mathbf{x})\|^2}$. The next assumption ensures that the integral of the complex integrand $x^n f(x)$ over the region of integration \mathbf{s} is zero. Finally, the conclusion models the condition, which says that the size of the set containing all the values $\mathbf{x} \in \mathbf{s}$ at which the function f is zero is negligible. Alternatively, it means that the function f is zero at every $\mathbf{x} \in \mathbf{s}$. We proceed with the proof process of Theorem 4.1 by transforming the HOL Light function `negligible` into its counterpart for the real-valued functions, i.e., `real_negligible`, which mainly requires the properties of vectors and negligible sets. Next, its proof is mainly based on the properties of integration along with the real-valued version of Theorem 4.1, i.e., for the functions of data type $\mathbb{R} \rightarrow \mathbb{R}$, which is represented as:

Theorem 4.2. Lemma for a Real-valued Function

$$\begin{aligned} &\vdash \forall \mathbf{f} \mathbf{s}. \\ &\quad \text{real_bounded } \mathbf{s} \wedge \\ &\quad [\mathbf{f}(\mathbf{x})]^2 \text{ real_integrable_on } \mathbf{s} \wedge \\ &\quad \left(\forall n. \int_{\mathbf{s}} x^n \mathbf{f}(\mathbf{x}) = 0 \right) \\ &\quad \Rightarrow \text{real_negligible } \{x \mid x \text{ IN } \mathbf{s} \wedge \mathbf{f}(x) \neq 0\} \end{aligned}$$

where all the assumptions of the above theorem are same as that of Theorem 4.1. However, they hold for the real-valued function $\mathbf{f} : \mathbb{R} \rightarrow \mathbb{R}$. We start the proof process of the above theorem by converting the set \mathbf{s} in to an interval, which directly implies from the first assumption of Theorem 4.2, i.e., `real_bounded s` and it results into the following subgoal:

Subgoal 4.1. `real_negligible {x | x IN [a,b] ∧ f(x) ≠ 0}`

Next, we assume $\overline{\mathbf{f}(\underline{\mathbf{x}})} = \mathbf{f}'$ and verify that the function \mathbf{f}' belongs to the L^2 space, which is represented in HOL Light as:

$$\mathbf{f}' \text{ IN } \text{lspace } ([\bar{a}, \bar{b}]) \text{ (2)}$$

where the predicate `lspace` accepts a set (interval) s and a real number p , which represents the order of the space and returns the corresponding L^p space, i.e., it returns

the set of functions f such that each f is measurable and $\overline{\|f(x)\|^2}$ is integrable on s . Its verification requires the properties of integration along with some real arithmetic reasoning and it serves as an assumption for the verification of Subgoal 4.1. Next, the following subgoal directly implies from Subgoal 4.1 as:

Subgoal 4.2. $\text{real_negligible } \{x \mid x \text{ IN } [a,b] \wedge [f(x)]^2 \neq \vec{0}^2\}$

After applying the properties of the integrals and negligible sets along with some real arithmetic reasoning, it results into the following subgoal:

Subgoal 4.3. $\int_a^b [f(x)]^2 \leq e$

Now, the function f can be approximated by a polynomial $p(x)$ with respect to L^2 norm and we further verify:

$\int_a^b p(x)f(x) = 0$

The above result after verification also serves as an assumption for Subgoal 4.3. After applying transitivity property of real numbers, Subgoal 4.3 results into the following subgoal:

Subgoal 4.4. $\int_a^b [f(x)]^2 dx \leq \int_a^b ([f(x)]^2 - p(x)f(x)) dx \wedge$
 $\int_a^b ([f(x)]^2 - p(x)f(x)) dx \leq e$

The proof of the above subgoal is based on the properties of the integrals, L^p spaces along with some real arithmetic reasoning. This concludes our proof of Theorem 4.2 and thus the lemma for Lerch's theorem. The details about the proof of the lemma can be found in the proof script [25].

5 Formalization/ Formal Proof of Lerch's Theorem

This section presents our formalization of Lerch's theorem using the HOL Light theorem prover. We formally verify the statement of Lerch's theorem as the following HOL Light theorem:

Theorem 5.1. Lerch's Theorem

$$\begin{aligned}
 &\vdash \forall f\ g\ r. \\
 &\quad 0 < \text{Re}(r) \wedge \\
 &\quad (\forall s. \text{Re}(r) \leq \text{Re}(s) \Rightarrow \text{laplace_exists } f\ s) \wedge \\
 &\quad (\forall s. \text{Re}(r) \leq \text{Re}(s) \Rightarrow \text{laplace_exists } g\ s) \wedge \\
 &\quad (\forall s. \text{Re}(r) \leq \text{Re}(s) \Rightarrow \\
 &\quad \quad \text{laplace_transform } f\ s = \text{laplace_transform } g\ s) \\
 &\quad \Rightarrow \left(\forall t. 0 \leq t \Rightarrow f(t) = g(t) \right)
 \end{aligned}$$

where f and g are vector-valued functions with data type $\mathbb{R}^1 \rightarrow \mathbb{R}^2$. Similarly, r and s are complex variables. The first assumption of Theorem 5.1 ensures the non-negativity of the real part of the Laplace variable r . The next two assumptions provide the Laplace existence conditions for the functions f and g , respectively. The last assumption presents the condition that the Laplace transforms of the two complex-valued functions f and g are equal. Finally, the conclusion of Theorem 5.1 presents the equivalence of the functions f and g for all values of their argument t in $0 \leq t$ since t represents time that is always non-negative. The proof of Theorem 5.1 mainly depends on the alternate representation of Lerch's theorem, which is verified as the following HOL Light theorem:

Theorem 5.2. Alternate Representation of Lerch's Theorem

$$\begin{aligned}
 &\vdash \forall f\ g\ N\ a\ b\ c. \\
 &\quad a + 1 < N \wedge \\
 &\quad f\ \text{continuous_on } \{t \mid 0 \leq t\} \wedge \\
 &\quad g\ \text{continuous_on } \{t \mid 0 \leq t\} \wedge \\
 &\quad \left(\forall t. 0 \leq t \Rightarrow \|f(t)\| \leq be^{at} \wedge \|g(t)\| \leq ce^{at} \right) \wedge \\
 &\quad (\forall n. N \leq n \Rightarrow \\
 &\quad \quad \text{laplace_transform } f\ \vec{n}^2 = \text{laplace_transform } g\ \vec{n}^2) \\
 &\quad \Rightarrow \left(\forall t. 0 \leq t \Rightarrow f(t) = g(t) \right)
 \end{aligned}$$

where the first assumption models the upper bound of the exponent a of the exponential function. The next two assumptions provide the continuity of the complex-valued functions f and g over the interval $[0, \infty)$, respectively. The next assumption presents the upper bounds of the functions f and g , which is very similar to the exponential order condition (Definition 3.3). The last assumption describes the condition that the Laplace transforms of the two functions f and g are equal. Finally,

the conclusion presents the equivalence of the functions f and g . We proceed with the proof of Theorem 5.2 by applying the properties of sets along with some complex arithmetic simplification, which results into the following subgoal:

Subgoal 5.1. $\forall t. t \text{ IN } \{x \mid 0 \leq x\} \Rightarrow f(t) - g(t) = \vec{0}^2$

The proof of the above subgoal is mainly based on the following lemma:

Lemma 5.1. $\vdash \forall f \ s \ a. \text{convex } s \wedge$
 $(\text{interior } s = \{\} \Rightarrow s = \{\}) \wedge$
 $f \text{ continuous_on } s \wedge$
 $\text{negligible } \{x \mid x \text{ IN } s \wedge f(x) \neq a\}$
 $\Rightarrow (\forall x. x \text{ IN } s \Rightarrow f(x) = a)$

The application of the above lemma on Subgoal 5.1 results into a subgoal, where it is required to verify all the assumptions of Lemma 5.1. The first three assumptions are verified using the properties of continuity and sets along with some complex arithmetic reasoning. Finally, the fourth assumption results into the following subgoal:

Subgoal 5.2. $\text{negligible } \{t \mid 0 \leq t \wedge (f(t) - g(t)) \neq \vec{0}^2\}$

The proof of the above subgoal is mainly based on the following theorem by setting the value of the function $h(t) = f(t) - g(t)$:

Theorem 5.3. Generalization of Lerch's Theorem
 $\vdash \forall h \ s \ a.$
 $h \text{ measurable_on } \{t \mid 0 \leq t\} \wedge$
 $a + 1 < N \wedge$
 $(\forall t. 0 \leq t \Rightarrow \|h(t)\| \leq be^{at}) \wedge$
 $(\forall n. N \leq n \Rightarrow \text{laplace_transform } h \ \vec{n}^2 = \vec{0}^2)$
 $\Rightarrow \text{negligible } \{t \mid 0 \leq t \wedge h(t) \neq \vec{0}^2\}$

where the first assumption models the condition that the function h is measurable on the interval $[0, \infty)$. The next two assumptions provide the upper bounds of the exponent a and the complex-valued function h . The last assumption describes the condition that the Laplace transform of the function h is equal to zero. Finally, the conclusion uses the predicate `negligible` to model the condition that the function $h(t)$ is equal to zero. We proceed with the proof of Theorem 5.3 by verifying the following subgoal:

Subgoal 5.3. $\left(\forall n. N \leq n \Rightarrow g \ n \ \text{measurable_on} \ (\bar{0}, \bar{1}) \right) \wedge$
 $\left(\forall n \ x. N \leq n \wedge x \ \text{IN} \ (\bar{0}, \bar{1}) \Rightarrow \|g \ n \ x\| \leq b \right)$

where,

$$g = h\left(-\overline{(\log(x))}\right) \left(\overline{(x)^2}\right)^{\overline{n-1}^2}$$

The proof of the above subgoal is mainly based on applying cases on $N \leq n$ along with the following lemma:

Lemma 5.2. $\vdash \forall h \ a \ b \ s.$
 $h \ \text{measurable_on} \ \{t \mid 0 \leq t\} \wedge$
 $a + 1 < \text{Re}(s) \wedge$
 $(\forall t. 0 \leq t \Rightarrow \|h(t)\| \leq be^{at})$
 $\Rightarrow h\left(-\overline{(\log(x))}\right) \left(\overline{(x)^2}\right)^{s-\overline{1}^2} \ \text{measurable_on} \ (\bar{0}, \bar{1}) \wedge$
 $\left(\forall x. \ x \ \text{IN} \ (\bar{0}, \bar{1}) \Rightarrow \left\| h\left(-\overline{(\log(x))}\right) \left(\overline{(x)^2}\right)^{s-\overline{1}^2} \right\| \leq b \right)$

The singularity of the logarithm function at value 0 in the above lemma is handled by taking the measurability of the function $h(-\log x)x^{(s-1)}$ over the interval $(0, 1)$. The verification of Subgoal 5.3 serves as one of the assumption for the verification of Theorem 5.3. Next, we simplify the conclusion of Theorem 5.3 using all the assumptions and properties of the sets, to obtain the following subgoal:

Subgoal 5.4. negligible $\{x \mid x \text{ IN } (\bar{0} \ \bar{1}) \wedge g \text{ N } x \neq \bar{0}^2\}$

The proof of the above subgoal is mainly based on the main lemma (Theorem 4.1), properties of integration and sets along with some complex arithmetic reasoning. This concludes our formal proof of Lerch's theorem.

Our proof script of the formalization, presented in Sections 4 and 5, consists of about 700 lines-of-code and it took about 45 man-hours for the verification. One of the major difficulties faced in the reported formalization was the unavailability of a formal proof for Lerch's theorem. Most of the mathematical texts on Laplace transform, e.g., [9] and [35], mention the uniqueness property of the Laplace transform without presenting its proof. We only found a couple of analytical paper-and-pencil proofs [12, 22] of Lerch's theorem, which formed the basis of the reported formalization. Secondly, we verified Lerch's theorem for the complex-valued function ($\mathcal{L}[f(t)]$ or $F(s)$), whereas the available paper-and-pencil proofs [12, 22] were based on a real-valued function. The formalization of Lerch's theorem enabled us to formally verify the solutions of the differential equations, which was not possible using the formalization of the Laplace transform presented in [38, 27]. We illustrate the practical effectiveness of our formalized Laplace transform theory by presenting the formal analysis of a $4\text{-}\pi$ soft error crosstalk model for ICs in the following section.

6 Formal Analysis of a $4\text{-}\pi$ Soft Error Crosstalk Model for Nanometer Technologies

With the advancement in the Complementary Metal-oxide Semiconductor (CMOS) technologies, nanometer circuits are becoming more vulnerable to soft errors, such as, clock jitters [37], soft delays [13], coupling noise, crosstalk noise pulses that are caused by Single Event (SE) particles [32], signal cross-coupling effects [8, 34] and voltage drops in power supply, and can badly effect the integrity of the signals. These circuits usually contain a huge amount of interconnection lines, in addition to the transistors, due to the scaling down of the deep submicron CMOS technology. Moreover, these lines can interfere with each other, contributing to the degradation of the performance of the circuit and thus cannot be considered as electrically isolated components. The increase in the heights of wires and reduction in the distances between the adjacent wires are the main causes of this interference, which can result in to crosstalk noise and signal delays. Modeling of these crosstalk noise and delays caused by SE particles and other sources can be helpful in identifying them and also in rectifying their effects on the CMOS technology. It also enables the designers to

develop a low-power and energy efficient CMOS circuit. Due to the wider utility of CMOS technologies in safety and mission critical applications, such as medicine [11], military [36] and avionics [21], the formal modeling and analysis of the soft error crosstalk in these technologies is of utmost importance as the verification of these models enhances the reliability and security of the overall system.

A $4-\pi$ interconnect circuit, depicted in Figure 1, models the SE crosstalk effect in the CMOS technologies [32, 33]. It mainly consists of two $2-\pi$ circuits that model the aggressor and victim lines (nets), respectively. Here, R_{1a} and R_{2a} are the resistors corresponding to the aggressor net, whereas, C_{1a} , C_{2a} and C_{3a} are the respective capacitors. Similarly, in the case of the victim net, R_{1v} and R_{2v} are the resistors, and C_{1v} , C_{2v} and C_{3v} are the respective capacitors. Also, C_c is the coupling capacitor used between the aggressor and the victim nets.

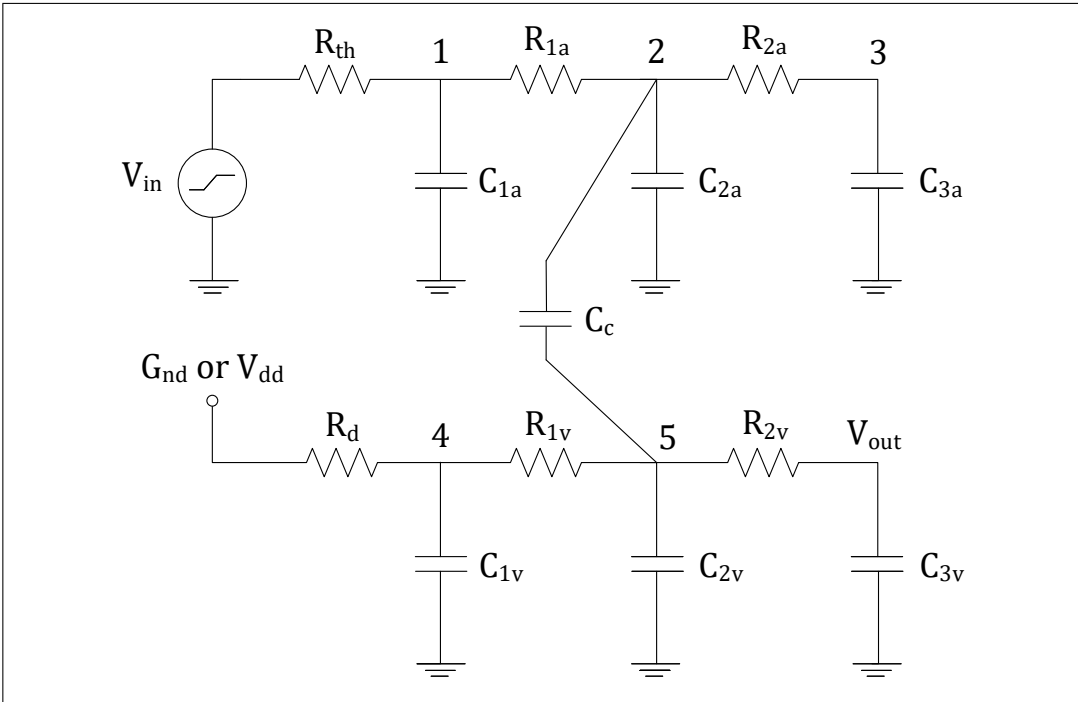


Figure 1: $4-\pi$ Interconnect Circuit Modeling the SE Crosstalk Effect [32]

6.1 Formal Analysis of Passive Aggressor

Based on the $4-\pi$ interconnect circuit (Figure 1), the passive aggressive model for analyzing the crosstalk noise and delay, is depicted in Figure 2, which is obtained

as a result of applying the decoupling approach [32, 33]. The resistance R_{th} is the effective resistance of the aggressor driver [32]. For the analysis of the passive aggressor, we first need to formalize its dynamical behaviour in the form of its governing differential equation in higher-order logic. We use the generic differential equation of order n , to model the differential equation of the passive aggressor as follows:

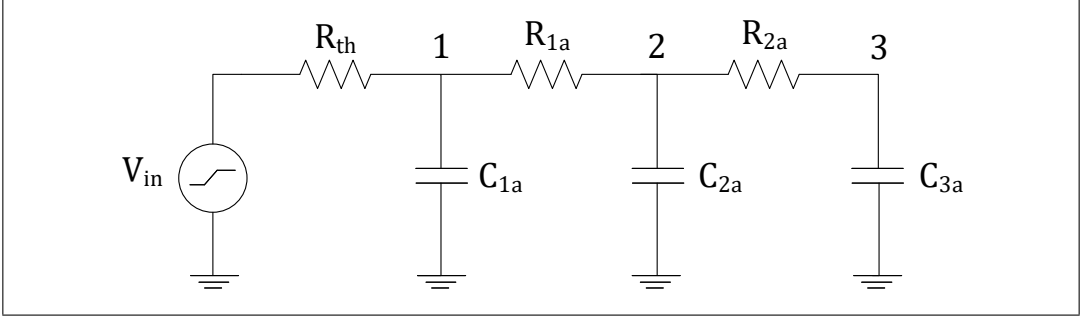


Figure 2: Passive Aggressor Model [33]

Definition 6.1. Behavioural Specification of Passive Aggressor

$\vdash \forall R_{1a} R_{2a} C_{2a} C_{3a}.$

$\text{inlst_pass_aggres } R_{1a} R_{2a} C_{2a} C_{3a} = [\vec{1}^2; \vec{A}^2; \vec{B}^2; \vec{C}^2]$

$\vdash \forall R_{1a} R_{2a} R_{th} C_{1a} C_{2a} C_{3a}.$

$\text{outlst_pass_aggres } R_{1a} R_{2a} R_{th} C_{1a} C_{2a} C_{3a} =$
 $[\vec{1}^2; \vec{D}^2; \vec{E}^2; \vec{F}^2; \vec{G}^2; \vec{H}^2]$

$\vdash \forall R_{th} C_{1a} V_2 R_{1a} R_{2a} C_{2a} C_{3a} V_{in} t.$

$\text{pass_aggressor_behav_spec } R_{1a} R_{2a} R_{th} C_{1a} C_{2a} C_{3a} V_{in} V_2 t \Leftrightarrow$
 $\text{diff_eq_n_order } 5$

$(\text{outlst_pass_aggres } R_{1a} R_{2a} R_{th} C_{1a} C_{2a} C_{3a}) V_2 t =$
 $\text{diff_eq_n_order } 3$

$(\text{inlst_pass_aggres } R_{1a} R_{2a} C_{2a} C_{3a}) V_{in} t$

where V_{in} is the input voltage having data type $\mathbb{R}^1 \rightarrow \mathbb{R}^2$. Similarly, V_2 is the voltage at node 2 and is considered as the output voltage. The elements A, B, C, D, E, F, G and H of the lists inlst_pass_aggres and $\text{outlst_pass_aggres}$ are:

$$\begin{aligned}
 A &= R_{1a}(C_{2a} + C_{3a}) + 2R_{2a}C_{3a} \\
 B &= R_{2a}C_{3a}(2R_{1a}C_{2a} + R_{1a}C_{3a} + R_{2a}C_{3a}) \\
 C &= R_{1a}R_{2a}^2C_{2a}C_{3a}^2 \\
 D &= 2R_{1a}(C_{2a} + C_{3a}) + 2R_{2a}C_{3a} + R_{th}(C_{1a} + C_{2a} + C_{3a}) \\
 E &= 2R_{1a}C_{2a}(2R_{2a}C_{3a} + R_{th}C_{1a} + R_{th}C_{3a}) + 2R_{th}C_{3a}(R_{1a}C_{1a} + R_{2a}C_{1a} + R_{2a}C_{2a}) + \\
 &\quad (R_{1a}^2 + R_{1a}R_{th})(C_{2a}^2 + C_{3a}^2) + R_{2a}C_{3a}^2(R_{2a} + R_{th}) + 2R_{1a}C_{3a}(R_{1a}C_{2a} + R_{2a}C_{3a}) \\
 F &= 2R_{1a}R_{2a}C_{2a}C_{3a}(R_{1a}C_{2a} + R_{1a}C_{3a} + R_{2a}C_{3a} + 2R_{th}C_{1a}) + \\
 &\quad 2R_{1a}R_{th}C_{3a}(R_{2a}C_{2a}^2 + R_{2a}C_{2a}C_{3a} + R_{1a}C_{1a}C_{2a} + R_{2a}C_{1a}C_{3a}) + \\
 &\quad R_{1a}^2R_{th}C_{1a}(C_{2a}^2 + C_{3a}^2) + R_{2a}^2R_{th}C_{3a}^2(C_{1a} + C_{2a}) \\
 G &= R_{1a}R_{2a}^2C_{2a}^2C_{3a}^2(R_{1a} + R_{th}) + 2R_{1a}R_{2a}R_{th}C_{1a}C_{2a}C_{3a}(R_{1a}C_{2a} + R_{1a}C_{3a} + R_{2a}C_{3a}) \\
 H &= R_{1a}^2R_{2a}^2R_{th}C_{1a}C_{2a}^2C_{3a}^2
 \end{aligned}$$

We verified the transfer function of the passive aggressor as follows:

Theorem 6.1. Transfer Function Verification of Passive Aggressor

$$\begin{aligned}
 &\vdash \forall R_{1a} R_{2a} R_{th} C_{1a} C_{2a} C_{3a} V_{in} V_2 s. \\
 &\quad 0 < R_{1a} \wedge 0 < R_{2a} \wedge 0 < R_{th} \wedge \\
 &\quad 0 < C_{1a} \wedge 0 < C_{2a} \wedge 0 < C_{3a} \wedge \\
 &\quad \text{laplace_transform } V_{in} s \neq \vec{0}^2 \wedge \\
 &\quad \vec{H}^2 s^5 + \vec{G}^2 s^4 + \vec{F}^2 s^3 + \vec{E}^2 s^2 + \vec{D}^2 s + \vec{1}^2 \neq \vec{0}^2 \wedge \\
 &\quad \text{zero_initial_conditions } V_{in} V_2 \wedge \\
 &\quad (\forall t. \text{differentiable_higher_derivative } V_{in} V_2 t) \wedge \\
 &\quad \text{laplace_exists_higher_deriv } V_{in} V_2 s \wedge \\
 &\quad (\forall t. \text{pass_aggressor_behav_spec } R_{1a} R_{2a} R_{th} C_{1a} C_{2a} C_{3a} V_{in} V_2 t) \\
 &\quad \Rightarrow \frac{\text{laplace_transform } V_2 s}{\text{laplace_transform } V_{in} s} = \\
 &\quad \quad \frac{\vec{C}^2 s^3 + \vec{B}^2 s^2 + \vec{A}^2 s + \vec{1}^2}{\vec{H}^2 s^5 + \vec{G}^2 s^4 + \vec{F}^2 s^3 + \vec{E}^2 s^2 + \vec{D}^2 s + \vec{1}^2}
 \end{aligned}$$

The first eight assumptions present the design requirements for the underlying system. The next assumption models the *zero initial conditions* for the voltage functions V_{in} and V_2 . The next two assumptions provide the differentiability and the Laplace existence condition for the higher-order derivatives of V_{in} and V_2 up to the orders 3 and 5, respectively. The last assumption presents the behavioural specification of the passive aggressor. Finally, the conclusion of Theorem 6.1 presents its required transfer function. A notable feature is that the verification of Theorem 6.1 is done

almost automatically using the automatic tactic `DIFF_EQ_2_TRANS_FUN_TAC`, which is developed in our proposed formalization.

Next, we verified the differential equation of the passive aggressor based on its transfer function using the following HOL Light theorem:

Theorem 6.2. Differential Equation Verification of Passive Aggressor

$$\begin{aligned}
 & \vdash \forall R_{1a} R_{2a} R_{th} C_{1a} C_{2a} C_{3a} V_{in} V_2 r. \\
 & \quad 0 < R_{1a} \wedge 0 < R_{2a} \wedge 0 < R_{th} \wedge 0 < C_{1a} \wedge 0 < C_{2a} \wedge 0 < C_{3a} \wedge \\
 & \quad (\forall s. \text{Re}(r) \leq \text{Re}(s) \Rightarrow \text{laplace_transform } V_{in} \ s \neq \vec{0}^2) \wedge \\
 & \quad \left(\forall s. \text{Re}(r) \leq \text{Re}(s) \Rightarrow \right. \\
 & \quad \quad \left. \vec{H}^2 s^5 + \vec{G}^2 s^4 + \vec{F}^2 s^3 + \vec{E}^2 s^2 + \vec{D}^2 s + \vec{1}^2 \neq \vec{0}^2 \right) \wedge \\
 & \quad \text{zero_initial_conditions } V_{in} \ V_2 \wedge \\
 & \quad (\forall t. \text{differentiable_higher_derivative } V_{in} \ V_2 \ t) \wedge \\
 & \quad 0 < \text{Re}(r) \wedge \\
 & \quad (\forall s. \text{Re}(r) \leq \text{Re}(s) \Rightarrow \text{laplace_exists_higher_deriv } 3 \ V_{in} \ s) \wedge \\
 & \quad (\forall s. \text{Re}(r) \leq \text{Re}(s) \Rightarrow \text{laplace_exists_higher_deriv } 5 \ V_2 \ s) \wedge \\
 & \quad \left(\forall s. \text{Re}(r) \leq \text{Re}(s) \Rightarrow \right. \\
 & \quad \quad \frac{\text{laplace_transform } V_2 \ s}{\text{laplace_transform } V_{in} \ s} = \\
 & \quad \quad \left. \frac{\vec{C}^2 s^3 + \vec{B}^2 s^2 + \vec{A}^2 s + \vec{1}^2}{\vec{H}^2 s^5 + \vec{G}^2 s^4 + \vec{F}^2 s^3 + \vec{E}^2 s^2 + \vec{D}^2 s + \vec{1}^2} \right) \\
 & \quad \Rightarrow \left(\forall t. \ 0 \leq t \Rightarrow \text{pass_aggressor_behav spec} \right. \\
 & \quad \quad \left. R_{1a} \ R_{2a} \ R_{th} \ C_{1a} \ C_{2a} \ C_{3a} \ V_{in} \ V_2 \ t \right)
 \end{aligned}$$

The first ten assumptions are the same as that of Theorem 6.1. The next assumption ensures that the real part of the Laplace variable r is always positive. The next two assumptions describe the differentiability condition for the functions V_{in} and V_2 and their higher derivatives up to the order 3 and 5, respectively. The last assumption provides the transfer function of the passive aggressor. Finally, the conclusion presents the corresponding differential equation of the passive aggressor. The verification of Theorem 6.2 is done almost automatically using the automatic tactic `TRANS_FUN_2_DIFF_EQ_TAC`, which is also developed in our proposed formalization.

6.2 Formal Analysis of Passive Victim

Based on the $4-\pi$ interconnect circuit, Figure 3 depicts the passive victim model for analyzing the crosstalk noise and delay. The resistance R_d is the effective resistance of the victim driver [32, 33].

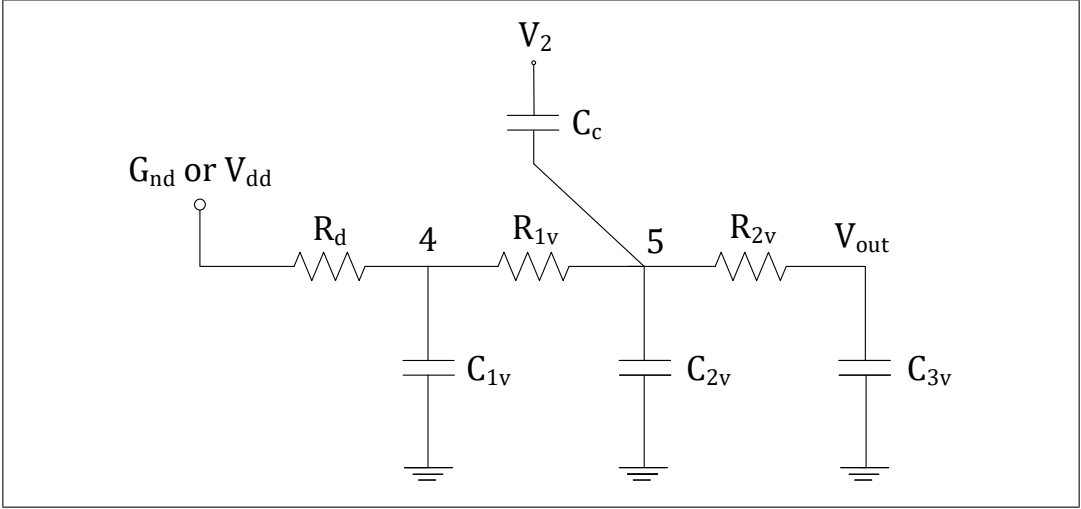


Figure 3: Passive Victim Model [33]

We model the dynamical behaviour, i.e., the modeling differential equation of the passive victim using the n -order differential equation as follows:

Definition 6.2. Behavioural Specification of Passive Victim

```

 $\vdash \forall R_{1v} R_{2v} R_d C_c C_{1v} C_{3v}.$ 
  inlst_pass_victim R1v R2v Rd Cc C1v C3v = [  $\vec{0}^2$ ;  $\vec{A}'^2$ ;  $\vec{B}'^2$ ;  $\vec{C}'^2$  ]
 $\vdash \forall R_{1v} R_{2v} R_d C_c C_{1v} C_{2v} C_{3v}.$ 
  outlst_pass_victim Rd R1v R2v Cc C1v C2v C3v =
  [  $\vec{1}^2$ ;  $\vec{D}'^2$ ;  $\vec{E}'^2$ ;  $\vec{F}'^2$ ;  $\vec{G}'^2$  ]
 $\vdash \forall V_2 V_{out} C_c C_{1v} C_{2v} C_{3v} R_d R_{1v} R_{2v} t.$ 
  pass_victim_behav_spec R1v R2v Rd Cc C1v C2v C3v V2 Vout t  $\Leftrightarrow$ 
  diff_eq_n_order 4
  (outlst_pass_victim R1v R2v Rd Cc C1v C2v C3v) Vout t =
  diff_eq_n_order 3
  (inlst_pass_victim R1v R2v Rd Cc C1v C3v) V2 t
  
```

where V_2 and V_{out} are the input and output voltages, respectively, having data types $\mathbb{R}^1 \rightarrow \mathbb{R}^2$. The elements A' , B' , C' , D' , E' , F' and G' of the lists `inlst_pass_victim`

and `outlst_pass_victim` are:

$$\begin{aligned}
 A' &= (R_d + R_{1v})C_c \\
 B' &= R_{2v}C_cC_{3v}(R_d + R_{1v}) + R_dR_{1v}C_cC_{1v} \\
 C' &= R_dR_{1v}R_{2v}C_cC_{1v}C_{3v} \\
 D' &= R_d(C_c + C_{1v} + C_{2v} + C_{3v}) + R_{1v}(C_c + C_{2v} + C_{3v}) + 2R_{2v}C_{3v} \\
 E' &= R_dR_{2v}C_{3v}(2C_c + 2C_{1v} + 2C_{2v} + C_{3v}) + R_{1v}R_{2v}C_{3v}(2C_{2v} + 2C_c + C_{3v}) + \\
 &\quad R_dR_{1v}C_{1v}(C_c + C_{2v} + C_{3v}) + R_{2v}^2C_{3v}^2 \\
 F' &= R_dR_{1v}R_{2v}C_{1v}C_{3v}(2C_c + 2C_{2v} + C_{3v}) + \\
 &\quad R_{2v}^2C_{3v}^2[R_d(C_c + C_{1v} + C_{2v}) + R_{1v}(C_c + C_{2v})] \\
 G' &= R_dR_{1v}R_{2v}^2C_{1v}C_{3v}^2(C_c + C_{2v})
 \end{aligned}$$

We verified the transfer function of the passive victim as follows:

Theorem 6.3. Transfer Function Verification of Passive Victim

$$\begin{aligned}
 &\vdash \forall R_{1v} R_{2v} R_d C_c C_{1v} C_{2v} C_{3v} V_2 V_{out} s. \\
 &\quad 0 < R_{1v} \wedge 0 < R_{2v} \wedge 0 < R_d \wedge \\
 &\quad 0 < C_{1v} \wedge 0 < C_{2v} \wedge 0 < C_{3v} \wedge 0 < C_c \wedge \\
 &\quad \text{laplace_transform } V_2 \ s \neq \overrightarrow{0}^2 \wedge \\
 &\quad \overrightarrow{G'}^2 s^4 + \overrightarrow{F'}^2 s^3 + \overrightarrow{E'}^2 s^2 + \overrightarrow{D'}^2 s + \overrightarrow{1'}^2 \neq \overrightarrow{0}^2 \wedge \\
 &\quad \text{zero_initial_conditions } V_2 \ V_{out} \wedge \\
 &\quad (\forall t. \text{differentiable_higher_derivative } V_2 \ V_{out} \ t) \wedge \\
 &\quad \text{laplace_exists_higher_deriv } V_2 \ V_{out} \ s \wedge \\
 &\quad (\forall t. \text{pass_victim_behav_spec } R_{1v} \ R_{2v} \ R_d \ C_c \ C_{1v} \ C_{2v} \ C_{3v} \ V_2 \ V_{out} \ t) \\
 &\Rightarrow \frac{\text{laplace_transform } V_{out} \ s}{\text{laplace_transform } V_2 \ s} = \frac{s(\overrightarrow{C'}^2 s^2 + \overrightarrow{B'}^2 s + \overrightarrow{A'}^2)}{\overrightarrow{G'}^2 s^4 + \overrightarrow{F'}^2 s^3 + \overrightarrow{E'}^2 s^2 + \overrightarrow{D'}^2 s + \overrightarrow{1'}^2}
 \end{aligned}$$

The first nine assumptions present the design requirements for the underlying system. The next assumption models the *zero initial conditions* for the voltage functions V_2 and V_{out} . The next two assumptions provide the differentiability and the Laplace existence condition for the higher-order derivatives of V_2 and V_{out} up to the orders 3 and 4, respectively. The last assumption presents the behavioural specification of the passive victim. Finally, the conclusion of Theorem 6.3 presents its required transfer function.

Now, we verified the differential equation of the passive victim based on its transfer function using the following HOL Light theorem:

Theorem 6.4. Differential Equation Verification of Passive Victim

$$\begin{aligned}
 & \vdash \forall V_{\text{out}} V_2 R_{1v} R_{2v} R_d C_{1v} C_{2v} C_{3v} C_c r. \\
 & \quad 0 < R_{1v} \wedge 0 < R_{2v} \wedge 0 < R_d \wedge 0 < C_c \wedge \\
 & \quad 0 < C_{1v} \wedge 0 < C_{2v} \wedge 0 < C_{3v} \wedge \\
 & \quad (\forall s. \text{Re}(r) \leq \text{Re}(s) \Rightarrow \text{laplace_transform } V_2 \ s \neq \vec{0}^2) \wedge \\
 & \quad \left(\forall s. \text{Re}(r) \leq \text{Re}(s) \Rightarrow \right. \\
 & \quad \quad \left. \vec{G}^2 s^4 + \vec{F}^2 s^3 + \vec{E}^2 s^2 + \vec{D}^2 s + \vec{1}^2 \neq \vec{0}^2 \right) \wedge \\
 & \quad \text{zero_initial_conditions } V_2 \ V_{\text{out}} \wedge \\
 & \quad (\forall t. \text{differentiable_higher_derivative } V_2 \ V_{\text{out}} \ t) \wedge \\
 & \quad 0 < \text{Re}(r) \wedge \\
 & \quad (\forall s. \text{Re}(r) \leq \text{Re}(s) \Rightarrow \text{laplace_exists_higher_deriv } 2 \ V_2 \ s) \wedge \\
 & \quad (\forall s. \text{Re}(r) \leq \text{Re}(s) \Rightarrow \text{laplace_exists_higher_deriv } 4 \ V_{\text{out}} \ s) \wedge \\
 & \quad \left(\forall s. \text{Re}(r) \leq \text{Re}(s) \Rightarrow \right. \\
 & \quad \quad \frac{\text{laplace_transform } V_{\text{out}} \ s}{\text{laplace_transform } V_2 \ s} = \frac{s(\vec{C}^2 s^2 + \vec{B}^2 s + \vec{A}^2)}{\vec{G}^2 s^4 + \vec{F}^2 s^3 + \vec{E}^2 s^2 + \vec{D}^2 s + \vec{1}^2} \left. \right) \\
 & \quad \Rightarrow \left(\forall t. \ 0 \leq t \Rightarrow \text{pass_aggressor_behav_spec} \right. \\
 & \quad \quad \left. V_2 \ V_{\text{out}} \ C_c \ C_{1v} \ C_{2v} \ C_{3v} \ R_{1v} \ R_{2v} \ R_d \ t \right)
 \end{aligned}$$

The first eleven assumptions of the above theorem are the same as that of Theorem 6.3. The next assumption ensures that the real part of the Laplace variable r is always positive. The next two assumptions model the existence condition of the Laplace transform for the functions V_2 , V_{out} and their higher derivatives up to the order 3 and 4, respectively. The last assumption provides the transfer function of the passive victim. Finally, the conclusion presents its corresponding differential equation. The verification of Theorem 6.4 is done almost automatically using the automatic tactic `TRANS_FUN_2_DIFF_EQ_TAC`.

Finally, the transfer function of the overall system is represented by the following mathematical equation.

$$\frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{V_{\text{out}}(s)}{V_2(s)} \times \frac{V_2(s)}{V_{\text{in}}(s)} \tag{8}$$

We also verified the above transfer function and its corresponding differential equation based on our formalization and the details about their verification can be found

in the proof script [25]. The formal analysis of the $4\text{-}\pi$ soft error crosstalk model is done almost automatically, thanks to our automatic tactics `DIFF_EQ_2_TRANS_FUN_TAC` and `TRANS_FUN_2_DIFF_EQ_TAC`, which are developed as part of the reported work and illustrate the usefulness of our proposed formalization of the Laplace transform in the analysis of safety-critical systems. The distinguishing feature of Theorems 6.2 and 6.4 is the relationship between the differential equation, which is expressed in the time domain, and the corresponding transfer function, which is expressed in frequency domain. However, Theorems 6.1 and 6.3 verified using our earlier formalization [38, 27] are completely based on the frequency domain and no relation with the commonly used differential equation is established. The formally verified Lerch's theorem allowed us to transform the problem of solving a differential equation in time domain to a problem of solving a linear equation in the frequency domain. This linear equation can be solved to determine constraints on the values of the components to ensure a low-power and energy efficient designing of the ICs. Moreover, all the verified theorems are of generic nature, i.e., all the variables and functions are universally quantified and can thus be specialized to any particular value for the analysis of a system. Similarly, the high expressiveness of the higher-order logic enabled us to model the dynamical behaviour of the system, i.e., the differential equation in its true form and to perform its corresponding analysis.

7 Conclusions

This paper presents a formalization of Lerch's theorem using the HOL Light theorem prover. This result extends our formalization of the Laplace transform, which includes the formal definition of the Laplace transform and verification of its various classical properties such as linearity, frequency shifting, differentiation and integration in time domain, time shifting, time scaling, modulation and the Laplace transform of a n -order differential equation. Lerch's theorem describes the uniqueness of the Laplace transform and thus can be used to find solutions of linear differential equations in the time domain, which was not possible with our earlier formalization of the Laplace transform. We used our proposed formalization for formally analyzing a $4\text{-}\pi$ soft error crosstalk model for the nanometer technologies.

In the future, we aim to formally verify the uniqueness of the Fourier transform using the reported formalization of Lerch's theorem. The region of integration for the case of Fourier transform is $(-\infty, \infty)$ [26], whereas, the one in the case of Laplace is from $[0, \infty)$. We can split region of the integration for the integral of the Fourier transform into two sub-intervals: $(-\infty, 0]$ and $[0, \infty)$. The uniqueness of the first integral can be directly handled by Lerch's theorem, whereas, for the case

of $(-\infty, 0]$, the integral can be first reflected and then the formally verified Lerch's theorem can be used to verify its uniqueness as well. Another future direction is to use this formalization in our project on system biology [30], for finding the analytical solutions of the differential equation based reaction kinetic models of the biological systems.

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BILATERALISM DOES PROVIDE A PROOF THEORETIC TREATMENT OF CLASSICAL LOGIC (FOR NON-TECHNICAL REASONS)

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1 Introduction

In [6], Michael Gabbay attempts to refute the claim, believed by many adherers of *Proof-Theoretic Semantics* (PTS) [4], that in being harmonious, the bilateral presentation¹ of classical natural deduction (ND) ([4]) produces a proof theoretic “kosher” certificate for classical logic. To the contrary, Gabbay claims that in spite of being harmonious, bilateralism *does not* provide a PTS-acceptable justification of classical logic. I assume here familiarity with Bilateralism, and with the above mentioned bilateral ND system.

In this short note, I argue that Gabbay’s alleged refutation fails, and the bilateral ND system mentioned above *does* justify classical logic from the point of view of PTS. I would like to stress that this claim, for me, does not constitute an endorsement of classical logic, that, I believe, should be rejected, though for other reasons, *together* with intuitionistic logic.

2 A bilateral bullet

The basis of Gabbay’s refutation of the above mentioned claim is that in spite of being harmonious, the bilateral ND system is *inconsistent*. To substantiate this claim, Gabbay considers a polarised version of the 0-ary connective ‘•’. The unilateral version of this connective was introduced in [10]. This connective is paradoxical,

I thank an *insistent* referee, whose constructive pedantry led to a considerable improvement of the presentation.

¹A slight variation of Rumfitt’s original presentation in [12].

since it has I/E -rules while still forming an atomic proposition, in contrast to standard atomic propositions having *no* I/E -rules internal to the system. A criticism of \bullet 's "credentials" as a connective can be found in [2]. For the sake of the current argument, I will accept ' \bullet ' as a connective.

Gabbay's polarised bullet is governed by the following polarised I/E -rules, regarded by Gabbay as bilateral rules. I present the rules in a slightly modified notation. The meta-variable φ ranges over object language formulas.

$$\frac{+\varphi \quad -\varphi}{-\bullet} (\bullet I^-) \quad \frac{+\varphi \quad -\varphi}{+\bullet} (\bullet I^+) \quad (1)$$

$$\frac{+\bullet}{+\varphi} (\bullet E_1^+) \quad \frac{+\bullet}{-\varphi} (\bullet E_2^+) \quad \frac{-\bullet}{+\varphi} (\bullet E_1^-) \quad \frac{-\bullet}{-\varphi} (\bullet E_2^-) \quad (2)$$

Those rules are clearly harmonious (under Dummett's original conception of harmony [1], namely, the presence of a derivation reduction removing maximal formulas); furthermore, those rules are shown by Gabbay as *derived rules* (in the bilateral classic ND system) by means of Rumfitt's structural "bilateral reductio" rules (SR):

$$\frac{\begin{array}{c} [-\varphi]_i \quad [-\varphi]_i \\ \vdots \quad \vdots \\ +\psi \quad -\psi \end{array}}{+\varphi} (SR^i) \quad \frac{\begin{array}{c} [+ \varphi]_i \quad [+ \varphi]_i \\ \vdots \quad \vdots \\ +\psi \quad -\psi \end{array}}{-\varphi} (SR^i) \quad (3)$$

In those rules, square brackets embrace a *discharged assumption*, indexed by a *discharge index*; the latter marks the instance of a rule application discharging the assumption.

To conclude Gabbay's argument, he now presents the following derivations, establishing inconsistency.

$$\frac{\frac{[+\bullet]_1}{+\varphi} (\bullet E_1^+) \quad \frac{[+\bullet]_1}{-\varphi} (\bullet E_2^+)}{\frac{-\bullet}{+\varphi} (\bullet E_1^-)} (SR^1) \quad \frac{\frac{[+\bullet]_1}{+\varphi} (\bullet E_2^+) \quad \frac{[+\bullet]_1}{-\varphi} (\bullet E_1^+)}{\frac{-\bullet}{-\varphi} (\bullet E_1^-)} (SR^1) \quad (4)$$

3 Bilateralism and harmony

The following (rhetoric) question immediately poses itself: *Are the I/E -rules above indeed bilateral I/E -rules (for '+/ - \bullet ')?*

More generally, does an *arbitrary* marking of premises and conclusions by ‘+/-’ (polarising) constitute legitimate bilateral rules? The negative answer to this question, together with a criterion for “proper polarisation”, based on ‘bilateral harmony’ (explained below) constitute the main basis for my rejection of Gabbay’s attack.

The first hint towards a negative answer to the above questions is the observation of the “suspicious” pair of rules ($\bullet I^-$) and ($\bullet I^+$), introducing both the acceptance and the rejection of ‘ \bullet ’ *from the same premises!* Similarly, the two pairs of rules ($\bullet E_i^+$) and ($\bullet E_i^-$) conclude both the acceptance and the rejection of the same formula from identical premises.

No wonder this system produces inconsistency. Inconsistency is built into it to start with.

So, what is going on here?

If ‘accepting’ and ‘denying’, the primitives of bilateralism underlying polarisation, deserve their names, no φ can have identical grounds for assertion and grounds for denial, or both asserted and denied conclusion. Recall that grounds, both for assertion and for denial, are defined in terms (mainly) of the I -rules, based on the notion of *canonical derivations* (see [4], Section 5.2.1). Here ‘ \bullet ’, in contrast to “regular” atomic sentences, has such grounds.

Avoiding model-theoretic considerations (a methodological point about semantics shared by Gabbay and myself), what can be a yardstick to which such grounds can be compared? A key concept here is *coherence* of a bilateral position, introduced by Restall [11]. A position is a sequent of the form $\Pi = [\Gamma : \Delta]$, where Γ contains the sentences (formulas) asserted in the position Π , while Δ contains the sentences denied in Π . The position Π is *incoherent* in case $\Gamma \cap \Delta \neq \emptyset$, and is coherent otherwise. In other words, it is incoherent to assert and deny the same sentence!

It is coherence which underlies the (SR)-rule: if denying φ in some context leads to incoherence, φ can only be asserted in that context, and vice versa, if asserting φ in a context leads to incoherence, φ can only be denied in that context.

Gabbay’s polarised rules lead to incoherence, as seen immediately from the observations above.

So, no arbitrary polarisation of rules can reflect bivalence correctly.

Let me return to harmony. When Dummett first proposed this criterion as a condition of an ND system to qualify as meaning conferring, he did not think of bilateral rules. His only concern was what I call the “vertical balance” between I/E -rules. The latter is indeed captured by the presence of a reduction eliminating maximal formulas (together with stability).

However, once polarised rules enter the picture, this notion of harmony is no more adequate. In order for ‘+’ and ‘-’ to capture, respectively, acceptance and denying, there must be some “horizontal balance” between any pair of (I^+) and

(I^-) for the same connective! Only such a balance can constitute a proof theoretic justification of the polarisation and make polarisation reflecting properly acceptance and denial. A formalisation of this horizontal balance is briefly delineated in the next section.

I have presented such an extended bilateral harmony in [4] (Section 4,4.1.5) (first presented in [3]). It will be too space consuming to repeat the full details here, but in the next section some more details are presented; but let me note that Gabbay's proposal of bilateral rules for ' \bullet ' fail to meet this criterion, as hinted by the above observations, while Rumfitt's original bilateral classical ND system meets both vertical and horizontal harmony. As far as the horizontal harmony is concerned, in a sense it is implicit in [12] for the classical connectives (Rumfitt did not consider arbitrary bilateral ND-systems as I do). This can be seen from his noticing that in the presence of his bilateral structural rule, not *all* the rules of his original system are needed, and he presents a smaller system still generating classical logic harmoniously. He sees that introducing a rejection of a conjunction and introducing its acceptance are tightly related, as are their respective eliminations.

4 Horizontal harmony

The material in this section is a reformulation of an extraction from [3].

In continuation to Gentzen's remark to the effect that (unilateral) I -rules are self-justifying, and E -rules are justified by the I -rules by the vertical inversion principle, one can see the positive I -rules as a definition that should justify also the negative I -rules. To that end, I introduce a *horizontal inversion principle*, the definition of which is based on the form of the positive I -rules.

To simplify, the formulation of the horizontal inversion principle below is for the case where the positive I/E -rules for a connective originate from some unilateral ND-system, to be extended into a bilateral one.

A positive rule is *combining* if it has more than one premise and is *splitting* otherwise. It is *categorical* if it does not discharge any assumptions, and is *hypothetical* otherwise..

The horizontal inversion principle:

constants with categorical I -rules: Let ' $*$ ' be any constant.

1. Any denial of a premise of a positive categorical combining I -rule for ' $*$ ' is a premise for a negative I -rule of ' $*$ '.

Example 1. Conjunction ‘ \wedge ’ has a positive categorical combining I -rule

$$\frac{+\varphi \quad +\psi}{+(\varphi \wedge \psi)} (+\wedge I)$$

Therefore, its negative I -rules are

$$\frac{-\varphi}{-(\varphi \wedge \psi)} (-\wedge I_1) \quad \frac{-\psi}{-(\varphi \wedge \psi)} (-\wedge I_2)$$

For example, the omission of one of those two rules, or a rule in which *both premises* of $(+\wedge I)$ are negated in an alleged $(-\wedge I)$ rule – violate the horizontal balance of the conjunction I -rules.

2. The collection of denials of the premises of a categorical positive splitting I -rules for ‘ $*$ ’ are joint premises of a negative I -rule of ‘ $*$ ’.

Example 2. Disjunction ‘ \vee ’ has positive categorical splitting I -rules.

$$\frac{+\varphi}{+(\varphi \vee \psi)} (+\vee I_1) \quad \frac{+\psi}{+(\varphi \vee \psi)} (+\vee I_2)$$

Therefore, its negative I -rule is

$$\frac{-\varphi \quad -\psi}{-(\varphi \vee \psi)} (-\vee I)$$

3. The collection of conclusions of the denials of the conclusions of the positive categorical splitting E -rules for ‘ $*$ ’ are a joint conclusion of the negative E -rule for ‘ $*$ ’.

Example 3. Conjunction has two positive categorical splitting E -rules

$$\frac{+(\varphi \wedge \psi)}{+\varphi} (+\wedge E_1) \quad \frac{+(\varphi \wedge \psi)}{+\psi} (+\wedge E_2)$$

Therefore, its negative I -rule is

$$\frac{-[\varphi]_i \quad -[\psi]_j}{\xi} (-\wedge E^{i,j})$$

4. Any denial of a conclusion of a categorical combining E -rule for ‘ $*$ ’ is a conclusion of a negative E -rule for ‘ $*$ ’.

The first principle indicates that the negative I -rules of ‘*’ a categorical combining ‘*’ are neither too weak nor too strong relative to the positive I -rules of ‘*’. The second principle assures the same for a categorical splitting logical constant. This balance can be seen as a generalization of (classical) De-Morgan’s laws, specifically relating negation and the dual pair of conjunction/disjunction. The other two principles relate the positive and negative E -rules to each other.

The fact that the required balance between positive and negative rules is a generalization of De-Morgan’s rules comes as no surprise. This fact just reflects the connection between the force of denial and top-level negation. The same holds for the ability to *define* negation in terms of denial, as done by Rumfitt in [12]. When defining negation this way, De-Morgan’s rule are imposed by the horizontal inversion principle merely due to the form of the ‘ \wedge ’-rules and ‘ \vee ’-rules, a form also responsible to their duality as discussed above.

constants with a hypothetical I/E -rule: The discharge of assumptions involves an additional technicality, and I skip the details, available in [4] (Section 4.4.1.5) or [3].

Definition 1. (bilateral harmony) A bilateral ND-system is *bilaterally harmonious* iff it satisfies both the vertical² and the horizontal inversion principles.

The notion of ‘bilateral harmony’ is summarized by the following diagram.

$$\begin{array}{ccc}
 (I^+) & \xrightarrow[\text{inversion}]{\text{horizontal}} & (I^-) \\
 \text{vertical} \downarrow \text{inversion} & & \text{vertical} \downarrow \text{inversion} \\
 (E^+) & \xrightarrow[\text{inversion}]{\text{horizontal}} & (E^-)
 \end{array} \tag{5}$$

The arrows in the diagram represent justification (in Dummett’s original sense in the positive, unilateral case). Note that similar diagrams, but with different orientations of the arrows, are possible (see [5]). I chose here to consider the (I^+) -node as the root to be consistent with Gentzen’s own view (in the unilateral case, of course).

A simple inspection of the positive and negative I -rules for ‘•’ shows that, having identical premises, they do not meet the horizontal inversion principle and are not bilaterally harmonious.

²Note that this requires the negative I/E -rules to be vertically harmonious too, not discussed above.

This can be seen³ as follows. Suppose we take as the point of departure is Gabbay’s $(\bullet I^+)$ in (1):

$$\frac{+\varphi \quad -\varphi}{+\bullet} (\bullet I^+)$$

By horizontal inversion from $(\bullet I^+)$, we get the following $(\bullet I^-)$ -rules:

$$\frac{+\varphi}{-\bullet} (\bullet I_1^-) \quad \frac{-\varphi}{-\bullet} (\bullet I_2^-)$$

By vertical inversion from $(\bullet I^+)$, we get the following $(\bullet E^+)$ -rules:

$$\frac{+\bullet}{+\varphi} (\bullet E_1^+) \quad \frac{-\bullet}{+\varphi} (\bullet E_2^+)$$

Finally, by horizontal inversion from $(\bullet E^+)$ (and vertical inversion from $(\bullet I^-)$), we get the following $(\bullet E^-)$ -rule:

$$\frac{\begin{array}{cc} [+ \varphi]_i & [- \varphi]_j \\ \vdots & \vdots \\ -\bullet & \chi \end{array}}{\chi} (\bullet E^{i,j})$$

This *is* a bilaterally harmonious set of I/E -rules for ‘ \bullet ’, essentially differing from Gabbay’s rules. It certainly does not support Gabbay’s “derivation” in (4).

5 Bilateral tonk

Another argument Gabbay brings up does not depend on the bilateral ‘ \bullet ’; rather he considers a bilateral version of Prior’s ‘tonk’, brought forward in [9] as a counter argument to the view that *any* ND system can serve as meaning conferring.

The original I/E -rules for ‘tonk’

$$\frac{\varphi}{\varphi \text{ tonk } \psi} (\text{tonk } I) \quad \frac{\varphi \text{ tonk } \psi}{\psi} (\text{tonk } E) \tag{6}$$

had a devastating effect on the proof system, trivialising it via the following derivation, that has an irreducible maximal formula.

$$\frac{\frac{\varphi}{\varphi \text{ tonk } \psi} (\text{tonk } I)}{\psi} (\text{tonk } E) \tag{7}$$

³I thank an anonymous referee who worked out those edetails, that I have initially omitted.

with the effect that $\varphi \vdash \psi$ for *any* φ and ψ .

Consider next a bilateral version of this connective. In a natural polarisation, the original rules are turned into assertion rules. Gabbay does not present the denial rules, but a natural choice is the following, which also satisfy horizontal harmony.

$$\frac{+\varphi}{+(\varphi \text{ tonk } \psi)} (\text{tonk } I^+) \quad \frac{+(\varphi \text{ tonk } \psi)}{+\psi} (\text{tonk } E^+) \quad (8)$$

$$\frac{-\varphi}{-(\varphi \text{ tonk } \psi)} (\text{tonk } I^-) \quad \frac{-(\varphi \text{ tonk } \psi)}{-\psi} (\text{tonk } E^-) \quad (9)$$

Now we have both $+\varphi \vdash +\psi$ and $-\varphi \vdash -\psi$.

Now Gabbay comes to ‘a more general problem with harmony and bilateralism’. He claims the derivation below provides the “missing reduction” for (7).

$$\frac{\frac{+\varphi}{+(\varphi \text{ tonk } \psi)} (\text{tonk } I^+) \quad \frac{\frac{[\text{+}(\varphi \text{ tonk } \psi)]_1}{+\psi} (\text{tonk } E^+) \quad \frac{[-\psi]_2}{-(\varphi \text{ tonk } \psi)} (SR^1)}{-(\varphi \text{ tonk } \psi)} (SR^2)}{+\psi} \quad (10)$$

As noted by Gabbay, this derivation involves two vacuous discharges in both applications of (SR) .

I claim that the derivation (10) does not qualify as a reduction of a maximal formula. The argument has two parts.

- First, as noted by Gabbay himself, the maximality involved in the derivation of $+\psi$ from $+\varphi$ does not vanish; rather it is just “spread out” in the derivation. It still constitutes a detour and is not canonical. As a result, Gabbay asks that

A bilateralist must show that somehow such uses of SR are not legitimate.

I claim that there is no need to show that such uses are illegitimate, they just do not constitute reductions of maximal formulas.

- Furthermore, let us remember that the notion of ‘reduction’ (in the context of harmony, not just a normalisation step) originates from Prawitz’ *inversion principle* [7]:

Let ρ be an application of an elimination rule that has ψ as conclusion. Then, the derivation that justifies the sufficient condition

[...] for deriving the major premiss of ρ , when combined with the derivations of the minor premisses of ρ (if any), already “contain” a derivation of ψ ; the derivation of ψ is thus obtainable directly from the given derivations without the addition of ρ .

In a later paper [8], Prawitz makes the notion of containment in this principle more precise. I repeat this explication below, adapted to natural deduction derivations⁴ and to my notation.

- the derivation \mathcal{D} is *immediately extracted* from the set \mathbf{D} of derivations if and only if either
 1. \mathcal{D} is a derivation in \mathbf{D} or a sub-derivation of some derivation in \mathbf{D} , or
 2. \mathcal{D} is the result of substituting a term for the occurrences of a free variable in a derivation in \mathbf{D} , or
 3. \mathcal{D} is the result of composing two derivations \mathcal{D}_1 and \mathcal{D}_2 in \mathbf{D} , that is, \mathcal{D} is the result of replacing some open assumptions φ in \mathcal{D}_2 by \mathcal{D}_1 , the latter having φ as a conclusion.
- $\hat{\mathcal{D}}$ is contained in a set \mathbf{D} of derivations, iff there is a sequence of derivations $\mathcal{D}'_1 \cdots, \mathcal{D}'_n$, where $\mathcal{D}'_n = \hat{\mathcal{D}}$, and for each $i \leq n$, \mathcal{D}'_i is immediately extracted from $\mathbf{D} \cup \{\mathcal{D}'_1 \cdots, \mathcal{D}'_{i-1}\}$.

I see no way of understanding ‘contains’ in that principle (as defined above) as applying to the derivation (10) being contained in a derivation of $+\varphi$!

6 Conclusion

I conclude that Gabbay’s attempt to reject the bilateral classical ND presentation (the polarised rules of which do meet the horizontal harmony criterion) by means of a bilateral ‘ \bullet ’ and a bilateral ‘tonk’ - fails. Bilateralism does not constitute an arbitrary polarisation of I/E -rules. Any such polarisation has to respect coherence via horizontal harmony.

Acknowledgment

I thank Michael Gabbay for his comments on a earlier draft of this note and discussions on the topic of this note.

⁴Prawitz’s explication is for a more general notion of an argument.

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A COMPLEXITY DICHOTOMY FOR POSET CONSTRAINT SATISFACTION

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Abstract

In this paper we determine the complexity of a broad class of problems that extend the temporal constraint satisfaction problems classified by Bodirsky and Kára. To be more precise, we study problems $\text{Poset-SAT}(\Phi)$ where Φ is a given set of quantifier-free \leq -formulas. An instance of $\text{Poset-SAT}(\Phi)$ then consists of finitely many variables and constraints on them expressible in Φ ; the question is then whether this input can be satisfied in some partial order or not. We show that every such problem is either NP-complete or in P, depending on the constraint language Φ .

All Poset-SAT problems can be formalized as constraint satisfaction problems of reducts of the random partial order. We use model-theoretic concepts and techniques from universal algebra to study these reducts. In the course of this analysis we establish a dichotomy that we believe is of independent interest in universal algebra and model theory.

Keywords: constraint satisfaction, random partial order, homogeneous structures, model theory, polymorphism clones

*Supported by project P27600 of the Austrian Science Fund (FWF) and grant PRIMUS/SCI/12 of Charles University, Prague.

†Supported by project P27600 of the Austrian Science Fund (FWF).

1 Introduction

1.1 Poset-SAT

Reasoning about temporal knowledge is a common task in various areas of computer science, for instance in Artificial Intelligence, Scheduling, Computational Linguistics and Operations Research. A lot of research in those areas only concerns linear models of time. Knowledge about temporal constraints is then expressed as collections of relations between time points or time intervals and a typical computational problem is to determine whether such a collection is satisfied by some linear order. A complete complexity classification of all such temporal constraint satisfaction problems was achieved in [13].

However, it has been observed many times that more complex time models are helpful, for instance in the analysis of concurrent and distributed systems or certain planning domains. A possible generalisation is to model time by partial orders as discussed in [24] and [3]. Some cases of the arising satisfiability problems have already been studied in [20]. We will give a complete classification in this paper.

Speaking more formally, let $\Phi = \{\phi_1, \phi_2, \dots, \phi_k\}$ be a finite set of quantifier free \leq -formulas. Our aim is then to determine the complexity of deciding whether constraints expressible in Φ can be satisfied by some partial order. This computational problem will be denoted by $\text{Poset-SAT}(\Phi)$ and is defined as follows:

Poset-SAT(Φ):
INSTANCE: A finite set of variables $X = \{x_1, \dots, x_n\}$ and a formula $\Psi(x_1, \dots, x_n) = \psi_1 \wedge \dots \wedge \psi_l$ where each ψ_i for $1 \leq i \leq l$ is obtained from one of the formulas ϕ in Φ by substituting the variables of ϕ by variables from X .
QUESTION: Is there a partial order $(A; \leq)$ and an assignment of variables $f : X \rightarrow A$ such that $\Psi(f(x_1), \dots, f(x_n))$ holds in $(A; \leq)$?

Note that in the above definition we can assume without loss of generality that $(A; \leq)$ has at most n many elements. Thus every $\text{Poset-SAT}(\Phi)$ problem is in NP: For an input with n many variables, we can “guess” a partial order $(A; \leq)$ with $|A| \leq n$ and an assignment $f : X \rightarrow A$ and then check in polynomial time if $\Psi(f(x_1), \dots, f(x_n))$ holds in $(A; \leq)$ or not. The main result of our paper is to give a full complexity classification of the $\text{Poset-SAT}(\Phi)$ problems, proving the following dichotomy.

Theorem 1.1. *Let Φ be a finite set of quantifier-free \leq -formulas. Then $\text{Poset-SAT}(\Phi)$ is either in P or NP-complete.*

$\text{SAT}(\Phi)$ is in P or NP -complete.

The proof of our result is based on a variety of methods and results. A first step is that we give a description of every Poset-SAT problem as constraint satisfaction problem over a homogeneous structure, the random partial order.

1.2 The random partial order and its reducts

The notion of homogeneity plays a key role when tackling constraint satisfaction problems over partial orders or also other classes of structures. A relational structure is called *homogeneous* if every isomorphism between finite substructures can be extended to an automorphism of the entire structure. Countable homogeneous structures are uniquely determined by the finite structures that embed into them; two prominent examples of such structures are the rationals as totally ordered set $(\mathbb{Q}; <)$ (given by the class of linear orders) and the random graph (given by finite graphs). The *random partial order* $\mathbb{P} = (P; \leq)$ is defined as the unique countable homogeneous partial order that embeds all finite partial orders. As the “generic” order, representing all finite partial orders it is an object that is both of theoretical and practical interest.

What does this imply for our Poset-SAT(Φ) problems? Let $\Phi = \{\phi_1, \dots, \phi_k\}$ be a finite set of quantifier free \leq -formulas. We can associate with Φ the structure $\mathbb{P}_\Phi = (P; R_1, \dots, R_k)$ that we obtain by defining the relations $R_i = \{(a_1, \dots, a_l) : \mathbb{P} \models \phi_i(a_1, \dots, a_l)\}$. Then, as \mathbb{P} embeds all finite partial orders, it is straightforward to see that an input $\phi_{i_1}(\dots) \wedge \dots \wedge \phi_{i_t}(\dots)$ in variables x_1, \dots, x_n is accepted by Poset-SAT(Φ) if and only if $\mathbb{P}_\Phi \models \exists x_1, \dots, x_n (R_{i_1}(\dots) \wedge \dots \wedge R_{i_t}(\dots))$. Hence Poset-SAT(Φ) translates directly to the *constraint satisfaction problem (CSP)* of \mathbb{P}_Φ . i.e. the problem of deciding primitive positive sentences in \mathbb{P}_Φ .

Following an established convention from [32] we call a structure whose relations are all first order definable in \mathbb{P} a *reduct of \mathbb{P}* . In this terminology the present article provides a complexity classification of the CSPs of reducts of the random partial order. It might seem at first that rephrasing Poset-SAT problems in this way is just an artificial complication. However this approach has already proven successful in other similar classification, for instance for the class of linear orders [13], the class of graphs [16], and the class of leaf structures of binary trees [11]. This relies on the fact that homogeneous structures are highly symmetric and come with several pleasant model theoretical properties. In particular homogenous structures in a finite relational language have *quantifier-elimination*, i.e. every formula is equivalent to a quantifier-free formula, and they are *ω -categorical*, i.e. the relations that are first-order definable are exactly those that are invariant under the action of the

automorphism group. For background on homogeneous and ω -categorical structures we refer to the survey [26] and the textbook [22].

1.3 Proof strategy

Our proof follows to a large extent the method invented in [16] to study the constraint satisfaction problems of reducts of the random graph. The first key component to this strategy is the use of the universal algebraic approach: instead of studying the reducts themselves, we investigate their polymorphism clones, i.e. the algebras consisting of all the operations preserving the structure. This approach originally stems from the study of finite CSPs, where it recently resulted in the proof of the long standing CSP dichotomy conjecture in [33] and [21].

The other foundation is the usage of Ramsey theory - in our case, a result of Paoli, Trotter and Walker [29] - and the concept of canonical functions introduced in [15]. This allows us to systematically investigate the polymorphism clones of reducts of \mathbb{P} , focussing only on operations with very regular behaviour. We remark that a helpful result has also already been established in the form of the classification of the closed supergroups of the automorphism group of the random partial order in [27]. This will be the starting point for our analysis and help to discuss the unary part of the appearing polymorphism clones.

On a technical level, there is however some novelty in the present proof. It turned out that all our hardness results can be traced back to the NP-hardness of the constraint relation $\text{Low}(x, y, z) := (x < y \wedge y \perp xy) \vee (x < z \wedge y \perp xz)$, we are going to discuss this part of the proof in Section 6. Using this fact we could avoid tedious case distinctions that we expected due to the already quite involved proofs in [27]. Similar observations could be also simplify future classification proofs. However we remark that this fact might be tied to particular properties of the random partial order; a similar statement does not hold for the random graph [16].

1.4 Overview

This paper has the following structure: In Section 2 we introduce basic notation and give an introduction to the universal-algebraic approach and canonical functions. In Section 3 we study the unary part of closed clones containing $\text{Aut}(\mathbb{P})$, which will allow us to reduce several cases to already known complexity classifications. In Section 4 and 5 we classify the reducts where $<$ and the incomparability relation \perp are pp-definable. In Section 6 we show that all the remaining reducts induce NP-complete CSPs. In Section 7 we then summarize our results and discuss how our complexity dichotomy corresponds to a stronger algebraic dichotomy regarding

the structural properties of reducts of \mathbb{P} .

2 Preliminaries

2.1 General notational conventions

In this section we fix some standard terminology and notation. A *relational language* τ is a set of symbols together with an arity function $ar: \tau \rightarrow \mathbb{N}$. A *relational structure* in language τ is an object of the form $\Gamma = (D; (R^\Gamma)_{R \in \tau})$ where D is a set (the *domain* of Γ) and all $R^\Gamma \subseteq D^{ar(R)}$ are relations of arity $ar(R)$ over D . When Δ and Γ are two τ -structures, then a *homomorphism* from Δ to Γ is a mapping h from the domain of Δ to the domain of Γ such that for all $R \in \tau$ and for all $(x_1, \dots, x_j) \in R^\Delta$ we have $h(x_1, \dots, x_j) \in R^\Gamma$. Injective homomorphisms for which also the converse implication holds are called *embeddings*. Bijective embeddings from Δ onto itself are called *automorphisms* of Δ . When working with relational structures it is often convenient not to distinguish between a relation and its relational symbol. We will also do so on several occasions, but this should never cause any confusion.

The relation symbol \leq will always denote a partial order, i.e. a binary relation that is reflexive, antisymmetric and transitive. Let $<$ be the corresponding strict order defined by $x \leq y \wedge x \neq y$. Further $x \perp y$ will always denote the incomparability relation defined by $\neg(x \leq y) \wedge \neg(y \leq x)$. Sometimes we will write $x < y_1 \cdots y_n$ for the conjunction of the formulas $x < y_i$ for all $1 \leq i \leq n$. Similarly we will use $x \perp y_1 \cdots y_n$ if $x \perp y_i$ holds for all $1 \leq i \leq n$.

2.2 Constraint satisfaction problems

A first-order formula $\phi(x_1, \dots, x_n)$ in the language τ is called *primitive positive* if it is of the form $\exists x_1, \dots, x_k (\psi_1 \wedge \cdots \wedge \psi_m)$ where ψ_1, \dots, ψ_m are atomic, i.e. of the form $y_1 = y_2$, or $R(y_1, \dots, y_n)$, for not necessarily distinct variables y_i .

For a structure Γ in finite relational language τ , the *constraint satisfaction problem of Γ* , or short $\text{CSP}(\Gamma)$ is the problem of deciding whether a given primitive positive sentence is true in Γ or not.

We say a relation R is *primitively positive definable* or *pp-definable* in Γ if there is a primitive positive formula $\phi(x_1, \dots, x_n)$ such that $(a_1, \dots, a_n) \in R$ if and only if $\phi(a_1, \dots, a_n)$ holds in Γ . Then the following observation holds:

Lemma 2.1 (from [23]). *Let Γ be a relational structure in finite language, and let Γ' be the structure obtained from Γ by adding a relation R . If R is primitive positive definable in Γ , then $\text{CSP}(\Gamma)$ and $\text{CSP}(\Gamma')$ are polynomial-time equivalent. \square*

By $\langle \Gamma \rangle_{pp}$ we denote the set of all primitively positive definable relations on Γ . So for two structures Γ and Δ the problems $\text{CSP}(\Gamma)$ and $\text{CSP}(\Delta)$ have the same complexity if $\langle \Gamma \rangle_{pp} = \langle \Delta \rangle_{pp}$. This means that in our analysis we only have to study reduct of the random poset up to primitive positive definability.

2.3 Polymorphism clones

Let Γ be a relational structure with domain D . By Γ^n we denote the direct product of n -copies of Γ . This is, we take a structure on D^n with same signature Γ . Then for n -tuples $\bar{x}^{(1)}, \dots, \bar{x}^{(k)}$ we set that $(\bar{x}^{(1)}, \dots, \bar{x}^{(k)}) \in R$ if and only if $(x_i^{(1)}, \dots, x_i^{(k)}) \in R$ holds in Γ for every coordinate $1 \leq i \leq n$.

Then an n -ary operation f is called a *polymorphism* of Γ if f is a homomorphism from Γ^n to Γ . Unary polymorphisms are called *endomorphisms*. For every relation R on D we say f *preserves* R if f is a polymorphism of $(D; R)$. Otherwise we say f *violates* R .

For a given structure Γ the set of all polymorphisms $\text{Pol}(\Gamma)$ contains all the projections $\pi_i^n(x_1, \dots, x_n) = x_i$ and is closed under composition. Every set of operation with these properties is called a *clone* or *function clone* (cf. [31]). $\text{Pol}(\Gamma)$ is called the *polymorphism clone* of Γ . We write $\text{Pol}(\Gamma)^{(k)}$ for the set of k -ary functions in $\text{Pol}(\Gamma)$ and we write $\text{End}(\Gamma)$ for the monoid consisting of all endomorphisms of Γ . The clone $\text{Pol}(\Gamma)$ is furthermore closed in the *topology of pointwise convergence*. In this topology, the closure \bar{F} of a set of operations F is given by all function $f: D^k \rightarrow D$, such that for every finite subset $A \subseteq D^k$ there is a function $g \in F$ with $f \upharpoonright A = g \upharpoonright A$. We say a set of operation F *generates* an operation g if g is in the smallest closed clone containing F .

Now primitive positive definability in ω -categorical (and finite) structures can be characterized by preservation under polymorphisms:

Theorem 2.2 (from [14]). *Let Γ be an ω -categorical structure. Then a relation is pp-definable in Γ if and only if it is preserved by the polymorphisms of Γ .*

Thus, by Lemma 2.1 the complexity of $\text{CSP}(\Gamma)$ only depend on the polymorphism clone $\text{Pol}(\Gamma)$ for ω -categorical Γ . We finish this section by the following observation that states that whenever a relation is not pp-definable, this can be witnessed by polymorphisms of bounded arity.

Theorem 2.3 (from [13]). *Let Γ be a relational structure and let R be a k -ary relation that is a union of at most m orbits of $\text{Aut}(\Gamma)$ on D^k . If Γ has a polymorphism f that violates R , then Γ also has an at most m -ary polymorphism that violates R .*

□

2.4 Canonical functions and Ramsey theory

As pointed out in the previous section, the complexity of $\text{CSP}(\Gamma)$ only depends on $\text{Pol}(\Gamma)$. It further turns out that in our analysis of such clones only functions that are highly symmetric, i.e. *canonical*, are relevant. We define canonical functions and explain their usage in this section.

Definition 2.4. Let Γ be a structure. Then the *type of an n -tuple* $\bar{a} = (a_1, \dots, a_n)$ in Γ is the set of all formulas $\phi(\bar{x}) = \phi(x_1, \dots, x_n)$ such that $\phi(\bar{a})$ holds in Γ .

In countable ω -categorical structure Γ two n -tuples have the same type if and only if they are in the same orbit with respect to $\text{Aut}(\Gamma)$ acting on the set of n -tuples.

Definition 2.5. Let Γ and Δ be two structures and f a function from the domain of Γ to the domain of Δ . The *behaviour of f* is defined as the set of all pairs (p, q) , where p is an n -type of Γ , q is an n -type of Δ , and there is a tuple \bar{a} of type p such that $f(\bar{a})$ is of type q . More generally we define the behaviour of an m -ary function f as all tuples of types (p_1, \dots, p_m, q) , such that there are tuples $\bar{a}_1 \in p_1, \dots, \bar{a}_m \in p_m$ such that $f(\bar{a}_1, \dots, \bar{a}_m) \in q$. Whenever the behaviour of f is a total function, i.e. tuples of the same type are mapped to tuples of the same type, we call f *canonical*.

Now Ramsey theory allows us to build canonical functions starting from an arbitrary function. This will simplify our analysis, reducing the task of classifying reducts to a mere combinatorial analysis. Besides its application to CSPs (e.g. [13], [16], [11]), this method has also been applied many times to the classification of first order reducts (e.g. [25], [1], [2]). As it is not needed for our proof we are going to skip the definition of Ramsey structures - for background on Ramsey theory we refer to the survey [9]. For Ramsey structures the following holds:

Theorem 2.6 (from [18]). *Let Γ be an ordered homogeneous Ramsey structure with domain D and let Δ be ω -categorical with domain F . Then, for every function $f: D^n \rightarrow F$, there is a function $g: D^n \rightarrow F$, such that $g \in \text{Aut}(\Delta) \circ f \circ (\text{Aut}(\Gamma))^n$ and g is canonical from Γ to Δ . \square*

The random partial order $\mathbb{P} = (P; \leq)$ itself is not a Ramsey structure. However, it is well known that there exists an expansion of \mathbb{P} in finite language that is Ramsey:

Theorem 2.7 (from [29]). *Let $(P; \leq, \prec)$ be the homogeneous structures embedding all structures $(A; \leq, \prec)$ such that $(A; \leq)$ is a partial order and \prec is an extension of \leq to a linear order. Then $(P; \leq)$ is isomorphic to the random partial order and $(P; \leq, \prec)$ has the Ramsey property. \square*

Furthermore also every expansion of $(P; \leq, \prec)$ by finitely many constants is a Ramsey structure. From that and Theorem 2.6 we draw the following conclusion for the random partial order:

Lemma 2.8. *Let c_1, \dots, c_n be elements of P and let $f: P^m \rightarrow P$ be an arbitrary function. Then there is a function $g: P^m \rightarrow P$, such that*

- *g is canonical from $(P; \leq, \prec, c_1, \dots, c_n)$ to \mathbb{P} ,*
- *$g \in \overline{\text{Aut}(\mathbb{P}) \circ f \circ (\text{Aut}(P; \leq, \prec, c_1, \dots, c_n))^m}$.* □

In practice we will use Lemma 2.8 as follows: Assume that a polymorphism clone $\text{Pol}(\Gamma)$ does not preserve a certain relation R . This can be witnessed by a function $f \in \text{Pol}(\Gamma)$ on a finite set $\{c_1, \dots, c_n\}$. Now Lemma 2.8 guarantees that there is a function $g \in \text{Pol}(\Gamma)$ witnessing it, which is furthermore canonical from $(P; \leq, \prec, c_1, \dots, c_n)$ to \mathbb{P} .

2.5 Model-complete cores

Let Δ and Γ be two structures with the same signature. We say Δ is *homomorphically equivalent* to Γ if there is a homomorphism from Δ to Γ and a homomorphism from Γ to Δ . Since homomorphisms preserve primitive positive formulas, the constraint satisfaction problems $\text{CSP}(\Delta)$ and $\text{CSP}(\Gamma)$ encode the same computational problem for homomorphically equivalent structures Δ and Γ .

A structure Δ is called a *model-complete core* if its endomorphism monoid is equal to the topological closure of the automorphism group. Now every CSP with ω -categorical template can be reformulated as a CSP on a template with model-complete core by the following theorem:

Theorem 2.9 (from [7]). *Every ω -categorical structure Γ is homomorphically equivalent to a model-complete core which is unique up to isomorphism. This core is ω -categorical or finite.* □

Thus model-complete cores can be thought of as “minimal” representants of a class of homomorphic equivalent structures. To identify the model-complete core of Γ can be a very helpful simplification in analysing the complexity of $\text{CSP}(\Gamma)$. Furthermore, by the following theorem of Bodirsky we can add constants to a model-complete core without increasing the complexity (this is however not true for general structures).

Theorem 2.10 (from [7]). *Let Γ be a model-complete ω -categorical or finite core, and let c be an element of Γ . Then $\text{CSP}(\Gamma)$ and $\text{CSP}(\Gamma, c)$ have the same complexity, up to polynomial-time many-one reductions.* □

2.6 Primitive positive interpretations

A further tool to compare the complexity of CSPs of structures, that generalises the concept of pp-definitions, to structures on possibly different domains are pp-interpretations. We say Δ is *pp-interpretable* in Γ if there is a $n \geq 1$ and a partial map $I : \Gamma^n \rightarrow \Delta$ such that

- I is surjective,
- the domain of I is pp-definable in Γ ,
- the preimage of the equality relation in Δ is pp-definable in Γ ,
- the preimage of every relation in Δ is pp-definable in Γ .

Then the following result holds:

Theorem 2.11 (see [8]). *If Δ is pp-interpretable in Γ then $\text{CSP}(\Delta)$ can be reduced to $\text{CSP}(\Gamma)$ in polynomial time.*

In summary, for ω -categorical and finite structures Δ and Γ we have a polynomial reduction from $\text{CSP}(\Delta)$ to $\text{CSP}(\Gamma)$ if

1. Δ is the model-complete core of Γ .
2. Γ is a model-complete core and Δ is obtained by adding finitely many constants to the signature of Γ .
3. Δ is pp-interpretable in Γ .

In our proof we are going to use the reductions (1)-(3) to prove NP-hardness. In particular we are going to use the NP-complete problem NAE-SAT (see [30]), which can be written as $\text{CSP}(\{0, 1\}; \text{NAE}, 0, 1)$ with $\text{NAE} := \{0, 1\}^3 \setminus \{(0, 0, 0), (1, 1, 1)\}$. For the remaining cases we are going to use the method of canonical functions to obtain structural information about the underlying reducts and show tractability. This dichotomy corresponds also to an algebraic dichotomy, which we will discuss in the summary in Section 7.2.

3 A pre-classification by model-complete cores

In this section we start our analysis of reducts of the random partial order $\mathbb{P} = (P; \leq)$. Our aim is to determine the model-complete core for every reduct Γ of \mathbb{P} , therefore we are going to study the endomorphism monoids $\text{End}(\Gamma) \supseteq \text{Aut}(\mathbb{P})$. Part of the work was already done in [27] where all the automorphism groups $\text{Aut}(\Gamma) \supseteq \text{Aut}(\mathbb{P})$ were determined. Whenever our proof just replicates arguments therefrom, we are going to directly refer to the corresponding proof steps in [27].

We start by giving descriptions of the group reducts. If we turn the partial order \mathbb{P} upside-down, then the obtained partial order is again isomorphic to \mathbb{P} . Hence there exists a bijection $\uparrow: P \rightarrow P$ such that for all $x, y \in P$ we have $x < y$ if and

only if $\downarrow(y) < \downarrow(x)$. By the homogeneity of \mathbb{P} it is easy to see that the monoid generated by \downarrow and $\text{Aut}(\mathbb{P})$ does not depend on the choice of the bijection \downarrow .

The class of all finite structures $(A; \leq, F)$, where $(A; \leq)$ is a partial order and F is an upwards closed set, is an amalgamation class. It induces a homogenous structure that is isomorphic to \mathbb{P} with an additional unary relation F . We say F is a *random filter* on \mathbb{P} . Note that F and $I = P \setminus F$ are both isomorphic to the random partial order. Furthermore for every pair $x \in I$ and $y \in F$ either $x < y$ or $x \perp y$ holds.

We define a new order relation $<_F$ on by setting $x <_F y$ if and only if

- $x, y \in F$ and $x < y$ or,
- $x, y \in I$ and $x < y$ or,
- $x \in F, y \in I$ and $x \perp y$.

It is shown in [27] that the resulting structure $(P; <_F)$ is isomorphic to $(P, <)$. Let us fix a random filter F and a function $\circ: P \rightarrow P$ that maps $(P; <)$ isomorphically to $(P, <_F)$. Using the homogeneity of \mathbb{P} one can prove that the smallest closed group generated by \circ and $\text{Aut}(\mathbb{P})$ does not depend on the choice of the random filter F .

For $B \subseteq \text{Sym}(P)$, let $\langle B \rangle$ denote the smallest closed subgroup of $\text{Sym}(P)$ containing B . For brevity, when it is clear we are discussing supergroups of $\text{Aut}(\mathbb{P})$, we may abuse notation and write $\langle B \rangle$ to mean $\langle B \cup \text{Aut}(\mathbb{P}) \rangle$.

Theorem 3.1 (Theorem 1 from [27]). *Let Γ be a reduct of \mathbb{P} . Then $\text{Aut}(\Gamma) \supseteq \text{Aut}(\mathbb{P})$ is equal to one of the five groups $\text{Aut}(\mathbb{P}), \langle \downarrow \rangle, \langle \circ \rangle, \langle \downarrow, \circ \rangle$ or $\text{Sym}(P)$. \square*

In this section we are going to show the following extension of Theorem 3.1:

Proposition 3.2. *Let Γ be a reduct of \mathbb{P} . Then for $\text{End}(\Gamma)$ at least one of the following cases applies:*

1. $\text{End}(\Gamma)$ contains a constant function,
2. $\text{End}(\Gamma)$ contains a function $g_{<}$ that preserves $<$ and maps P onto a chain,
3. $\text{End}(\Gamma)$ contains a function g_{\perp} that preserves \perp and maps P onto an antichain,
4. The automorphism group $\text{Aut}(\Gamma)$ is dense in $\text{End}(\Gamma)$, i.e. Γ is a model-complete core. So by the classification in Theorem 3.1, $\text{End}(\Gamma)$ is the topological closure of $\text{Aut}(\mathbb{P}), \langle \downarrow \rangle, \langle \circ \rangle, \langle \downarrow, \circ \rangle$ or $\text{Sym}(P)$ in the space of all functions P^P .

Before we start with the proof of Proposition 3.2 we want to point out its relevance for the complexity analysis of the CSPs on reducts of \mathbb{P} . Constraint satisfaction problems on reducts of $(\mathbb{Q}; <)$ are called *temporal constraint satisfaction problems*. The CSPs on reducts of a countable set with a predicate for equality $(\omega; =)$ are

called *equality satisfaction problems*. For both classes a full complexity dichotomy is known, see [13] and [12]. As a corollary of Proposition 3.2 we get the following pre-classification of CSPs reducing all the cases where Γ is not a model-complete core to temporal or equality satisfaction problems:

Corollary 3.3. *Let Γ be a reduct of \mathbb{P} . Then one of the following holds*

1. *The model-complete core of Γ has one element and $\text{CSP}(\Gamma)$ is in P;*
2. *The model-complete core of Γ is a reduct of $(\omega; =)$,
so $\text{CSP}(\Gamma)$ is equal to an equality satisfaction problem;*
3. *The model-complete core of Γ is a reduct of $(\mathbb{Q}; <)$,
so $\text{CSP}(\Gamma)$ is equal to a temporal constraint satisfaction problem;*
4. *$\text{End}(\Gamma)$ is the topological closure of $\text{Aut}(\mathbb{P})$, $\langle \uparrow \rangle$, $\langle \circ \rangle$ or $\langle \uparrow, \circ \rangle$.*

Proof. If there is a constant function in $\text{End}(\Gamma)$, then Γ is homomorphically equivalent to a one-element structure. In this case, an instance of $\text{CSP}(\Gamma)$ is accepted if and only if it contains no relation symbol that corresponds to an empty relation. Hence $\text{CSP}(\Gamma)$ is in P. So let us from now on assume that $\text{End}(\Gamma)$ contain no constant function.

Assume that $g_{\perp} \in \text{End}(\Gamma)$. Since g_{\perp} preserves \perp , the image of $(P; \perp)$ under g_{\perp} is isomorphic to a countable antichain, or in other word, a countable set ω with a predicate for inequality $(\omega; \neq)$. Thus, for every reduct of Γ the image $g_{\perp}(\Gamma)$ can be seen as a reduct of $(\omega; \neq)$. Now clearly Γ and $g_{\perp}(\Gamma)$ are homomorphically equivalent. It is shown in [12] that every reduct of $(\omega; \neq)$ without constant endomorphisms is a model-complete core. So we are in the second case.

Now assume that $g_{<} \in \text{End}(\Gamma)$ but $g_{\perp} \notin \text{End}(\Gamma)$. Since $g_{<}$ preserves $<$ and is a chain, the image of $(P; <)$ under $g_{<}$ has to be isomorphic to the rational order $(\mathbb{Q}; <)$. Thus for every reduct of Γ the image $g_{<}(\Gamma)$ can be seen as a reduct of \mathbb{Q} . Now clearly Γ and $g_{<}(\Gamma)$ are homomorphically equivalent. It is shown in [13] that the model-complete core of every reduct of $(\mathbb{Q}, <)$ is either trivial, definable in (ω, \neq) or the reduct itself. So we are in the third case.

Note that also in the case where $\text{End}(\Gamma) = \overline{\text{Sym}(P)}$ we have that $e_{\perp} \in \text{End}(\Gamma)$. So by Proposition 3.2 we are only left with the cases where $\text{End}(\Gamma)$ is the topological closure of $\text{Aut}(\mathbb{P})$, $\langle \uparrow \rangle$, $\langle \circ \rangle$ or $\langle \uparrow, \circ \rangle$. □

Let us define the following relations on P :

$$\begin{aligned} \text{Betw}(x, y, z) &:= (x < y \wedge y < z) \vee (z < y \wedge y < x). \\ \text{Cycl}(x, y, z) &:= (x < y \wedge y < z) \vee (y < z \wedge z < x) \vee (z < x \wedge x < y) \vee \\ &\quad (x < y \wedge z \perp xy) \vee (y < z \wedge x \perp yz) \vee (z < x \wedge y \perp zx). \end{aligned}$$

$$\begin{aligned} \text{Par}(x, y, z) := & (x < yz \wedge y \perp z) \vee (y < xz \wedge y \perp z) \vee (z < xy \wedge x \perp y) \vee \\ & (x > yz \wedge y \perp z) \vee (y > xz \wedge y \perp z) \vee (z > xy \wedge x \perp y) \vee \\ & (x \perp y \wedge y \perp z \wedge y \perp z). \end{aligned}$$

$$\begin{aligned} \text{Sep}(x, y, z, t) := & (\text{Cycl}(x, y, z) \wedge \text{Cycl}(y, z, t) \wedge \text{Cycl}(x, y, t) \wedge \text{Cycl}(x, z, t)) \vee \\ & (\text{Cycl}(z, y, x) \wedge \text{Cycl}(t, z, y) \wedge \text{Cycl}(t, y, x) \wedge \text{Cycl}(t, z, x)). \end{aligned}$$

In Lemma 3.5 we are going to give a description of the monoids $\overline{\langle \uparrow \rangle}$, $\overline{\langle \circ \rangle}$ and $\overline{\langle \uparrow, \circ \rangle}$ as endomorphism monoids with the help of the above relations. We remark that Cycl and Par describes the orbits of triples under $\langle \circ \rangle$ and Sep describes the orbit of a linearly ordered 4-tuple under $\langle \uparrow, \circ \rangle$.

Lemma 3.4. *The incomparability relation \perp is pp-definable in $(P; <, \text{Cycl})$ and Par is pp-definable in $(P; \text{Cycl})$.*

Proof. To prove the first part of the lemma, let $\psi(x, y, a, b, c, d)$ by the following formula:

$$\begin{aligned} x < a < c \wedge x < b < d \wedge y < c \wedge y < d \wedge \text{Cycl}(x, a, y) \wedge \text{Cycl}(x, b, y) \\ \wedge \text{Cycl}(y, c, b) \wedge \text{Cycl}(y, d, a) \wedge \text{Cycl}(b, d, c) \wedge \text{Cycl}(a, c, d). \end{aligned}$$

We claim that $x \perp y$ is equivalent to $\exists a, b, c, d \psi(x, y, a, b, c, d)$. It is not hard to verify that $x \perp y$ implies $\exists a, b, c, d \psi(x, y, a, b, c, d)$. For the other direction note that $\psi(x, y, a, b, c, d)$ implies that $x \neq y$ because $\text{Cycl}(x, a, y)$ is part of the conjunction ψ .

Let us first assume that $x < y$ and $\psi(x, y, a, b, c, d)$ holds for some elements $a, b, c, d \in P$. Then $\text{Cycl}(x, a, y)$ implies that $a < y$, symmetrically we have $b < y$. Since $y < c, d$ we have that $a < d$ and $b < c$. Then $\text{Cycl}(b, d, c)$ implies $d < c$ and $\text{Cycl}(a, c, d)$ implies $c < d$, which is a contradiction.

Now assume that $y < x$ and $\psi(x, y, a, b, c, d)$ holds for some elements $a, b, c, d \in P$. Then we have $y < a, b$ by the transitivity of the order. Then $\text{Cycl}(y, c, b)$ implies $c < b$ and $\text{Cycl}(y, d, a)$ implies $d < a$. But this leads to the contradiction $a < c < b$ and $b < d < a$.

For the second part of the lemma let $s, t \in P$ be two elements with $s < t$. Then the set $X = \{x \in P : s < x < t\}$ is pp-definable in $(P; \text{Cycl}, s, t)$ by the formula $\phi(x) := \text{Cycl}(s, x, t)$. By a back-and-forth argument one can show the two structures $(X; \leq)$ and $(P; \leq)$ are isomorphic. The order relation, restricted to X is also pp-definable in $(P; \text{Cycl}, s, t)$ by the equivalence

$$y <_{|X} z \leftrightarrow \phi(x) \wedge \phi(z) \wedge \text{Cycl}(y, z, t).$$

Since \perp is pp-definable in $(P; <, \text{Cycl})$, we have that its restriction to X has a pp-definition in $(P; \text{Cycl}, s, t)$. Therefore also the relation $R = \{(x, y, z) \in X^3 : x \perp y \wedge x \perp z \wedge z \perp y\}$ is pp-definable in $(P; \text{Cycl}, s, t)$. Let $\psi(s, t, u, v, w)$ be a primitive positive formula defining R .

We claim that $\exists x, y \psi(x, y, u, v, w)$ is equivalent to $(u, v, w) \in \text{Par}$. Let $(u, v, w) \in \text{Par}$. The relation Par describes the orbit of a 3-element antichain under the action of $\langle \circ \rangle \subseteq \text{End}(P; \text{Cycl})$. So we can assume that (u, v, w) is a 3-antichain, otherwise we take an image under a suitable function from $\langle \circ \rangle$. Now let us take elements $s < t$ such that $s < uvw$ and $uvw < t$. Then clearly $\psi(s, t, u, v, w)$ has to hold.

Conversely let (s, t, u, v, w) be a tuple such that $\psi(s, t, u, v, w)$ holds. We can assume that $s < t$ (otherwise we take the image of (s, t, u, v, w) under a suitable function in $\langle \circ \rangle$). By what we proved above, (u, v, w) is antichain, hence it satisfies Par . □

Lemma 3.5.

1. $\text{End}(P; <, \perp) = \overline{\text{Aut}(\mathbb{P})}$
2. $\text{End}(P; \text{Betw}, \perp) = \overline{\langle \updownarrow \rangle}$
3. $\text{End}(P; \text{Cycl}) = \overline{\langle \circ \rangle}$
4. $\text{End}(P; \text{Sep}) = \overline{\langle \updownarrow, \circ \rangle}$

Proof.

1. Clearly $\text{Aut}(\mathbb{P}) \subseteq \text{End}(P; <, \perp)$. For the other inclusion let $f \in \text{End}(P; <, \perp)$. Let $A \subseteq P$ be an arbitrary finite set. The restriction of f to a finite subset $A \subseteq P$ is an isomorphism between posets. By the homogeneity of \mathbb{P} there is an automorphism $\alpha \in \text{Aut}(\mathbb{P})$ such that $f \upharpoonright A = \alpha \upharpoonright A$.
2. Since \updownarrow preserves Betw and \perp , we know that $\overline{\langle \updownarrow \rangle} \subseteq \text{End}(P; \text{Betw}, \perp)$ holds. For the opposite inclusion let $f \in \text{End}(P; \text{Betw}, \perp)$. If f preserves $<$, then $f \in \text{End}(P; <, \perp)$ and we are done. Otherwise there is a pair of elements $c_1 < c_2$ with $f(c_1) > f(c_2)$. Let $d_1 < d_2$ be an other pair of points in P . Then there are $a_1, a_2 \in P$ such that $c_1 < c_2 < a_1 < a_2$ and $d_1 < d_2 < a_1 < a_2$. Since f preserves Betw , $f(a_1) > f(a_2)$ holds and hence also $f(d_1) > f(d_2)$. So f inverts the order, while preserving \perp . Therefore $\updownarrow \circ f \in \text{End}(P; <, \perp)$. We conclude that $f \in \overline{\langle \updownarrow \rangle}$.
3. It is easy to see that $\overline{\langle \circ \rangle} \subseteq \text{End}(P; \text{Cycl})$. So let $f \in \text{End}(P; \text{Cycl})$. Clearly f is injective and preserves also the relation $\text{Cycl}'(x, y, z) := \text{Cycl}(y, x, z)$. By Lemma 3.4, f also preserves the relation Par . It follows that $\text{End}(P; \text{Cycl})$ also preserves the negation of Cycl . In other words, f is a self-embedding of $(P; \text{Cycl})$. So, when restricted to a finite $A \subset P$, f is a partial isomorphism. By the results in [28] we know that $(P; \text{Cycl})$ is a homogeneous structure. Hence

for every finite $A \subset P$ we find an automorphism $\alpha \in \text{Aut}(P; \text{Cycl}) = \langle \cup \rangle$ such that $f \upharpoonright A = \alpha \upharpoonright A$.

4. Let $f \in \text{End}(P; \text{Sep})$. We claim that either f or $\uparrow \circ f$ preserves Cycl. If we can prove our claim we are done by (3). First of all note that $\text{Sep}(x, y, z, u)$ implies $\text{Cycl}(x, y, z) \leftrightarrow \text{Cycl}(y, z, u)$.

Without loss of generality we can assume that there are elements $x, y, z \in P$ with $\text{Cycl}(x, y, z)$ and $\text{Cycl}(f(x), f(y), f(z))$; otherwise we look at $\uparrow \circ f$ instead of f . Let (r, s, t) be arbitrary tuple satisfying Cycl. We can always find elements $a < b < c$ in P that are incomparable with all entries of (x, y, z) and (r, s, t) . Further we can choose elements $u, v \in P$ that are incomparable with (a, b, c) such that $z < u < v$ and $\text{Sep}(x, y, z, u) \wedge \text{Sep}(y, z, u, v)$ holds. This can be done by a case distinction and is left to the reader. By construction we have

$$\text{Sep}(x, y, z, u) \wedge \text{Sep}(y, z, u, v) \wedge \text{Sep}(z, u, v, a) \wedge \text{Sep}(u, v, a, b) \wedge \text{Sep}(v, a, b, c).$$

So we have that $(f(x), f(y), f(z)) \in \text{Cycl}$ if and only if $(f(a), f(b), f(c)) \in \text{Cycl}$. Repeating the same argument for (r, s, t) gives us that $(f(r), f(s), f(t)) \in \text{Cycl}$. So f preserves Cycl. □

Now we can return to the actual proof of Proposition 3.2. Recall that we obtain an ordered homogeneous Ramsey structure $(P; \leq, \prec)$ given by the class of finite structures $(A; \leq, \prec)$, where $(A; \leq)$ is a partial order on A and \prec an extension of $<$ to a total order. We can regard this structure to be an expansion of \mathbb{P} by a total order. By Lemma 2.8 the following holds. Let $g: P \rightarrow P$ be an arbitrary function and $c_1, \dots, c_n \in P$ be any points. Then there exists a function $f: P \rightarrow P$ such that

1. $f \in \text{Aut}(\mathbb{P}) \circ g \circ \text{Aut}(\mathbb{P})$.
2. $g(c_i) = f(c_i)$ for $i = 1, \dots, n$.
3. Regarded as a function from $(P; \leq, \prec, \bar{c})$ to $(P; \leq)$, f is a canonical function.

Let Γ be a reduct of \mathbb{P} . We are going to study all feasible behaviours of a canonical function $f: (P; \leq, \prec, \bar{c}) \rightarrow (P; \leq)$ when $f \in \text{End}(\Gamma)$. Note that the behaviour of such f only depends on the behaviour on the 2-types because $(P; \leq, \prec, \bar{c})$ is homogeneous and its signature contains at most 2-ary relation symbols. Since there are only finitely many 2-types, the study of all possible behaviours of such canonical functions is a combinatorial problem. We introduce the following notation:

Notation 3.6. Let $A, B \subseteq P$ be definable subsets of $(P; \leq, \prec, \bar{c})$ and let $\phi_1(x, y), \dots, \phi_n(x, y)$ be formulas. We then let $p_{A, B, \phi_1, \dots, \phi_n}(x, y)$ denote the (partial) type determined by the formula $x \in A \wedge y \in B \wedge \phi_1(x, y) \wedge \dots \wedge \phi_n(x, y)$. Using this

notation, we can describe the 2-types of $(P; \leq, \prec, \bar{c})$. They are all of the form $p_{X,Y,\phi,\psi} = \{(a, b) \in P^2 : a \in X, b \in Y, \phi(a, b) \text{ and } \psi(a, b)\}$, where X and Y are 1-orbits of $\text{Aut}(P; \leq, \prec, \bar{c})$, $\phi \in \{=, <, >, \perp\}$ and $\psi \in \{=, \prec, \succ\}$. For a definable subset A of $(P; \leq, \prec, \bar{c})$ and functions f and g we are going to say that f behaves like g on A , if the restriction of their behaviour to all types involving only elements of A is equal.

If X and Y are two subsets of P we write $X \underset{<}{\perp} Y$ if there are pairs $(x, y), (x', y') \in X \times Y$ with $x < y$ and $x' \perp y'$. When it is convenient for us we will abuse notation and write \bar{c} to describe the set containing all entries of the tuple \bar{c} .

Observation 3.7. The structure $(P; \leq, \prec, \bar{c})$ is homogeneous. If X is an 1-orbit of $\text{Aut}(P; \leq, \prec, \bar{c})$ with infinitely many elements, we claim that the induced substructure $(X; \leq, \prec)$ is isomorphic to $(P; \leq, \prec)$. From the homogeneity of $(P; \leq, \prec, \bar{c})$ it follows that also $(X; \leq, \prec)$ is homogeneous. So to prove the claim, we only need to show that $(X; \leq, \prec)$ embeds all finite linearly extended partial orders $(A; \leq, \prec)$. Note that the type of the elements x of X is completely determined by a conjunction $\phi(x, \bar{c})$ of formulas of the form $x < c_i, x > c_k, x \perp c_l \wedge x \prec c_l, x \perp c_m \wedge x \succ c_m$. Now if some extended partial order $(A; \leq, \prec)$ would not embed into $(X; \leq, \prec)$, this would imply that the structure $(A \cup \bar{c}; \leq, \prec)$ with $\phi(a, \bar{c})$ for all $a \in A$ is not an extended partial order. However it is not hard to see that this only happens if already $(A; \leq, \prec)$ or $(\bar{c}; \leq, \prec)$ is not a linearly extended extended partial order - contradiction.

Similarly, if X and Y are 1-orbit of $(P; \leq, \prec, \bar{c})$ such that $X \underset{<}{\perp} Y$ holds, then one can show that $X \cup Y$ is isomorphic to $(P; \leq)$ with Y being a random filter. By $X \leq Y \leftrightarrow \exists x \in X, y \in Y : x \leq y$ we get a partial order on the 1-orbits of $(P; \leq, \prec, \bar{c})$ (cf. Lemma 18 of [27]). But note that the 1-orbits of $(P; \leq, \prec, \bar{c})$ are not necessarily linearly ordered by \prec : There can be infinite 1-orbits X, Y and $(x, y), (x', y') \in X \times Y$ with $x \prec y, x \perp y$ and $y' \prec x', x' \perp y'$.

In the following lemmas let Γ be always be a reduct of \mathbb{P} and let $f \in \text{End}(\Gamma)$ be a canonical function from $(P; \leq, \prec, \bar{c})$ to $(P; \leq)$.

Lemma 3.8. *Let X be a 1-orbit of $\text{Aut}(P; \leq, \prec, \bar{c})$ with infinitely many elements. Then f behaves like id or \uparrow on X , otherwise $\text{End}(\Gamma)$ contains a constant function, $g_{<}$ or g_{\perp} .*

Proof. Note that $(X; \leq, \prec)$ is isomorphic to $(P; \leq, \prec)$. Then the statement follows from the same arguments as in Lemma 8 of [27]. □

Lemma 3.9. *Let X, Y two infinite 1-orbits of $\text{Aut}(P; \leq, \prec, \bar{c})$ with $X \underset{<}{\perp} Y$. Assume f behaves like id on X . Then f behaves like id or \circlearrowleft_X on $X \cup Y$, otherwise $\text{End}(\Gamma)$ contains a constant function, $g_{<}$ or g_{\perp} .*

Proof. Since $X \perp_{<} Y$ there are at most three types of pairs in $X \times Y$, namely $p_{X,Y,<}$, $p_{X,Y,\perp,<}$ and $p_{X,Y,\perp,>}$. Assume that f does not contains a constant function, $g_{<}$ or g_{\perp} . Note that the union of X and Y is isomorphic to \mathbb{P} with X being a random filter of $X \cup Y$. By following the arguments of Lemma 22 in [27] one can show that we only have the two possibilities that

1. $f(p_{X,Y,<}) = p_{<}$ and $f(p_{X,Y,\perp,<}) = p_{\perp}$ or
2. $f(p_{X,Y,<}) = p_{\perp}$ and $f(p_{X,Y,\perp,<}) = p_{>}$.

By Lemma 3.8 we may assume that f behaves like id or \uparrow on Y . But if f behaves like \uparrow on Y , the image of $y_1, y_2 \in Y$ and $x \in X$ with $x \prec y_1 < y_2$, $x \perp y_1$ and $x < y_2$ would be a non partially ordered set. So if the type $p_{X,Y,\perp,>}$ is empty, f behaves like id or \circlearrowleft_X on $X \cup Y$ and we are done.

If $p_{X,Y,\perp,>}$ is not empty, there are $x \in X$ and $y \in Y$ with $x \succ y$ and $x \perp y$. We claim that in this case $f(p_{X,Y,\perp,>}) = f(p_{X,Y,\perp,<})$. We only prove this claim for (1), the proof for (2) is the same.

Assume that $f(p_{X,Y,\perp,>}) = p_{<}$. Then let $x' \in X$ be an element such that $y \prec x'$ and $x < x'$ and $y \perp x'$. The fact that such an element indeed exists can be verified by checking that extension of $\{x, y\} \cup \bar{c}$ by such an element x' still lies in the age of $(P; \leq, \prec, \bar{c})$. By our assumption we then have $f(x) < f(x') < f(y)$, which contradicts to $f(x) \perp f(y)$.

Now assume that $f(p_{X,Y,\perp,>}) = p_{>}$. Then let $x' \in X$ be such that $x \prec y \prec x'$ and $x < y$ and $x' \perp xy$. Again the fact that x' exists can be verified by the homogeneity of $(P; \leq, \prec, \bar{c})$. Then $f(x) < f(y) < f(x')$, which contradicts to $f(x') \perp f(x)$. \square

Lemma 3.10. *Either f behaves like id or \uparrow on every single 1-orbit of $\text{Aut}(P; \leq, \prec, \bar{c})$ or $\text{End}(\Gamma)$ contains a constant function, $g_{<}$ or g_{\perp} .*

Proof. For every two infinite orbits X, Y of $\text{Aut}(P; \leq, \prec, \bar{c})$ such that $X < Y$ there is a infinite orbit Z with $X \perp_{<} Z$ and $Z \perp_{<} Y$. For every two infinite orbits $X \perp Y$ there is an infinite orbit Z with $X < Z$ and $Y < Z$. So this statement holds by Lemma 3.9. (cf. Lemma 23 of [27]) \square

Lemma 3.11. *Assume $\text{End}(\Gamma)$ does not contains constant functions, $g_{<}$ or g_{\perp} . Then there is a $g \in \overline{(\circlearrowleft, \uparrow)} \cap \text{End}(\Gamma)$ such that $g \circ f$ is canonical from $(P; \leq, \prec, \bar{c})$ to $(P; \leq)$ and behaves like id on every set $(P \setminus \bar{c}) \cup \{c\}$, with $c \in \bar{c}$.*

Proof. By Lemma 3.10, f behaves like id or \uparrow on every infinite orbit. Without loss of generality we can assume that the first case holds, otherwise consider $\uparrow \circ f$.

Let $X \perp_{<} Y$, $Y \perp_{<} Z$ and $X \perp_{<} Z$ or $X < Z$. If f behaves like id on $X \cup Y$ and $Y \cup Z$ it also has to behave like id on $X \cup Z$; otherwise the image of a triple $(x, y, z) \in X \times Y \times Z$ with $x < y < z$ would not be partially ordered. Let $X < Z$,

$Y < Z$ and $X \perp Z$. Again, if f behaves like id on $X \cup Y$ and $Y \cup Z$ it also has to behave like id on $X \cup Z$, otherwise we get a contradiction.

By Lemma 3.9 f either behaves like id or like \circlearrowleft_X on the union of two orbits $X \perp_{<} Y$. In the second case $\circlearrowleft \in \text{End}(\Gamma)$. The set $A = \{x \in P : y < x \vee y \perp x \text{ for all } y \in f(Y)\}$ is a union of orbits of $\text{Aut}(P; \leq, \prec, \bar{c})$ and a random filter of P . So $\circlearrowleft_A \circ f$ is canonical and behaves like id on $X \cup Y$. Repeating this step finitely many times gives us a function $g \in \langle \circlearrowleft \rangle$ such that $g \circ f$ behaves like id on the union of infinite orbits, by the observations in the paragraph above.

It is only left to show that $g \circ f$ behaves like id between a given constant c in \bar{c} and an infinite orbit X . Assume for example that $c < X$ and $g \circ f(p_{c, X, <}) = p_{\perp}$. Let $A \subseteq P$ with $a \in A$. By homogeneity of \mathbb{P} we find an automorphism of \mathbb{P} that maps a to c and all points that are greater than a to X . If we then apply $g \circ f$ and repeat this process at most $|A|$ -times we can map A to an antichain. Thus $g_{\perp} \in \text{End}(\Gamma)$ which contradicts to our assumption.

Similarly all other cases where $g \circ f$ does not behave like id between c and X contradict our assumptions. We leave the proof to the reader. Hence $g \circ f$ behaves like id everywhere except on \bar{c} . □

Now we are ready to proof the main result of the section.

Proof of Proposition 3.2. Let Γ be a reduct of \mathbb{P} such that $\text{End}(\Gamma)$ does not contains constant functions, $g_{<}$ or g_{\perp} . We show that then $\text{End}(\Gamma)$ is equal to $\overline{\text{Aut}(\mathbb{P})}$, $\langle \updownarrow \rangle$, $\langle \circlearrowleft \rangle$ or $\langle \updownarrow, \circlearrowleft \rangle$.

First assume that $\text{End}(\Gamma)$ contains a non injective function. This can be witnessed by constants $c_1 \neq c_2$ and a function $f \in \text{End}(\Gamma)$ with $f(c_1) = f(c_2)$ that is canonical as function $f: (P; \leq, \prec, c_1, c_2) \rightarrow (P; \leq)$. By Lemma 3.11 we can assume that f behaves like id everywhere except from c_1, c_2 . But this is not possible, since there is a point in $a \in P$ with $a \perp c_1$ but $\neg(a \perp c_2)$. Since $f(c_1) = f(c_2)$ either $<$ or \perp is violated, which contradicts to f behaving like id everywhere except on $\{c_1, c_2\}$. So from now on let $\text{End}(\Gamma)$ only contain injective functions.

Assume $\text{End}(\Gamma)$ violates Sep. This can also be witnessed by a canonical function $f: (P; \leq, \prec, \bar{c}) \rightarrow (P; \leq)$ such that $\bar{c} \in \text{Sep}$ but $f(\bar{c}) \notin \text{Sep}$. By Lemma 3.11 we can assume that f behaves like id on every set $(P \setminus \bar{c}) \cup \{c\}$, with $c \in \bar{c}$. If there are $c_i < c_j$ with $f(c_i) \perp f(c_j)$ it is easy to see that $\text{End}(\Gamma)$ generates g_{\perp} which contradicts to our assumptions. If there are $c_i < c_j$ or $c_i \perp c_j$ with $f(c_i) > f(c_j)$ let a be an element of $(P \setminus \bar{c})$ with $a < c_j$ and $a \perp c_i$. Then the image of a, c_i, c_j under f induces a non partially ordered structure - contradiction.

So $\text{End}(\Gamma)$ preserves Sep. By Lemma 3.5 we know that $\text{End}(\Gamma) \subseteq \overline{\langle \updownarrow, \circlearrowleft \rangle}$. If $\text{End}(\Gamma)$ violates Cycl and Betw or Cycl and \perp we can proof as in the paragraph

above that $\text{End}(\Gamma) = \overline{\langle \downarrow, \circ \rangle}$. Similarly, if $\text{End}(\Gamma)$ preserves Cycl but violates Betw or \perp we can show that $\text{End}(\Gamma) = \overline{\langle \circ \rangle}$. If $\text{End}(\Gamma)$ preserves Betw and \perp but violates Cycl , then $\text{End}(\Gamma) = \overline{\langle \uparrow \rangle}$. Finally, if $\text{End}(\Gamma)$ preserves Betw , \perp and Cycl we have $\text{End}(\Gamma) = \overline{\text{Aut}(\mathbb{P})}$. \square

4 The case where $<$ and \perp are pp-definable

Throughout the remaining parts of this paper we are going to study the complexity of $\text{CSP}(\Gamma)$ for reducts Γ of \mathbb{P} that are model-complete cores. We start with the case where $\text{End}(\Gamma)$ is the topological closure of $\text{Aut}(\mathbb{P})$. In this case the two relations $<$ and \perp are pp-definable by Theorem 2.3. In the following we are going to discuss some of the CSPs that can appear then: In Section 4.1 we describe two classes of tractable problems. In Section 4.2 we show that whenever $\text{Low}(x, y, z) := (x < y \wedge z \perp xy) \vee (x < z \wedge y \perp xz)$ is pp-definable we have an NP-complete CSP. As we will prove later on in Section 5 these are in fact the only two cases that can appear.

Observation 4.1. The binary relation $x \perp\!\!\!\perp y$ defined by $x < y \vee x \perp y$ is equivalent to the primitive positive formula $\exists z (z < y \wedge z \perp x)$. Hence $x \perp\!\!\!\perp y$ is pp-definable in Γ .

4.1 Horn tractable CSPs given by $e_<$ and e_\leq

By $e_<$ we denote an embedding of the structure $(P; <)^2$ into $(P; <)$ and by e_\leq we denote an embedding of $(P; \leq)^2$ into $(P; \leq)$ (using the homogeneity of the random partial order it is not hard to verify that such maps indeed exists). Both the maps $e_<$ and e_\leq are canonical from $(P; \leq)^2 \rightarrow (P; \leq)$, we can even pick them to be canonical from $(P; \leq, \prec)^2 \rightarrow (P; \leq)$. They have the following behaviour on binary types:

$e_<$	$=$	$<$	$>$	\perp	e_\leq	$=$	$<$	$>$	\perp
$=$	$=$	\perp	\perp	\perp	$=$	$=$	$<$	$>$	\perp
$<$	\perp	$<$	\perp	\perp	$<$	$<$	$<$	\perp	\perp
$>$	\perp	\perp	$>$	\perp	$>$	$>$	\perp	$>$	\perp
\perp	\perp	\perp	\perp	\perp	\perp	\perp	\perp	\perp	\perp

We will show in this section that if one of them is a polymorphisms of Γ , then the problem $\text{CSP}(\Gamma)$ is tractable. In both cases $\text{CSP}(\Gamma)$ belongs to the class of Horn-tractable problems described in [10].

Let Δ and Λ be relational structures of the same signature. We say a map $h : \Delta \rightarrow \Lambda$ is a *strong homomorphism* if $\bar{x} \in R \leftrightarrow h(\bar{x}) \in R$. By $\hat{\Delta}$ we denote the expansion of Δ that contains the negation $\neg R$ for every R is in Δ .

Theorem 4.2 (from [10]). *Let Δ be an ω -categorical structure and let Γ be a reduct of Δ . Suppose $\text{CSP}(\hat{\Delta})$ is tractable. If Γ has a polymorphism that is a strong homomorphism from Δ^2 to Δ , then also Γ is tractable. \square*

By definition $e_<$ is a strong homomorphism from $(P; <)^2 \rightarrow (P; <)$ and e_\leq is a strong homomorphism from $(P; \leq)^2 \rightarrow (P; \leq)$. Let $\not<$ respectively $\not\leq$ denote the negation of the order relation $<$ respectively \leq . One can see that every input to $\text{CSP}(P; <, \not<)$ and $\text{CSP}(P; \leq, \not\leq)$ is accepted as long as it does not contradict to the transitivity of $<$ respectively \leq . But this can be checked in polynomial time, thus the two problems are tractable. So by Theorem 4.2 every template Γ with polymorphism $e_<$ or e_\leq gives us a tractable problem.

In addition we can also give a syntactic characterisation of these tractable problems via Horn formulas, we refer also to [10] for the proof.

Proposition 4.3. *Let Γ be a reduct of \mathbb{P} . Suppose that $e_\leq \in \text{Pol}(\Gamma)$. Then $\text{CSP}(\Gamma)$ is tractable and every relation in Γ is equivalent to Horn formula in $(P; \leq)$:*

$$\begin{aligned} x_{i_1} \leq x_{j_1} \wedge x_{i_2} \leq x_{j_2} \wedge \dots \wedge x_{i_k} \leq x_{j_k} &\rightarrow x_{i_{k+1}} \leq x_{j_{k+1}} \text{ or} \\ x_{i_1} \leq x_{j_1} \wedge x_{i_2} \leq x_{j_2} \wedge \dots \wedge x_{i_k} \leq x_{j_k} &\rightarrow \perp \end{aligned}$$

Suppose that $e_< \in \text{Pol}(\Gamma)$. Then $\text{CSP}(\Gamma)$ is tractable and every relation in Γ is equivalent to a Horn formula in $(P; <)$, i.e. a formula of the form:

$$\begin{aligned} x_{i_1} \triangleleft_1 x_{j_1} \wedge x_{i_2} \triangleleft_2 x_{j_2} \wedge \dots \wedge x_{i_k} \triangleleft_k x_{j_k} &\rightarrow x_{i_{k+1}} \triangleleft_{k+1} x_{j_{k+1}} \text{ or} \\ x_{i_1} \triangleleft_1 x_{j_1} \wedge x_{i_2} \triangleleft_2 x_{j_2} \wedge \dots \wedge x_{i_k} \triangleleft_k x_{j_k} &\rightarrow \perp, \end{aligned}$$

where $\triangleleft_i \in \{<, =\}$ for all $i = 1, \dots, k + 1$. \square

4.2 The NP-hardness of Low

Let Low be the ternary relation defined by

$$\text{Low}(x, y, z) := (x < y \wedge z \perp xy) \vee (x < z \wedge y \perp xz).$$

It is not hard to see that \perp and $<$ are pp-definable in Low, hence $(\mathbb{P}; \text{Low})$ is a model-complete core with endomorphism monoid $\overline{\text{Aut}(\mathbb{P})}$. In this section we prove the NP-hardness of $\text{CSP}(P; \text{Low})$.

Lemma 4.4. *Let us define the relations*

$$\begin{aligned} \text{Abv}(x, y, z) &:= (x > y \wedge xy \perp z) \vee (x > z \wedge xz \perp y) \\ U(x, y, z) &:= (y < x \vee z < x) \wedge (y \perp z) \end{aligned}$$

Then Abv and U are pp-definable in Low.

Proof. Note that the formula $\phi(x, y, z, v) := \exists u (u \perp v \wedge \text{Low}(u, y, z) \wedge \text{Low}(y, x, v) \wedge \text{Low}(z, x, v))$ holds if and only if $v \perp x$ and $y \perp z$, y is smaller than exactly one element of $\{x, v\}$, z is smaller than exactly one element of $\{x, v\}$, and v is not greater than both z and y . With that in mind one can see that $\exists v_1, v_2 (\phi(x, y, z, v_1) \wedge \phi(x, y, v_2, z))$ is equivalent to $\text{Abv}(x, y, z)$ and $\exists v \phi(x, y, z, v)$ is equivalent to $U(x, y, z)$. \square

Proposition 4.5. *Let $a, b \in P$ with $a \perp b$. There is a pp-interpretation of the structure $(\{0, 1\}; \text{NAE}, 0, 1)$ in $(P; \text{Low}, a, b)$. Thus $\text{CSP}(P; \text{Low})$ is NP-hard.*

Proof. Let $D := \{x \in P : \text{Low}(x, a, b)\}$, $D_0 := \{x \in D : x < a\}$, $D_1 := \{x \in D : x < b\}$. Note that $D_0 \perp D_1$. Let $I : D \rightarrow \{0, 1\}$ be given by:

$$I(x) := \begin{cases} 0 & \text{if } x \in D_0 \\ 1 & \text{if } x \in D_1 \end{cases}.$$

Clearly the domain D of I is pp-definable in $(P; \text{Low}, a, b)$. Since the order relation $<$ is pp-definable in Low also the sets D_0 and D_1 are pp-definable. Let $R = \{(x, y, z, t) \in P^4 : (x > y \vee x > z \vee x > t) \wedge y \neq z \wedge y \neq t \wedge z \neq t\}$. We claim that the relation R is pp-definable in Low . Observe that $(x, y, z, t) \in R$ is equivalent to

$$\exists u, v (\text{Abv}(x, u, v) \wedge U(x, y, u) \wedge U(x, z, u) \wedge U(x, t, v) \wedge y \neq z \wedge y \neq t \wedge z \neq t).$$

Inequality is pp-definable in Low by the fact that $(P; \text{Low})$ is a model-complete core. Therefore R is pp-definable in Low by Lemma 4.4. By the definition of R we have that $I(c_1, c_2, c_3) \in \text{NAE}$ if and only if $(a, c_1, c_2, c_3) \in R$ and $(b, c_1, c_2, c_3) \in R$. Thus the preimage of NAE is pp-definable in $(P; \text{Low}, a, b)$. \square

5 Violating the relation Low

We saw in Proposition 4.3 that $\text{CSP}(\Gamma)$ is tractable if $e_{<}$ or e_{\leq} is a polymorphism of Γ . By Proposition 4.5 we know that $\text{CSP}(\Gamma)$ is NP-complete if Low is pp-definable in Γ . In this section we are going to show that these results already cover all possible reducts where $<$ and \perp are pp-definable.

Proposition 5.1. *Let Γ be a reduct of \mathbb{P} such that \perp and $<$ are pp-definable in Γ . Then Low is not pp-definable in Γ if and only if $\text{Pol}(\Gamma)$ contains one of the functions $e_{<}$ or e_{\leq} .*

Proof outline. Note that by Theorem 2.3, Low is not pp-definable in Γ if and only if there is a binary $g \in \text{Pol}(\Gamma)$ violating Low . In other words, there are tuples

$(a, b, c) \in \text{Low}$ and $(a', b', c') \in \text{Low}$, such that $(g(a, a'), g(b, b'), g(c, c')) \notin \text{Low}$. Without loss of generality we can assume that $a < b \wedge ab \perp c$. Then $a' < c' \wedge a' c' \perp b'$ has to hold, otherwise this would contradict to the fact that $<$ and \perp are preserved under $\text{Pol}(\Gamma)$. By the homogeneity of \mathbb{P} there is an automorphism $\beta \in \text{Aut}(\Gamma)$ mapping a to a' , b to c' and c to b' . Let us define the polymorphism $f(x, y) := g(x, \beta(y))$.

Then we either have $f(a, a) < f(b, c) \wedge f(a, a) < f(c, b)$ or $f(a, a) \perp f(b, c) \wedge f(a, a) \perp f(c, b)$. Only these two cases appear since f preserves $\frac{1}{2}$ and \perp , see Observation 4.1. Furthermore we can assume that $a \prec b \prec c$; otherwise, by the homogeneity of \mathbb{P} , we can find an automorphism $\alpha \in \text{Aut}(\mathbb{P})$ such that $\alpha(a) \prec \alpha(b) \prec \alpha(c)$. Then we consider the map $(x, y) \mapsto f(\alpha^{-1}(x), \alpha^{-1}(y))$ with constants $\alpha(a)$, $\alpha(b)$ and $\alpha(c)$ instead of a, b, c .

By Lemma 2.8 we can assume that f is canonical as a function from $(P; <, \prec, a, b, c)^2$ to $(P; <)$. We are going to show that the existence of such a canonical function implies that $\text{Pol}(\Gamma)$ contains $e_{<}$ or e_{\leq} . In Subsection 5.1 we start with the analysis, by studying canonical functions from $(P; <, \prec)^2$ to $(P; <)$, which will turn out to be helpful further on. We will then prove Proposition 5.1 for the two cases in Lemma 5.10 and Lemma 5.17 respectively. \square

5.1 Canonical binary functions on $(P; \leq, \prec)$

A first step in analysing the binary part of $\text{Pol}(\Gamma)$ is to look at the special case of canonical functions. So in the following subsection we are going to study the behaviour of binary functions $f \in \text{Pol}(\Gamma)$ that are canonical seen as functions from $(P; \leq, \prec)^2$ to $(P; \leq)$. We are going to specify conditions for which $\text{Pol}(\Gamma)$ contains $e_{<}$ or e_{\leq} .

Definition 5.2. Let $f: \mathbb{P}^2 \rightarrow \mathbb{P}$ be a function. Then f is called *dominated by the first argument* if

- $f(x, y) < f(x', y')$ for all $x < x'$ and
- $f(x, y) \perp f(x', y')$ for all $x \perp x'$.

We say f is *dominated* if either f or $(x, y) \mapsto f(y, x)$ is dominated by the first argument.

In this subsection we are going to prove the following lemma:

Lemma 5.3. *Let Γ be a reduct of \mathbb{P} in which $<$ and \perp are pp-definable. Let $f(x, y) \in \text{Pol}(\Gamma)$ be canonical when seen as a function from $(P; \leq, \prec)^2$ to $(P; \leq)$. Then at least one of the following cases holds:*

- f is dominated
- $\text{Pol}(\Gamma)$ contains $e_{<}$

- $\text{Pol}(\Gamma)$ contains e_{\leq}

First of all we make some general observations for binary canonical functions preserving $<$ and \perp . We are again going to use the notation introduced in Notation 3.6. Let us fix a function $- : (P; \leq, <) \rightarrow (P; \leq, <)$ such that $x < y \leftrightarrow -y < -x$ holds. It is easy to see that such a function exists.

Lemma 5.4. *Let $f : (P; \leq, <)^2 \rightarrow (P; \leq)$ be canonical and $f \in \text{Pol}(\Gamma)$. Then the following statements hold:*

1. $f(p_{<}, p_{<}) = p_{<}$, $f(p_{\perp}, p_{\perp}) = p_{\perp}$
2. $f(p, q) = -f(-p, -q)$, for all types p, q .
3. $f(p_{<}, p_{\perp, <})$, $f(p_{<}, p_{\perp, >})$, $f(p_{\perp, >}, p_{<})$ and $f(p_{\perp, <}, p_{<})$ can only be equal to $p_{<}$ or p_{\perp} .
4. At least one of $f(p_{<}, p_{\perp, <})$ and $f(p_{\perp, <}, p_{<})$ is equal to p_{\perp} .
5. At least one of $f(p_{<}, p_{\perp, >})$ and $f(p_{\perp, <}, p_{>})$ is equal to p_{\perp} .
6. It is not possible that $f(p_{<}, p_{>}) = p_{=}$ holds.
7. $f(p_{\perp, <}, p_{<}) = p_{\perp} \rightarrow f(p_{\perp}, p_{=}) = p_{\perp}$

Proof.

1. This is clear, since f is a polymorphism of Γ and hence preserves $<$ and \perp .
2. This is true by definition of $-$.
3. This is true since f preserves the relation $\frac{\perp}{<}$, see Observation 4.1.
4. Assume $f(p_{<}, p_{\perp, <}) = f(p_{\perp, <}, p_{<}) = p_{<}$. Let $a_1 < a_2 < a_3$ with $a_1 \perp a_2$, $a_3 \perp a_1 a_2$ and $b_1 < b_2 < b_3$ with $b_2 < b_3$, $b_1 \perp b_2 b_3$. By our assumption $f(a_1, b_1) < f(a_2, b_2) < f(a_3, b_3)$ holds, which contradicts to f preserving \perp .
5. This can be proven similarly to (4).
6. Assume that $f(p_{<}, p_{>}) = p_{=}$ holds. Let $a_1 < a_2 < a_3$ with $a_1 < a_3$, $a_2 < a_3$, $a_1 \perp a_2$ and $b_1 > b_2 > b_3$ with $b_1 > b_3$, $b_2 > b_3$, $b_1 \perp b_2$. Then $f(a_1, b_1) \perp f(a_2, b_2)$ but also $f(a_1, b_1) = f(a_3, b_3) = f(a_2, a_2)$ have to hold, which is a contradiction.
7. Assume that there are $a_1 \perp a_2$ and b such that $f(a_1, b) \leq f(a_2, b)$ holds. Then we take elements a_3 and b' with $a_2 < a_3$, $a_1 \perp a_3$, $a_1 < a_3$ and $b' > b$. Then $f(a_1, b) \leq f(a_2, b) < f(a_3, b')$ holds. But $f(p_{\perp, <}, p_{<}) = p_{\perp}$ implies that $f(a_1, b) \perp f(a_3, b')$, a contradiction!

□

By Lemma 5.4 (2) we only have to consider pairs of types where the first entry is $p_{=}$, $p_{<}$ or $p_{\perp, <}$ when studying the behaviour of f . Further Lemma 5.4 implies that $f(x, y) \neq f(x', y')$ always holds for $x \neq x'$ and $y \neq y'$.

Lemma 5.5. *Let $f \in \text{Pol}(\Gamma)$. Then the following are equivalent:*

1. $f(p_{<}, p_{>}) = p_{<}$

2. $f(p_{<}, p_{\perp, \succ}) = p_{<}$
3. $f(p_{<}, q) = p_{<}$ for all 2-types q
4. f is dominated by the first argument

Proof. It is clear that the implications (4) \rightarrow (3) \rightarrow (2) and (3) \rightarrow (1) are true.

(1) \rightarrow (3): Let $a_1 < a_2 < a_3$ and $b_1 b_3 < b_2$. Then $f(a_1, b_1) < f(a_2, b_2) < f(a_3, b_3)$ has to hold regardless if the type of (b_1, b_3) is $p_{\perp, \prec}$, $p_{\perp, \succ}$ or $p_{=}$. So $f(p_{<}, q) = p_{<}$ for all 2-types q .

(2) \rightarrow (1): Let $a_1 < a_2 < a_3$ and $b_1 \succ b_2 \succ b_3$ with $b_1 > b_3$, $b_2 \perp b_1 b_3$. Then $f(a_1, b_1) < f(a_2, b_2) < f(a_3, b_3)$ implies $f(a_1, b_1) < f(a_3, b_3)$ and so $f(p_{<}, p_{>}) = p_{<}$.

(3) \rightarrow (4): We have to consider all the pairs of 2-types where the first entry is $p_{\perp, \prec}$. By Lemma 5.4 (4) and (5) we know that $f(p_{\perp, \prec}, p_{<}) = f(p_{\perp, \prec}, p_{>}) = p_{\perp}$. From Lemma 5.4(7) follows that $f(p_{\perp}, p_{=}) = p_{\perp}$.

We want to point out that we did not require f to be canonical; it can be easily verified that all proof steps also work for general binary functions. □

Lemma 5.6. *Let $f: (P; \leq, \prec)^2 \rightarrow (P; \leq)$ be canonical and $f \in \text{Pol}(\Gamma)$. If f is not dominated the following statements hold:*

1. $f(p_{<}, p_{>}) = f(p_{<}, p_{\perp, \succ}) = f(p_{\perp, \prec}, p_{>}) = p_{\perp}$.
2. $f(p_{<}, p_{=}) = p_{<}$ or $f(p_{<}, p_{=}) = p_{\perp}$.
3. $f(p_{\perp, \prec}, p_{=}) = p_{\perp}$ or $f(p_{\perp, \prec}, p_{=}) = p_{<}$.

Proof.

1. is a direct consequence of Lemma 5.5.
2. Suppose there are $a_1 < a_2$ and b such that $f(a_1, b) \geq f(a_2, b)$. Then we take elements $a_3, b' \in P$ with $a_2 \perp a_3$, $a_2 \succ a_3$, $a_1 < a_3$ and a $b' > b$. Then $f(a_2, b) \leq f(a_1, b) < f(a_3, b')$ holds, which is a contradiction to $f(a_2, b) \perp f(a_3, b')$.
3. Assume that there are $a_1 \perp a_2$, $a_1 \prec a_2$ and b such that $f(a_1, b) \geq f(a_2, b)$ holds. There are elements a_3 and b' with $a_2 > a_3$, $a_1 \perp a_3$, $a_1 \prec a_3$ and $b' < b$. Then $f(a_2, b) > f(a_3, b')$ and $f(a_1, b) \perp f(a_3, b')$. But this contradicts to our assumption. □

Definition 5.7. Let us say a binary function is \perp -falling, if it has the same behaviour as $e_{<}$ respectively e_{\leq} on pairs of partial type (p_{\neq}, p_{\neq}) .

Lemma 5.8. *Let $f \in \text{Pol}(\Gamma)$ be a canonical function $f: (P; \leq, \prec)^2 \rightarrow (P; \leq)$ of \perp -falling behaviour. Then $\text{Pol}(\Gamma)$ contains $e_{<}$ or e_{\leq} .*

Proof. From Lemma 5.4 (7) follows that $f(p_{\perp}, p_{=}) = p_{\perp}$ and $f(p_{=}, p_{\perp}) = p_{\perp}$. By Lemma 5.6 we further know that $f(p_{<}, p_{=}), f(p_{=}, p_{<}) \in \{p_{\perp}, p_{<}\}$. So we have to do a simple case distinction:

- If $f(p_-, p_<) = f(p_<, p_-) = p_\perp$, then f behaves like $e_<$, hence $e_< \in \text{Pol}(\Gamma)$.
- If $f(p_-, p_<) = p_<$ and $f(p_<, p_-) = p_\perp$, the function $(x, y) \rightarrow f(f(x, y), x)$ has the same behaviour as $e_<$, thus $e_< \in \text{Pol}(\Gamma)$.
- Symmetrically if $f(p_-, p_<) = p_\perp$ and $f(p_<, p_-) = p_<$, the function $(x, y) \rightarrow f(f(y, x), y)$ has the same behaviour as $e_<$, thus $e_< \in \text{Pol}(\Gamma)$.
- If $f(p_-, p_<) = p_< = f(p_<, p_-) = p_<$, then f has the same behaviour as e_{\leq} , thus $e_{\leq} \in \text{Pol}(\Gamma)$.

□

We now give a criterion for the existence of a canonical \perp -falling function in $\text{Pol}(\Gamma)$. This criterion will allow us to finish the proof of Lemma 5.3 and also help in the proof of Proposition 5.1.

Lemma 5.9. *Assume that for every $k > 1$, every pair of tuples $\bar{a}, \bar{b} \in P^k$ and every indices $p, q \in \{1, 2, \dots, k\}$ with $a_p < a_q$ and $\neg(b_p \leq b_q)$ there exists a binary function $g \in \text{Pol}(\Gamma)$ such that $g(a_p, b_p) \perp g(a_q, b_q)$ and for all $i, j \in \{1, 2, \dots, k\}$:*

1. $a_i < a_j$ implies $g(a_i, b_i) < g(a_j, b_j)$ or $g(a_i, b_i) \perp g(a_j, b_j)$,
2. $a_i \perp a_j$ implies $g(a_i, b_i) \perp g(a_j, b_j)$.

Then $\text{Pol}(\Gamma)$ contains $e_<$ and e_{\leq} .

Proof. We only need to show that for all $\bar{a}, \bar{b} \in P^k$ there is a binary function $f \in \text{Pol}(\Gamma)$ that is \perp -falling on (\bar{a}, \bar{b}) . To be more precise we want to construct an $f \in \text{Pol}(\Gamma)$ such that:

- $f(a_i, b_i) < f(a_j, b_j)$ if $a_i < a_j$ and $b_i < b_j$,
- $f(a_i, b_i) \perp f(a_j, b_j)$ if $a_i < a_j$ and $\neg(b_p \leq b_q)$.
- $f(a_i, b_i) \perp f(a_j, b_j)$ if $a_i \perp a_j$ and $b_i \neq b_j$

By a compactness argument there exists a $h \in \text{Pol}(\Gamma)$ that is \perp -falling on P^2 . By Lemma 2.8 this function generates a canonical functions, which then clearly also has to be \perp -falling. By Lemma 5.8 we have that $e_<$ or e_{\leq} is an element of $\text{Pol}(\Gamma)$. In order to prove the above claim let $\bar{a}, \bar{b} \in P^k$ be arbitrary tuples. We are going to construct f by a recursive argument.

Let $f^{(0)}(x, y) = g^{(0)}(x, y) = x$ and $\bar{a}^{(0)} = f^{(0)}(\bar{a}, \bar{b})$. If already $f^{(0)}$ has the desired properties we set $f(x, y) = f^{(0)}(x, y)$ and are done. Otherwise, in the $(k+1)$ -th recursion step, we are given a function $f^{(k)}(x, y)$ and a tuple $\bar{a}^{(k)} = f^{(k)}(\bar{a}, \bar{b})$. Let us assume that there are indices p, q with $a_p < a_q$, $\neg(b_p \leq b_q)$ and $a_p^{(k)} < a_q^{(k)}$. Then by our assumption there is a function $g^{(k+1)}(x, y) \in \text{Pol}(\Gamma)$ such that $g^{(k+1)}(a_p^{(k)}, b_p^{(k)}) \perp g^{(k+1)}(a_q^{(k)}, b_q^{(k)})$. We set $f^{(k+1)}(x, y) = g^{(k+1)}(f^{(k)}(x, y), y)$ and $\bar{a}^{(k+1)} = f^{(k+1)}(\bar{a}, \bar{b})$.

Note that by the properties (1) and (2) of the function g^k the only possible cases for f^k being not \perp -falling is the case above. It is clear that the recursion ends after finitely many steps. \square

Proof of Lemma 5.3. Let $f: (P; \leq, \prec)^2 \rightarrow (P; \leq)$ be canonical and $f \in \text{Pol}(\Gamma)$. Let us assume that f is not dominated. By Lemma 5.6 we know $f(p_{<}, p_{>}) = f(p_{<}, p_{\perp, >}) = f(p_{\perp, <}, p_{>}) = p_{\perp}$.

By Lemma 5.4 (3) and (4) we have to look at the following cases:

1. $f(p_{<}, p_{\perp, <}) = f(p_{\perp, <}, p_{<}) = p_{\perp}$.
2. $f(p_{<}, p_{\perp, <}) = p_{<}$ and $f(p_{\perp, <}, p_{<}) = p_{\perp}$.
3. $f(p_{<}, p_{\perp, <}) = p_{\perp}$ and $f(p_{\perp, <}, p_{<}) = p_{<}$.

In the first case f has \perp -falling behaviour therefore we are done by Lemma 5.8. For the remaining cases we can restrict ourselves to (2), otherwise we take the function that maps (x, y) to $f(y, x)$. From Lemma 5.4 (7) follows that $f(p_{\perp}, p_{=}) = p_{\perp}$. Thus $f(p_{\perp}, q) = p_{\perp}$ holds for every 2-type q .

We are going to show that then the conditions in Lemma 5.9 are satisfied. Let $\bar{a}, \bar{b} \in P^k$ be two tuples of arbitrary length k and let $p, q \in \{1, 2, \dots, k\}$ such that $a_p < a_q$, $b_p \prec b_q$ and $b_p \perp b_q$ hold. Then let $\alpha \in \text{Aut}(\mathbb{P})$ with $\alpha(b_p) \succ \alpha(b_q)$. Such an automorphism exists by the homogeneity of \mathbb{P} . Then we set $g(x, y) = f(x, \alpha(y))$. Clearly $g(a_p, b_p) \perp g(a_q, b_q)$, since $\alpha(b_p) \succ \alpha(b_q)$. Also the other conditions in Lemma 5.9 are satisfied, by the properties of f . Therefore $\text{Pol}(\Gamma)$ contains $e_{<}$ or e_{\leq} . \square

5.2 $f(a, a) < f(b, c) \wedge f(a, a) < f(c, b)$

The aim of this subsection is to prove the following lemma.

Lemma 5.10. *Let $f \in \text{Pol}(\Gamma)$ be canonical as a function from $(P; <, \prec, a, b, c)^2$ to $(P; <)$. If $f(a, a) < f(b, c) \wedge f(a, a) < f(c, b)$ then $\text{Pol}(\Gamma)$ contains $e_{<}$ or e_{\leq} .*

We are going to prove Lemma 5.10 by contradiction; so assume that Γ contains such an f , but neither $e_{<}$ nor e_{\leq} . By Lemma 5.3 we can assume that every binary function in $\text{Pol}(\Gamma)$, which is canonical from $(P; <, \prec)$ to $(P; <)$, has to be dominated. Every infinite 1-orbit of $\text{Aut}(P; <, \prec, a, b, c)$ induces a substructure of $(P; <, \prec)$ that is isomorphic to $(P; <, \prec)$ by Observation 3.7. Thus every restriction of f to an infinite 1-orbit of $\text{Aut}(P; <, \prec, a, b, c)$ has to be dominated, which motivates the following notation:

Notation 5.11. Let $f: P^2 \rightarrow P$ be a function and X, Y, X', Y' be subsets of P such that both the restriction of f to $X \times Y$ and to $X' \times Y'$ are dominated. We say that f has the *same domination* on $X \times Y$ and $X' \times Y'$ if f is dominated by the first argument on both $X \times Y$ and $X' \times Y'$ or dominated by the second argument

on both $X \times Y$ and $X' \times Y'$. Otherwise, we say that f has *different domination* on $X \times Y$ and $X' \times Y'$.

We define the following two sets:

- $B_1 := \{x \in P : x > c \wedge x \perp a \wedge x \perp b\}$,
- $B_2 := \{x \in P : x > b \wedge x > c\}$.

Then observe that B_1 and B_2 are both 1-orbits of $\text{Aut}(P; <, \prec, a, b, c)$ with infinitely many elements. By Observation 3.7 both $(B_1; \leq, \prec)$ and $(B_2; \leq, \prec)$ are isomorphic to $(P; \leq, \prec)$. Furthermore $B_1 \stackrel{\perp}{<} B_2$, thus by Observation 3.7 also the union of B_1 and B_2 is an isomorphic copy of $(P; \leq, \prec)$, in which B_1 forms a random filter.

Lemma 5.12. *The restriction of f to $B_i \times B_j$ is dominated for every $i, j \in \{1, 2\}$.*

Proof. For a contradiction we assume that the restriction of f to some $B_i \times B_j$ is not dominated. Since $(B_i; \leq, \prec)$ and $(B_j; \leq, \prec)$ are isomorphic to $(P; \leq, \prec)$ there are $\alpha : P \rightarrow B_i$ and $\beta : P \rightarrow B_j$ such that α is an isomorphism from $(P; \leq, \prec)$ to $(B_i; \leq, \prec)$ and β is an isomorphism from $(P; \leq, \prec)$ to $(B_j; \leq, \prec)$. Let $g : P^2 \rightarrow P$ be given by $g(x, y) := f(\alpha(x), \beta(y))$. Such g would be canonical from $(P; \leq, \prec)$ to $(P; \leq)$ and not dominated, which contradicts to Lemma 5.3. \square

Lemma 5.13. *f has the same domination on all sets $B_i \times B_j$, $i, j \in \{1, 2\}$.*

Proof. We claim that f has the same domination on $B_1 \times B_k$ and $B_2 \times B_k$ for any $k \in \{1, 2\}$. For a contradiction we assume that f does not have the same domination on $B_1 \times B_k$ and $B_2 \times B_k$. Without loss of generality we can assume that f is dominated by the first argument on $B_1 \times B_k$ and dominated by the second argument on $B_2 \times B_k$. Let $x, y \in B_1, z, t \in B_2$ be such that $x < y \wedge y < z \wedge x \perp t$. Let $x', y', z', t' \in B_k$ be such that $x' \perp t' \wedge y' < z' \wedge z' < t'$. Since f is dominated by the first argument on $B_1 \times B_k$ we have $f(x, x') < f(y, y')$. Since f is dominated by the second argument on $B_2 \times B_k$ we have $f(z, z') < f(t, t')$. Since f preserves $<$ we have $f(y, y') < f(z, z')$. Thus $f(x, x') < f(t, t')$, a contradiction to the fact that f preserves \perp .

By considering the map $(x, y) \mapsto f(y, x)$ we have that f has the same domination on $B_k \times B_1$ and $B_k \times B_2$ for every $k \in \{1, 2\}$. This implies that f has the same dominations on all products $B_i \times B_j, i, j \in \{1, 2\}$. \square

In the rest of this section we assume that f is dominated by the first argument on $B_i \times B_j$ for every $i, j \in \{1, 2\}$. The other case can be reduced to this case by considering the map $(x, y) \mapsto f(y, x)$.

Lemma 5.14. *Let $u, v \in B_1$ and $u' \in B_2, v' \in B_1$ be such that $u < v \vee u \perp v$. Then $f(u, u') \perp f(v, v')$.*

Proof. First we claim that $f(u, u') > f(v, v') \vee f(u, u') \perp f(v, v')$. For a contradiction we assume that $f(u, u') \leq f(v, v')$. Since f preserves $<$ we have $f(c, b) < f(u, u')$. Therefore $f(a, a) < f(c, b) < f(u, u') < f(v, v')$, a contradiction to the \perp -preservation of f . Thus the claim follows.

The proof is completed by showing that $f(u, u') > f(v, v')$ is impossible. For a contradiction we assume that $f(u, u') > f(v, v')$. Let $s, t \in B_1$ be such that $s \perp t \wedge s < v \wedge u < t$. Let $s' \in B_1, t' \in B_2$ be such that $s' \perp t'$. By the domination of f we have $f(s, s') < f(v, v') \wedge f(u, u') < f(t, t')$. It follows from $f(u, u') > f(v, v')$ that $f(s, s') < f(t, t')$, a contradiction to \perp -preservation of f . \square

Lemma 5.15. *Let $u, v \in B_1$ be such that $u \perp v$. Then for every $u', v' \in B_1 \cup B_2$ we have $f(u, u') \perp f(v, v')$.*

Proof. For a contradiction we assume that $\neg(f(u, u') \perp f(v, v'))$. Without loss of generality we assume that $f(u, u') \leq f(v, v')$. Let $s, t \in B_1$ be such that $s < u \wedge v < t \wedge s \perp t$. Let $s', t' \in B_1 \cup B_2$ be such that $s' \perp t'$, s', u' are in the same orbit and t', v' are in the same orbit. By the domination of f we have $f(s, s') < f(u, u') \wedge f(v, v') < f(t, t')$. Since $f(u, u') < f(v, v')$ we have $f(s, s') < f(t, t')$, a contradiction to the \perp -preservation of f . \square

Lemma 5.16. *Let $u, v \in B_1$ and $u', v' \in B_1 \cup B_2$ be such that $u < v$. Then $f(u, u') < f(v, v') \vee f(u, u') \perp f(v, v')$.*

Proof. For a contradiction we assume that $f(v, v') \leq f(u, u')$. Let $s, t \in B_1$ be such that $t < v \wedge u < s \wedge s \perp t$. Let $s', t' \in B_1 \cup B_2$ be such that $s' \perp t'$, s', u' are in the same orbit and t', v' are in the same orbit. By the domination of f we have $f(t, t') < f(v, v') \wedge f(u, u') < f(s, s')$. Since $f(v, v') < f(u, u')$ we have $f(t, t') < f(s, s')$, a contradiction to the \perp -preservation of f . \square

Now we are ready for the proof of Lemma 5.10:

Proof of Lemma 5.10. We are going to show that $\text{Pol}(\Gamma)$ contains a function that behaves like $e_<$ or like e_\leq by checking the conditions of Lemma 5.9.

So let $\bar{a}, \bar{b} \in P^k$ with $a_p < a_q$ and $\neg(b_p \leq b_q)$. We set $Y := \{b_i : b_i \geq b_p\}, Z := \{b_i : \neg(b_i \geq b_p)\}$. By definition we have $b_q \in Z$. By the homogeneity of \mathbb{P} there is $\alpha \in \text{Aut}(\mathbb{P})$ such that $\alpha(Y) \subseteq B_2$ and $\alpha(Z) \subseteq B_1$. Let $\beta \in \text{Aut}(\mathbb{P})$ such that $\beta(\{a_i : i \in \{1, 2, \dots, k\}\}) \subseteq B_1$. Let $g(x, y) := f(\beta(x), \alpha(y))$. Clearly $g \in \text{Pol}(\Gamma)$. By Lemma 5.14 we have that $g(a_p, b_p) \perp g(a_q, b_q)$. Further we know by Lemma 5.16

that $g(a_i, b_i) < g(a_j, b_j)$ or $g(a_i, b_i) \perp g(a_j, b_j)$ holds for all $a_i < a_j$. By Lemma 5.15 we know that $g(a_i, b_i) \perp g(a_j, b_j)$ holds for all $a_i \perp a_j$. So the conditions of Lemma 5.9 are satisfied. Hence $e_{<}$ or e_{\leq} is a polymorphism of Γ . \square

5.3 $f(a, a) \perp f(b, c) \wedge f(a, a) \perp f(c, b)$

The aim of this subsection is to prove the following.

Lemma 5.17. *Let $f \in \text{Pol}(\Gamma)$ be canonical as a function from $(P; <, \prec, a, b, c)^2$ to $(P; <)$. If $f(a, a) \perp f(b, c) \wedge f(a, a) \perp f(c, b)$, then $\text{Pol}^{(2)}(\Gamma)$ contains $e_{<}$ or e_{\leq} .*

We define the following sets.

$$B_1 := \{x \in P : a < x < b \wedge x \perp c\}$$

$$B_2 := \{x \in P : x < b \wedge x < c \wedge x \perp a \wedge x \prec a\}.$$

Throughout the lemmata and corollaries below in this section we assume that every binary canonical function in Γ is dominated and $f(a, a) \perp f(b, c) \wedge f(a, a) \perp f(c, b)$. Observe again that by Observation 3.7 the structure $(B_1 \cup B_2; \leq, \prec)$ is isomorphic to $(P; \leq, \prec)$ with B_2 being a random filter. By Lemma 5.3 we can again assume that every binary canonical function in $\text{Pol}(\Gamma)$ is dominated.

Lemma 5.18. *f is dominated when restricted to $B_i \times B_j$ and has the same domination on all $B_i \times B_j, i, j \in \{1, 2\}$.*

Proof. This can be shown as in the proof of Lemma 5.12 and Lemma 5.13. \square

In the rest of this section we assume that f is dominated by the first argument on $B_i \times B_j$ for every $i, j \in \{1, 2\}$. Similarly to Lemma 5.14 we have the following.

Lemma 5.19. *Let $u, v \in B_1$ and $u' \in B_1, v' \in B_2$ be such that $u < v \vee u \perp v$. Then $f(u, u') \perp f(v, v')$.*

Proof. First we prove that $f(v, v') < f(u, u') \vee f(v, v') \perp f(u, u')$. For a contradiction we assume that $f(u, u') \leq f(v, v')$. Since $a < u \wedge a < u'$ we have $f(a, a) < f(u, u')$. Since $v < b \wedge v' < c$ we have $f(v, v') < f(b, c)$. Thus $f(a, a) < f(b, c)$, a contradiction to the fact that $f(a, a) \perp f(b, c)$. Thus $f(v, v') < f(u, u') \vee f(v, v') \perp f(u, u')$.

The proof is completed by showing that $f(u, u') > f(v, v')$ is impossible. For a contradiction we assume that $f(u, u') > f(v, v')$. Let $s, t \in B_1$ be such that $s \perp t \wedge s < v \wedge u < t$. Let $s' \in B_2, t' \in B_1$ be such that $s' \perp t'$. By the domination of f we have $f(s, s') < f(v, v') \wedge f(u, u') < f(t, t')$. It follows from $f(u, u') > f(v, v')$ that we have $f(s, s') < f(t, t')$, a contradiction to \perp -preservation of f . \square

Lemma 5.20. *Let $u, v \in B_1$ be such that $u \perp v$. Then for every $u', v' \in B_1 \cup B_2$ we have $f(u, u') \perp f(v, v')$.*

Proof. analogous to Lemma 5.15. □

Lemma 5.21. *Let $u, v \in B_1$ and $u', v' \in B_1 \cup B_2$ be such that $u < v$. Then $f(u, u') < f(v, v') \vee f(u, u') \perp f(v, v')$.*

Proof. analogous to Lemma 5.16. □

Now we are ready for the proof of Lemma 5.17

Proof of Lemma 5.17. We are again going to show that $\text{Pol}(\Gamma)$ contains a function that behaves like $e_{<}$ or like e_{\leq} by checking the conditions of Lemma 5.9.

So let $\bar{a}, \bar{b} \in P^k$ with $a_p < a_q$ and $\neg(b_p \leq b_q)$. We set $Y := \{b_i : b_i \geq b_p\}$ and $Z := \{b_i : \neg(b_i \geq b_p)\}$. By definition we have $b_q \in Z$. By the homogeneity of \mathbb{P} there is an $\alpha \in \text{Aut}(\mathbb{P})$ such that $\alpha(Y) \subseteq B_1$ and $\alpha(Z) \subseteq B_2$. Let β be an automorphism such that $\beta(\{a_i : i \in \{1, 2, \dots, k\}\}) \subseteq B_1$ and let us define $g(x, y) := f(\beta(x), \alpha(y))$. Clearly $g \in \text{Pol}(\Gamma)$.

By Lemma 5.14 we have that $g(a_p, b_p) \perp g(a_q, b_q)$. Further by Lemma 5.16 we know that $g(a_i, b_i) < g(a_j, b_j)$ or $g(a_i, b_i) \perp g(a_j, b_j)$ holds for all $a_i < a_j$. By Lemma 5.15 we know that $g(a_i, b_i) \perp g(a_j, b_j)$ holds for all $a_i \perp a_j$. So the conditions of Lemma 5.9 are satisfied. Hence $e_{<}$ or e_{\leq} is a polymorphism of Γ . □

6 The NP-hardness of Betw, Sep and Cycl

In our analysis of reducts Γ of \mathbb{P} we are only left with the cases where $\text{End}(\Gamma)$ is equal to $\overline{\langle \updownarrow \rangle}$, $\overline{\langle \circ \rangle}$ or $\overline{\langle \updownarrow, \circ \rangle}$. By Lemma 3.5 and Theorem 2.3 in those cases one of the relations Betw, Sep or Cycl is pp-definable in Γ . In this section we are going to show that these relations induce NP-complete CSPs.

Interestingly, we can treat all cases similarly: By fixing finitely many constants c_1, \dots, c_n on Γ we obtain definable subsets of $(\Gamma, c_1, \dots, c_n)$ on which $<$ and Low are pp-definable. This enables us to reduce every such case to the NP-completeness of Low.

Lemma 6.1. *Let $u, v \in P$ with $u < v$. Then the relations $<$ and Low are pp-definable in $(P, \text{Betw}, \perp, u, v)$.*

Proof. It is easy to verify that there is a pp-definition of the strict order relation $x < y$ by the following formula:

$$\exists a, b (\text{Betw}(a, u, v) \wedge \text{Betw}(u, v, b) \wedge \text{Betw}(a, x, b) \wedge \text{Betw}(a, y, b) \wedge \text{Betw}(a, x, y)).$$

The two maps $e_{<} : P^2 \rightarrow P$ and $e_{\leq} : P^2 \rightarrow P$ do not preserve Betw , since for every triple $\bar{a} = (a_1, a_2, a_3)$ with $a_1 < a_2 < a_3$ and $\bar{b} = (b_1, b_2, b_3)$ with $b_1 > b_2 > b_3$, the image of (\bar{a}, \bar{b}) forms an antichain. By Proposition 5.1 we have that Low is pp-definable in $(P, \text{Betw}, \perp, u, v)$. \square

Lemma 6.2. *Let c, d be two constants in P such that $c < d$. Then there is a pp-interpretation of $(P; \text{Low})$ in $(P; \text{Cycl}, c, d)$*

Proof. Let $X := \{x \in P : c < x < d\}$. By using back-and-forth argument one can show easily that $(P; <)$ and $(X; <|_X)$ are isomorphic. We first show that X (as a unary predicate) and $<|_X$ are pp-definable in $(P; \text{Cycl}, c, d)$. It is easy to verify that the set X can be defined in $(P; \text{Cycl}, c, d)$ by $\phi(x) := \text{Cycl}(c, x, d)$ and that $x <|_X y \leftrightarrow \phi(x) \wedge \phi(y) \wedge \text{Cycl}(c, x, y)$. Now a pp-interpretation of $(P; <, \text{Cycl})$ in $(P; \text{Cycl}, c, d)$ is simply given by the identity on X .

By Lemma 3.4 we have that \perp is pp-definable in $(P; <, \text{Cycl})$. It is easy to verify that $e_{<}$ and e_{\leq} do not preserve Cycl . Therefore, by Proposition 5.1, Low is pp-definable in $(P; <, \text{Cycl})$, which concludes the proof of the Lemma. \square

Lemma 6.3. *Let c, d, u be constants in P such that $c < d < u$. Then $(P; \text{Low})$ has a pp-interpretation in $(P; \text{Sep}, c, d, u)$.*

Proof. Let $X := \{x \in P : d < x < u\}$. By using a back-and-forth argument, one can show easily that $(X; \leq)$ and \mathbb{P} are isomorphic. Similarly as in the proof of Lemma 6.2, X and $<|_X$ are pp-definable in $(P; \text{Sep}, c, d, u)$ as follows.

The set X can be defined by the formula $\phi(x) := \text{Sep}(c, d, x, u)$, and $<|_X$ can be defined by $x <|_X y :\Leftrightarrow \phi(x) \wedge \phi(y) \wedge \text{Sep}(c, d, x, y)$. Also $\text{Cycl}(x, y, z)|_X$ can be defined by the primitive positive formula $\phi(x) \wedge \phi(y) \wedge \phi(z) \wedge \text{Sep}(c, x, y, z)$. So a pp-interpretation of $(P; <, \text{Cycl})$ in $(P; \text{Sep}, c, d, u)$ is simply given by the identity, restricted to X . By Lemma 6.2, Low is pp-definable in $(P; <, \text{Cycl})$, which concludes the proof of the Lemma. \square

Note that by the transitivity of pp-interpretations and Proposition 4.5 also $(\{0, 1\}, \text{NAE}, 0, 1)$ has a pp-interpretation in $(P; \text{Betw})$, $(P; \text{Cycl})$ and $(P; \text{Sep})$ extended by finitely many constants.

7 Main result and discussion

In this section we summarize the proof of the complexity dichotomy stated in Theorem 1.1 and give criteria that allow us to decide whether a given reduct Γ of \mathbb{P} has a tractable or an NP-hard CSP. Furthermore we phrase a stronger, structural

dichotomy on the reducts of the random partial order that is in accordance with the algebraic dichotomy conjecture for CSPs over finitely bounded homogeneous structures stated by Bodirsky and Pinsker.

7.1 A complexity dichotomy

The main challenge of this paper was to determine the complexity for reducts of \mathbb{P} that are model-complete cores. We were able to show that the following dichotomy holds:

Theorem 7.1. *Let Γ be a reduct of \mathbb{P} in a finite relational language that is a model-complete core. Then either*

- *one of the relations Low, Betw, Cycl, Sep is pp-definable in Γ and $\text{CSP}(\Gamma)$ is NP-complete or*
- *$\text{CSP}(\Gamma)$ is in P.*

Proof. If Low, Betw, Cycl or Sep are pp-definable in Γ , then $\text{CSP}(\Gamma)$ is NP-complete by Proposition 4.5 and Lemma 6.1, 6.2 and 6.3. By Theorem 3.2 the only remaining case is the one, where $<$ and \perp are pp-definable, but Low is not. In this case $e_{<}$ or e_{\perp} is a polymorphism of Γ by Proposition 5.1. Proposition 4.3 then implies that the problem is tractable. \square

As an immediate consequence of Theorem 7.1 and Corollary 3.3 we get the proof of the dichotomy in Theorem 1.1. However even more holds, since we obtained a finite list of relations that describe all NP-complete CSPs: We can algorithmically determine whether for a given set Φ the problem $\text{Poset-SAT}(\Phi)$ is NP-complete or in P.

Corollary 7.2. *Let Γ be a reduct of \mathbb{P} in a finite relational language. Under the assumption $P \neq NP$ the problem $\text{CSP}(\Gamma)$ is either NP-complete or solvable in polynomial time. Further the “meta-problem” of determining the complexity of $\text{CSP}(\Gamma)$ for a give Γ is decidable.*

Proof. By Proposition 3.2 we know that either Γ is a model-complete core or is has a constant function, $g_{<}$ or g_{\perp} as endomorphism. In the first case the dichotomy holds by Theorem 7.1. In the other cases Γ is homomorphically equivalent to a reduct of $(\mathbb{Q}, <)$ and the dichotomy holds by the dichotomy result in [13] respectively [12], see Corollary 3.3.

In order to decide the “meta-problem” note first that it is decidable to tell whether Γ is a model-complete core or not: we just need to check if all its relations are preserved by a constant function, $g_{<}$, or g_{\perp} . If Γ is not a model-complete core

we can decide the meta-problem due to the results in [13]. In the other case the main result of [19] implies that it is decidable to tell whether Low, Betw, Cycl or Sep are pp-definable in Γ . By Theorem 7.1 these are the only cases in which $\text{CSP}(\Gamma)$ is NP-complete. Hence the meta-problem is decidable. \square

7.2 An algebraic dichotomy

To state the structural dichotomy, we would like to recall that all complexity reductions we used were of three types: For finite or ω -categorical structures Δ and Γ the complexity of $\text{CSP}(\Delta)$ reduces to $\text{CSP}(\Gamma)$ if

1. Δ is the model-complete core of Γ .
2. Γ is a model-complete core and Δ is obtained by adding finitely many constants to the signature of Γ .
3. Δ is pp-interpretable in Γ .

It was conjectured by Bodirsky and Pinsker that a CSP of a reduct Γ of finitely bounded homogeneous structures is NP-complete if and only if $(\{0, 1\}; \text{NAE}, 0, 1)$ and consequently *all* finite structures can be reduced to Γ by the reductions (1)-(3) and that $\text{CSP}(\Gamma)$ is in P otherwise (see for instance [8]).

As pp-definability is characterised by inclusion of clones, pp-interpretation can be characterized by clone homomorphisms. In particular Γ pp-interprets the structure $(\{0, 1\}; \text{NAE}, 0, 1)$ if and only if there is a uniformly continuous clone homomorphism to $\mathbf{1} := \text{Pol}(\{0, 1\}; \text{NAE}, 0, 1)$, the clone consisting of all projections on a two element set (see [17]). It was shown in [6] that also the opposite case, where no stabiliser of the model-complete core maps homomorphically to the projection clone $\mathbf{1}$, has a positive algebraic characterisation: In this case the clone $\text{Pol}(\Delta)$ satisfies a non-trivial equational condition, namely the existence of a pseudo Siggers operation.

In our special case of the reducts of the random partial order we are able to give an additional equational conditions that are equivalent to the existence of a pseudo Siggers operation: In our analysis we saw that in the only tractable model-complete cases the binary operations $e_{<}$ or e_{\leq} appear. These that are symmetric modulo composition with embeddings from the outside.

Furthermore $(\{0, 1\}; \text{NAE}, 0, 1)$ can be reduced to Δ by the complexity reductions (1)-(3) *in arbitrary order* if and only if there is a uniformly continuous h1-clone homomorphism to $\mathbf{1}$ (see [5]). This lead to another conjecture stating that the only source of NP-hardness for $\text{CSP}(\Gamma)$ is that there is a uniformly continuous h1-clone homomorphism from $\text{Pol}(\Gamma)$ to $\mathbf{1}$. In the following theorem we summarize all the algebraic characterisations of the tractable cases of reducts of \mathbb{P} :

Theorem 7.3. *Let Γ be a reduct of \mathbb{P} and let Δ be the model-complete core of Γ . Then the following are equivalent:*

1. For all $c_1, \dots, c_n \in \Delta$ there is no uniformly continuous clone homomorphism from $\text{Pol}(\Delta)$ onto $\mathbf{1}$.
2. There is a pseudo Siggers polymorphism, i.e. a function $f \in \text{Pol}(\Delta)^{(6)}$ and endomorphism $e_1, e_2 \in \text{End}(\Delta)$ such that for all x, y, z :

$$e_1(f(x, y, x, z, y, z)) = e_2(f(y, x, z, x, z, y)).$$

3. There is a binary $f \in \text{Pol}(\Delta)$ and endomorphisms $e_1, e_2 \in \text{End}(\Delta)$ such that for all x, y :

$$e_1(f(x, y)) = e_2(f(y, x))$$

or there is a ternary $f \in \text{Pol}(\Delta)$ and endomorphisms $e_1, e_2, e_3 \in \text{End}(\Delta)$ such that for all x, y :

$$e_1(f(x, x, y)) = e_2(f(x, y, x)) = e_3(f(y, x, x)).$$

4. There is no uniformly continuous h1-clone homomorphism from $\text{Pol}(\Gamma)$ to $\mathbf{1}$.
5. There is no h1-clone homomorphism from $\text{Pol}(\Gamma)$ to $\mathbf{1}$.

Proof. For all ω -categorical structures (1) and (2) are equivalent by the results of [6], furthermore they are equivalent to (4) for reducts of finitely bounded homogeneous structures by [4].

If Γ is not a model-complete core, then the equivalence of (1) to (3) follows from the analysis in [13]. If Γ is a model-complete core for which (1) holds, by our analysis it contains the binary operations $f = e_{<}$ or $f = e_{\leq}$. But it is not hard to verify that those operations satisfy the equation $e_1(f(x, y)) = e_2(f(y, x))$ for some $e_1, e_2 \in \text{End}(\Gamma)$.

At last the equivalence of (4) to (5) is shown in [4], where explicit linear equations are constructed from $e_{<}$ and e_{\leq} that prevent h1-clone homomorphisms to $\mathbf{1}$. □

We finish with an algebraic version of our dichotomy that is a direct implication of our complexity analysis and Theorem 7.3:

Corollary 7.4. *Let Γ be a reduct of \mathbb{P} in a finite relational language and let Δ be its model-complete core. Then either*

- all finite structures are pp-interpretable in Δ extended by finitely many constant and $\text{CSP}(\Gamma)$ is NP-complete; or
- one of the conditions (1)-(6) in Theorem 7.3 holds and $\text{CSP}(\Gamma)$ is in P. □

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ABDUCTION IN AKKADIAN MEDICAL DIAGNOSIS

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Abstract

Ancient medical diagnosis has been studied from different perspectives. Analysis and translations of texts done by Asiriologists and Physicians shed the light on ancient practices. Although their works is amazing, several aspects remain mysterious. I propose here to study Akkadian medical diagnosis from the perspective of philosophy and argumentation, and to compare it with the inferences at stake in modern medical diagnosis. If we focus on inference, without any preconceived thinking about rationality or irrationality of the Mesopotamians, whether we talk about science or magic regarding ancient texts, it seems that the two inferences at stake have the same structure. Modern medical diagnosis, in some cases or some phases, fits better within an abductive reasoning than an inductive or a deductive one. The same holds for several examples of Ancient medical diagnosis I will put forward in this paper. In these cases, we face abductive inferences that work as ignorance preserving reasoning. A number of problems are left unsolved and a deeper study of the inference in Ancient Mesopotamian texts would help us understand how the medical practice in early medical texts in History works.

1 Ancient Medical Practice

When we talk about Ancient medicine, we are usually referring to Greek medicine as the start point of medical care in the History. Almost everyone would agree that the Father of Medicine was the Greek Hippocrates. Most of the work attributed to Hippocrates has been written between 430 and 330 B.C., and later [25]. Nevertheless,

I am grateful to the two anonymous referees and the editors for their constructive inputs. I thank Matthieu Fontaine, Teresa Lopez-Soto and Angel Nepomuceno for their comments, corrections and help and the participants of the conferences Issues in Medical Epistemology, Cologne, December 14-16, 2017 and Explanatory Practices: Interaction, Dialogues, and Cognitive Processes Workshop, Lisbon, February 15-16, 2018.

*Sponsored by VPPI-US (Contrato de acceso al Sistema Español de Ciencia, Tecnología e Innovación para el desarrollo del programa propio I+D+i de la Universidad de Sevilla)

there are texts of medical practice in Mesopotamia far before the most known text of Hippocrates or Galen. It is true that there are no treatises of medicine as such in Babylonian medicine (in the modern sense of medical treatise) and the imputation of the concept of disease to Mesopotamian physicians is perhaps an anachronism [1]. These are questions that still need to be answered and studied deeply. Despite different concepts, there is a clear medical practice in Mesopotamia. In fact, the first time the physician (in Sumerian *a-zu* or AZU) is attested is in texts as early as Fāra period (mid-third mill. B.C.) and the first medical prescriptions seem to appear in the Ur III period (end- third mill B.C.). There are many medical texts during the Mesopotamian period.

The ancient medicine in Mesopotamia has been widely studied [1, 19, 13]. The main source has been the Diagnostic Handbook [16, 24], written in Akkadian language (a Semitic syllabic language written in cuneiform that also uses Sumerian pictograms). This is a medical treatise created in Babylon in the middle of the eleventh century B.C. and recopied through the first millenium B.C. as part of the cuneiform tradition (see Heessel, N.P. in [19]). We also have different comments on the Diagnostic Handbook that try to specify or complete the medical diagnosis compendium [21, 14], these Uruk commentaries help the *asûm*-physician/*āšipûtum* or *mašmaššum* (different names for practitioners involved in medical diagnosis) to better understand the symptoms ([13] p. 147), for example. All these medical diagnosis texts are usually written as omen with conditional structures and in clay tablet. The many thousands of tablets have survived because of the durability of the clay, which makes Mesopotamian texts better documented than any other ancient society.

A question to be asked is whether Ancient medicine can be considered a science or a magical practice. In fact, Mesopotamian medical practice is completely “contaminated” by magical practices. The therapeutic medicine in the Akkadian texts seems to be complemented with magical medicine and both aspects coexist. Sometimes, the physician *asûm* is called to treat the patient with the figure of exorcist *mašmaššum* or *wāšipum* and the diviners *baarum*¹ Sometimes the illness is considered as a divine punishment for a transgression, other times from a contact with an ill man or an animal bite [22]. Usually, the name of the illness is of the form “Hand of a God”, even if it does not mean a supernatural origin. Diagnostic texts are organized as other divination texts. The presentation of symptoms follows the identification of the illness and/or the prognosis. Regarding the pharmacological texts, they are organized as mathematical texts, with the list of procedures. In the tablets, they are sometimes mixed with incantations, too.

So, with this information in mind, we could consider that the Mesopotamian

¹See “Maladies et mediciens” in [10].

medicine is more a magical practice than a rational one. In fact, it has usually been said that Babylonians have hardly indulged in theory and that they have limited themselves to practical speculation about the physical world, making no reference to hypothetical generalizations or methodological rules. Babylonian medicine is un-scientific and relates to magical practice. The problem has always been how to recognize Babylonian epistemology and theory with the biased sources at our disposal ([13] p. 11-12). Nevertheless, the question remains whether Babylonian medicine should be considered as unscientific as it might seem.

Authors such as Geller[13] state that Babylonian medicine is actually a "science". According to him, three prerequisites to scientific thinking - i.e. imagination, logical deduction and observation² - are also present in Ancient medicine. However, my point is not to discuss scientificity properly, but rather rationality. By understanding medical diagnosis, not in terms of deduction and/or induction reasoning, but in terms of abductive reasoning, we grasp similarities between Ancient and Modern diagnosis. In other words, despite their differences, Ancient and Modern diagnoses rely on the same inferences, namely abductions. In fact the medicine in Babylon is a real practice, they do not have a theory as the Greek medicine, they experiment even if they do not realize clinical trials. For that, I will quote Claude Bernard that explain clearly the medical practice:

“Un médecin qui observe une maladie dans diverses circonstances, qui raisonne sur l’influence de ces circonstances, et qui en tire des conséquences qui se trouvent contrôlées par d’autres observations; ce médecin fera un raisonnement expérimental quoiqu’il ne fasse pas d’expériences. Mais s’il veut aller plus loin et connaître le mécanisme intérieur de la maladie, il aura affaire à des phénomènes cachés, alors il devra expérimenter; mais il raisonnera toujours de même” [4],p. 46³.

If we focus on the inference, and not in the magical aspect, does the Babylonian medical argumentation really differ from the medical diagnosis that we know nowadays? How to differentiate the ailment from the symptoms? What is the status of the symptoms in relation with health and wellness (there are also treatises of health and wellness in Babylon, physiognomy[6])⁴? How to chose a hypothetical disease?

²I will not enter into details about this characteristics. More information, see [13], p. 10 and ff.

³My translation: A physician who observes a disease in several circumstances, who reasons on the influence of these circumstances, and who draws consequences which are controlled by other observations; this doctor will make an experimental reasoning even though he does not experiment. But if he wants to go further and know the inner mechanism of the disease, he will have to deal with hidden phenomena, so he will have to experiment; but he will always reason anyway.

⁴Notice that for authors as Claude Bernard medical practice needs physiology, pathology and

Is the inference in modern medicine so different from the one in Mesopotamia? For example, it is just the formulation of the hypothesis that changes from malaria to the ⁴LUGAL.GÌR.RA⁵? How to design one hypothesis or another? The background theory, the medical knowledge and/or the cultural aspect? Do we face a difference between rational and irrational or a difference between hypothesis and information at the moment of the inference?

I will study the inference in Mesopotamian medical diagnosis from a descriptive viewpoint. In what follows, I will explain the inference at stake in modern medical diagnosis and why I consider it as an abductive inference. Then, I will put forward different examples of Akkadian medical diagnosis and I will confront them with abductive reasoning for medical diagnosis⁶. At the end I will analyse the possibility to differentiate the modern general physician diagnosis and the Akkadian diagnosis at the inference level.

2 Inference in Medical Diagnosis

Diagnosis is the art/science of recognizing possible diseases⁷ from their symptoms/signs⁸, distinguishing between them and indicating its prognosis and treatment [23, 20]. We could differentiate two methods of medical diagnosis⁹. One is based

therapeutics. Of course the ancient medical practices differs from the modern one, but there are these three parts in Akkadian medicine too. There are treatises of health and wellness, the pathology as ancient etiology and the treatment. “Por embrasser le problem medicale dans son entier, la médecine expérimentale doit comprendre trois parties fondamentales: la physiologie, la pathologie et la thérapeutique” ([4], p. 26) (To embrace the medical problem as a whole, experimental medicine must include three fundamental parts: physiology, pathology and therapeutics, my translation). I just want to enfatize that we are in front of a medical practice not a theoretical approach to medicine. Nevertheless, as I said before, these aspects of ancient medicine as science or experimental science need to be treated in depth and it is beyond the propouse of this paper.

⁵Name attributed to cerebral malaria in Mesopotamia, it is a god associated with the god Nergal, the god of pestilence. See example 3 in this paper.

⁶I want to clarify that this is a philosophical approach, not a Semitician approach. There are really good studies of Mesopotamian medicine made by Orientalistic researcher, even in collaboration with modern doctors. See for example [39]. My point in this paper is to study the medical diagnosis in Akkadian from another perspective. Nevertheless, I will use the translation and studies made by Assyriologists as a basis of my analysis.

⁷I will not enter into debate about the concept of disease and its subcategories. For the moment, I will use the general concept, even if it is not so clear. Nevertheless, a deeper study will be necessary to take into account these different subcategories as for example disease, illness or sickness. See [40] for more details.

⁸I will not establish a difference between signs and symptoms for the moment.

⁹I do not mention the pathophysiological reasoning used by other sciences as biophysics, genetics, etc. because I will not use it for approaching Ancient Near Eastern medicine.

on probabilistic induction and even if it could be successful, it is very hardly used directly in everyday medical practice [23, 20]. The other is based on causal reasoning for etiological diagnosis, which is the attribution of an observed symptom/sign to a cause identified [20]. This medical diagnosis has often been modeled as hypothetical-deductive process [7]: the doctors generate a set of diagnostic hypotheses and test them by gathering further data. Unless the problem is solved, they generate new hypotheses and test them again. However, the methodology would be better understood in term of non deductive inference. I will use as an example the case of malaria diagnosis. A and B instantiate a common subject matters.

A = fever, headache, shivering, vomiting, jaundice, convulsions. . .

B = malaria.

__ → __ = if__ then__

Here: If A, then B.

- So if a patient has symptoms A, then he has the ailment B. From the viewpoint of deduction, the reasoning would be like $A \rightarrow B$, A / B

But it would be more accurate to see the following: if a patient has malaria (B) he will have the symptoms (A).

- In fact, in medical diagnosis, the reasoning is rather of the following form: $B \rightarrow A$, A / B . If a patient has malaria (B), he has the symptoms (A). He has the symptoms (A), so he has malaria (B).

If we consider that this reasoning is a deduction, it would be no more than a fallacy, namely the fallacy of affirming the consequent to put it Aristotle's words. Indeed, it might be the case that a patient had the symptoms described, but not the malaria, e.g. yellow fever, enteric fever, etc. Nevertheless, this kind of reasoning is useful, from the perspective of the economy of reasoning or when we seek explanation. Hence, if this reasoning should be accepted, it is according to another criteria of correct inference, i.e., a non-deductive criteria. In that case, the hypothetical and defeasible status of the conclusion should be emphasized. Indeed, in our example, we might be led to revise the conclusion that the patient has malaria on the basis of new information (e.g. if he has enteric fever).

In fact, it seems that we are in front of a reasoning that has been called Abduction, since Peirce [33]. Abduction is a kind of reasoning based on hypothesis. It is a defeasible reasoning because what we obtain is just a conjecture. We are not sure of the result and it can be questioned or denied, in contrast to deduction that has a sure conclusion. Abduction follows, in Peirce words, the next schema:

“The surprising fact, C, is observed;

But if A were true, C would be a matter of course,

Hence, there is reason to suspect that A is true". (Peirce CP 5.189)[33]

The aim of this kind of reasoning is to explain surprising facts. E.g., I see that the road is wet. If it had been raining, this would have explained it. So, I abductively infer it has been raining. However, the explanation might be different, e.g. the gardener of the city hall might have watered the road. This is a defeasible reasoning, we could go back because the conclusion is not definitive, it is a defeasible hypothesis. Abduction is triggered by a question arising from a lack of information, a question for which the agent has no answer in the background theory. The description of a fact is not an explanation by itself; thus, we need a hypothesis that, together with the background theory, would provide the explanation. The following schema explains the abductive process¹⁰:

1. $T!Q(\alpha)$
2. $\sim (R(K, T))$ [Fact]
3. $\sim (R(K*, T))$ [Fact]
4. $H \notin K$ [Fact]
5. $H \notin K*$ [Fact]
6. $\sim R(H, T)$ [Fact]
7. $\sim R(K(H), T)$ [Fact]
8. $H \rightsquigarrow R(K(H), T)$ [Fact]
9. H satisfies conditions S_1, \dots, S_n [Fact]
10. Then, $C(H)$ [Sub-conclusion, 1-7]
11. Then, H^C [Conclusion 1-8]

Here, the first issue is the starting point that triggers the abductive process, $T!Q(\alpha)$. Then, an unanswered question remains: which illness produces this situation? T

¹⁰I will follow G-W schema for abduction because I consider that this schema give us more features to analyze the process than the standard schema. For example, the emphasis on defeasibility and the importance of the action. All these characteristic features of what Woods calls "third way reasoning", see [42, 11, 3, 28]. Besides, this abduction as inference from the best explanaton, ignorance preserving inference and activation of the hypothesis fits better with Akkadian diagnosis. I am interested in abduction as a process from a pragmatic perspective.

would be something that we would like to reach, in order to solve the problem, and α would be the answer, the name of the malady. In fact, we are in front of an ignorance problem. We do not know for sure what is causing the situation, in our case what is triggering the vomiting, shivering, etc. This is a cognitive irritant, unpleasant situation that we would like to solve. We are in a lack of knowledge situation and we would like to overcome it, nevertheless we do not find a solution. Even if we do not have the solution, we need to act, to do something, to overcome the situation. We try to do something for the patient. Here, the *step 2* which explains that the current knowledge is not enough to solve the problem. The *step 3* means that there is no immediate successor of K that helps us to solve the ignorance problem. If we would have this, we would just add some knowledge and we would solve the problem. It would be something like checking and adding new information (maybe by making some tests with modern methods, we would know that we are in front of a malaria ailment). In the case that new information would be added, we would not be in an abductive process. In fact, we would be in what Woods[42] calls subduance: new knowledge would remove the initial ignorance. If, for example, we decide to give up the problem, we will be in a surrender point. Abduction is triggered when we are in a situation in which we do not have any answer, we still are in a lack of knowledge, but we still want to continue. Then, to solve or better say, to overcome the ignorance, we set a plausible solution which is a hypothesis. So, we hypothesize something, but this is not a solution, it is just a hypothesis (*step 6* and *7*). This is why the hypothesis relates only subjunctively to the cognitive-target (*step 8*). In this case, we hypothesize the malaria illness and if this would be the case, it could be an acceptable solution. The subjunctive relation means that If H was the case, it would provide a solution. But abduction does not consist in providing a new knowledge, only a hypothesis. *Step 9* represents that some conditions could be added to the hypothesis¹¹. So we conjecture the hypothesis that the patient has malaria ($C(H)$). Nevertheless, we do not stop here because we activate the hypothesis H^C and we act in consequence by using it in further reasoning, we give a prognosis and a treatment. Here, it is what Woods calls full abduction, while if we stop in *step 10* we will have a partial abduction. If for example, we make the test to know if we are in front of a malaria ailment and this confirms our hypothesis, we will stop in a partial abduction and we will have a similar situation that we had in K^* . New information is added¹². This will not trigger a full abduction because we do not act in a ignorance-preserving way. Other situation would be the not-confirmation

¹¹Aliseda, for example, adds the following conditions: plain, consistent, explanatory, minimal and preferential, see [2] .

¹²See also the ST-Model where it is explained how induction and deduction inferences are used at this point. More details in [26]. See also section 5 below.

and at this point, we could give up or continue in an ignorance preserving situation by reaching a full abduction. In fact, Gabbay and Woods's abduction would be better understood in terms of inference from the best explanation than in terms of inference to the best explanation. We act from the best explanation that in this case is malaria disease.

Medical diagnosis seems to have an abductive reasoning structure more than a deductive or an inductive one, even if the two latter ones could also be used¹³. In fact, an illness starts with the symptoms/signs. We define what the symptoms/signs exactly are in opposition to health and wellness. This is the starting point, the starting question. These symptoms or signs act as a surprising fact that has to be inserted in the background theory. This element is the beginning of a problem. We can not explain the signs/symptoms with the knowledge at our disposal (we start from a healthy state). We trigger a reasoning that helps us continue [12, 42, 3], what leads to the abductive process. Pieces of evidence or just an evidence (symptoms, surprising facts) start a process to solve a problem. But we do not really have the answer, so we guess an hypothesis. This hypothesis is the conjecture of an illness. One of the biggest problems in abduction is to know how to choose the correct hypothesis or if it is possible to choose at all. What makes the doctor to choose one over another hypothesis, disease? When the abductive inference is done and we conjecture a hypothetical ailment, the prognosis and the treatment usually follow, and the most of the time, this action is taken in an ignorance-preserving way¹⁴. Does it mean that the epistemological status of the medical diagnosis is based more in a lack of knowledge than in a knowledge? In fact it seems that sometimes it is¹⁵.

Essentially, medical diagnosis seems to be nearer to what is call abductive reasoning or abduction. Although the analysis of medical diagnosis in terms of abductive inference could clarify some aspects of it, abduction comes with its own difficulties.

¹³In fact, we could not say that medical diagnosis relies on only one kind of reasoning. Different kind of reasoning could have a role in medical diagnosis, see for example the role of analogy in [15]. Here we just treat the general abductive schema, but it would be interesting to check the mix of reasoning or the role for example that analogy has inside abductive reasoning, see [41].

¹⁴Unless the hypothesis is verified with modern techniques, in which case we would be in a subduance point that add new information. This would be partial abduction [42]. This is not the case in the example coming from Ancient Medicine we will study in this paper. And if we check general modern medicine most of the times the doctor treats the patient making a full abduction. They do not realize the test, they just act in an ignorance preserving way by hypotetizing the illness. In fact, testability is not intrinsic to the making of succesful abductive hypthoheses, [12], p. 44.

¹⁵See [5] for two kind of ignorances depending on selective or creative abduction. See [29], Chapter 3, for a discusion between ignorance and knowledge enhancing and the difference between abducing fiction in literature and scientific models. And [27], chapter 2.1 and [29], chapter 1.1, for the ignorance preserving character of abduction (or ignorance mitigating).

Nevertheless, if modern medical diagnosis seems to be an abductive inference¹⁶, which kind of inference/inferences are used in medical diagnosis texts in Akkadian literature [19, 39, 13]? Are both inferences the same? What are their differences?

3 Akkadian Medical Diagnosis

Nowadays tools of modern medicine allow us to approach diagnosis in a more accurate way. In fact, modern doctors have tools to confirm or verify the hypothesis and more precise methods to recognize symptoms/signs. In ancient Mesopotamia the resources were limited, they did not make autopsies¹⁷ and their knowledge of infectious diseases, microorganisms, human body, etc. was different. Besides, the immune defense mechanisms and the organisms may have mutated along time. In spite of these differences, we could recognize descriptions of signs and symptoms that closely seem to correspond to modern diseases¹⁸. The *āšipu* was still an intellectual and he tried to recognize and named symptoms and signs or patterns of disease gathering what we may call ancient etiologies. Nevertheless, the exact identification of the disease is still doubtful. Sometimes the symptoms/signs are the same and nowadays we know that they can be provoked by two or more different illness. Other times, one illness provokes different symptoms/signs and in Ancient Mesopotamia they were considered as different illness. In what follows, I will present different medical diagnosis. In the next section, I approach Ancient medical diagnosis from a philosophical point of view by focussing in the inference at stake. The next four examples of medical diagnosis are quite complete and elaborated:

EXAMPLE 1. GALL BLADDER DISEASE¹⁹

- ORIGINAL TEXT IN AKKADIAN: DIŠ NA GABA-*su u šá-šal-la-šú* KÚM.MEŠ ZÚ.MEŠ-*šú i-hi-la e-piš* KA-*šú* DUGUD NA BI ZÉ GIG *ana* TI-*šú*... (BAM

¹⁶See for example [34, 7, 26] or [8] for a compararison between medical and nursing diagnosis.

¹⁷At least we do not know for sure about autopsies [13], p. 21 and ff. “(...) Although autopsies were never written about (except for animals), we believe that some must have been performed in fatal cases of pneumonia in order to gain more information.” [39]p. 43

¹⁸For a complete study about this see [39].

¹⁹Notice that the approach to ancient medicine comes from Assyriologists and Doctors. Concretely, I use [39]. This a great book that study the medical diagnosis in Akkadian trying to understand the illness from a modern perspective. It does not mean that the modern medicine is the same as the Akkadian one but their analysis allows them to recognize ancient etiology with the modern name’s illness. The evidences of this analysis could be found in their book. Here, I just used their examples.

578 i 50//BAM 159 i 38-39)²⁰²¹

- TRANSLATION: “If a person’s breast and back are warm and his teeth ooze blood and opening his mouth is difficult, that person (has) gall bladder disease, to cure him...”
- COMMENTS: Gallbladder disease is a term used for several conditions that affect the gallbladder. We do not know how (animal or human autopsy) but the *āšīpus* were aware of the existence of the gall bladder²². The gallbladder is a small pear-shaped sac located underneath the liver. The gallbladder’s main function is to store the bile produced by the liver and pass it along to the small intestine. Most of the problems are caused by inflammation of the gallbladder wall (cholecystitis), gallbladder polyps, gallbladder cancer, gallbladder stones, etc. Some of the symptoms are fever, nausea, vomiting, jaundice, pain in the gallbladder region, disorientation, low blood pressure, etc. In fact, we do not know how the *āšīpus* reaches the conclusion, but following the conclusion of Scurlock, J. A. and Andersen, B. R. [39], it seems that they refer to this type of disease.

EXAMPLE 2. CONJUNCTIVA SUFFUSED WITH BLOOD - “HAND “ OF THE GOD MARDUK - HEMORRHAGIC VIRAL INFECTION²³

- ORIGINAL TEXT IN AKKADIAN: DIŠ UB.MEŠ-[⌈]šú DU₈⌋.MEŠ SAG ŠÀ-šú *di-ik-šá* TUKU[⌋] *pi-qam la pi-qam* MÚD *ina* KA/KIR₄-šú Á^{II}-šú SIG.MEŠ NÍG.ZI.[IR] ŠUB.ŠUB-*su* IGI^{II}-šú MÚD *šu-un-nu-’-a* ŠU^dAMAR.UTU *a-dir-ma* GAM (DPS XXII:34-35[AOAT 43.254])²⁴.
- TRANSLATION: “If his limbs are supple, his upper abdomen (epigastrium) has a needling pain, blood incessantly flows from his mouth/nose, his arms are continually weak, [⌈]depression[⌋] continually falls upon him, (and) his eyes are suffused with blood, “Hand of Marduk”; he will be worried and die.”
- COMMENTS: It seems to be due to hemorrhagic viral infection “Hand of Marduk”. These are infections that cause bleeding problems and usually they come

²⁰I follow the conventions of Assyriology in the text writing. Akkadian words or phonetic complements are in italics separated by hyphens. Sumerian word-signs are in capital letters separated by periods.

²¹See [39], p. 138, example 6.113; and [30]

²²I follow Scurlock, J. A. and Andersen, B. R. affirmation, see [39], p. 136-138.

²³Even if our knowledge of the viruses date from the end of the XIX century, it does not mean they do not exist. In fact they are as old as animal and they were already present in Ancient Mesopotamia. The ancient doctor recognizes their symptoms and treats them.

²⁴See [39], p.189,190; [16].

from other animals like monkeys as primary host. We know that monkeys are kept as pets in Ancient Mesopotamia. These fevers are Lassa fever, Ebola, Marburg virus, Crimean-Congo hemorrhagic fever and yellow fever²⁵. The symptoms are high fever, bleeding and often death. The *āšīpu* attributes the illness to different gods/goddesses or spirits. In this case, it is attributed to “Hand of Marduk”. This god is always associated with the heart and this variety is distinguished by the others by a heart pain (needling pain) and bleeding.

EXAMPLE 3. MALARIA - *^dLUGAL.GÌR.RA*

- ORIGINAL TEXT IN AKKADIAN: DIŠ U₄ UD. ʾDU-šū IGI-šú šá 15/150 GIN₇ *GIŠ*BAL *i-lam-[mi]* IGI-šú šá 150/ZAG MÚD DIRI-*al* KA-šú BAD.BAD-*te* EME-[šú] ú-na-šak *^dLUGAL.GÌR.RA* DIB-*su* (STT 89:103-105a, 109-112 Stol, Epilepsy 91-92)
- TRANSLATION: “If when it ʾcomesʾ over him, his left/right eye (makes bobbing movements similar to what) a spindle (does) when it spins (and) his right/left eye is full of blood, he continually opens his mouth (and) he bites [his] tongue, *^dLUGAL.GÌR.RA* afflicts him.”²⁶”
- COMMENTS: Malaria is an infectious disease caused by parasitic protozoans to the Plasmodium type. It comes from a mosquito bite and it is probable that malaria has existed in ancient Mesopotamia. Some texts told about an infectious disease associated with water “He must not go into the lowlands by the river or an infectious disease will infect him”²⁷ There are different types of malaria: Plasmodium Vivax, P. Ovale and P. Malariae; but most of deaths are caused by Plasmodium Falciparum. This one is the cerebral malaria that involves encephalopathy and it could be the one of the example. Some of the symptoms are headache, fever, shivering, joint pain, vomiting, hemolytic anemia, jaundice, retinal damage and convulsions. The cerebral malaria includes among others horizontal and vertical nystagmus (jerking eye movement). It seems that the ancient etiology name for this is *^dLUGAL.GÌR.RA*. It refers to the patient which eyes are dropping, indicating the appearance of neurological symptoms relating to the eyes. This divinity and his twins *^dMES.LAM.TA.É.A* are associated to the god Nergal, the god of pestilence.

These examples show us how the common structure in a Akkadian medical diagnosis is. In fact, we usually have the following form:

²⁵See[39], p. 77,189-190, and [9].

²⁶The neurological signs could be produced by malaria, but the type of cerebral malaria Plasmodium Falciparum. See [39], p. 37, n 3.47.

²⁷See [39], p. 36-37.

1. Symptoms/signs. In the ancient texts, these two terms are not always differentiated. In modern medicine, we differentiate between signs, that the doctor's findings are, and the symptoms, that are the patient's complaints. The ancient physicians are aware of this distinction and they know that sometimes they do not coincide. They have even used sometimes different words²⁸. Nevertheless, the distinction is not always made. For the moment and in a first step of analyzing the inference I will consider them as the same part without making the difference between them. A deeper study of the inference will be needed to approach the nuances of this two terms.
2. Hypothesis/ailment. When we approach to the ancient etiology there is a broad amount of various syndrome that comes from gods/goddesses, ghost, demons/demonesses and demonic forces, courses and sorcery. We could say, as it was mentioned before, that this is more a magical practice than a rational one. But, in fact, it seems that this is a way to deal and organize a broad category of disease as fevers, traumas, mental illness, circulatory problems, viral infections, etc. As the Ancient Mesopotamian religion was polytheistic, the ancient doctor has a wide and flexible system to organize the diseases. Sometimes, syndromes which signs or symptoms did not coincide with the gods/goddesses, ghost or demons/demonesses became and evolve into demonic forces. Some of the syndromes are easier to relate with the nature of the magical creature, but other times it escapes our understanding. What is sure is that the *āšipu* knew better his magical world than we do now. Sometimes we can not explain them. I do not analyze this cultural world as magical or rational, I focus on the inference so I will use this categorization as names of the illness whether they were given a magical explanation or not. What is more or less clear is that he had a practical or empirical approach more than a theoretical one. First, he recognized the signs/symptoms and then he named the ailment.
3. Prognosis. This is something that requires medical knowledge and intuition. In Mesopotamia it has a significant impact on therapy²⁹ because if the *āšipu* considers that it is hopeless, there was no treatment. Why would they perform expensive sacrifices or magical rituals if the patient would die anyway? Only a favorable prognosis yields a treatment.
4. Treatment. Mesopotamian medicine was really a practical approach. It is based on practice and they do not construct untested theoretical principles.

²⁸See [39], p. xix for more details.

²⁹See [39], p. 529 and ff.

Instead, they systematically try and observe. They test different plants and minerals and record the results³⁰. Usually the treatment is based on medicines and surgery. Even if some rituals should be realized, they were a supplement. Once the ailment was established and the prognosis favorable, they passed to the treatment. I will not enter into detail of the therapies, just mention that they used bandages and topical wound treatments and they administered medicines by different ways such as inhalation, orally, rectally, etc.

These four parts of the medical diagnosis are not always present in the texts, but they are quite common. They could be considered as the four basic parts in Akkadian medical diagnosis. We also have therapeutic texts that record and specify in detail the correct treatment for each malady.

4 Inference in Akkadian Medical Diagnosis

In this section, I will analyze the inference at stake in Akkadian medical diagnosis. First, I will mention how it has been understood until now. Later, I will suggest another approach to medical diagnosis in Mesopotamia from a philosophical point of view, by relying on previous Assyriologists' studies and various examples.

Usually the medical texts have a specific pattern, characteristic of scribal practice. Formally, they take the appearance of omen texts and they are organized in collections³¹.

“If a man is sick (and has the following symptoms)...”

“If a man suffers from (such and such) pain in his head (or other part of the body)”

The established pattern of the omen in ancient Mesopotamia has the form of a conditional structure. This is the basic formulation of all Babylonian omens “if so - and -so happens, then so - and - so can result”, “if (or, “whenever”) x then y”³². They use a possible or probable conditional with a protasis or antecedent and an apodosis or consequent. This omen structure offers a warning about the future, that it is not yet realized. The possibility nuance constitutes the core of the structure. Nevertheless, as it has already been emphasized³³, the difference between the literary omens and the medical omens is to be found in the status of the protasis/antecedent.

³⁰It is true that we do not have modern tests, but the ancient doctor tries and observes in the sense of experimental medicine: observation and experience, see [4], p. 42

³¹See for example [31], p. 289 and ff.

³²See [36], p. 257-9.

³³See [31], p. 294 and [13], p. 15.

Even if both of them express a warning about the future, the medical one refers to signs/symptoms from practice, an empirical fact.

This omen structure is explained by Geller in terms of “quasi-causality”. The relation is of the type: “If omen A was associated with event B at some time in the past, inference would suggest that the reoccurrence of A might lead to the reoccurrence of B”³⁴. He considers this relation in terms of deductive logic:

“Babylonian epistemology bases itself upon logical deduction in which general patterns can be inferred from a myriad of details. A typical example of this kind of deduction can be seen in the relationship between the protasis and apodosis clauses of omens. (...) we must admit that this type of deductive logic in Babylonia was not entirely without purpose, since establishing a relationship between data and inferences is after all the beginning of methodical and scientific thinking.” (Geller[13], p. 12-13.)

However, according to Geller such an explanation would rely on a fallacy:

“From a modern perspective, the entire logic is fallacious, and we would dismiss the association of two seemingly unrelated events as the fallacy of “post hoc ergo propter hoc”, namely the fallacy of assuming that where two events are in sequence the second is caused by the first.” (Geller [13], p. 12.)

Geller is right in pointing out that this would constitute a fallacy. In addition, from the viewpoint of modern deductive logic, we might add that the quoted examples rely on another fallacy, namely the fallacy of affirming the consequent. Indeed, affirming the consequent consists in inferring *A (illness)* from *if A (illness), then B (symptoms)* and *B (symptoms)*. We could also better understand Ancient medical diagnosis in terms of abduction. We formally represent the symptoms by *B* and the illness by *A*. The inference would have the structure of an abduction explained as before. Let us illustrate the point by the following examples:

EXAMPLE 4. HEPATITIS B - TÛN.GIG (“SICK LIVER”)

- ORIGINAL TEXT IN AKKADIAN: DIŠ NA [TÛN] ŠÀ-šú DIB-s[u x x] di-kiš ŠÀ GIG ŠÀ-šú KÚ-šú ŠÀ-šú ru-uq-šú ŠÀ. MEŠ-šú i-sa-bu-u’ DÛ UZU.MEŠ-šú tab-ku ni-kim-tú ŠÀ-bi TUKU-ši na-aš-pa-ak bir-ki u a-ḫi GIG NA BI TÛN.GIG ana TI-šú...(BAM 87:1-5)³⁵.
- TRANSLATION: “If a person’s ‘liver’ afflicts ‘him’ [...] he is sick with a needling pain in the abdomen, his abdomen hurts him, his heart is distant from him, his stomach churns, all of his flesh is tense, he has bloated stomach, (and) he is sick with tenseness in the arms and legs, that person (has) TÛN.GIG (“sick liver”), to cure him...”

³⁴See [13], p. 12.

³⁵See [39], p. 212, example 10.20.

In fact, the element A would be the TÛN.GIG (“sick liver”), while the consequences would be the signs/symptoms (B) that produce the illness. The “quasi-causality” would be in this sense. But in a diagnosis we do not have the A element as starting point, but the B . So, if we have the signs/symptoms (B), we have the element B and what we hypothesize is the A . My point is that the inference at stake in the Akkadian medical diagnosis is not a deductive inference, but an abductive inference, as it happens in modern medical diagnosis. Abduction is a different kind of inference³⁶, triggered by a surprising fact, and by which a hypothesis is introduced. It is neither induction, nor deduction. I will use the previous G-W model for abduction to explain Akkadian medical diagnosis inference.

1. $T!Q(\alpha)$

The starting point is a question for which if we have the answer, this answer would be α . The question is the following: which illness? So our cognitive target is to know the ailment to be able to treat the patient. In fact, we are in front of an ignorance problem, we do not know the malady and we would like to achieve the target to solve the ignorance problem. In this case, we talk about signs/symptoms (a needling pain in the abdomen, his abdomen hurts him, his heart is distant from him, his stomach churns, all of his flesh is tense, he has bloated stomach, (and) he is sick with tenseness in the arms and legs). That makes us think the patient is ill and need to be treated. This surprising fact could be evidence (or pieces of evidence) that are not normal or out of current healthy state³⁷.

2. $\sim (R(K, T))$ [Fact]

We do not have a relation between our current knowledge (background knowledge) and the target. We do not know which kind of illness has the patient, we just have the signs/ symptoms.

3. $\sim (R(K*, T))$ [Fact]

We do not have an immediate successor in our base of knowledge that allows us to know the answer. That is to say, we do not have a relation between the successor of K and our target. There is nothing that we could add, as a new information, that allows us to solve the problem. In this case, we could discover the answer by some kind of test, but this would lead to subduance. In ancient Mesopotamian medicine, the direct test are rare and there are usually

³⁶Even if some authors consider that we are not in front of an inference at all, see [17, 35].

³⁷Here we could ask and we should study deeply what a healthy state is in ancient Mesopotamia. One study about that could give us information about is [6], but I will not enter into details in this paper.

no such subduance. In the persisting state of ignorance, abduction runs in view of the target.

4. $H \notin K$ [Fact]

So, what to do to solve the problem? The agent, in order to set a plausible solution to the ignorance problem, makes a hypothesis (Hepatitis B - TÙN.GIG (“sick liver”). The hypothesis is not part of the base of knowledge. The ailment as hypothesis is not something that we had before in our base of knowledge.

5. $H \notin K^*$ [Fact]

The hypothesis or ailment is not part of an immediate successor of our base of knowledge either.

6. $\sim R(H, T)$ [Fact]

And there is no relation between the hypothesis and the target.

7. $\sim R(K(H), T)$ [Fact]

There is neither a relation of both (hypothesis and target) combined with the base of knowledge.

8. $H \rightsquigarrow R(K(H), T)$ [Fact]

The relation is only subjunctive. That is, if H was the case, it would provide an answer to reach the target. The hypothesis relates subjunctively the target in combination with our knowledge base. In fact, this might account for the “quasi-causality” referred to before. This subjunctive relation is expressed in Akkadian language by a conditional one and in an omen structure. But what does “subjunctive relation” exactly mean? This relates to the “hence” in the scheme of Peirce and it is one of the biggest problem of abduction. This relation means that H is not a true sentence, it is not a piece of knowledge either, but if it were, it would be an acceptable solution to our problem. In our case, we could hypotetize Hepatitis B, but it could be another malady with similars symptoms/signs. We do not have the proper test to verify if it is Hepatitis B or not. Notice that we do not have α , which would be the answer to our problem, we just have the hypothesis.

9. H satisfies conditions S_1, \dots, S_n [Fact]

Nevertheless, our hypothesis is plausible. It should satisfy several conditions to be plausible. In our case, Hepatitis B - TÙN.GIG (“sick liver”) usually produces this kind of symptoms/signs, it is possible that Hepatitis B existed at that time, etc.

10. Then, $C(H)$ [Sub-conclusion, 1-7]

So, at the end, we decide to conjecture the Hepatitis B - TÛN.GIG (“sick liver”) as the malady that causes the signs/symptoms displayed by our patient. Until here, abduction is only partial. At this point, the modern test could be really useful, as in the *step 3* (if we would have been an immediate successor of our base of knowledge). A modern test for Hepatitis B could tell us if we are in front of a Hepatitis B virus or not. So, here we would have three possibilities. First, we test and we discover that we are really in a case of Hepatitis B. This is a subduance. Second, we test and we discover that we are not. In this case, we could continue with another hypothesis and maybe reach a full abduction. Third, even if we do not test, we continue our reasoning and we act in consequence.

11. Then, H^C [Conclusion 1-8]

Here is the point in which we arrive to by a full abduction. Even if we do not have knowledge, we continue with our hypothesis as it is, a hypothetical statement. We consider it as a plausible explanation. Then, we activate the hypothesis and we use it in a further reasoning. We act as if it were true. In the Akkadian medical diagnosis, even if we do not have proof of the illness conjectured, we treat the patient as it would be the case. The following example shows this hypothesis activation:

EXAMPLE 5- POTION FOR ŠĒTU - ENTERIC FEVER- BAM145 (=KAR 199)³⁸

ORIGINAL TEXT IN AKKADIAN:

- 1.[DIŠ NA U] D.DA TAB.ĀBA⁷ - [(*ma* SÍG SAG.DU-šú)]
- 2.[(GU)]B.MEŠ IGI.MEŠ-šú NIGIN.MEŠ-^rdu⁷-šú-[*ma*]
- 3.*i-ta-na-aš-ra-ḥu*
- 4.SU-šú *ta-ni-ḥu* TUKU.TUKU-š*i*
- 5.KÚM *la ḥa-aḥ-ḥaš* <(TUKU.T[UKU])>
- 6.*su-a-lam* TUKU.TUKU-š*i*
- 7.ŠÀ-šú *e-ta-na-áš-šá-áš*
- 8.*il-la-tu-šú* DU¹-*ku*
- 9.ŠÀ-šú *ig-da-na-ru-ur*
- 10.*re-du-ut ir-ri* GIG *u ú-šar-d[(a)]*
- 11.*e-le-nu* UZU-šú ŠED₇
- 12.[(*ša*)*p-l*]*a-nu* ĜÌR.PAD.MEŠ-šú *šar-ḥa*
- 13.[(*ina*^r *ša*^r-*la*)]-*li-šú* <(i-*ní*-⁷*i*)> GI.GÍD *ḥa-še-šú*
- 14.[(*it*)]-^r*tí-ni*⁴⁰-*is-kìr*

³⁸See reference text and translation in [38], p. 423-424.

15. [(ú)]-ga-na-ah⁴¹
 16. [(ši-ri-ih-t)]i KÚM ŠÀ TUKU.MEŠ
 17. [(NA BI UD.DA.TAB.BA ana TI-šú)]
 18. [(^ÚGA)MUN] ka-^rta^r-[r]a
 19. [^Úk]am-man-tú ^ÚŠ[(E.L)]ILLAN
 20. ^Úkám-ka-du ^ÚNIM.NIM⁴²^Ú GEŠTIN.KA₅.A
 21. 7 (variant:8) Ú.MEŠ ŠEŠ 1-niš GAZ SIM
 22. ina KAŠ <<KAŠ>> NAG.MEŠ
 23. ina Ì EŠ.MEŠ-su-ma TI

24-29 effaced colophon

- TRANSLATION: “(1-23) [If] ^ršētu^r burns [a person] so that the air of his head continually stands on end, his face seems continually to spin [and] he continually feels burning hot, his body is continually tired (and) <(he continually has)> a lukewarm temperature, he continually has *su*^lālu-cough, his stomach is continually upset, his saliva flows, his stomach turns over and over, he is sick with “flowing” of the intestines and makes (one bowel movement) ^rfollow^r (another), the flesh above is cold (but) his bones below (feel) burning hot, when he tries to sleep, (his breath) turns back (and) his wind pipe continually closes up, he coughs (variant: belches), (and) he continually has burning intestinal fever, that person is burned by šētu.

To cure him, you crush together (and) sift these seven (variant: eight) plants: ^rkamūnu^r-cumin, *katarru*-fungus (variant: *kamūn* šadê-fungus), *kamantu*-henna (?), *lillānu* (ripe grain), *kamkādu*, *samānu* (and) “fox grape”. You have him repeatedly drink (it mixed) with beer. If you repeatedly rub him gently (with it mixed) with oil, he should recover.”

Here, a clear example of the activation of the hypothesis is given. The hypothetical ailment *šētu*-fever is conjectured and we activate it in a further reasoning. We act in consequence and treat the patient. In fact, as it was mentioned, once the hypothesis is settled in ancient Mesopotamian medicine, there are different options. First, the hopeless prognosis and the *āšipu* decide not to treat the patient. Even this “not-doing something” is the activation of the hypothesis in a further reasoning because he is acting in consequence on the basis of a hypothesis. Second, the prognosis is favorable and the *āšipu* treats the patient. The treatment could be as the one shown in the *example 6*, just with herbs, plants, minerals, etc. as a recipe or prescription or with a supplemented ritual. Both cases show how the hypothesis is activated in a further reasoning, in both cases an action is followed from a conjectural malady.

5 Inference in modern and Akkadian medical diagnosis: a compared approach

I have analyzed inferential aspects of modern medical diagnosis and Akkadian medical diagnosis in terms of abduction. For the sake of precision, we should distinguish between three kinds of diagnosis: modern medical diagnosis, general physician modern medical diagnosis, and Akkadian diagnosis. First, modern medical diagnosis is committed with hypothesis testing, unlike general physician diagnosis. Actually, medical diagnosis reasoning as abductive reasoning has been considered as inference to the best explanation (or to the best diagnosis)[26]. This conception is focused not just in the generation of hypothesis (abduction), but also on the evaluation of the hypothesis. This reasoning to the best explanation also uses induction and deduction inferences, as Magnani pointed out³⁹. He considers medical reasoning may be broken down into two different phases: the selection of the hypothesis (abduction) and the evaluation of the hypothesis (corroborate or eliminate, deduction-induction phase)⁴⁰. In the context of the GW model, we may consider that this evaluation stops the abductive process and leads to Woods's subduence. Such a diagnosis consists of a partial abduction. This partial abduction is just a different procedure. On the contrary, the general physician medical diagnosis usually does not evaluate hypothesis and they do not use to make tests. In fact, this is the reason why I have used G-W model and not a model based on inference to the best explanation (IBE). This kind of diagnosis is an inference that preserves ignorance. The conjecture is activated without testing. That is, a full abduction in the context of the GW-model. Another question is to establish if we are in front of a creative or selective abduction (in Magnani words). It seems that general physician modern medical diagnosis relies on a selective process. We select the illness between the known illnesses. Creative hypothesis would occur when a new malady is discovered and a name is introduced for it⁴¹.

Going back to Akkadian medical diagnosis, I have compared it with modern general physician medical diagnosis because they used to treat the patient without test and the schema of reasoning seems to fit with an inference that preserves the ignorance. In fact, they also activate the hypothesis without tests. The question now will be, are we in front of a selective or a creative abduction? In most of the cases we

³⁹There are two main epistemological meaning of the word abduction: 1.abduction that only generate "plausible" hypothesis (selective or creative) and 2.abduction considered as the result as inference to the best explanation, which also evaluates hypothesis.[26], p. 19.

⁴⁰See abduction/deduction-induction cycle in Magnani's ST-MODEL for medical reasoning in [26], Chapter 4; and [27], section 1.3.1. p. 9 and ff.

⁴¹See also [27], p. 10.

are facing a selective abduction. The *āšīpu* has his medical Handbook and he chooses the malady which corresponds to the sign/symptoms. Nevertheless, is it always like that? As in modern medical diagnosis there are cases of non-selective abduction. Following Scurlock and Andersen⁴², the *āšīpu* sometimes uses a name that evolves into a demon to name a malady for which they did not have a demon. That is to say, even if they use the divine or magical category to explain the maladies, sometimes it goes on the other way round. It is not always a selection between options, the options sometimes are created.

To sum up, this means that we do not really have knowledge, at least in this kind of diagnosis (I mean general diagnosis in modern medicine and Akkadian medical diagnosis) but we overcome our ignorance and we reason from the best explanation. Modern medical diagnosis is at some point different from modern general physician and Akkadian diagnosis because it makes use of tests, whereas the latter do not. Now, is there any difference between the general modern diagnosis and the Akkadian one? Each one relies on ignorance-preserving abductive inference. But, is it enough to claim they are the same type of abductions?

If we follow Schurz classification for abduction⁴³ we could consider that both inferences fit with an hypothetical (common) cause abduction. This kind of abduction is the most fundamental kind of conceptually creative abduction⁴⁴. This abductive conjecture postulates a new unobservable entity together with new laws connecting it with the observable properties (without lowering it by analogy). There are two types of this kind: scientific common cause abduction (for Schurz's is the causal unification) and speculative (cause) abduction:

a) Speculative. It happens when the phenomenon is an effect of hypothetical and unobservable cause. Schurz use the following example for a speculative abduction:

John got ill,
some power wanted that John get ill,
and whatever this power wants, happens.

The point of this abduction is that it does not have a predictive power (it is

⁴²See [39], p. 505.

⁴³See [37]. Notice that Schurz classification is made for an abduction as inference to the best explanation (IBE) and the G-W model is better understood as inference from the best explanation, abduction is not intrinsically explanatory, but merely radically instrumental. Nevertheless, I will use Schurz classification to try to understand the differences between these two medical diagnosis. I use the evaluation criteria (IBE) as conditions to fulfill for the hypothesis to be (step 9 in G-W model).

⁴⁴Here we have the point that in medical diagnosis sometimes it is the first time to name an illness and it is creative, but the most of times is selective as I have already explained. Even if Schurz does not talk about the selective one, his classification could help us to analyze the different kinds. See [37], p. 218 y ss. for details about hypothetical (common) cause abduction.

post-hoc), and does not offer proper unification. There are a specific God for each event ⁴⁵

b) Unification. It happens when the phenomenon are effects of that cause. Many elementary phenomena are explained by a few basic principles and it leads to new predictions. Their hypotheses are independently testable because they produce predictions.

If we use this classification, general physician modern medical diagnosis could be part of a causal unification while Akkadian medical diagnosis could be a speculative abduction. General physician medical diagnosis unifies different sign/symptoms as coming from a common cause. It has a predictable character and it is testable (even if we do not test, we can do it) as well as discovering new kinds of properties enlarge our causal understanding. The fact to assign a god, or a “Hand of God” in Akkadian medical diagnosis, could make us think that this diagnosis is a purely speculative abduction. Nevertheless, some examples make us doubt about the purely speculative character of the reasoning. The lack of the intrinsic virtue of unification is not really clear, because the Akkadian doctor tries to unify different signs/symptoms with one cause. This cause it is not always a god or demon, sometimes it is a part of the body⁴⁶. Besides, Akkadian medical diagnosis sometimes has a predictable character as in the case of malaria “He must not go into the lowlands by the river or an infectious disease will infect him⁴⁷”. The fact that discovering new kind of properties enlarges our causal understanding could be exemplified on the commentaries to the Handbook. The *āšīpu* is constantly trying to enlarge this understanding. Nevertheless, something that surely could differentiate them could be the possibility of test. At the time of Akkadian medical diagnosis, there are not modern tests, but the testability is also evaluated through their virtue of producing new predictions and this is also possible in Akkadian medicine. My point is that, even if Akkadian medical diagnosis seems to be a speculative and unscientific abduction, it might be at least in a middle way to a scientific unification abduction.

Finally, is causal unification the only scientific abduction? Following Hoffmann [18], there are cases of speculative abduction in scientific reasoning as the case of “Possible Production of Elements of the atomic number higher than 92” by Ida Noddack⁴⁸. In fact, we do not really have a clear classification of abduction because it depends on the criteria that we choose and more deeply to our under-

⁴⁵See [37] for more details. He explain how the probability of the “scientific” hypothesis is much higher than that of the “speculative” one.

⁴⁶See my example of sick liver number 4 and [39]p. 504.

⁴⁷See [39], p. 36-37.

⁴⁸See [18].

standing of abduction⁴⁹. The point is that if, as Park[32] said, classifying abduction is yet to get off the ground, how we could make a difference between modern general physician diagnosis and Akkadian diagnosis just at the level of the inference? I think this point needs a deeper study and for the moment the classification of the diagnosis (modern general physician and akkadian) as different kind of abduction is yet to get of the ground.

6 Conclusion

I have started this study about Ancient Mesopotamian Practice by tackling the wide problems inherent to this kind of texts. Is it a science-rational thinking or a magical practice? Without making any choice in this respect, I have focused on the inferential aspect of medical diagnosis. In fact, I just try to analyze the reasoning at stake, not the scientific or magical aspect. First, I treat modern medical diagnosis as an inference closer to what we call abduction. Modern general physician medical diagnosis seems to better fit within an abductive reasoning than a deductive or inductive one, even if they could also be used in the process. Then, I come back to Ancient Mesopotamian medicine and I show several examples of Ancient medical diagnosis. By taking into account the inference at stake in Akkadian medical diagnosis, we see how the inference and the reasoning are not so different from the modern one. In fact, it seems that ancient medical texts use the same kind of reasoning than the modern ones. This reasoning is an ignorance preserving reasoning and it is different from induction and deduction. In fact, we try to reach a cognitive target and we reason from a lack of knowledge state that continues as such in the process. We are not led to a new belief or knowledge, but we work with hypotheses that continue being conjectural, even if we use them in a further reasoning. My first point is to answer the assiriologists as Geller [13]that stated the ancient medical diagnosis is a logical deduction. No, we are not in front of a logical deduction. The reasoning at stake in medical diagnosis modern and ancient is an abductive reasoning. Here, we could go back to my starting point and set the following questions. If we consider that modern medical diagnosis is a rational thinking and that it is an abductive reasoning, why would we consider that the Ancient medical diagnosis that uses the same kind of inference is an irrational thinking? If we only check the inference at stake, both rely on the same schema. So, here my second point, the ancient medical diagnosis is a rational thinking as it is the modern medical diagnosis. It is not just if_then rules and they do not use test. This is exactly what is expressed in G-W model for abduction and this is why I use this model. Third, we consider that

⁴⁹See [32] for an analysis of the classification of abduction.

modern medical diagnosis is an abduction as it is the ancient medical diagnosis. Nevertheless, what is the different between them? Is it a different kind of abduction? After analyzing the possible types of abduction I consider that maybe Akkadian medical diagnosis will be in a middle way between unification causal hypothesis and speculative abduction. I said maybe, because I consider that further analysis will be needed. We can not establish a clear classification nowadays without doubt of abduction, so to differentiate the two diagnosis as two different kinds of abduction is not possible. Besides, further analysis would be needed to really clarify the role of the different elements in an abductive schema in Ancient Mesopotamian diagnosis. A deeper study of the linguistic forms and its role on the inference schema would help us better understand the inference in Ancient Akkadian Practice. Of course, abductive reasoning is problematic. For example, what is the criteria for a correct abduction? Referring to medical practice, what makes a diagnosis a good diagnosis? What is the role of the different elements inside the inference schema? Answering these questions provides the basis for a deeper understanding of medical practice. It could also shed light on the G-W model for abduction, in particular what they call “full abduction”, by applying it to a concrete practical case.

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RIESZ SPACE-VALUED STATES ON PSEUDO MV-ALGEBRAS

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Abstract

We introduce Riesz space-valued states, called $(R, 1_R)$ -states, on a pseudo MV-algebra, where R is a Riesz space with a fixed strong unit 1_R . Pseudo MV-algebras are a non-commutative generalization of MV-algebras. Such a Riesz space-valued state is a generalization of usual states on MV-algebras. Any $(R, 1_R)$ -state is an additive mapping preserving a partial addition in pseudo MV-algebras. We introduce $(R, 1_R)$ -state-morphisms and extremal $(R, 1_R)$ -states, and we study relations between them. We study metrical completion of unital ℓ -groups with respect to an $(R, 1_R)$ -state. If the unital Riesz space is Dedekind complete, we study when the space of $(R, 1_R)$ -states is a Choquet simplex or even a Bauer simplex.

Keywords: MV-algebra, pseudo MV-algebra, state, state-morphism, unital Riesz space, $(R, 1_R)$ -state, extremal $(R, 1_R)$ -state, $(R, 1_R)$ -state-morphism, R -Jordan signed measure, Choquet simplex, Bauer simplex

AMS Mathematics Subject Classification: 06D35, 06C15

1 Introduction

The notion of a state is a basic one in the theory of quantum structures, see e.g. [10]. It is an analogue of a finitely additive probability measure. MV-algebras as well as its non-commutative generalization, pseudo MV-algebras, introduced in [16, 30], form an important subclass of quantum structures. Mundici defined a notion

Sponsored by grant of the Slovak Research and Development Agency under contract APVV-16-0073, by the grant VEGA No. 2/0069/16 SAV and GAČR 15-15286S

of a state on an MV-algebra in [27] as averaging the truth-value in Łukasiewicz logic. States on MV-algebras are studied very intensively last 10–15 years when many important results as an integral representation of states by regular σ -additive probability measures, [21, 29], or the MV-algebraic approach to de Finetti’s notion of coherence have been established, see [23]. Some applications of states on MV-algebras can be found in [31].

In the last period, the so-called Riesz MV-algebras have been studied in the frames of MV-algebras, see [6]. A prototypical example of Riesz MV-algebras is an interval in a unital Riesz space, when we use Mundici’s representation functor Γ , see [26] or [5, Chap 2]. The converse is also true: For any Riesz MV-algebra M , there is a unital Riesz space $(R, 1_R)$ such that $M \cong \Gamma(R, 1_R)$, [6, Thm 3]. Whereas MV-algebras are algebraic semantic of the Łukasiewicz logic, [4], Riesz MV-algebras are an extension of the Łukasiewicz logic: The propositional calculus that has Riesz MV-algebras as models is a conservative extension of Łukasiewicz infinite-valued propositional calculus, [6]. Moreover, these structures have also several applications, among which we mention artificial neural networks, image compression, game theory, etc., see [2, 22]. Fuzzy logics with noncommutative conjunctions inspired by pseudo MV-algebras were studied in [19].

For more information about MV-algebras, see [5] and about states on MV-algebras, see [28], and for the most fresh survey on states on MV-algebras, see [14].

States on pseudo MV-algebras have been studied in [8]. For pseudo MV-algebras there is a basic representation by unital ℓ -groups not necessarily Abelian, [9], which generalizes Mundici’s representation of MV-algebras, see [27]. A state on a pseudo MV-algebra is defined as an additive functional with non-negative real values which at the top element of the pseudo MV-algebra attains the value 1. Whereas every MV-algebra (with $0 \neq 1$) admits at least one state, this is not a case for pseudo MV-algebras, because as it was shown in [8], there are stateless pseudo MV-algebras. Moreover, a pseudo MV-algebra admits at least one state if and only if it has at least one normal ideal that is also maximal. Therefore, all linearly ordered pseudo MV-algebras, representable pseudo MV-algebras or normal valued ones have at least one state.

Riesz space-valued states on MV-algebras have been firstly introduced in [3] where they are called generalized states. Their main result on generalized states concerns with a representation theorem for semisimple MV-algebra via Riesz space-valued states for a Dedekind complete Riesz space. The standard states reflect only an additive structure of numerical averaging the truth-value in Łukasiewicz logic. More information about the MV-algebra or a pseudo MV-algebra we obtain if its structure allows also multiplication by reals which goes to Riesz MV-algebras and

such events can be better describe numerically by states where multiplication by reals is involved and it goes to Riesz space-valued states. Riesz space-valued states are a quite natural generalization of well-studied states on ℓ -groups and MV-algebras (see also [18, 27]). In this paper we provide a framework in which it is possible to encode and decode more information than usual. We note that in mathematics we study also different generalizations of probability measures as signed measures, group-valued measures, operator-valued measures, vector-measures, hyper-states [25], etc., and this contribution gives new information about Riesz space-valued states in Łukasiewicz type logic.

A Riesz space-valued state is introduced in this paper as an $(R, 1_R)$ -state on a pseudo MV-algebra M which is an additive mapping on M attaining values in the interval $[0, 1_R]$ of the unital Riesz space $(R, 1_R)$, where R is a Riesz space and 1_R is a fixed strong unit of R , Section 3. The main aim of this study, in contrast to one in [3], is a detailed study of the $(R, 1_R)$ -state space for a pseudo MV-algebra. This study contains both situations, MV-algebras and non-commutative pseudo MV-algebras. We show that there are many parallels with the standard state space of MV-algebras, [27], as well as of pseudo MV-algebras, see [8].

We introduce also extremal $(R, 1_R)$ -states and $(R, 1_R)$ -state-morphisms as homomorphisms of pseudo MV-algebras into the interval $[0, 1_R]$. We show relations between them and we discuss when the latter two kinds of $(R, 1_R)$ -states coincide and when not. Whereas according to [8], there is a one-to-one correspondence among extremal states, state-morphisms and maximal ideals that are normal, respectively, we show that for $(R, 1_R)$ -states this is not a case, in general. We will study cases when $(R, 1_R)$ is an Archimedean unital Riesz space or even a Dedekind complete unital Riesz space. In Section 4, we present metrical completion of a unital ℓ -group by an $(R, 1_R)$ -state. In Section 5, we introduce also R -measures and R -Jordan signed measures and we study situations when the $(R, 1_R)$ -state space is a simplex, or a Choquet simplex or even a Bauer simplex and when every $(R, 1_R)$ -state lies in the weak closure of the convex hull of extremal $(R, 1_R)$ -states.

The paper is endowed with a couple of illustrating examples.

2 Pseudo MV-algebras and Riesz Spaces

In the present section we gather basic notions and results on pseudo MV-algebras and Riesz spaces.

Pseudo MV-algebras as a non-commutative generalization of MV-algebras were defined independently in [16] as pseudo MV-algebras and in [30] as generalized MV-algebras.

Definition 2.1. A pseudo MV-algebra is an algebra $(M; \oplus, ^-, \sim, 0, 1)$ of type $(2, 1, 1, 0, 0)$ such that the following axioms hold for all $x, y, z \in M$ with an additional binary operation \odot defined via

$$y \odot x = (x^- \oplus y^-)^\sim$$

$$(A1) \quad x \oplus (y \oplus z) = (x \oplus y) \oplus z;$$

$$(A2) \quad x \oplus 0 = 0 \oplus x = x;$$

$$(A3) \quad x \oplus 1 = 1 \oplus x = 1;$$

$$(A4) \quad 1^\sim = 0; \quad 1^- = 0;$$

$$(A5) \quad (x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-;$$

$$(A6) \quad x \oplus (x^\sim \odot y) = y \oplus (y^\sim \odot x) = x \odot (y^- \oplus y) = y \odot (x^- \oplus x);$$

$$(A7) \quad x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y;$$

$$(A8) \quad (x^-)^\sim = x.$$

We shall assume that $0 \neq 1$. If we define $x \leq y$ iff $x^- \oplus y = 1$, then \leq is a partial order such that M is a distributive lattice with $x \vee y = x \oplus (x^\sim \odot y)$ and $x \wedge y = x \odot (x^- \oplus y)$. We recall that a pseudo MV-algebra is an MV-algebra iff \oplus is a commutative binary operation. As usually, we assume that \odot has higher binding priority than \wedge and \oplus , and \oplus is higher than \vee .

A non-void subset I of M is an *ideal* of M if (i) $a \leq b \in I$ implies $a \in I$, and (ii) if $a, b \in I$, then $a \oplus b \in I$. The sets M and $\{0\}$ are ideals of M . An ideal $I \neq M$ of M is *maximal* if it is not a proper subset of any proper ideal of M . An ideal I of M is *normal* if $a \oplus I := \{a \oplus b : b \in I\} = \{c \oplus a : c \in I\} =: I \oplus a$ for any $a \in M$. For basic properties of pseudo MV-algebras see [16].

Pseudo MV-algebras are intimately connected with ℓ -groups. We remind that a *po-group* is a group $(G; +, -, 0)$ written additively endowed with a partial order \leq such that, for $g, h \in G$ with $g \leq h$ we have $a + g + b \leq a + h + b$ for all $a, b \in G$. If the partial order \leq is a lattice order, G is said to be an ℓ -group. The *positive cone* of a po-group G is the set $G^+ = \{g \in G : 0 \leq g\}$. A po-group G satisfies *interpolation* if, for $x_1, x_2, y_1, y_2 \in G$ with $x_1, x_2 \leq y_1, y_2$, there is an element $z \in G$ such that $x_1, x_2 \leq z \leq y_1, y_2$. An element $u \geq 0$ is a *strong unit* of G if, given $g \in G$, there is an integer $n \geq 1$ such that $g \leq nu$. A couple (G, u) , where G is an ℓ -group and u is a fixed strong unit of G , is said to be a *unital ℓ -group*. An ℓ -group G is (i) *Archimedean* if, for $a, b \in G$, $na \leq b$ for each integer $n \geq 1$ implies $a \leq 0$, (ii) *Dedekind σ -complete* if any sequence $\{g_n\}$ of elements of G that is bounded from

above by an element $g_0 \in G$ has supremum $\bigvee_n g_n \in G$, and (iii) *Dedekind complete* if any family $\{g_t : t \in T\}$ of elements of G which is bounded from above by an element $g_0 \in G$ has supremum $\bigvee_{t \in T} g_t \in G$. An ℓ -ideal of an ℓ -group G is any ℓ -subgroup P of G such that $a \in P$ and $|a| \leq |b|$ yield $a \in P$. Here $|g| = g^+ + g^-$, $g^+ = g \vee 0$ and $g^- = -(g \wedge 0)$ for each $g \in G$. For non-explained notions about ℓ -groups, please, consult e.g. [15, 17].

A prototypical example of pseudo MV-algebras is from ℓ -groups: If u is a strong unit of a (not necessarily Abelian) ℓ -group G ,

$$\Gamma(G, u) := [0, u]$$

and

$$\begin{aligned} x \oplus y &:= (x + y) \wedge u, \\ x^- &:= u - x, \\ x^\sim &:= -x + u, \\ x \odot y &:= (x - u + y) \vee 0, \end{aligned}$$

then $\Gamma(G, u) := ([0, u]; \oplus, ^-, \sim, 0, u)$ is a pseudo MV-algebra [16]. Conversely, for every pseudo MV-algebra M , there is a unique unital ℓ -group (G, u) (up to isomorphism of unital ℓ -groups) such that $M \cong \Gamma(G, u)$, and there is a categorical equivalence between the category of pseudo MV-algebras and the category of unital ℓ -groups as it follows from the basic representation theorem [9] given by the functor $(G, u) \mapsto \Gamma(G, u)$.

We define a partial operation, $+$, on M in such a way: $x + y$ is defined in M iff $x \odot y = 0$, and in such a case, we set $x + y := x \oplus y$. Using the representation of pseudo MV-algebras by unital ℓ -groups, we see that the partial operation $+$ coincides with the group addition restricted to M . The operation $+$ is associative. We note, that if $x \leq y$ for $x, y \in M$, then there are two unique elements $z_1, z_2 \in M$ such that $z_1 + x = y = x + z_2$. We denote them by $z_1 = y - x$ and $z_2 = -x + y$, and using the group representation, $-$ coincides with the group subtraction. Since $+$ is associative, we say that a finite system $(a_i)_{i=1}^n$ of M is *summable* if there is an element $a = a_1 + \dots + a_n \in M$; a is said to be the *sum* of $(a_i)_{i=1}^n$ and the sequence $(a_i)_{i=1}^n$ is said to be *summable*.

For the partial addition $+$ on any pseudo MV-algebra the following form of the Riesz Decomposition property, called the *strong Riesz Decomposition Property*, RDP_2 for short, holds: If for any $a_1, a_2, b_1, b_2 \in M$ such that $a_1 + a_2 = b_1 + b_2$ there are four elements $c_{11}, c_{12}, c_{21}, c_{22} \in M$ such that $a_1 = c_{11} + c_{12}$, $a_2 = c_{21} + c_{22}$, $b_1 = c_{11} + c_{21}$, $b_2 = c_{12} + c_{22}$ and $c_{12} \wedge c_{21} = 0$. It is derived from such a decomposition

holding in ℓ -groups, see [15, Thm V.1] and [11, 12]. Equivalently, if a_1, \dots, a_m and b_1, \dots, b_n are elements of a pseudo MV-algebra M such that $a_1 + \dots + a_m = b_1 + \dots + b_n$, there is a system $\{c_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ of elements of M satisfying

$$a_i = c_{i1} + \dots + c_{in}, \quad b_j = c_{1j} + \dots + c_{mj}, \tag{2.1}$$

for all $1 \leq i \leq m, 1 \leq j \leq n$, and

$$(c_{i+1,j} + \dots + c_{mj}) \wedge (c_{i,j+1} + \dots + c_{in}) = 0, \quad i < m, j < n. \tag{2.2}$$

For any $x \in M$ and any integer $n \geq 0$, we define

$$\begin{aligned} x^0 &= 1, x^{n+1} = x^n \odot x, \quad n \geq 1, \\ 0 \odot x &= 0, (n+1) \odot x = (n \odot x) \odot x, \quad n \geq 1, \\ 0x &= 0, (n+1)x = nx + x, \quad n \geq 1, \text{ if } nx + x \text{ exists in } M. \end{aligned}$$

A real vector space R with a fixed partial order \leq is a *Riesz space* if

- (i) R with respect to \leq is an ℓ -group;
- (ii) $f \in R^+$ implies $af \in R^+$ for every real number $a \geq 0$.

A Riesz space R is *Archimedean* if it is Archimedean as an ℓ -group, analogously R is *Dedekind σ -complete* or *Dedekind complete* if so is R as an ℓ -group. We note that if R is Dedekind complete, then it is Dedekind σ -complete, and if R is Dedekind σ -complete then it is Archimedean, see [24, Thm 25.1]. A *Riesz ideal* of R is any ℓ -ideal of R . We note that any Riesz ideal of R is a Riesz subspace of R .

A *unital Riesz space* is a couple $(R, 1_R)$ where R is a Riesz space and 1_R is a fixed strong unit of R (i.e. 1_R is a strong unit of the ℓ -group R). Important examples of unital Archimedean Riesz spaces are spaces of real-valued functions on a topological space: Let $T \neq \emptyset$ be a compact Hausdorff topological space. We denote by $C(T)$ and $C_b(T)$ the system of all continuous real-valued functions of T and the system of all bounded real-valued functions on T , respectively. Then $C(T)$ and $C_b(T)$ are Archimedean Riesz spaces with respect to the partial order of functions $f \leq g$ iff $f(t) \leq g(t)$ for each $t \in T$. The function 1_T defined by $1_T(t) = 1$ for each $t \in T$ is a strong unit for both $C(T)$ and $C_b(T)$. Both spaces are closed under usual product of two functions, and if we endow both spaces with the sup norms, $C(T)$ and $C_b(T)$ are Banach algebras. The space $C(T)$ has an important property: If T' is another non-void compact Hausdorff space, then there is an isomorphism of Riesz spaces between $(C(T), 1_T)$ and $(C(T'), 1_{T'})$ preserving product of functions iff T and T' are homeomorphic, see [7, Thm IV.6.26]. In addition, let $T' = \mathcal{M}(C(T))$

denote the space of maximal ideals of $C(T)$. Under the hull-kernel topology, T' is homeomorphic to T and $C(T)$ and $C(T')$ are isomorphic Riesz spaces and isometric Banach spaces, see [24, Ex 45.7].

In the last period, there has appeared a class of important MV-algebras, Riesz MV-algebras, which are connected with Riesz spaces, for more details, see [6]. We note that if $(R, 1_R)$ is a unital Riesz space, then the MV-algebra $\Gamma(R, 1_R)$ is a characteristic example of a Riesz MV-algebra.

For Archimedean unital Riesz spaces there is a representation theorem by Yosida, see [32] or [24, Thm 45.3]:

Theorem 2.2. [Yosida Theorem] *Let $(R, 1_R)$ be an Archimedean unital Riesz space. Then there is a compact Hausdorff topological space T such that R can be embedded as a Riesz subspace into $C(T)$, the Riesz space of continuous real-valued functions on T , such that 1_R maps to the constant function 1_T , where $1_T(t) = 1$, $t \in T$. In addition, T can be chosen such that the image of the embedding of R into $C(T)$ is uniformly dense in $C(T)$, i.e. the uniform closure of the image of R is $C(T)$.*

We remind that there are nice topological characterizations, Nakano's theorems, when the Riesz space $C(T)$ ($T \neq \emptyset$ compact and Hausdorff) is Dedekind σ -complete and Dedekind complete, respectively: (1) $C(T)$ is Dedekind σ -complete iff T is a *basically disconnected space*, that is, the closure of every open F_σ subset of T is open, see e.g. [18, Cor 9.3], [24, Thm 43.9]. (2) $C(T)$ is Dedekind complete iff T is *extremally disconnected*, that is, the closure of every open set of T is open, see [24, Thm 43.11]. We note that the same characterizations hold also for the Riesz space $C_b(T)$ of bounded real-valued functions on T .

General Dedekind σ -complete unital Riesz spaces are characterized as follows, see [24, Thm 45.4]:

Theorem 2.3. *If $(R, 1_R)$ is a Dedekind σ -complete Riesz space, then it is isomorphic to some $(C(T), 1_T)$, where $T \neq \emptyset$ is a compact basically disconnected Hausdorff topological space.*

In addition, if $(R, 1_R)$ is even Dedekind complete, then T is extremally disconnected. In both cases, the space T can be chosen as the set of maximal ideals of R topologized by the hull-kernel topology.

More about Riesz spaces can be found in [24] and some information about representations of Archimedean Riesz spaces by systems of functions attaining also infinite values are in the survey [13].

3 $(R, 1_R)$ -states on Pseudo MV-algebras

In the present section, we define states on pseudo MV-algebras and then we define $(R, 1_R)$ -states as additively defined mappings on a pseudo MV-algebra M which preserve the partial addition $+$ on M and have values in the interval $[0, 1_R]$ of a unital Riesz space $(R, 1_R)$ mapping the top element $1 \in M$ onto the strong unit $1_R \in R$. We introduce also extremal $(R, 1_R)$ -states, $(R, 1_R)$ -state-morphisms, and we show relationships between them.

States, analogues of finitely additive measures, on pseudo MV-algebras were introduced in [8] as follows: Let M be a pseudo MV-algebra. A *state* on M is any real-valued mapping $s : M \rightarrow [0, 1]$ such that (i) $s(1) = 1$, and (ii) $s(x + y) = s(x) + s(y)$ whenever $x + y$ is defined in M . According to [8, Prop 4.1], if s is a state on M , then (i) $s(0) = 0$, (ii) $s(a) \leq s(b)$ if $a \leq b$, (iii) $s(x^-) = 1 - s(x) = s(x^\sim)$, (iv) $s(x^\bar{=}) = s(x) = s(x^\approx)$, (v) $s(x \vee y) + s(x \wedge y) = s(x) + s(y) = s(x \oplus y) + s(x \odot y)$, (vi) $s(x \oplus y) = s(y \oplus x)$. A state s is *extremal* if from $s = \lambda s_1 + (1 - \lambda)s_2$ for states s_1, s_2 on M and $\lambda \in (0, 1)$ we have $s_1 = s_2$. Let $\mathcal{S}(M)$ and $\mathcal{S}_\partial(M)$ denote the set of all states and extremal states, respectively, on M . It can happen that $\mathcal{S}(M) = \emptyset$, see [8, Cor 7.4], however, if M is an MV-algebra, M has at least one state, [18, Cor 4.4]. We note that a pseudo MV-algebra possesses at least one state iff M has at least one maximal ideal that is also normal, see [8]. We say that a net $\{s_\alpha\}_\alpha$ of states on M *converges weakly* to a state s if $s(x) = \lim_\alpha s_\alpha(x)$ for each $x \in M$. Then $\mathcal{S}(M)$ and $\mathcal{S}_\partial(M)$ are either simultaneously the empty sets or non-void compact Hausdorff topological spaces, and due to the Krein–Mil’man Theorem, every state s on M is a weak limit of a net of convex combinations of extremal states.

Now we extend the notion of a state to a Riesz space-valued mapping.

Definition 3.1. *Let 1_R be a strong unit of a Riesz space R . An $(R, 1_R)$ -state on a pseudo MV-algebra M is any mapping $s : M \rightarrow [0, 1_R]$ such that (i) $s(1) = 1_R$ and (ii) $s(x + y) = s(x) + s(y)$ whenever $x + y$ is defined in M .*

An $(R, 1_R)$ -state-morphism on a pseudo MV-algebra M is any homomorphism of pseudo MV-algebras $s : M \rightarrow \Gamma(R, 1_R)$. We denote by \mathbb{R} the group of real numbers. It is evident that any $(\mathbb{R}, 1)$ -state is a state on M . An $(\mathbb{R}, 1)$ -state-morphism on M is said to be a *state-morphism*. It is clear that any $(R, 1_R)$ -state-morphism is in fact an $(R, 1_R)$ -state on M . The converse is not true, in general.

We can define also an $(R, 1_R)$ -state on every unital ℓ -group (G, u) as follows: It is a mapping $s : G \rightarrow R$ such that (i) $s(g) \geq 0$ if $g \geq 0$, (ii) $s(g + h) = s(g) + s(h)$ for all $g, h \in G$, and (iii) $s(u) = 1_R$. The restriction of any $(R, 1_R)$ -state on (G, u) onto the pseudo MV-algebra $\Gamma(G, u)$ gives an $(R, 1_R)$ -state on $\Gamma(G, u)$, and using the categorical equivalence between pseudo MV-algebras and unital ℓ -groups, see [9,

Thm 6.4], every $(R, 1_R)$ -state on $\Gamma(G, u)$ can be extended to a unique $(R, 1_R)$ -state on (G, u) .

It is worthy of recalling that if s is an $(R, 1_R)$ -state on $\Gamma(S, 1_S)$, where $(S, 1_S)$ is a unital Riesz space, then $s(tx) = ts(x)$ for each $x \in \Gamma(S, 1_S)$ and any real number $t \in [0, 1]$. Indeed, since $s(x) = s(n\frac{1}{n}x) = ns(\frac{1}{n}x)$, i.e. $s(\frac{1}{n}x) = \frac{1}{n}s(x)$. Then for each integer $m = 0, 1, \dots, n$, we have $s(\frac{m}{n}x) = s(m\frac{1}{n}x) = ms(\frac{1}{n}x) = \frac{m}{n}s(x)$. The statement is trivially satisfied if $t = 0, 1$. Thus let $t \in (0, 1)$. There are two sequences of rational numbers $\{p_n\}$ and $\{q_n\}$ from the interval $(0, 1)$ such that $\{s_n\} \nearrow t$ and $\{q_n\} \searrow t$ which implies $p_ns(x) = s(p_nx) \leq s(tx) \leq s(q_nx) = q_ns(x)$, so that $s(tx) = ts(x)$.

In addition, if s is an $(R, 1_R)$ -state on a unital Riesz space $(S, 1_S)$, we can show that $s(ta) = ts(a)$ for each $a \in S$ and $t \in \mathbb{R}$.

Proposition 3.2. *Let s be an $(R, 1_R)$ -state on a pseudo MV-algebra M . Then*

- (i) $s(0) = 0$.
- (ii) *If $x \leq y$, then $s(x) \leq s(y)$, and*

$$s(y \odot x^-) = s(y) - s(x) = s(x^\sim \odot y).$$
- (iii) $s(x^-) = 1 - s(x) = s(x^\sim)$.
- (iv) $s(x^=) = s(x) = s(x^\approx)$.
- (v) $s(x \vee y) + s(x \wedge y) = s(x) + s(y)$.
- (vi) $s(x \oplus y) + s(x \odot y) = s(x) + s(y)$.
- (vii) $s(x \oplus y) \oplus s(x \odot y) = s(x) \oplus s(y)$.
- (viii) *The kernel of s , $\text{Ker}(s) := \{x \in M : s(x) = 0\}$, is a normal ideal of M .*
- (ix) $[x] = [y]$ *if and only if* $s(x) = s(x \wedge y) = s(y)$, *where where $[x]$ and $[y]$ are the cosets in $M/\text{Ker}(s)$ determined by $x, y \in M$.*
- (x) *There is a unique $(R, 1_R)$ -state \tilde{s} on $M/\text{Ker}(s)$ such that $\tilde{s}([x]) = s(x)$ for each $[x] \in M/\text{Ker}(s)$.*
- (xi) $\tilde{s}([x]) = 0$ *if and only if* $[x] = [0]$.
- (xii) $s(x \oplus y) = s(y \oplus x)$ *whenever R is an Archimedean Riesz space. In addition, $M/\text{Ker}(s)$ is an Archimedean MV-algebra.*

Proof. Assume $M = \Gamma(G, u)$ for a unital ℓ -group (G, u) . Properties (i)–(iv) follow directly from definition of pseudo MV-algebras and $(R, 1_R)$ -states.

(v) It follows from equalities $(x \vee y) \odot y^- = (x \vee y) - y = x - (x \wedge y) = x \odot (x \wedge y)^-$ which hold in the ℓ -group G and the pseudo MV-algebra M .

(vi) It follows from the identity $x = (x \oplus y) \odot y^- + (y \odot x)$, see [16, Prop 1.25] and (ii).

(vii) It follows from (vi) and from the identity $r_1 \oplus r_1 = (r_1 + r_2) \wedge 1_R$ for $r_1, r_2 \in [0, 1_R]$.

(viii) If $s(x), s(y) = 0$, then by (vi), we have $s(x \oplus y) = 0$. By (ii) we conclude $\text{Ker}(s)$ is an ideal of M . To show that $\text{Ker}(s)$ is normal, let $a \in M$ and $x \in \text{Ker}(s)$. Then $s(a \oplus x) = s(a) = s(x \oplus a)$ and $a \oplus x = (a \oplus x) \odot a^- \oplus a$ so that, $s((a \oplus x) \odot a^-) = 0$. In a similar way, we prove that $x \oplus a = a \oplus (a^\sim \odot (x \oplus a))$ and $a^\sim \odot (x \oplus a) \in \text{Ker}(s)$.

(ix) It is evident.

(x) Let $[x] \leq [y]^-$. We define $x_0 = x \wedge y^-$. Then $x_0 \leq y^-$ and $[x_0] = [x \wedge y^-]$, so that

$$\begin{aligned} \tilde{s}([x] + [y]) &= \tilde{s}([x \oplus y]) = \tilde{s}([x_0 + y]) = s(x_0 + y) = s(x_0) + s(y) \\ &= \tilde{s}([x_0]) + \tilde{s}([y]) = \tilde{s}([x]) + \tilde{s}([y]) \end{aligned}$$

which proves that \tilde{s} is an $(R, 1_R)$ -state on $M/\text{Ker}(s)$. By (ix), $[x] = [y]$ implies $s(x) = s(y)$.

(xi) It follows from (ix).

(xii) Due to (xi), $\tilde{s}([x]) = 0$ iff $[x] = [0]$. We claim that $M/\text{Ker}(s)$ is an Archimedean pseudo MV-algebra. Indeed, let $n[x]$ be defined in $M/\text{Ker}(s)$ for any integer $n \geq 1$. Then $\tilde{s}(n[x]) = n\tilde{s}([x]) = ns(x) \leq 1_R$ for any n . Therefore, $\tilde{s}([x]) = s(x) = 0$. The Archimedeanity of $M/\text{Ker}(s)$ entails the commutativity of $M/\text{Ker}(s)$, see [8, Thm 4.2]. Therefore, $s(x \oplus y) = \tilde{s}([x \oplus y]) = \tilde{s}([x] \oplus [y]) = \tilde{s}([y] \oplus [x]) = \tilde{s}([y \oplus x]) = s(y \oplus x)$. \square

From (xii) of the latter proposition we conclude that if we apply an $(R, 1_R)$ -state s to information contained in the pseudo MV-algebra M , then the non-commutativity of M is "killed" by the $(R, 1_R)$ -state s whenever R is Archimedean. That is, non-commuting pairs of elements of M cannot be distinguished by any $(R, 1_R)$ -state if R is Archimedean. We do not know whether this is true for each unital Riesz space $(R, 1_R)$.

Example 3.3. Let G be the group of all matrices of the form

$$A = \begin{pmatrix} \xi & \alpha \\ 0 & 1 \end{pmatrix},$$

where ξ and α are real numbers such that $\xi > 0$; the group-operation is the usual multiplication of matrices. If we denote $A = (\xi, \alpha)$, then $A^{-1} = (1/\xi, -\alpha/\xi)$, $(1, 0)$ is the neutral element, and $G^+ := \{(\xi, \alpha) : \text{where (i) } \xi > 1, \text{ or (ii) } \xi = 1 \text{ and } \alpha \geq 0\}$. Then G with the positive cone G^+ is a linearly ordered ℓ -group with a strong unit $U = (2, 0)$. If we define $M = \Gamma(G, U)$, M is a non-commutative pseudo MV-algebra. Given a unital Riesz space $(R, 1_R)$, let

$$s((\xi, \alpha)) = \log_2(\xi)1_R, \quad (\xi, \alpha) \in M.$$

Then s is an $(R, 1_R)$ -state, and if R is Archimedean, due to Proposition 3.6 below, it is a unique $(R, 1_R)$ -state on M .

In the same way as for states, we can define extremal $(R, 1_R)$ -states. Let $\mathcal{S}(M, R, 1_R)$, $\mathcal{SM}(M, R, 1_R)$, and $\mathcal{S}_\partial(M, R, 1_R)$ denote the set of $(R, 1_R)$ -states, $(R, 1_R)$ -state-morphisms and extremal $(R, 1_R)$ -states, respectively, on M . Analogously, we can define extremal $(R, 1_R)$ -states on unital ℓ -groups. In addition, using the categorical equivalence [9, Thm 6.4], if $M = \Gamma(G, u)$, then an $(R, 1_R)$ -state s on M is extremal if and only if the unique extension of s to an $(R, 1_R)$ -state on (G, u) is extremal and vice-versa, that is, an $(R, 1_R)$ -state on (G, u) is extremal iff its restriction to $\Gamma(G, u)$ is extremal.

Lemma 3.4. *An $(R, 1_R)$ -state s on M is extremal if and only if \tilde{s} is extremal on $M/\text{Ker}(s)$.*

Proof. (1) Let s be extremal and let $\tilde{s} = \lambda m_1 + (1 - \lambda)m_2$, where m_1, m_2 are $(R, 1_R)$ -states on $M/\text{Ker}(s)$ and $\lambda \in (0, 1)$. Then $s_i(x) := m_i([x])$, $x \in M$, $i = 1, 2$, is an $(R, 1_R)$ -state on M , and $s = \lambda s_1 + (1 - \lambda)s_2$ so that $s_1 = s_2$ and $m_1 = m_2$ proving \tilde{s} is extremal.

Conversely, let \tilde{s} be extremal and let $s = \lambda s_1 + (1 - \lambda)s_2$ for $(R, 1_R)$ -states s_1, s_2 on M and $\lambda \in (0, 1)$. Then $\text{Ker}(s) = \text{Ker}(s_1) \cap \text{Ker}(s_2)$. We assert that $m_i([x]) := s_i(x)$, $[x] \in M/\text{Ker}(s)$, is an $(R, 1_R)$ -state on $M/\text{Ker}(s)$ for $i = 1, 2$. Indeed, first we show that m_i is correctly defined. Thus let $[x] = [y]$. By (ix) of Proposition 3.2, we have $s(x) = s(x \wedge y) = s(y)$ so that $s(x \odot y^-) = 0 = s(y \odot x^-)$ which yields $s_i(x \odot y^-) = 0 = s_i(y \odot x^-)$ and $s_i(x) = s_i(x \wedge y) = s_i(y)$ for $i = 1, 2$. So that, we have $m_i([x]) = m_i([y])$. Therefore, m_i is an $(R, 1_R)$ -state. Then $\tilde{s} = \lambda m_1 + (1 - \lambda)m_2$ implying $m_1 = m_2$ and $s_1 = s_2$. □

We note that it can happen that on M there is no $(R, 1_R)$ -state even for $(R, 1_R) = (\mathbb{R}, 1)$ as it was already mentioned. In the following proposition, we show that if M is an MV-algebra, then $\mathcal{S}(M, R, 1_R)$ is non-void.

Proposition 3.5. *Every MV-algebra has at least one $(R, 1_R)$ -state for every unital Riesz space $(R, 1_R)$.*

Proof. Due to [18, Cor 4.4], any MV-algebra M has at least one state; denote it by s_0 . Then the mapping $s : M \rightarrow [0, 1_R]$ defined by $s(x) := s_0(x)1_R$, $x \in M$, is an $(R, 1_R)$ -state on M . \square

The following result was established in [8] for states on MV-algebras. In the following proposition, we extend it for $(R, 1_R)$ -states.

Proposition 3.6. *Let $(R, 1_R)$ be a unital Riesz space. The following statements are equivalent:*

- (i) *The pseudo MV-algebra M possesses at least one $(R, 1_R)$ -state.*
- (ii) *M has at least one maximal ideal that is normal.*
- (iii) *M has at least one state.*

(1) *Every linearly ordered pseudo MV-algebra possesses at least one $(R, 1_R)$ -state. The same is true if M is representable, i.e. it is representable as a subdirect product of linearly ordered pseudo MV-algebras.*

(2) *If M is a linearly ordered pseudo MV-algebra and $(R, 1_R)$ is an Archimedean unital Riesz space, then M possesses only a unique $(R, 1_R)$ -state.*

Proof. By [8, Prop 4.3], (ii) and (iii) are equivalent.

(i) \Rightarrow (ii). Let s be an $(R, 1_R)$ -state on M . Since $\Gamma(R, 1_R)$ is an MV-algebra, it has at least one state, say s_0 . Then the mapping $s_0 \circ s$ is a state on M , which by [8, Prop 4.3] means that M has at least one maximal ideal that is normal.

(ii) \Rightarrow (i). Let I be a maximal ideal of M that is normal. By [8, Cor 3.5], M/I is an MV-algebra. Due to Proposition 3.5, M/I possesses at least one $(R, 1_R)$ -state, say s_0 . Then the mapping $s(x) := s_0(x/I)$, $x \in M$, is an $(R, 1_R)$ -state on M .

(1) Now suppose that M is linearly ordered. By [8, Prop 5.4], M possesses a unique maximal ideal and this ideal is normal. Applying just proved equivalences, M possesses an $(R, 1_R)$ -state.

If M is representable, then it can be embedded into a direct product $\prod_t M_{t \in T}$ of linearly ordered pseudo MV-algebras $\{M_t : t \in T\}$, i.e. there is an embedding of pseudo MV-algebras $h : M \rightarrow \prod_t M_t$ such that $\pi_t \circ h : M \rightarrow M_t$ is a surjective homomorphism, where $\pi_t : \prod_{t \in T} M_t \rightarrow M_t$ is the t -th projection for each $t \in T$. Every M_t possesses an $(R, 1_R)$ -state s_t , so that $s_t \circ \pi_t \circ h$ is an $(R, 1_R)$ -state on M .

(2) Let M be linearly ordered and $(R, 1_R)$ be an Archimedean unital Riesz space. By the Yosida Representation Theorem, Theorem 2.2, there is a compact Hausdorff

topological space such that R can be embedded into the Riesz space $C(T)$ of continuous real-valued functions on T as its Riesz subspace. Given $t \in T$, define a mapping $s_t : \Gamma(C(T), 1_T) \rightarrow [0, 1]$ defined by $s_t(f) = f(t)$, $f \in \Gamma(C(T), 1_T)$; it is a state on $\Gamma(C(T), 1_T)$. Due to (1) of the present proof, M admits at least one $(R, 1_R)$ -state. Let s_1, s_2 be $(R, 1_R)$ -states on M . Then $s_t \circ s_i$ is a state on M for $i = 1, 2$ and each $t \in T$. According to [8, Thm 5.5], M admits only one state. Therefore, $s_t \circ s_1 = s_t \circ s_2$ for each $t \in T$, i.e. $s_1 = s_2$. \square

For additional relationships between $(R, 1_R)$ -state-morphisms and their kernels as maximal ideals, see Propositions 3.21–3.22 below.

We note that in (2) of the latter proposition, if $(R, 1_R)$ is not Archimedean, then it can happen that M has uncountably many $(R, 1_R)$ -states and each of these states is an $(R, 1_R)$ -state-morphism, see Example 3.9 below.

Proposition 3.7. (1) *An $(R, 1_R)$ -state s on a pseudo MV-algebra M is an $(R, 1_R)$ -state-morphism if and only if*

$$s(x \wedge y) = s(x) \wedge s(y), \quad x, y \in M. \tag{3.1}$$

(2) *An $(R, 1_R)$ -state s on M is an $(R, 1_R)$ -state-morphism if and only if the $(R, 1_R)$ -state \tilde{s} on $M/\text{Ker}(s)$ induced by s is an $(R, 1_R)$ -state-morphism on $M/\text{Ker}(s)$.*

Proof. (1) Assume that s is an $(R, 1_R)$ -state-morphism. Then $s(x \wedge y) = s(x \odot (x^- \oplus y)) = s(x) \odot (s(x^-) \oplus s(y)) = s(x) \wedge s(y)$.

Conversely, let (3.1) hold. Then $x \oplus y = x + (x^\sim \odot (x \oplus y)) = x + (x^\sim \wedge y)$, so that $s(x \oplus y) = s(x) + s(x^\sim \wedge y) = s(x) + (1_R - s(x)) \wedge s(y) = (s(x) + s(y)) \wedge 1_R = s(x) \oplus s(y)$ proving s is an $(R, 1_R)$ -state-morphism.

(2) Using Proposition 3.2, we have $s(x \wedge y) = \tilde{s}([x \wedge y]) = \tilde{s}([x] \wedge [y])$. Applying (1), we have the assertion in question. \square

By [8, Prop 4.3], a state s on a pseudo MV-algebra M is a state-morphism iff $\text{Ker}(s)$ is maximal. In what follows, we exhibit this criterion for the case of $(R, 1_R)$ -state-morphisms. We note that an element $r \in R$ is *strictly positive* if $r \geq 0$ and $r \neq 0$.

Proposition 3.8. (1) *Let s be an $(R, 1_R)$ -state on M . If $\text{Ker}(s)$ is a maximal ideal of M , then s is an $(R, 1_R)$ -state-morphism.*

(2) *In addition, let $(R, 1_R)$ be a unital Riesz space such that every element of $R^+ \setminus \{0\}$ is a strong unit. If s is an $(R, 1_R)$ -state-morphism, then $\text{Ker}(s)$ is a maximal ideal.*

Proof. (1) Assume $\text{Ker}(s)$ is a maximal ideal of M . By [GeIo, (i) Prop 1.25], $x \odot y^- \wedge y \odot x^- = 0$ for all $x, y \in M$, that is $[x] \odot [y]^- \wedge [y] \odot [x]^- = [0]$ and due to [8, Cor 3.5], $M/\text{Ker}(s)$ is an Archimedean linearly ordered MV-algebra which entails either $s(x \odot y^-) = 0$ or $s(y \odot x^-) = 0$. In the first case we have $0 = s(x \odot y^-) = s(x \odot (x \wedge y)^-) = s(x) - s(x \wedge y)$. Similarly, $0 = s(y \odot x^-)$ entails $s(y) - s(x \wedge y) = 0$, i.e., $s(x \wedge y) = \min\{s(x), s(y)\} = s(x) \wedge s(y)$, which by Proposition 3.7 means that s is an $(R, 1_R)$ -state-morphism.

(2) Let $(R, 1_R)$ be a Riesz space such that every strictly positive element of R is a strong unit, then R is Archimedean. Let s be an $(R, 1_R)$ -state-morphism on M and let $x \in M$ be an element such that $s(x) \neq 0$.

Denote by $\text{Ker}(s)_x$ the ideal of M generated by $\text{Ker}(s)$ and x . By [16, Lem 3.4], $\text{Ker}(s)_x = \{y \in M : y \leq n \odot x \oplus h \text{ for some } n \geq 1 \text{ and some } h \in \text{Ker}(s)\}$. Let z be an arbitrary element of M . Since $s(x)$ is a strong unit of R , there exists an integer $n \geq 1$ such that $s(z) < ns(x)$, so that $s(z) \leq n \odot s(x)$. Then $s((n \odot x)^\sim \odot z) = 0$. Since $z = (n \odot x) \wedge z \oplus ((n \odot x) \wedge z)^\sim \odot z = (n \odot x) \wedge z \oplus (n \odot x)^\sim \odot z \leq n \odot x \oplus (n \odot x)^\sim \odot z$, it proves that $z \in \text{Ker}(s)_x$, consequently, $M = \text{Ker}(s)_x$ which shows that $\text{Ker}(s)$ is a maximal ideal of M . □

We notify that (2) of the preceding proposition follows directly from Theorem 3.13. We have left here the proof of (2) only to present different used methods.

We note that if s is an $(R, 1_R)$ -state-morphism and $(R, 1_R)$ is not Archimedean, then $\text{Ker}(s)$ is not necessarily maximal as the following example shows. In addition, it can happen that every $(R, 1_R)$ -state is an $(R, 1_R)$ -state-morphism but not every $(R, 1_R)$ -state-morphism is extremal.

Example 3.9. Let $M = \Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (1, 0))$, where \mathbb{Z} is the group of integers, $R = \mathbb{R} \overrightarrow{\times} \mathbb{R}$ be the lexicographic product of the real line \mathbb{R} with itself, and choose $1_R = (1, 0)$. Then R is a linearly ordered Riesz space that is not Archimedean, and every element of the form $(0, x)$, where $x > 0$, is strictly positive but no strong unit for R . The mapping $s : M \rightarrow [0, 1_R]$ defined by $s(a, b) = (a, b)$ for $(a, b) \in M$ is an $(R, 1_R)$ -state-morphism and $\text{Ker}(s) = \{(0, 0)\}$ is an ideal that is not a maximal ideal of M because it is properly contained in the maximal ideal $I = \{(0, n) : n \geq 0\}$ which is a unique maximal ideal of M .

In addition, M has uncountably many $(R, 1_R)$ -states, any $(R, 1_R)$ -state on M is an $(R, 1_R)$ -state-morphism, and there is a unique $(R, 1_R)$ -state having maximal kernel and it is a unique extremal $(R, 1_R)$ -state M .

Proof. Let s be any $(R, 1_R)$ -state on M . Then $s(0, 1) = (a, b)$ for a unique $(a, b) \in \Gamma(\mathbb{R} \overrightarrow{\times} \mathbb{R}, (1, 0))$, where $a \geq 0$. Since $s(0, n) = (na, nb) \leq (1, 0)$, $n \geq 0$, we have $a = 0$ and $b \geq 0$. Therefore, $s(1, -n) = (1, -nb)$. We denote this $(R, 1_R)$ -state

by s_b . Hence, there is a one-to-one correspondence between $(R, 1_R)$ -states on M and the positive real axis $[0, \infty)$ given by $b \mapsto s_b$, $b \in [0, \infty)$. Then every $(R, 1_R)$ -state s_b is an $(R, 1_R)$ -state-morphism, $\text{Ker}(s_b) = \{(0, 0)\}$ for $b > 0$ which is an ideal of M but not maximal, and only s_0 is an extremal $(R, 1_R)$ -state on M and $\text{Ker}(s_0) = \{(0, n) : n \geq 0\}$ is a maximal ideal. \square

We note that in Example 3.18 and Proposition 3.19 we will show also cases of $(R, 1_R)$ -state-morphisms for an Archimedean Riesz space $(R, 1_R)$ whose kernel is not maximal.

We remind the following result from [8, Lem 4.4] which follows e.g. from [5, Prop 7.2.5].

Lemma 3.10. (i) *Let G_1 and G_2 be two subgroups of $(\mathbb{R}; +)$ each containing a common non-zero element g_0 . If there is an injective group-homomorphism ϕ of G_1 into G_2 preserving the order such that $\phi(g_0) = g_0$, then $G_1 \subseteq G_2$ and ϕ is the identity on G_1 . If, in addition, ϕ is surjective, then $G_1 = G_2$.*

(ii) *Let M_1 and M_2 be two MV-subalgebras of the standard MV-algebra $[0, 1]$. If there is an MV-isomorphism ψ from M_1 onto M_2 , then $M_1 = M_2$, and ψ is the identity.*

Theorem 3.11. *Let $(R, 1_R)$ be an Archimedean unital Riesz space. Let s_1, s_2 be two $(R, 1_R)$ -state-morphisms on a pseudo MV-algebra M such that their kernels are maximal ideals of M and $\text{Ker}(s_1) = \text{Ker}(s_2)$. Then $s_1 = s_2$.*

Proof. Since R is an Archimedean Riesz space with a strong unit 1_R , due to the Yosida Representation Theorem, Theorem 2.2, there is a compact Hausdorff topological space $T \neq \emptyset$ and an injective homomorphism of Riesz spaces $\phi : R \rightarrow C(T)$ with $\phi(1_R) = 1_T$, where $(C(T), 1_T)$ is the unital Riesz space of continuous real-valued functions on T . Then $M_i := s_i(M)$ are MV-subalgebras of the MV-algebra $\Gamma(R, 1_R)$ for $i = 1, 2$. Define a mapping $s_t^i : \phi(M_i) \rightarrow [0, 1]$ for each $t \in T$ by $s_t^i(\phi(s_i(x))) = (\phi(s_i(x)))(t)$ for each $t \in T$. Then the mapping $\hat{s}_t^i := s_t^i \circ \phi \circ s_i$ is a state-morphism on M , and by [8, Prop 4.3], each $\text{Ker}(\hat{s}_t^i)$ is a maximal ideal of M . Since $\text{Ker}(s_i) = \bigcap_{t \in T} \text{Ker}(\hat{s}_t^i)$ and $\text{Ker}(s_i) \subseteq \text{Ker}(\hat{s}_t^i)$ are also maximal ideals of M , we conclude that $\text{Ker}(\hat{s}_t^1) = \text{Ker}(s_1) = \text{Ker}(s_2) = \text{Ker}(\hat{s}_t^2)$ for each $t \in T$ which by [8, Prop 4.5] means that $s_t^1 = s_t^2$ for each $t \in T$. Then $\phi(s_1(x))(t) = \phi(s_2(x))(t)$, $t \in T$, i.e. $s_1 = s_2$. \square

We note that if $(R, 1_R)$ is not Archimedean, then Theorem 3.11 is not necessarily valid, see Example 3.9.

Proposition 3.12. *Let I be a maximal and normal ideal of a pseudo MV-algebra M and let $(R, 1_R)$ be a unital Riesz space. Then there is an $(R, 1_R)$ -state-morphism s such that $\text{Ker}(s) = I$. If, in addition, R is Archimedean, there is a unique $(R, 1_R)$ -state-morphism s such that $\text{Ker}(s) = I$.*

Proof. Since M/I is by [8, Prop 3.4] an Archimedean linearly ordered MV-algebra, it is isomorphic by Lemma 3.10 to a unique MV-subalgebra of $\Gamma(\mathbb{R}, 1)$; identify it with its image in \mathbb{R} . Define a mapping $s : M \rightarrow R$ as follows: $s(x) = x/I1_R$, $x \in M$. Then s is an $(R, 1_R)$ -state-morphism on M such that $\text{Ker}(s) = I$.

Let, in addition, R be an Archimedean Riesz space. By Theorem 3.11, if s' is another $(R, 1_R)$ -state-morphism on M with $\text{Ker}(s') = I$, then $s = s'$. \square

The following result deals with a one-to-one correspondence between $(R, 1_R)$ -states and states on pseudo MV-algebras for a special kind of Archimedean unital Riesz spaces.

Theorem 3.13. *Let M be a pseudo MV-algebra and let $(R, 1_R)$ be a unital Riesz space such that every strictly positive element of R is a strong unit for R . Given a state m on M , the mapping*

$$s(x) := m(x)1_R, \quad x \in M,$$

is an $(R, 1_R)$ -state on M .

Conversely, if s is an $(R, 1_R)$ -state on a pseudo MV-algebra M , there is a unique state m_s on M such

$$s(x) := m_s(x)1_R, \quad x \in M.$$

The mapping $s \mapsto m_s$ is a bijective affine mapping from $\mathcal{S}(M)$ onto $\mathcal{S}(M, R, 1_R)$.

In addition, the following statements are equivalent:

- (i) *s is an extremal $(R, 1_R)$ -state on M .*
- (ii) *s is an $(R, 1_R)$ -state-morphism on M .*
- (iii) *$s(x \wedge y) = s(x) \wedge s(y)$, $x, y \in M$.*
- (iv) *s is an $(R, 1_R)$ -state-morphism on M if and only if m_s is a state-morphism on M .*

Proof. First, we characterize Riesz spaces from the assumptions of the theorem: Since every strictly positive element of R is a strong unit, by [18, Lem 14.1], R is a simple ℓ -group, i.e., the only ℓ -ideals of R are $\{0\}$ and R . Therefore, the ideal $\{0\}$ is a unique maximal ideal of $\Gamma(R, 1_R)$, so that by Proposition 3.8(1) and Proposition

3.12, there is a unique state μ on $\Gamma(R, 1_R)$, it is a state-morphism as well as an extremal state, so that, $\text{Ker}(\mu) = \{0\}$ and $\mu(a) = \mu(b)$ for $a, b \in \Gamma(R, 1_R)$ if and only if $a = b$.

Now, let m be a state on M . Then the mapping $s(x) := m(x)1_R$, $x \in M$, is trivially an $(R, 1_R)$ -state on M . This is true for each unital Riesz space.

Conversely, let s be an arbitrary $(R, 1_R)$ -state on M . By the Yosida Theorem 2.2, there is a compact Hausdorff topological space $T \neq \emptyset$ such that the unital Riesz space $(R, 1_R)$ can be injectively embedded into the unital Riesz space $(C(T), 1_T)$ of continuous functions on T as its Riesz subspace. If ϕ is this embedding, then $(\phi(R), 1_R)$ is a unital Riesz space whose every strictly positive element is a strong unit for $\phi(R)$.

For any $t \in T$, let us define $s_t : \Gamma(\phi(R), 1_T) \rightarrow [0, 1]$ by $s_t(f) := f(t)$, $f \in \Gamma(\phi(R), 1_T)$. Then each s_t is a state-morphism on $\Gamma(\phi(R), 1_T)$. Define a mapping $\hat{s}_t : M \rightarrow [0, 1]$ as $\hat{s}_t = s_t \circ \phi \circ s$. Since every strictly positive element of $\phi(R)$ is a strong unit for it, by the above first note from the beginning of our proof, we conclude that $s_t = s_{t'}$ for all $t, t' \in T$. Hence, $\hat{s}_t = \hat{s}_{t'}$. Thus we denote by $\hat{s} = \hat{s}_{t_0}$ for an arbitrary $t_0 \in T$. Consequently, $\hat{s}(x)$ is a constant function on T for each $x \in M$, and the range of \hat{s} is a linearly ordered set, therefore, the range of $s(M)$ is a linearly ordered set in R .

If we define $m_s(x) := s_{t_0}(\phi(s(x)))$, $x \in M$, then m_s is a state on M . Consequently, $s(x) := m_s(x)1_R$, $x \in M$.

Now it is clear that the mapping $\Phi : \mathcal{S}(M, R, 1_R) \rightarrow \mathcal{S}(M)$ defined by $\Phi(s) := m_s$, $s \in \mathcal{S}(M, R, 1_R)$, is a bijective affine mapping.

Due to this bijective affine mapping Φ , we see that in view of [8, Prop 4.7], statements (i)–(iv) are mutually equivalent. □

We note that the results of the precedent theorem are not surprising because if $(R, 1_R)$ is a unital Riesz space, whose every strictly positive element is a strong unit for R , then $(R, 1_R) \cong (\mathbb{R}, 1)$. Indeed, since every ℓ -ideal of R is also a Riesz ideal of R and vice versa (this is true for each Riesz space), then $\{0\}$ is a unique proper Riesz ideal of R , see [18, Lem 14.1], therefore, it is maximal, and since R is Archimedean, by the proof of the Yosida Representation Theorem, see [24, Chap 45], T is in fact the set of maximal Riesz ideals of $(R, 1_R)$ which is topologized by the hull-kernel topology and in our case, T is a singleton. Consequently, $(R, 1_R) \cong (\mathbb{R}, 1)$, and we can apply [8, Prop 4.7].

In the next proposition we show that if every strictly positive element of a unital ℓ -group (G, u) is a strong unit for G and $(R, 1_R)$ is Archimedean, then $M = \Gamma(G, u)$ possesses a unique $(R, 1_R)$ -state.

Proposition 3.14. *Let $M = \Gamma(G, u)$ and every non-zero element of M be a strong unit for G . Then every $(R, 1_R)$ -state s on M is an $(R, 1_R)$ -state-morphism for every unital Riesz space $(R, 1_R)$, $\text{Ker}(s)$ is a maximal ideal of M , M is an MV-algebra, and $\mathcal{S}(M, R, 1_R) \neq \emptyset$. If, in addition, $(R, 1_R)$ is an Archimedean unital Riesz space, then $|\mathcal{S}(M, R, 1_R)| = 1$.*

Proof. The hypotheses imply that M is Archimedean, and by [9, Thm 4.2], M is an MV-algebra. Whence, every strictly positive element of G is a strong unit of G , where (G, u) is a unital ℓ -group such that $M \cong \Gamma(G, u)$. Therefore, (G, u) is an Abelian unital ℓ -group. By [18, Lem 14.1], G is a simple ℓ -group, so that G has a unique proper ℓ -ideal, namely the zero ℓ -ideal. Then $\{0\}$ is a unique maximal ℓ -ideal of M . Hence, by Proposition 3.5, M possesses an $(R, 1_R)$ -state for every unital Riesz space $(R, 1_R)$.

Let s be an $(R, 1_R)$ -state on M . Since $\text{Ker}(s) \neq M$, $\text{Ker}(s) = \{0\}$, and hence $\text{Ker}(s)$ is a maximal ideal of M . By Proposition 3.8(1), s is an $(R, 1_R)$ -state-morphism on M .

Now let $(R, 1_R)$ be an Archimedean unital Riesz space. Let s_1 and s_2 be $(R, 1_R)$ -states on M . Then $s := 1/2s_1 + 1/2s_2$ is also an $(R, 1_R)$ -state on M and in view of the first part of the present proof, s, s_1, s_2 are $(R, 1_R)$ -state-morphisms such that $\text{Ker}(s) = \text{Ker}(s_1) = \text{Ker}(s_2) = \{0\}$ are maximal ideals of M , which by Theorem 3.11 yields $s_1 = s = s_2$. In particular, the unique $(R, 1_R)$ -state on M is extremal. \square

Proposition 3.15. *Let T be a non-void set and $R = C_b(T)$ be the Riesz space of bounded real-valued functions on T , and let 1_T be the constant function $1_T(t) = 1$, $t \in T$. Then $1_R = 1_T$ is a strong unit for R . Let s be an $(R, 1_R)$ -state on a pseudo MV-algebra M . The following statements are equivalent:*

- (i) s is an extremal $(R, 1_R)$ -state on M .
- (ii) s is an $(R, 1_R)$ -state-morphism on M .
- (iii) $s(x \wedge y) = s(x) \wedge s(y)$, $x, y \in M$.

In addition,

$$\mathcal{S}_\partial(M, R, 1_R) = \mathcal{SM}(M, R, 1_R). \tag{3.2}$$

Proof. (i) \Rightarrow (ii). Let s be an extremal $(R, 1_R)$ -state on M . If we define an MV-algebra $\Gamma(R, 1_R)$, then every mapping $s_t : \Gamma(R, 1_R) \rightarrow [0, 1]$ defined by $s_t(f) = f(t)$, $f \in \Gamma(R, 1_R)$, is a state-morphism on $\Gamma(R, 1_R)$. We assert that $s_t \circ s$ is an extremal state on M for each $t \in T$. Indeed, let m_1^t, m_2^t be states on M and $\lambda \in (0, 1)$ such that $s_t \circ s = \lambda m_1^t + (1 - \lambda)m_2^t$. Define a function $m_i : M \rightarrow [0, 1_R]$ such that $(m_i(x))(t) =$

$m_i^t(x)$ for $t \in T$ where $x \in M$ and $i = 1, 2$. Then $m_1(x), m_2(x) \in C_b(T)$ for each $x \in M$, so that m_1, m_2 are $(R, 1_R)$ -states on M such that $s = \lambda m_1 + (1 - \lambda)m_2$. Since s is extremal, $s = s_1 = s_2$ which gives $s_t \circ s = m_1^t = m_2^t$ for each $t \in T$, and $s_t \circ s$ is an extremal state on M .

By [8, Prop 4.7], $s_t \circ s$ is a state-morphism on M , therefore, $s_t(s(x \wedge y)) = \min\{s_t(s(x)), s_t(s(y))\}$, that is $s(x \wedge y) = s(x) \wedge s(y)$ for all $x, y \in M$ and s is an $(R, 1_R)$ -state morphism, see Proposition 3.7.

(ii) \Rightarrow (i). Let s be an $(R, 1_R)$ -state-morphism on M and let s_1, s_2 be $(R, 1_R)$ -states on M such that $s = \lambda s_1 + (1 - \lambda)s_2$ for some $\lambda \in (0, 1)$. Let s_t be a state-morphism from the foregoing implication for every $t \in T$. Then $s_t \circ s = \lambda s_t \circ s_1 + (1 - \lambda)s_t \circ s_2$. Applying [8, Prop 4.7], we have $s_t \circ s$ is an extremal state because $s_t \circ s$ is a state-morphism on M . Therefore, $s_t \circ s = s_t \circ s_1 = s_t \circ s_2$ for each $t \in T$ which in other words means that $s = s_1 = s_2$, that is, s is an extremal $(R, 1_R)$ -state on M .

The equivalence of (ii) and (iii) was established in Proposition 3.7. Equation (3.2) follows from the equivalence of (i) and (ii). □

Corollary 3.16. *Let $(R, 1_R) = (\mathbb{R}^n, 1_{\mathbb{R}^n})$, $n \geq 1$, where $1_{\mathbb{R}} = (1, \dots, 1)$, and let s be an $(R, 1_R)$ -state on a pseudo MV-algebra M . The following statements are equivalent:*

- (i) s is an extremal $(R, 1_R)$ -state on M .
- (ii) s is an $(R, 1_R)$ -state-morphism on M .
- (iii) $s(x \wedge y) = s(x) \wedge s(y)$, $x, y \in M$.

Moreover, (3.2) holds.

Proof. It follows from Proposition 3.15 because $(\mathbb{R}^n, 1_{\mathbb{R}^n}) \cong (C_b(T), 1_T)$, where $|T| = n$. □

The latter result extends a characterization of state-morphisms and extremal states from [8, Prop 4.7], because if T is a singleton, then $(C(T), 1_T)$ corresponds in fact to $(\mathbb{R}, 1)$.

Theorem 3.17. *Let $(R, 1_R)$ be an Archimedean unital Riesz space, M a pseudo MV-algebra, and s, s_1, s_2 be $(R, 1_R)$ -states on M .*

- (1) *If s is an $(R, 1_R)$ -state-morphism on M , then s is an extremal $(R, 1_R)$ -state.*
- (2) *If s is an $(R, 1_R)$ -state such that $\text{Ker}(s)$ is a maximal ideal, then s is an $(R, 1_R)$ -state-morphism and an extremal $(R, 1_R)$ -state on M as well.*

(3) Let s_1, s_2 be $(R, 1_R)$ -states on M such that $\text{Ker}(s_1) = \text{Ker}(s_2)$ and $\text{Ker}(s_1)$ is a maximal ideal. Then s_1 and s_2 are $(R, 1_R)$ -state-morphisms and extremal $(R, 1_R)$ -states such that $s_1 = s_2$.

(4) Let s be an $(R, 1_R)$ -state on M such that $M_s = M/\text{Ker}(s)$ is linearly ordered. Then s is an $(R, 1_R)$ -state-morphism and an extremal $(R, 1_R)$ -state, and $\text{Ker}(s)$ is a maximal ideal of M .

Proof. (1) Due to the Yosida Representation Theorem 2.2, there is a compact Hausdorff topological space $T \neq \emptyset$ such that $(R, 1_R)$ can be embedded into $(C(T), 1_T)$ as its Riesz subspace; let ϕ be the embedding. For each $t \in T$, the function $s_t : \Gamma(C(T), 1_T) \rightarrow [0, 1]$ defined by $s_t(f) := f(t)$, $f \in C(T)$, is a state-morphism on $\Gamma(C(T), 1_T)$ for each $t \in T$. Then the mapping $m_t := s_t \circ \phi \circ s$ is a state-morphism on M , so that by [8, Prop 4.7], m_t is an extremal state on M . Let $s = \lambda s_1 + (1 - \lambda)s_2$, where s_1, s_2 are $(R, 1_R)$ -states on M and $\lambda \in (0, 1)$. Then

$$m_t = s_t \circ \phi \circ s = \lambda s_t \circ \phi \circ s_1 + (1 - \lambda) \circ \phi s_2,$$

which implies $m_t = s_t \circ \phi \circ s_1 = s_t \circ \phi \circ s_2$ for each $t \in T$. Hence, $\phi(s(x)) = \phi(s_1(x)) = \phi(s_2(x))$ for every $x \in M$, that is $s(x) = s_1(x) = s_2(x)$, and finally $s = s_1 = s_2$.

(2) Let s be an $(R, 1_R)$ -state on M such that $\text{Ker}(s)$ is a maximal ideal. By Proposition 3.8, s is an $(R, 1_R)$ -state-morphism which by the first part of the present proof entails s is extremal.

(3) Let s_1, s_2 be $(R, 1_R)$ -states on M such that $\text{Ker}(s_1) = \text{Ker}(s_2)$ and $\text{Ker}(s_1)$ is a maximal ideal. By (2), s_1 and s_2 are extremal, and by Proposition 3.8, s_1 and s_2 are also $(R, 1_R)$ -state-morphisms on M , so that Theorem 3.11 implies $s_1 = s_2$.

(4) Assume M_s is linearly ordered. By Proposition 3.2(xii), M_s is an MV-subalgebra of the standard MV-algebra $\Gamma(\mathbb{R}, 1)$ of the real line R . Then, for the $(R, 1_R)$ -state \tilde{s} on $M/\text{Ker}(s) = M_s$ induced by s , we have $\text{Ker}(\tilde{s}) = \{0\}$ is a maximal ideal of M_s and $\text{Ker}(s)$ is a maximal ideal of M . Hence, there is a subgroup \mathbb{R}_0 of \mathbb{R} such that $1 \in \mathbb{R}_0$ and $M_s \cong \Gamma(\mathbb{R}_0, 1)$. Then there is a unique state-morphism (= extremal state in this case) m on M_s . The mapping $s_0(x) := m(x)1_R$ for each $x \in M$ is an $(R, 1_R)$ -state-morphism on M . Since \mathbb{R}_0 is a simple ℓ -group, applying Proposition 3.14, we have $s = s_0$ and s is an $(R, 1_R)$ -state-morphism and an extremal $(R, 1_R)$ -state on M , as well. \square

By [8, Prop 4.3], a state s on a pseudo MV-algebra is a state-morphism iff $\text{Ker}(s)$ is a maximal ideal. In the following example and proposition we show that it can happen that an $(R, 1_R)$ -state-morphism s , consequently an extremal $(R, 1_R)$ -state, has the kernel $\text{Ker}(s)$ that is not maximal even for an Archimedean Riesz space $(R, 1_R)$. We note that in Example 3.9 we had an analogous counterexample for a non-Archimedean Riesz space.

Example 3.18. *There are a pseudo MV-algebra M , an Archimedean unital Riesz space $(R, 1_R)$, and an $(R, 1_R)$ -state-morphism s on M , consequently an extremal $(R, 1_R)$ -state, such that $\text{Ker}(s)$ is not maximal.*

Proof. Let $(R, 1_R) = (\mathbb{R}^n, 1_{\mathbb{R}^n})$, $M = \Gamma(\mathbb{R}^n, 1_{\mathbb{R}^n})$ and $s : M \rightarrow [0, 1_{\mathbb{R}^n}]$ be such that $s(x) = x$, $x \in M$. Then s is an $(R, 1_R)$ -state-morphism, and by Theorem 3.17(1) it is an extremal $(R, 1_R)$ -state. Because $\text{Ker}(s) = \{0\}$, then $\text{Ker}(s)$ is a maximal ideal of M iff $n = 1$. □

According to [16], we say that an element e of a pseudo MV-algebra M is *Boolean* if $e \wedge e^- = 0$, equivalently, $e \wedge e^\sim = 0$, equivalently $e \oplus e = e$. Let $B(M)$ be the set of Boolean elements, then $B(M)$ is a Boolean algebra that is an MV-algebra and subalgebra of M , and $e^- = e^\sim$; we put $e' = e^-$. If s is an $(R, 1_R)$ -state-morphism and e is a Boolean element of M , then $s(e)$ is a Boolean element of the MV-algebra $\Gamma(R, 1_R)$. We recall that for a sequence $(a_i)_{i=1}^n$ of Boolean elements of M we have that it is summable iff $a_i \wedge a_j = 0$ for $i \neq j$; in such a case, $a_1 + \dots + a_n = a_1 \vee \dots \vee a_n$. If we say that for an ordered finite system (a_1, \dots, a_n) of Boolean elements of M we assume that $a_1 + \dots + a_n = 1$, it can happen that some of a_i are zeros.

We exhibit the latter example in more details. We note that the MV-algebra $\Gamma(\mathbb{R}^n, 1_{\mathbb{R}^n})$ has each ideal I of M of the form $I = I_1 \times \dots \times I_n$, where $I_i \in \{\{0\}, \mathbb{R}\}$ for each $i = 1, \dots, n$. In particular, all maximal ideals are of the form $I_1 \times \dots \times I_n$ where exactly one $I_i = \{0\}$ and $I_j = \mathbb{R}$ for $j \neq i$.

Proposition 3.19. *Let $M = \Gamma(\mathbb{R}^n, 1_{\mathbb{R}^n})$, $(R, 1_R) = (\mathbb{R}^n, 1_{\mathbb{R}^n})$, and let $B(M)$ be the set of Boolean elements of M .*

(1) *Let (a_1, \dots, a_n) be an n -tuple of summable elements of $B(M)$ such $\sum_{i=1}^n a_i = 1$. Then the mapping*

$$s(x) = x_1 a_1 + \dots + x_n a_n, \quad x = (x_1, \dots, x_n) \in M, \tag{3.3}$$

is both an $(\mathbb{R}^n, 1_{\mathbb{R}^n})$ -state-morphism and an extremal $(\mathbb{R}^n, 1_{\mathbb{R}^n})$ -state on M as well. Conversely, each $(\mathbb{R}^n, 1_{\mathbb{R}^n})$ -state-morphism on M can be obtained in this way.

(2) *If σ is an arbitrary permutation of the set $\{1, \dots, n\}$, then $s_\sigma(x_1, \dots, x_n) = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$, $x = (x_1, \dots, x_n) \in M$, is an $(R, 1_R)$ -state-morphism such that $\text{Ker}(s_\sigma) = \{0\}$. If $s_0(x) = x$, $x \in M$, s_0 is an $(R, 1_R)$ -state-morphism on M corresponding to the identical permutation. Conversely, every $(\mathbb{R}^n, 1_{\mathbb{R}^n})$ -state-morphism s on M such that $\text{Ker}(s) = \{0\}$ can be obtained in this way.*

In addition,

$$\mathcal{S}(M, \mathbb{R}, 1_{\mathbb{R}}) = \mathcal{S}_\partial(M, \mathbb{R}, 1_{\mathbb{R}}) = \{s_0\} = \mathcal{SM}(M, \mathbb{R}, 1_{\mathbb{R}}).$$

(3) For each $i = 1, \dots, n$, let π_i be the i -th projection from M onto $[0, 1]$. Then each s_i , where $s_i(x) = \pi_i(x)1_{\mathbb{R}^n}$, $x \in M$, is an $(\mathbb{R}^n, 1_{\mathbb{R}^n})$ -state-morphism such that $\text{Ker}(s_i)$ is a maximal ideal of M , and conversely, every $(\mathbb{R}^n, 1_{\mathbb{R}^n})$ -state-morphism on M whose kernel is a maximal ideal of M is of this form.

(4)

$$|\mathcal{S}_\partial(M, \mathbb{R}^n, 1_{\mathbb{R}^n})| = n^n = |\mathcal{SM}(M, \mathbb{R}^n, 1_{\mathbb{R}^n})|$$

and

$$\mathcal{S}(\mathbb{R}^n, 1_{\mathbb{R}^n}) = \text{Conv}(\mathcal{S}_\partial(\mathbb{R}^n, 1_{\mathbb{R}^n})) = \text{Conv}(\mathcal{SM}(\mathbb{R}^n, 1_{\mathbb{R}^n})),$$

where Conv denotes the convex hull.

Proof. Since the Archimedean unital Riesz space $(\mathbb{R}^n, 1_{\mathbb{R}^n})$ satisfies the conditions of Corollary 3.16, we have that $\mathcal{S}_\partial(M, \mathbb{R}^n, 1_{\mathbb{R}^n}) = \mathcal{SM}(M, \mathbb{R}^n, 1_{\mathbb{R}^n})$. For each $i = 1, \dots, n$, let e_i be the vector of M whose all coordinates are zeros, only at the i -th place there is 1. The elements e_1, \dots, e_n are unique atoms of the Boolean algebra $B(M)$ and $e_1 + \dots + e_n = 1$. The set of Boolean elements of M has 2^n elements, and each Boolean element of M is a vector $e \in M$ whose coordinates are only 0 and 1. Let $s_0(x) = x$, $x \in M$. If $n = 1$, then $\mathcal{S}_\partial(M, \mathbb{R}, 1_{\mathbb{R}}) = \{s_0\}$.

(1) Let (a_1, \dots, a_n) be a summable sequence of Boolean elements of M such that $a_1 + \dots + a_n = 1$. We define a mapping s by (3.3). Then it is clear that s is an $(\mathbb{R}^n, 1_{\mathbb{R}^n})$ -state on M . To show that s is also an $(\mathbb{R}^n, 1_{\mathbb{R}^n})$ -state-morphism, we verify the criterion (iii) of Corollary 3.16. Let $x = (x_1, \dots, x_n) \in M$ and $y = (y_1, \dots, y_n) \in M$. Then $x \wedge y = (x_1 \wedge y_1, \dots, x_n \wedge y_n)$. Therefore,

$$\begin{aligned} s(x) \wedge s(y) &= (x_1 a_1 + \dots + x_n a_n) \wedge (y_1 a_1 + \dots + y_n a_n) \\ &= (x_1 a_1 \vee \dots \vee x_n a_n) \wedge (y_1 a_1 \vee \dots \vee y_n a_n) \\ &= (x_1 \wedge y_1) a_1 + \dots + (x_n \wedge y_n) a_n \\ &= s(x \wedge y), \end{aligned}$$

when we have used the fact that in ℓ -groups if, for $a, b \in G^+$, $a \wedge b = 0$, then $a + b = a \vee b$. Consequently, s is an $(\mathbb{R}^n, 1_{\mathbb{R}^n})$ -state-morphism on M .

Conversely, let $n \geq 2$ and let s be an $(\mathbb{R}^n, 1_{\mathbb{R}^n})$ -state-morphism on M . We define $f_i^s = s(e_i)$. For $i = 1$, we have $s(1, 0, \dots, 0) = s(n/n, 0, \dots, 0) = ns(1/n, 0, \dots, 0)$ so that $f_1^s/n = s(1/n, 0, \dots, 0)$. Now let $0 \leq m \leq n$. Then $s(m/n, 0, \dots, 0) = ms(1/n, 0, \dots, 0) = \frac{m}{n} f_1^s$. Now let t be a real number from $(0, 1)$. Passing to two monotone sequences of rational numbers $\{p_n\} \nearrow t$ and $\{q_n\} \searrow t$, we obtain $p_n s(1, 0, \dots, 0) \leq s(t, 0, \dots, 0) \leq q_n s(1, 0, \dots, 0)$, so that $s(t, 0, \dots, 0) = t f_1^s$. The same is true for each i , i.e. $s(0, \dots, t, \dots, 0) = t f_i^s$. Hence, for each $x =$

$(x_1, \dots, x_n) \in M$, we have

$$s(x) = \sum_{i=1}^n x_i f_i^s. \tag{3.4}$$

From (3.4) we see that if we put $a_i = f_i^s$ for each $i = 1, \dots, n$, then we obtain formula (3.3).

(2) Assume $\text{Ker}(s) = \{0\}$. Then each $f_i^s \neq 0$ and all f_i^s 's are mutually different. Since $1_{\mathbb{R}^n} = f_1^s + \dots + f_n^s$, we assert that each $f_i^s \in \{e_1, \dots, e_n\}$. Indeed, if some f_i^s has two coordinates equal 1, then one of f_j^s for $j \neq i$ has to be zero. Therefore, for each f_i^s , there is a unique e_{j_i} such that $f_i^s = e_{j_i}$. This defines a permutation σ such that $s = s_\sigma$. Conversely, each s_σ is an $(\mathbb{R}^n, 1_{\mathbb{R}^n})$ -state-morphism, consequently an extremal $(\mathbb{R}^n, 1_{\mathbb{R}^n})$ -state whose kernel is $\{0\}$.

(3) Now we exhibit $(\mathbb{R}^n, 1_{\mathbb{R}^n})$ -state-morphisms s whose kernel is a maximal ideal of M . It is easy to verify that every s_1, \dots, s_n defined in the proposition is an $(\mathbb{R}^n, 1_{\mathbb{R}^n})$ -state-morphism on M whose kernel is a maximal ideal. Conversely, let s be an $(\mathbb{R}^n, 1_{\mathbb{R}^n})$ -state-morphism on M such that $\text{Ker}(s)$ is a maximal ideal. From (3.3) we conclude, that there is a unique $i = 1, \dots, n$ such that $a_i = 1$ and $a_j = 0$ for each $j \neq i$ which entails the result.

(4) Let $\tau : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$, i.e. $\tau \in \{1, \dots, n\}^{\{1, \dots, n\}}$. Then the mapping $s_\tau : M \rightarrow [0, 1_{\mathbb{R}^n}]$ defined by $s_\tau(x_1, \dots, x_n) = (x_{\tau(1)}, \dots, x_{\tau(n)})$, $x = (x_1, \dots, x_n) \in M$, is an $(\mathbb{R}^n, 1_{\mathbb{R}^n})$ -state-morphism on M . Since every a_i for $i = 1, \dots, n$ is a finite sum of Boolean elements of e_1, \dots, e_n , using (3.3), we see that every $(\mathbb{R}^n, 1_{\mathbb{R}^n})$ -state-morphism on M is of this form.

Finally, we say that a net $\{s_\alpha\}_\alpha$ of $(\mathbb{R}^n, 1_{\mathbb{R}^n})$ -states on M converges weakly to an $(\mathbb{R}^n, 1_{\mathbb{R}^n})$ -state s , and we write $\{s_\alpha\}_\alpha \xrightarrow{w} s$, if, for each $i = 1, \dots, n$, $\lim_\alpha \pi_i(s_\alpha(x)) = \pi_i(s(x))$ for each $x \in M$. Since $\pi_i \circ s$ is in fact a state on M , it is easy to see that we have the weak convergence of states on M which gives a compact Hausdorff topology on the state space of M . Whence if, for a net $\{s_\alpha\}_\alpha$ of $(\mathbb{R}^n, 1_{\mathbb{R}^n})$ -states, we have that there is, for each $i = 1, \dots, n$, $\lim_\alpha \pi_i(s_\alpha(x)) = s_i(x)$, $x \in M$, then every s_i is a state on M , so that $s(x) := (s_1(x), \dots, s_n(x))$, $x \in M$, is an $(\mathbb{R}^n, 1_{\mathbb{R}^n})$ -state such that $\{s_\alpha\}_\alpha \xrightarrow{w} s$. Consequently, the space $\mathcal{S}(M, \mathbb{R}^n, 1_{\mathbb{R}^n})$ is a non-void convex compact Hausdorff space, so that by the Krein–Mil'man theorem, [18, Thm 5.17], we see that every $(\mathbb{R}^n, 1_{\mathbb{R}^n})$ -state lies in the weak closure of the convex hull of $\mathcal{S}_\partial(M, \mathbb{R}^n, 1_{\mathbb{R}^n}) = \mathcal{SM}(M, \mathbb{R}^n, 1_{\mathbb{R}^n})$. Since the space $\mathcal{SM}(M, \mathbb{R}^n, 1_{\mathbb{R}^n})$ has exactly n^n elements, let $\mathcal{SM}(M, \mathbb{R}^n, 1_{\mathbb{R}^n}) = \{s_1, \dots, s_{n^n}\}$, so that every element s of $\text{Conv}(\mathcal{SM}(M, \mathbb{R}^n, 1_{\mathbb{R}^n}))$ is of the form $s = \lambda_1 s_1 + \dots + \lambda_{n^n} s_{n^n}$, where each $\lambda_j \in [0, 1]$ and $\lambda_1 + \dots + \lambda_{n^n} = 1$. Hence, there is a net $\{\lambda_1^{\alpha} s_1 + \dots + \lambda_{n^n}^{\alpha} s_{n^n}\}_\alpha$ from the convex hull which converges weakly to the $(\mathbb{R}^n, 1_{\mathbb{R}^n})$ -state s . In addition, for each $i = 1, \dots, n^n$, there is a subnet $\{\lambda_i^{\alpha\beta}\}_\beta$ of the net $\{\lambda_i^{\alpha}\}_\alpha$ such that $\lim_\beta \lambda_i^{\alpha\beta} = \lambda_i$. Whence $\lambda_1 + \dots + \lambda_{n^n} = 1$ and

$s = \lambda_1 s_1 + \dots + \lambda_n s_n$ which finishes the proof. □

A more general type of the weak convergence of $(R, 1_R)$ -states for a Dedekind σ -complete Riesz space will be studied in Proposition 5.3 below.

The latter proposition can be extended for $(\mathbb{R}^m, 1_{\mathbb{R}^m})$ -state-morphisms on the MV-algebra $M_n = \Gamma(\mathbb{R}^n, 1_{\mathbb{R}^n})$ for all integers $m, n \geq 1$.

Proposition 3.20. *Let $n, m \geq 1$ be integers and let (a_1, \dots, a_n) be a summable sequence of Boolean elements from $M_m = \Gamma(\mathbb{R}^m, 1_{\mathbb{R}^m})$. Then the mapping $s(x) := x_1 a_1 + \dots + x_n a_n$, $x = (x_1, \dots, x_n) \in M_n$, is an $(\mathbb{R}^m, 1_{\mathbb{R}^m})$ -state-morphism on M_n , and conversely, each $(\mathbb{R}^m, 1_{\mathbb{R}^m})$ -state-morphism on M_n can be described in this way.*

Equivalently, let τ be any mapping from $\{1, \dots, m\}$ into $\{1, \dots, n\}$. The mapping $s_\tau(x_1, \dots, x_n) = (x_{\tau(1)}, \dots, x_{\tau(m)})$, $x = (x_1, \dots, x_n) \in M_n$, is an $(\mathbb{R}^m, 1_{\mathbb{R}^m})$ -state-morphism on M_n , and conversely, each $(\mathbb{R}^m, 1_{\mathbb{R}^m})$ -state-morphism on M_n can be described in this way.

In particular, $|\mathcal{SM}(M_n, \mathbb{R}^m, 1_{\mathbb{R}^m})| = n^m = |\mathcal{S}_\partial(M_n, \mathbb{R}^m, 1_{\mathbb{R}^m})|$.

An $(\mathbb{R}^m, 1_{\mathbb{R}^m})$ -state-morphism s on M_n has maximal kernel if and only if there is $i = 1, \dots, n$ such that $s(x) = \pi_i(x)1_{\mathbb{R}^m}$, $x \in M_n$, where $\pi_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is the i -th projection.

Proof. The proof follows methods of the proof of Proposition 3.19. □

Now we present a criterion for $(R, 1_R)$ when the kernel of every $(R, 1_R)$ -state-morphism on an arbitrary pseudo MV-algebra is a maximal ideal.

Proposition 3.21. *Let $(R, 1_R)$ be a unital Riesz space. Then every $(R, 1_R)$ -state-morphism on an arbitrary pseudo MV-algebra has the kernel a maximal ideal if and only if $(R, 1_R)$ is isomorphic to $(\mathbb{R}, 1)$.*

Proof. Let $(R, 1_R) \cong (\mathbb{R}, 1)$ and let M be an arbitrary pseudo MV-algebra. Let s be an $(R, 1_R)$ -state-morphism on M . According to [8, Prop 4.3], $\text{Ker}(s)$ is a maximal ideal of M .

Conversely, let M be an arbitrary pseudo MV-algebra with an $(R, 1_R)$ -state-morphism s such that $\text{Ker}(s)$ is a maximal ideal of M . Take the special MV-algebra $M = \Gamma(R, 1_R)$ and let $s(x) = x$, $x \in M$. Then clearly s is an $(R, 1_R)$ -state-morphism with $\text{Ker}(s) = \{0\}$, and by the assumption, $\text{Ker}(s) = \{0\}$ is a maximal ideal of M . Therefore, M has only the zero ideal and M . Due to the categorical equivalence of Abelian unital ℓ -groups and MV-algebras, the Riesz space $(R, 1_R)$ has only two ℓ -ideals, $\{0\}$ and R which by [18, Lem 14.1] yields that every strictly positive element of R is a strong unit of R . As it was shown just after Theorem 3.13, this means that $(R, 1_R) \cong (\mathbb{R}, 1)$.

The same result we obtain if take into account that the ℓ -ideal $\{0\}$ of R is a maximal ℓ -ideal of R generated by $\text{Ker}(s)$. Therefore, $R \cong R/\{0\}$ and the quotient unital Riesz space $(R/\{0\}, 1_{R/\{0\}})$ can be identify with a unital subgroup $(\mathbb{R}_0, 1)$ of $(\mathbb{R}, 1)$. Since $\alpha = \alpha 1 \in \mathbb{R}_0$ for each real number α , we conclude $\mathbb{R}_0 = \mathbb{R}$. \square

Now we present another criterion of maximality of $\text{Ker}(s)$ for an $(R, 1_R)$ -state-morphism s when $(R, 1_R)$ is an Archimedean unital Riesz space.

Proposition 3.22. *Let T be a non-void compact Hausdorff space and M be a pseudo MV-algebra. Then the kernel of a $(C(T), 1_T)$ -state-morphism s on M is a maximal ideal if and only if there is a state-morphism s_0 on M such that $s(x) = s_0(x)1_T$, $x \in M$.*

The same statement holds for a $(C_b(T), 1_T)$ -state-morphism on M .

Proof. Let s be a $(C(T), 1_T)$ -state-morphism on M , and for each $t \in T$, let $s_t : M \rightarrow [0, 1]$ be a mapping given by $s_t(x) := s(x)(t)$, $x \in M$. Then each s_t is a state-morphism on M , and $\text{Ker}(s) = \bigcap_{t \in T} \text{Ker}(s_t)$. Hence, if $\text{Ker}(s)$ is maximal, then from $\text{Ker}(s) \subseteq \text{Ker}(s_t)$ we conclude $\text{Ker}(s) = \text{Ker}(s_t)$ because every $\text{Ker}(s_t)$ is a maximal ideal of M , see [8, Prop 4.3], so that $s_t = s_{t'}$ for all $t, t' \in T$ which gives the desired result.

The converse statement is evident. \square

Using [18, Lem 8.10], it is possible to show that if s is an extremal state on an MV-algebra M , then for each Boolean element $e \in B(M)$, we have $s(e) \in \{0, 1\}$. Using the Proposition 3.19, we can show that this is not true for each extremal $(R, 1_R)$ -state.

Example 3.23. *There are an MV-algebra M , an Archimedean unital Riesz space $(R, 1_R)$, an $(R, 1_R)$ -state-morphism, i.e. an extremal $(R, 1_R)$ -state-morphism s and a Boolean element $e \in B(M)$ such that $s(e) \notin \{0, 1_R\}$.*

Proof. Take a particular case of Proposition 3.19, namely $(R, 1_R) = (\mathbb{R}^3, 1_{\mathbb{R}^3})$, $M = \Gamma(\mathbb{R}^3, 1_{\mathbb{R}^3})$ and let s be an $(\mathbb{R}^3, 1_{\mathbb{R}^3})$ -state-morphism given by (3.3), where $a_1 = (1, 0, 0)$, $a_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$. If we take the Boolean element $e = (1, 0, 0)$, then $s(e) = a_1 \notin \{0, (1, 1, 1)\}$. \square

4 Metrical Completion of Unital ℓ -groups with Respect $(R, 1_R)$ -States

In this section, we show that if $(R, 1_R)$ is a unital Archimedean or even a Dedekind complete Riesz space and s is an $(R, 1_R)$ -state on a unital ℓ -group (G, u) , then G

can be metrically completed with respect to a norm induced by s . If R is Dedekind complete, in particular, if $(R, 1_R) = (\mathbb{R}^n, 1_{\mathbb{R}^n})$, where $1_{\mathbb{R}^n} = (1, \dots, 1)$, then the metrical completion of G gives a Dedekind complete ℓ -group.

Let T be a compact Hausdorff topological space. We endow $C(T)$ with the uniform topology generated by the norm $\|\cdot\|_T$, i.e. $\|f\|_T = \sup\{|f(t)| : t \in T\}$.

Let $(R, 1_R)$ be an Archimedean unital Riesz space. By the Yosida Representation Theorem, Theorem 2.2, there is a compact Hausdorff topological space T such that $(R, 1_R)$ can be embedded into $(C(T), 1_T)$ as its Riesz subspace. If ϕ is this embedding, then the image $\phi(R)$ is uniformly dense in $C(T)$. More precisely, let T be the set of maximal ideals of $(R, 1_R)$. If we define the hull-kernel topology on T , T becomes a non-void compact Hausdorff topological space. If I is a maximal ideal of $(R, 1_R)$, then the quotient R/I can be identify with a unital subgroup $(\mathbb{R}_0, 1)$ of $(\mathbb{R}, 1)$ (in fact, $\mathbb{R}_0 = \mathbb{R}$). Moreover, given $x \in R$, the mapping $\hat{x} : T \rightarrow \mathbb{R}$ given by $\hat{x}(I) := x/I, I \in T$, is a continuous function on T , and the mapping $\phi : R \rightarrow C(T)$ defined by $\phi(x) = \hat{x}, x \in R$, is by [24, Thm 45.3] an isomorphic embedding of the unital Archimedean space $(R, 1_R)$ into $(C(T), 1_T)$ such that $\phi(R)$ is uniformly dense in $C(T)$. We call this embedding the *canonical embedding*, the triple $(C(T), 1_T, \phi)$ is said to be the *canonical representation* of $(R, 1_R)$, and we shall write $(R, 1_R) \sim (C(T), 1_T, \phi)$. We note that according to [24, Thm 45.4] if, in addition, $(R, 1_R)$ is Dedekind σ -complete, then $\phi(R) = C(T)$, and ϕ is an isomorphism of Riesz spaces.

Thus, let $(R, 1_R)$ be an Archimedean unital Riesz space with the canonical representation $(C(T), 1_T, \phi)$. Let s be an $(R, 1_R)$ -state on a pseudo MV-algebra M . Assume $M = \Gamma(G, u)$ for a unital ℓ -group (G, u) . Due to the categorical equivalence, s can be extended to a unique $(R, 1_R)$ -state on (G, u) ; we denote it by \hat{s} . Due to (ix) and (xii) of Proposition 3.2, we have that $M_s := M/\text{Ker}(s)$ is an Archimidean MV-algebra, and $M_s \cong \Gamma(G_s, u/I_s)$, where $G_s = G/I_s$ and I_s is an ℓ -ideal of G generated by $\text{Ker}(s)$. Then the $(R, 1_R)$ -state \tilde{s} on M_s can be uniquely extended to a unique $(R, 1_R)$ -state $\hat{\tilde{s}}$ on $(G_s, u/I_s)$. In addition, $I_s = \{x \in G : \hat{\tilde{s}}(|x|) = 0\}$, where $|x| = x^+ + x^-$, $x^+ = x \vee 0$ and $x^- = -(x \wedge 0)$. In general, if s is an $(R, 1_R)$ -state on a unital ℓ -group (G, u) , then the *kernel* of s is the set $\text{Ker}(s) = \{x \in G : s(|x|) = 0\}$.

We define a pseudo norm $|\cdot|_s$ on M as follows

$$|x|_s := \|\phi(s(x))\|_T := \sup\{|\phi(s(x))(t)| : t \in T\}, \quad x \in M,$$

This pseudo norm can be extended to a pseudo norm $|\cdot|_s$ on G as follows

$$|x|_s := \|\phi(\hat{\tilde{s}}(x))\|_T = \sup\{|\phi(\hat{\tilde{s}}(x))(t)| : t \in T\}, \quad x \in G.$$

The simple properties of $|\cdot|_s$ are as follows: For $x, y, \in G$, we have

- (i) $|x + y|_s \leq |x|_s + |y|_s$.
- (ii) $|\phi(\hat{s}(x))(t) - \phi(\hat{s}(y))(t)| \leq |x - y|_s, t \in T$.
- (ii) $|nx|_s \leq |n| \cdot |x|_s, n \in \mathbb{Z}$.
- (iv) $|-x|_s = |x|_s, |0|_s = 0$.
- (v) If $y \in G^+$ and $-y \leq x \leq y$, then $|x|_s \leq |y|_s$.

Since our aim is to study extremal $(R, 1_R)$ -states on M and in view of Lemma 3.4, an $(R, 1_R)$ -state s on M is extremal iff so is \tilde{s} on $M_s := M/\text{Ker}(s)$, without loss of generality we will assume that M is an Archimedean MV-algebra, (G, u) is an Archimedean (Abelian) unital ℓ -group, and $s(x) = 0$ for $x \in M$ iff $x = 0$ because $\tilde{s}([x]) = 0$ iff $[x] = 0$. Then $|\cdot|_s$ is a norm on M and (G, u) , respectively, and $d_s(x, y) := |x - y|_s$ defines a metric called the s -metric.

Because of triangle inequality (i), addition and subtraction in G are uniformly continuous with respect to d_s . Hence, the d_s -completion \overline{G} of G is a topological Abelian group, and the natural mapping $\psi : G \rightarrow \overline{G}$ is a continuous group homomorphism. We define a relation \leq on \overline{G} so that for any $x, y \in \overline{G}$, we put $x \leq y$ iff $y - x$ lies in the closure of $\psi(G^+)$. We note $\text{Ker}(\psi) = \{x \in G : |x|_s = 0\}$. In the following statement we show, in particular, that this relation is a translation-invariant partial order on \overline{G} , that is, given $x, y, z \in \overline{G}$, $x \leq y$ implies $x + z \leq y + z$. In what follows, we are inspired by ideas and proofs from [18, Chap 12, Thm 12.2] where this question was studied for $(\mathbb{R}, 1)$ -states.

Proposition 4.1. *Let $(R, 1_R)$ be an Archimedean unital Riesz space with $(R, 1_R) \sim (C(T), 1_T, \phi)$. Let (G, u) be an Archimedean unital ℓ -group, and s an $(R, 1_R)$ -state on (G, u) such that if $s(x) = 0$ for some $x \geq 0$, then $x = 0$. Let \overline{G} be the d_s -completion of G , $\psi : G \rightarrow \overline{G}$ be the natural embedding, and let \overline{d}_s denote the induced metric on \overline{G} . Then*

- (i) \overline{G} is a directed po-group with positive cone equal to the closure of $\psi(G^+)$.
- (ii) There is a unique continuous mapping $\overline{s} : \overline{G} \rightarrow C(T)$ such that $\phi \circ s = \overline{s} \circ \psi$, and \overline{s} is a positive homomorphism of po-groups.
- (iii) $\overline{d}_s(g, 0) = \|\overline{s}(g)\|_T$ for each $g \in \overline{G}$, and $\overline{d}_s(g, 0) = |g|_s = \|\phi(s(g))\|_T$ for each $g \in G$.

Proof. (i) Let C be the closure of $\psi(G^+)$ in \overline{G} . Since $\psi(0) = 0$, we have $0 \in C$. As $\psi(G^+)$ is closed under addition, the continuity of addition in \overline{G} entails that C is

closed under addition, so that C is a cone. Now let $x, -x \in C$. Take two sequences $\{x_n\}$ and $\{y_n\}$ in G^+ such that $\psi(x_n) \rightarrow x$ and $\psi(y_n) \rightarrow -x$. Then $\psi(x_n + y_n) \rightarrow 0$. Since

$$\bar{d}_s(\psi(x_n + y_n), 0) = d_s(x_n + y_n, 0) = |x_n + y_n|_s = \|\phi(s(x_n + y_n))\|_T \rightarrow 0$$

for all n , consequently, the sum $\phi(s(x_n)) + \phi(s(y_n))$ of positive real-valued continuous functions on T converges uniformly on T to the zero function 0_T on T . Then $0_T \leq \phi(s(x_n)) \leq \phi(s(x_n + y_n))$ for all n . Whence, $\phi(s(x_n)) \rightrightarrows 0_T$, and therefore, $x = 0$. Thus C is a strict cone, and \bar{G} becomes a po-group with positive cone C .

Now we show that \bar{G} is directed. Let $x \in \bar{G}$, and let us choose a sequence $\{x_n\}$ in G such that $|x_{n+1} - x_n|_s < 1/2^n$ for all n . Then $x_{n+1} - x_n = a_n - b_n$, where $a_n = (x_{n+1} - x_n) \vee 0 \in G^+$ and $b_n = -((x_{n+1} - x_n) \wedge 0) \in G^+$. Then $|x_{n+1} - x_n|_s = |a_n + b_n|_s < 1/2^n$.

Since $a_n \leq a_n + b_n$, we have $|a_n|_s \leq |a_n + b_n|_s < 1/2^n$ for all n . Therefore, the partial sums of the series $\sum_n a_n$ form a Cauchy sequence with respect to d_s . Consequently, the series $\sum_n \psi(a_n)$ converges to an element $a \in \bar{G}$. As the partial sums of this series all lie in $\psi(G^+)$, then $a \in \bar{G}^+$. In the same way, the series $\sum_n \psi(b_n)$ converges to an element $b \in \bar{G}^+$. We have

$$\begin{aligned} a - b &= \lim_k \sum_{n=1}^k \psi(a_n - b_n) = \lim_k \sum_{n=1}^k \psi(x_{n+1} - x_n) \\ &= \lim_k (\psi(x_{k+1}) - \psi(x_1)) = x - \psi(x_1). \end{aligned}$$

Since $x_1 = c - d$ for $c, d \in G^+$, we have $x = (a + \psi(c)) - (b + \psi(d))$ with $a + \psi(c)$ and $b + \psi(d) \in \bar{G}^+$ which established \bar{G} is directed.

(ii) Let $g \in \bar{G}$. Choose two sequences $\{f_n\}$ and $\{h_n\}$ in G such that $\bar{d}_s(f_n, g) \rightarrow 0$ and $\bar{d}_s(h_n, g) \rightarrow 0$. Then $\phi(s(f_n)) \rightrightarrows f \in C(T)$ and $\phi(s(h_n)) \rightrightarrows h \in C(T)$. We assert that $f = h$. Indeed

$$\begin{aligned} \|f - h\|_T &= \|f - \phi(s(f_n))\|_T + |\phi(s(f_n)) - \phi(s(h_n))|_s + \|\phi(s(h_n)) - h\|_T \\ &= \|f - \phi(s(f_n))\|_T + |f_n - h_n|_s + \|\phi(s(h_n)) - h\|_T \rightarrow 0, \end{aligned}$$

which yields $f = h$. Therefore, we can define unambiguously $\bar{s} : \bar{G} \rightarrow C(T)$ as follows: $\bar{s}(g) = \lim_n \phi(s(g_n))$, $g \in \bar{G}$, for any sequence $\{g_n\}$ of elements of G such that $\bar{d}_s(g_n, g) \rightarrow 0$. Then evidently $\bar{s}(g) \geq 0_T$ if $g \in \bar{G}^+$, and $\bar{s}(g + h) = \bar{s}(g) + \bar{s}(h)$ for all $f, g \in \bar{G}$ as well as $\phi \circ s = \bar{s} \circ \psi$. In addition, $\bar{d}_s(g, 0) = \|\bar{s}(g)\|_T$, $g \in \bar{G}$.

If $\bar{d}_s(g, g_n) \rightarrow 0$ for $g \in \bar{G}$ and for a sequence $\{g_n\}$ of elements of \bar{G} , then $\|\bar{s}(g) - \bar{s}(g_n)\|_T = \bar{d}_s(g, g_n) \rightarrow 0$, so that \bar{s} is continuous on \bar{G} . If $s' : \bar{G} \rightarrow C(T)$

is a continuous positive homomorphism of po-groups such that $\phi \circ s = s' \circ \psi$, then $s(g)$ and $s'(g)$ coincide for each $g \in G$, so that $s' = \bar{s}$.

(iii) It follows from the end of the proof of (ii). □

Remark 4.2. The po-group \overline{G} from the latter proposition is said to be the *metrical completion* of G with respect to an $(R, 1_R)$ -state s . Nevertheless that it was supposed that G is an Archimedean ℓ -group and s has the property $s(x) = 0$ for $x \in G^+$ implies $x = 0$, passing to the $(R, 1_R)$ -state \widehat{s} on G_s , the metrical completion of G_s from Proposition 4.1 with respect to the $(R, 1_R)$ -state \widehat{s} is also in fact a metrical completion of G with respect to the $(R, 1_R)$ -state s . In other words, G can be homomorphically embedded into an Abelian metrical complete po-group \overline{G} .

Proposition 4.3. *Let the conditions of Proposition 4.1 hold. If $\{x_\alpha\}_\alpha$ and $\{y_\alpha\}_\alpha$ be nets in \overline{G} such that $x_\alpha \rightarrow x$ and $y_\alpha \rightarrow y$. If $x_\alpha \leq y_\alpha$ for each α , then $x \leq y$.*

Proof. The differences $y_\alpha - x_\alpha$ form a net in \overline{G} which converges to $y - x$. As \overline{G}^+ is closed in \overline{G} , we have that $y - x$ lies in \overline{G}^+ , and consequently, $x \leq y$. □

In what follows, we show that the metrical completion \overline{G} of G enjoys also some lattice completeness properties. In order to do that, we have to strengthen conditions posed to the Riesz space $(R, 1_R)$ assuming R is Dedekind complete. Then due to Theorem 2.3, we have the canonical representation $(R, 1_R) \sim (C(T), 1_T, \phi)$ and $(R, 1_R) \cong (C(T), 1_T)$, where $T \neq \emptyset$ is a compact Hausdorff extremally disconnected topological space. Such a situation is e.g. when $(R, 1_R) = (\mathbb{R}^n, 1_{\mathbb{R}^n})$, $n \geq 1$, then $(R, 1_R) \cong (C(T), T)$, where $|T| = n$ and every singleton of T is clopen.

Proposition 4.4. *Let $(R, 1_R)$ be a Dedekind complete unital Riesz space with the canonical representation $(R, 1_R) \sim (C(T), 1_T, \phi)$, where $T \neq \emptyset$ is a Hausdorff compact extremally disconnect topological space. Let (G, u) be an Archimedean unital ℓ -group and let s be an $(R, 1_R)$ -state on (G, u) such that if $s(x) = 0$ for $x \geq 0$, then $x = 0$. Let \overline{G} be the d_s -completion of G , $\psi : G \rightarrow \overline{G}$ be the natural mapping, and let \overline{d}_s denote the induced metric on \overline{G} . Let $\{x_\alpha : \alpha \in A\}$ be a net of elements of \overline{G} which is bounded above and $x_\alpha \leq x_\beta$ whenever $\alpha \leq \beta$, $\alpha, \beta \in A$. Then there is an element $x^* \in \overline{G}$ such that $x_\alpha \rightarrow x^*$ and x^* is the supremum of $\{x_\alpha : \alpha \in A\}$ in \overline{G} .*

Proof. Since $C(T)$ is Dedekind complete, there is a continuous function $f \in C(T)$ such that $f = \bigvee_\alpha \overline{s}(x_\alpha)$. Then $f(t) = \sup_\alpha \overline{s}(x_\alpha)(t) = \lim_\alpha \overline{s}(x_\alpha)(t)$ for each $t \in T$. Applying the Dini Theorem, see e.g. [20, p. 239], for the net $\{\overline{s}(x_\alpha) : \alpha \in A\}$ of continuous functions on T , the net converges uniformly to f . Consequently, $\{x_\alpha : \alpha \in A\}$ is a Cauchy net in \overline{G} , so that it converges to some $x^* \in \overline{G}$.

For any x_α , the subnet $\{x_\beta : x_\beta \geq x_\alpha\}$ also converges to x^* , whence by Proposition 4.3, $x^* \geq x_\alpha$. Now, let $z \in \overline{G}$ be any upper bound for $\{x_\alpha : \alpha \in A\}$. Applying again Proposition 4.3, we conclude $x^* \leq z$, which proves that x^* is the supremum in question. \square

Proposition 4.5. *Let the conditions of Proposition 4.4 hold. Then \overline{G} has interpolation.*

Proof. We show that \overline{G} has interpolation, that is, if $x_1, x_2 \leq y_1, y_2$ for $x_1, x_2, y_1, y_2 \in \overline{G}$, there is a $z \in \overline{G}$ such that $x_1, x_2 \leq z \leq y_1, y_2$. To prove this we follow ideas of the proof of [18, Thm 12.7].

There are four sequences of elements of G , $\{x_{in}\}_n, \{y_{in}\}_n$ for $i = 1, 2$, such that

$$\overline{d}_s(\psi(x_{in}), x_i) < 1/2^{n+5} \quad \text{and} \quad \overline{d}_s(\psi(y_{in}), y_j) < 1/2^{n+5}$$

for all $i, j = 1, 2$ and all n . For all i, n , we have

$$\begin{aligned} |x_{i,n+1} - x_{in}|_s &= \overline{d}_s(\psi(x_{i,n+1}), \psi(x_{in})) \leq \overline{d}_s(\psi(x_{i,n+1}), x_i) + \overline{d}_s(x_i, \psi(x_{in})) \\ &< 1/2^{n+6} + 1/2^{n+5} < 1/2^{n+4}. \end{aligned}$$

Similarly, $|y_{j,n+1} - y_{jn}|_s < 1/2^{n+4}$. We shall construct Cauchy sequences $\{b_n\}$ and $\{z_n\}$ of elements in G such that $b_n \rightarrow 0$ and $x_{in} \leq z_n \leq y_{jn} + b_n$ for all i, j, n . The limit $\{\psi(z_n)\}$ provides an element in \overline{G} to interpolate between x_1, x_2 and y_1, y_2 .

We first construct elements a_1, a_2, \dots in G such that $|a_n|_s < 1/2^{n+2}$ for all n and also

$$x_{in} - a_n \leq x_{i,n+1} \leq x_{in} + a_n \quad \text{and} \quad y_{jn} - a_n \leq y_{j,n+1} \leq y_{jn} + a_n$$

for all i, j, n .

For each i, n , we have $\overline{d}_s(\psi(x_{i,n+1}), \psi(x_{in})) < 1/2^{n+4}$, so that

$$x_{i,n+1} - x_{in} = p_{in} - q_{in}$$

for $p_{in}, q_{in} \in G^+$ satisfying $|p_{in} + q_{in}|_s < 1/2^{n+4}$. Similarly, each

$$y_{j,n+1} - y_{jn} = r_{jn} - s_{jn}$$

for some $r_{jn}, s_{jn} \in G^+$ satisfying $|r_{jn} + s_{jn}|_s < 1/2^{n+4}$. Set

$$a_n := p_{1n} + q_{1n} + p_{2n} + q_{2n} + r_{1n} + s_{1n} + r_{2n} + s_{2n}$$

for all n . Then $|a_n|_s < 4/2^{n+4} = 1/2^{n+2}$. Moreover,

$$\begin{aligned}
 x_{in} - a_n &\leq x_{in} - q_{in} = x_{i,n+1} - p_{in} \leq x_{i,n+1} \\
 &\leq x_{i,n+1} + q_{in} = x_{in} + p_{in} \leq x_{in} + a_n, \\
 y_{jn} - a_n &\leq y_{jn} - s_{jn} = y_{j,n+1} - r_{jn} \leq y_{j,n+1} \\
 &\leq y_{j,n+1} + s_{jn} = y_{jn} + r_{jn} \leq y_{jn} + a_n
 \end{aligned}$$

for all i, j, n .

Next, we construct elements b_1, b_2, \dots in G^+ such that $|b_n|_s < 1/2^{n+1}$ for all n , while also $x_{in} \leq y_{jn} + b_n$ for all i, j, n .

Fix n for a while. Since each $y_j - x_i$ lies in \overline{G}^+ , we have

$$\overline{d}_s(\psi(t_{ij}), y_j - x_i) < 1/2^{n+4}$$

for some $t_{ij} \in G^+$. Then

$$\begin{aligned}
 |t_{ij} - (y_{jn} - x_{in})|_s &= \overline{d}_s(\psi(t_{ij}), \psi(y_{jn}) - \psi(x_{in})) \\
 &\leq \overline{d}_s(\psi(t_{ij}), y_j - x_i) + \overline{d}_s(y_j, \psi(y_{jn})) + \overline{d}_s(\psi(x_{in}), x_i) \\
 &< 1/2^{n+4} + 1/2^{n+5} + 1/2^{n+5} = 1/2^{n+3},
 \end{aligned}$$

and consequently,

$$t_{ij} - y_{jn} + x_{in} = u_{ij} - v_{ij}$$

for some $u_{ij}, v_{ij} \in G^+$ satisfying $|u_{ij} + v_{ij}|_s < 1/2^{n+3}$. Set

$$b_n := u_{11} + u_{12} + u_{21} + u_{22},$$

then $|b_n|_s \leq \sum |u_{ij} + v_{ij}|_s < 4/2^{n+3} = 1/2^{n+1}$. Moreover,

$$x_{in} \leq x_{in} + t_{ij} = y_{jn} + u_{ij} - v_{ij} \leq y_{jn} + u_{ij} \leq y_{jn} + b_n$$

for all i, j .

Finally, we construct elements z_1, z_2, \dots in G such that

$$x_{in} \leq z_n \leq y_{ij} + b_n$$

for all i, j, n , while also $|z_{n+1} - z_n|_s \leq 1/2^n$.

As $x_{i1} \leq y_{j1} + b_1$ for all i, j , interpolation in G immediately provides us an element z_1 . Now suppose that z_1, \dots, z_n have been constructed, for some n . Then

$$\begin{aligned}
 x_{i,n+1} &\leq y_{j,n+1} + b_{n+1}, & x_{i,n+1} &\leq x_{in} + a_n \leq z_n + a_n, \\
 z_n - b_n - a_n &\leq y_{jn} - a_n \leq y_{j,n+1} \leq y_{j,n+1} + b_{n+1}
 \end{aligned}$$

for all i, j . Hence, there exists $z_{n+1} \in G$ such that

$$\begin{aligned} x_{i,n+1} &\leq z_{n+1} \leq y_{1,n+1} + b_{n+1} \\ x_{2,n+1} &\leq z_{n+1} \leq y_{2,n+1} + b_{n+1} \\ z_n - b_n - a_n &\leq z_{n+1} \leq z_n + a_n. \end{aligned}$$

Since $-(a_n + b_n) \leq z_{n+1} - z_n \leq a_n \leq a_n + b_n$, we conclude from property (v) of $|\cdot|_s$ that

$$|z_{n+1} - z_n|_s \leq |a_n + b_n|_s = |a_n|_s + |b_n|_s < 1/2^{n+2} + 1/2^{n+1} < 1/2^n$$

which completes the induction.

The sequence $\{z_n\}$ is a Cauchy sequence in G , and hence, there is $z \in \overline{G}$ such that $\psi(z_n) \rightarrow z$. In view of

$$\overline{d}_s(\psi(b_n), 0) = |b_n|_s < 1/2^{n+1}$$

for all n , we also have $\psi(b_n) \rightarrow 0$. Since

$$\psi(x_{in}) \leq \psi(z_n) \leq \psi(y_{jn}) + \psi(b_n)$$

for all i, j, n , we have finally $x_i \leq z \leq y_j$ for all i, j which proves that \overline{G} has interpolation. \square

Theorem 4.6. *Let the conditions of Proposition 4.4 hold. Then \overline{G} is a Dedekind complete ℓ -group.*

Proof. Let $x, y \in \overline{G}$. Let A be the set of lower bounds for $\{x, y\}$. Then A is a non-empty set. In view of Proposition 4.5, \overline{G} has interpolation, so that A is an upwards directed set, and therefore, if A is indexed by itself, A satisfies condition of Proposition 4.4, so that A has supremum a in \overline{G} , and clearly, $a = x \wedge y$. Similarly, $(-x) \wedge (-y)$ exists in \overline{G} , and $-((-x) \wedge (-y)) = x \vee y$ exists in \overline{G} proving \overline{G} is an ℓ -group. Applying Proposition 4.4, we see \overline{G} is a Dedekind complete ℓ -group. \square

As an important corollary of the latter theorem we have that if $(R, 1_R) = (\mathbb{R}^n, 1_{\mathbb{R}^n})$, $n \geq 1$, then the metrical completion \overline{G} of G with respect to any $(R, 1_R)$ -state is a Dedekind complete ℓ -group which generalizes [18, Thm 12.7]:

Corollary 4.7. *Let s be any $(\mathbb{R}^n, 1_{\mathbb{R}^n})$ -state on a unital ℓ -group (G, u) , $n \geq 1$. There is a metrical completion \overline{G} of G with respect to s such that \overline{G} is a Dedekind complete ℓ -group.*

Proof. By Remark 4.2, the metrical completion \overline{G} of G_s with respect to \widehat{s} is in fact a metrical completion of G with respect to s . The desired result follows from Theorem 4.6. □

Theorem 4.8. *Let the conditions of Proposition 4.4 hold with an $(R, 1_R)$ -state s on (G, u) and let $(R, 1_R) = (C(T), 1_T)$, where $T \neq \emptyset$ is a Hausdorff compact extremally disconnected topological space. If G_0 is an ℓ -subgroup of \overline{G} generated by $\psi(u)$, then the restriction \overline{s}_0 of \overline{s} onto G_0 is an $(R, 1_R)$ -state on the unital Dedekind complete ℓ -group $(G_0, \psi(u))$, where \overline{s} is a continuous mapping defined in Proposition 4.1(ii). In addition, \overline{s}_0 is an extremal $(R, 1_R)$ -state on $(G_0, \psi(u))$ if and only if so is s on (G, u) .*

Proof. Let s be an $(R, 1_R)$ -state on (G, u) and let G_0 be the ℓ -subgroup of \overline{G} generated by $\psi(u)$. Due to Theorem 4.6, $(G_0, \psi(u))$ is an Abelian Dedekind complete unital ℓ -group. By Proposition 4.1(ii), there is a unique continuous mapping $\overline{s} : \overline{G} \rightarrow (R, 1_R) = (C(T), 1_T)$ such that $\phi \circ s = \overline{s} \circ \psi$ (the mapping $\phi : R \rightarrow C(T)$ is the identity). Consequently, \overline{s}_0 is an $(R, 1_R)$ -state on $(G_0, \psi(u))$.

Assume that s is an extremal $(R, 1_R)$ -state and let $\overline{s}_0 = \lambda m_1 + (1 - \lambda)m_2$, where m_1, m_2 are $(R, 1_R)$ -states on $(G_0, \psi(u))$ and $\lambda \in (0, 1)$. The mappings $s_i(x) := m_i(\psi(x))$, $x \in G$, are $(R, 1_R)$ -states on (G, u) for each $i = 1, 2$, and $s(x) = \lambda s_1(x) + (1 - \lambda)s_2(x)$, $x \in G$. The extremality of s entails $s(x) = s_1(x) = s_2(x)$ for each $x \in G$. We have to show that $\overline{s}_0 = m_1 = m_2$.

Since $\overline{s}_0(g), m_1(g), m_2(g)$ are in fact continuous functions on T , then they are positive functions for each $g \in \overline{G}_0^+$, and hence, $\overline{s}_0(g)/\lambda \geq m_1(g)$ and $\overline{s}_0(g)/(1 - \lambda) \geq m_2(g)$ for each $g \in \overline{G}_0^+$. For any $g \in G_0^+$, there is a sequence $\{x_n\}$ in G^+ such that $\psi(x_n) \leq \psi(x_{n+1}) \leq g$ and $\psi(x_n) \rightarrow g$. Then $\|m_1(g) - m_1(\psi(x_n))\|_T \leq \|\overline{s}(g) - \overline{s}(\psi(x_n))\|_T/\lambda \rightarrow 0$ and whence, $m_1(g) = \lim_n m_1(\psi(x_n)) = \lim_n s_1(x_n) = \lim_n s(x_n) = \overline{s}(g)$. In a similar way we have $m_2(g) = \lim_n s_2(x_n) = \lim_n s(x_n) = \overline{s}(g)$. Then $m_1(g) = m_2(g) = \overline{s}(g) = \overline{s}_0(g)$ for each $g \in G_0$ which shows that \overline{s}_0 is an extremal $(R, 1_R)$ -state on $(G_0, \psi(u))$.

Conversely, let \overline{s}_0 be an extremal $(R, 1_R)$ -state on $(G_0, \psi(u))$ and let $s = \lambda s_1 + (1 - \lambda)s_2$, where s_1, s_2 are $(R, 1_R)$ -states on (G, u) and $\lambda \in (0, 1)$. We define mappings $m_i : G_0 \rightarrow R$ for $i = 1, 2$ as follows. First, we put $m_i(\psi(x)) = s_i(x)$ for $x \in G^+$ and $i = 1, 2$. Then each m_i is a well-defined mapping on $\psi(G)^+$. Now let $g \in G_0^+$. There is a sequence $\{x_n\}$ in G^+ with $\psi(x_n) \leq \psi(x_{n+1}) \leq g$ such that $g = \lim_n \psi(x_n)$. Since for continuous functions we have $0 \leq \psi(x_m) - \psi(x_n)$ for each $m \geq n$, then $\|s_1(\psi(x_m)) - s_1(\psi(x_n))\|_T \leq \|\overline{s}_0(\psi(x_m)) - \overline{s}_0(\psi(x_n))\|_T/\lambda \rightarrow 0$ and $\|s_2(\psi(x_m)) - s_2(\psi(x_n))\|_T \leq \|\overline{s}_0(\psi(x_m)) - \overline{s}_0(\psi(x_n))\|_T/(1 - \lambda) \rightarrow 0$ when $m, n \rightarrow \infty$. Then $\{s_i(\psi(x_n))\}$ is a Cauchy sequence in $C(T) = R$, and there is

$f_i \in C(T)^+$ such that $s_i(\psi(x_n)) \rightrightarrows f_i$ for $i = 1, 2$. If $\{y_n\}$ in G^+ is another sequence in G_0^+ such that $\psi(y_n) \leq \psi(y_{n+1}) \leq g$ and $g = \lim_n \psi(y_n)$, then $\lim_n s_i(\psi(y_n)) = f_i = \lim_n s_i(\psi(x_n))$, and therefore, the extension of m_i to G_0^+ is defined by $m_i(g) := \lim_n m_i(\psi(y_n))$ whenever $\{y_n\}$ is a sequence in G^+ with $\psi(y_n) \leq \psi(y_{n+1}) \leq g$ such that $g = \lim_n \psi(y_n)$. Finally, m_i can be extended to the whole G_0 , so that every m_i is an $(R, 1_R)$ -state on (G, u) , and $\bar{s}_0 = \lambda m_1 + (1 - \lambda)m_2$. This yields $\bar{s}_0 = m_1 = m_2$ and consequently, $s = s_1 = s_2$ proving s is an extremal $(R, 1_R)$ -state on (G, u) . \square

5 Lattice Properties of R -measures and Simplices

In this section we extend the notion of an $(R, 1_R)$ -state to R -measures and R -Jordan signed measures on a pseudo MV-algebra. If R is a Dedekind complete Riesz space, we show that the space of R -Jordan signed measures can be converted into a Dedekind complete Riesz space. This allows us to show when the space of $(R, 1_R)$ -states on a pseudo MV-algebra is a Choquet simplex or even a Bauer simplex. In addition, we show when every state is a weak limit of a net of convex combinations of $(R, 1_R)$ -state-morphisms.

Thus let M be a pseudo MV-algebra and R be a Riesz space. A mapping $m : M \rightarrow R$ is said to be an R -signed measure if $m(x + y) = m(x) + m(y)$ whenever $x + y$ is defined in M . An R -signed state is (i) an R -measure if $m(x) \geq 0$ for each $x \in M$, (ii) an R -Jordan signed measure if m is a difference of two R -measures. It is clear that (i) every $(R, 1_R)$ -state is an R -measure, (ii) $m(0) = 0$ for each R -signed measure m , (iii) if $x \leq y$, then $m(x) \leq m(y)$ whenever $x \leq y$ for each R -measure m . We denote by $\mathcal{JSM}(M, R)$ and $\mathcal{M}(M, R)$ the set of R -Jordan signed measures and R -measures, respectively, on M . Then $\mathcal{JSM}(M, R)$ is a real vector space and if for two R -Jordan signed measures m_1 and m_2 we put $m_1 \leq^+ m_2$, then $\mathcal{JSM}(M)$ is an Abelian po-group with respect to the partial order \leq^+ with positive cone $\mathcal{M}(M, R)$. Using ideas from [18, p. 38–41], we show that $\mathcal{JSM}(M, R)$ is a Dedekind complete Riesz space whenever R is a Dedekind complete Riesz space. We note that in [18] this was established for Abelian interpolation po-groups G whereas we have functions on M with the partial operation $+$ that is not assumed to be commutative a priori.

In this section, let R be a Dedekind complete Riesz space and M be a pseudo MV-algebra.

A mapping $d : M \rightarrow R$ is said to be *subadditive* provided $d(0) = 0$ and $d(x + y) \leq d(x) + d(y)$ whenever $x + y \in M$.

Proposition 5.1. *Let M be a pseudo MV-algebra, R a Dedekind complete Riesz space, and let $d : M \rightarrow R$ be a subadditive mapping. For all $x \in M$, assume that the*

set

$$D(x) := \{d(x_1) + \cdots + d(x_n) : x = x_1 + \cdots + x_n, x_1, \dots, x_n \in M, n \geq 1\} \quad (5.1)$$

is bounded above in R . Then there is an R -signed measure $m : M \rightarrow R$ such that $m(x) = \bigvee D(x)$ for all $x \in M$.

Proof. The map $m(x) := \bigvee D(x)$ is a well-defined mapping for all $x \in M$. It is clear that $m(0) = 0$ and now we show that m is additive on M .

Let $x + y \in M$ be given. For all decompositions

$$x = x_1 + \cdots + x_n \text{ and } y = y_1 + \cdots + y_k$$

with all $x_i, y_j \in M$, we have $x + y = x_1 + \cdots + x_n + y_1 + \cdots + y_k$, which yields

$$\sum_i d(x_i) + \sum_j d(y_j) \leq m(x + y).$$

Therefore, $s + t \leq m(x + y)$ for all $s, t \in D(x)$. Since R is a Dedekind complete Abelian ℓ -group, \bigvee is distributive with respect to $+$, see [18, Prop 1.4]. Whence

$$\begin{aligned} m(x) + m(y) &= \left(\bigvee D(x) \right) + m(y) = \bigvee_{s \in D(x)} (s + m(y)) \\ &= \bigvee_{u \in D(x)} \left(s + \left(\bigvee D(y) \right) \right) = \bigvee_{s \in D(x)} \bigvee_{y \in D(y)} (s + t) \\ &\leq m(x + y). \end{aligned}$$

Conversely, let $x + y = z_1 + \cdots + z_n$ be a decomposition of $x + y$ where each $z_i \in M$. Then the strong Riesz decomposition Property RDP_2 with (2.1)–(2.2) implies that there are elements $x_1, \dots, x_n, y_1, \dots, y_n \in M$ such that $x = x_1 + \cdots + x_n$, $y = y_1 + \cdots + y_n$ and $z_i = x_i + y_i$ for $i = 1, \dots, n$. This yields

$$\sum_i d(z_i) \leq \sum_i (d(x_i) + d(y_i)) = \left(\sum_i d(x_i) \right) + \left(\sum_i d(y_i) \right) \leq m(x) + m(y),$$

and therefore, $m(x + y) \leq m(x) + m(y)$ and finally, $m(x + y) = m(x) + m(y)$ for all $x, y \in M$ such that $x + y$ is defined in M , so that m is an R -signed measure on M . □

Theorem 5.2. *Let M be a pseudo MV-algebra and R be a Dedekind complete Riesz space. For the set $\mathcal{J}(M, R)$ of R -Jordan signed measures on M we have:*

(a) $\mathcal{J}(M, R)$ is a Dedekind complete ℓ -group with respect to the partial order \leq^+ .

(b) If $\{m_i\}_{i \in I}$ is a non-empty set of $\mathcal{J}(M, R)$ that is bounded above, and if $d(x) = \bigvee_i m_i(x)$ for all $x \in M$, then

$$\left(\bigvee_i m_i\right)(x) = \bigvee\{d(x_1) + \cdots + d(x_n) : x = x_1 + \cdots + x_n, x_1, \dots, x_n \in M\}$$

for all $x \in M$.

(c) If $\{m_i\}_{i \in I}$ is a non-empty set of $\mathcal{J}(M, R)$ that is bounded below, and if $e(x) = \bigwedge_i m_i(x)$ for all $x \in M$, then

$$\left(\bigwedge_i m_i\right)(x) = \bigwedge\{e(x_1) + \cdots + e(x_n) : x = x_1 + \cdots + x_n, x_1, \dots, x_n \in M\}$$

for all $x \in M$.

(d) The set $\mathcal{J}(M, R)$ is a Dedekind complete Riesz space.

Proof. Let $m_0 \in \mathcal{J}(M, R)$ be an upper bound for $\{m_i\}_{i \in I}$. For any $x \in M$, we have $m_i(x) \leq m_0(x)$, so that the mapping $d(x) = \bigvee_i m_i(x)$ defined on M is a subadditive mapping on the pseudo MV-algebra M . For any $x \in M$ and any decomposition $x = x_1 + \cdots + x_n$ with all $x_i \in M$, we conclude $d(x_1) + \cdots + d(x_n) \leq m(x_1) + \cdots + m(x_n) \leq m_0(x)$. Hence, $m_0(x)$ is an upper bound for $D(x)$ defined by (5.1).

By Proposition 5.1, we conclude that there is an R -signed measure $m : M \rightarrow R$ such that $m(x) = \bigvee D(x)$. For every $x \in M$ and every m_i we have $m_i(x) \leq d(x) \leq m(x)$, which gives $m_i \leq^+ m$. The mappings $m - m_i$ are positive R -measures belonging to $\mathcal{J}(M, R)$, so that $m - m_i =: f_i \in \mathcal{M}(M, R)$, and $m = m_i^+ + f_i - m_i^-$, where $m_i^+, m_i^- \in \mathcal{M}(M, R)$ and $m_i = m_i^+ - m_i^-$. Consequently, $m \in \mathcal{J}(M, R)$. If $h \in \mathcal{J}(M, R)$ is an R -Jordan signed measure such that $m_i \leq^+ h$ for any $i \in I$, then $d(x) \leq h(x)$ for any $x \in M$. As in the preceding paragraph, we can show that $h(x)$ is also an upper bound for $D(x)$, whence $m(x) \leq h(x)$ for any $x \in M$, which gives $m \leq^+ h$. In other words, we have proved that m is the supremum of $\{m_i\}_{i \in I}$, and its form is given by (b).

Now if we apply the order anti-automorphism $z \mapsto -z$ holding in the Riesz space R , we see that if the set $\{m_i\}_{i \in I}$ in $\mathcal{J}(M, R)$ is bounded below, then it has an infimum given by (c).

It is clear that $\mathcal{J}(M, R)$ is directed. Combining (b) and (c), we see that $\mathcal{J}(M, R)$ is a Dedekind complete ℓ -group.

(d) If m is an R -Jordan signed measure on M and $\alpha \in \mathbb{R}$, then clearly $\alpha m \in \mathcal{J}(M, R)$ and if, in addition $\alpha \geq 0$, then αm an R -measure whenever m is an R -measure. Consequently, $\mathcal{J}(M, R)$ is a Dedekind complete Riesz space. \square

Now let, for an Archimedean unital Riesz space $(R, 1_R)$, $(C(T), 1_T, \phi)$ be its canonical representation, i.e. $(R, 1_R) \sim (C(T), 1_T, \phi)$. We say that a net of $(R, 1_R)$ -states $\{s_\alpha\}_\alpha$ on a pseudo MV-algebra M converges weakly to an $(R, 1_R)$ -state s on M , and we write $\{s_\alpha\}_\alpha \xrightarrow{w} s$, if $\|\phi \circ s_\alpha(x) - \phi \circ s(x)\|_T \rightarrow 0$ for each $x \in M$. We note that the weak convergence introduced in the proof of (4) of Proposition 3.19 is a special case of the present definition.

If $(R, 1_R) = (\mathbb{R}, 1)$, then $(R, 1_R)$ -states are usual states on pseudo MV-algebras, therefore, the weak convergence of $(R, 1_R)$ -states coincides with the weak convergence of states introduced in the beginning of Section 3.

First, we show that if, for a net of $(R, 1_R)$ -states on M , we have $\{s_\alpha\}_\alpha \xrightarrow{w} s$ and $\{s_\alpha\}_\alpha \xrightarrow{w} s'$, then $s = s'$. Indeed, if s' is another $(R, 1_R)$ -state on M such that $\|\phi \circ s_\alpha(x) - \phi \circ s'(x)\|_T \rightarrow 0$ for each $x \in M$, then $\|\phi(s(x) - s'(x))\|_T \leq \|\phi(s(x) - s_\alpha(x))\|_T + \|\phi(s_\alpha(x) - s'(x))\|_T \rightarrow 0$ so that $\phi(s(x)) = \phi(s'(x))$ which proves $s(x) = s'(x)$ for each $x \in M$ and finally, we have $s = s'$.

We note that the weak convergence of $(R, 1_R)$ -states on M can be defined also in another but equivalent form: Let $(R, 1_R)$ be an Archimedean unital Riesz space. For any $r \in R$, we set

$$\|r\|_{1_R} := \inf\{\alpha \in \mathbb{R}^+ : |r| \leq \alpha 1_R\}.$$

Then $\|\cdot\|_{1_R}$ is a norm on R . In particular, for each $f \in C(T)$, we have $\|f\|_T = \|f\|_{1_T}$. In addition, if $(R, 1_R) \sim (C(T), 1_T, \phi)$, then $\|x\|_{1_R} = \|\phi(x)\|_T$ for each $x \in R$. Therefore, a net $\{s_\alpha\}_\alpha$ of $(R, 1_R)$ -states converges weakly to an $(R, 1_R)$ -state s iff $\lim_\alpha \|s_\alpha(x) - s(x)\|_{1_R} = 0$ for each $x \in M$.

Proposition 5.3. *Let M be a pseudo MV-algebra and $(R, 1_R)$ be a Dedekind σ -complete unital Riesz space. Then the space $\mathcal{S}(M, R, 1_R)$ is either empty or a non-empty convex compact set under the weak convergence.*

Proof. By Proposition 3.6, M has at least one state iff M possesses at least one normal ideal that is also maximal. In particular, if M is an MV-algebra, with $0 \neq 1$, M admits at least one state.

Thus, let M have at least one $(R, 1_R)$ -state. Clearly, $\mathcal{S}(M, R, 1_R)$ is a convex set. Since R is Dedekind σ -complete, according to [24, Thm 45.4], see also Theorem 2.3, $(R, 1_R)$ has the canonical representation $(C(T), 1_T, \phi)$, and ϕ is bijective, where T is the set of maximal ideals of $(R, 1_R)$ with the hull-kernel topology.

Let $D := \{f \in C(T) : \|f\|_T \leq 1\}$. If s is an $(R, 1_R)$ -state, then $\phi \circ s \in D^M$. Since D is compact in the norm-topology $\|\cdot\|_T$, D^M is due to Tychonoff's theorem a compact Hausdorff topological space in the product topology of D^M . The set $\phi(\mathcal{S}(M, R, 1_R)) := \{\phi \circ s : s \in \mathcal{S}(M, R, 1_R)\}$ is a subset of the cube D^M . Let us assume that $\{s_\alpha\}_\alpha$ is a net of $(R, 1_R)$ -states on M such that there exists the limit $\mu(x) = \lim_\alpha \phi \circ s_\alpha(x) \in D \subset C(T)$ for each $x \in M$. Then $\mu : x \mapsto \mu(x)$, $x \in M$, is a $(C(T), 1_T)$ -state on M . Put $s(x) := \phi^{-1}(\mu(x))$ for each $x \in M$. Then $s : x \mapsto s(x)$, $x \in M$, is an $(R, 1_R)$ -state on M such that $\{s_\alpha\}_\alpha \xrightarrow{w} s$, which says that $\phi(\mathcal{S}(M, R, 1_R))$ is a closed subset of D^M . Since D is compact in the norm-topology $\|\cdot\|_T$, and $\phi(\mathcal{S}(M, R, 1_R))$ is a closed subset of D^M , $\phi(\mathcal{S}(M, R, 1_R))$ is compact. Consequently, $\mathcal{S}(M, R, 1_R)$ is a compact set in the weak topology of $(R, 1_R)$ -states. \square

Corollary 5.4. *Under the conditions of Proposition 5.3 every $(R, 1_R)$ -state on M lies in the closure of the convex hull of extremal $(R, 1_R)$ -states on M , where the closure is given in the weak topology of $(R, 1_R)$ -states, i.e.*

$$\mathcal{S}(M, R, 1_R) = (\text{Conv}(\mathcal{S}_\partial(M, R, 1_R)))^-.$$

Proof. It is a direct application of the Krein–Mil'man Theorem, Theorem [18, Thm 5.17], and Proposition 5.3. \square

Proposition 5.5. *Let M be a pseudo MV-algebra and $(R, 1_R)$ be a unital Riesz space isomorphic to the unital Riesz space $(C_b(T), 1_T)$ of bounded real-valued functions on T , where $T \neq \emptyset$ is a basically disconnected compact Hausdorff topological space. Then the set of extremal $(R, 1_R)$ -states on M is closed in the weak topology of $(R, 1_R)$ -states.*

Proof. According to Proposition 3.15, every extremal $(R, 1_R)$ -state on M is an $(R, 1_R)$ -state-morphism on M and vice-versa. Since T is basically disconnected, by Nakano's theorem $(C_b(T), 1_T)$ is a Dedekind σ -complete Riesz space, consequently, so is $(R, 1_R)$. By Proposition 5.3, we can introduce the weak topology of $(R, 1_R)$ -states on M which gives a compact space $\mathcal{S}(M, R, 1_R)$. Applying the criterion (iii) of Proposition 3.15, we see that the space $\mathcal{SM}(M, R, 1_R)$ of $(R, 1_R)$ -state-morphisms is closed and compact. Due to (3.2), we have $\mathcal{S}_\partial(M, R, 1_R) = \mathcal{SM}(M, R, 1_R)$ is also compact. \square

Corollary 5.6. *Under the conditions of Proposition 5.5 every $(R, 1_R)$ -state on M lies in the closure of the convex hull of $(R, 1_R)$ -state-morphisms on M , where the closure is given in the weak topology of $(R, 1_R)$ -states, i.e.*

$$\mathcal{S}(M, R, 1_R) = (\text{Conv}(\mathcal{SM}(M, R, 1_R)))^-.$$

Proof. Due to (3.2), we have $\mathcal{S}_\partial(M, R, 1_R) = \mathcal{SM}(M, R, 1_R)$. Applying Corollary 5.4, we have the result. \square

Now we present some results when the space of $(R, 1_R)$ -states on a pseudo MV-algebra is a Choquet simplex or even a Bauer simplex. Therefore, we introduce some notions about simplices. For more info about them see the books [1, 18].

We recall that a *convex cone* in a real linear space V is any subset C of V such that (i) $0 \in C$, (ii) if $x_1, x_2 \in C$, then $\alpha_1 x_1 + \alpha_2 x_2 \in C$ for any $\alpha_1, \alpha_2 \in \mathbb{R}^+$. A *strict cone* is any convex cone C such that $C \cap -C = \{0\}$, where $-C = \{-x : x \in C\}$. A *base* for a convex cone C is any convex subset K of C such that every non-zero element $y \in C$ may be uniquely expressed in the form $y = \alpha x$ for some $\alpha \in \mathbb{R}^+$ and some $x \in K$.

Any strict cone C of V defines a partial order \leq_C on V via $x \leq_C y$ if and only if $y - x \in C$. It is clear that $C = \{x \in V : 0 \leq_C x\}$. A *lattice cone* is any strict convex cone C in V such that C is a lattice with respect to \leq_C .

A *simplex* in a linear space V is any convex subset K of V that is affinely isomorphic to a base for a lattice cone in some real linear space. A simplex K in a locally convex Hausdorff space is said to be (i) *Choquet* if K is compact, and (ii) *Bauer* if K and K_∂ are compact, where K_∂ is the set of extreme points of K .

Theorem 5.7. *Let M be a pseudo MV-algebra and $(R, 1_R)$ be a Dedekind complete unital Riesz space. Then the set of $(R, 1_R)$ -states on M is either empty set or a non-void Choquet simplex.*

Proof. By Proposition 3.6, M has at least one $(R, 1_R)$ state iff M has at least one normal ideal that is also normal. Thus assume that M admits at least one $(R, 1_R)$ -state. According to Theorem 5.2, the space $\mathcal{J}(M, R)$ of R -Jordan signed measures on M is a Dedekind complete Riesz space. Since the positive cone of $\mathcal{J}(M, R)$ is the set $\mathcal{M}(M, R)$ of R -measures on M that is also a strict lattice cone of $\mathcal{J}(M, R)$, it is clear that the set $\mathcal{S}(M, R, 1_R)$ of $(R, 1_R)$ -states is a base for $\mathcal{J}(M, R)$. Whence, $\mathcal{S}(M, R, 1_R)$ is a simplex. Now applying Proposition 5.3, we see that $\mathcal{S}(M, R, 1_R)$ is compact in the weak topology of $(R, 1_R)$ -states, which gives the result. \square

Something more we can say when $(R, 1_R) \cong (C_b(T), 1_T)$ for some extremally disconnected space T .

Theorem 5.8. *Let M be a pseudo MV-algebra and $(R, 1_R) \cong (C_b(T), 1_T)$ for some extremally disconnected space $T \neq \emptyset$. Then $\mathcal{S}(M, R, 1_R)$ is either the empty set or a non-void a Bauer simplex.*

Proof. Assume that M possesses at least one $(R, 1_R)$ -state. Due to the Nakano theorem, $(C_b(T), 1_T)$ is a unital Dedekind complete Riesz space, consequently, so is $(R, 1_R)$. Applying Theorem 5.7, we have $\mathcal{S}(M, R, 1_R)$ is compact and by Proposition 5.5, the space of extremal states is compact in the weak topology of $(R, 1_R)$ -states, so that $\mathcal{S}(M, R, 1_R)$ is a Bauer simplex. \square

6 Conclusion

In the paper, we have introduced $(R, 1_R)$ -states on pseudo MV-algebras, where R is a Riesz space with a fixed strong unit 1_R , as additive functionals on the pseudo MV-algebra M with values in the interval $[0, 1_R]$ preserving partial addition $+$ and mapping the top element of M onto 1_R . $(R, 1_R)$ -states generalize usual states because every $(\mathbb{R}, 1)$ -state is a state and vice versa. Besides we have introduced $(R, 1_R)$ -state-morphisms and extremal $(R, 1_R)$ -states. If $(R, 1_R)$ is an Archimedean unital Riesz space, every $(R, 1_R)$ -state-morphism is an extremal $(R, 1_R)$ -state, Theorem 3.17. We note that there are $(R, 1_R)$ -state-morphisms whose kernel is not maximal ideal, Proposition 3.19, whereas, if an $(R, 1_R)$ -state has a maximal ideal, it is an $(R, 1_R)$ -state-morphism, Proposition 3.8. Metrical completion of a unital ℓ -group with respect to an $(R, 1_R)$ -state, when $(R, 1_R)$ is a Dedekind complete unital Riesz space, gives a Dedekind complete ℓ -group, Theorem 4.6. Theorem 5.2 shows that the space of R -Jordan signed measures, when R is a Dedekind complete Riesz space, can be converted into a Dedekind complete Riesz space. This allows us to show when the space of $(R, 1_R)$ -states is a compact set, Proposition 5.3, and when every $(R, 1_R)$ -state is in the weak closure of the convex hull of extremal $(R, 1_R)$ -states, Corollary 5.6. We have showed that the space of $(R, 1_R)$ -states, when $(R, 1_R)$ is Dedekind complete, is a Choquet simplex, Theorem 5.7, and we established when it is even a Bauer simplex, Theorem 5.8.

From our study we see that there are many parallels among the $(R, 1_R)$ -state spaces of MV-algebras and pseudo MV-algebras, respectively, if we use e.g. Archimedean $(R, 1_R)$ -states. It would be desirable to extend our results also for non-Archimedean spaces, or even to have "states" that distinguish non-commutative pairs of a pseudo MV-algebra.

Acknowledgement

The author is very indebted to an anonymous referee for his/her careful reading and suggestions which helped us to improve the presentation of the paper.

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ON THE UPDATE OPERATION IN SKEW LATTICES

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Abstract

Noncommutative lattices are present in the everyday tasks of computer scientists whenever they make use of common operations such as update and override. In this paper we shall focus on the algebraic properties of the update operation, in their natural context of skew lattices.

1 Introduction

Given nonempty sets A, B and partial functions f, g from A to B there are a few natural ways how to combine them. These include the following binary operations:

$$\begin{aligned} \textit{Restriction:} \quad f \wedge g &= g|_{\text{dom}(f) \cap \text{dom}(g)}, \\ \textit{Override:} \quad f \vee g &= f \cup g|_{\text{dom}(g) \setminus \text{dom}(f)}, \\ \textit{Update:} \quad f[g] &= g|_{\text{dom}(f) \cap \text{dom}(g)} \cup f|_{\text{dom}(f) \setminus \text{dom}(g)}. \end{aligned} \tag{1}$$

Algebras of partial functions with operations restriction, override and update appear in theoretical computer science and they were first studied in [1]. In [5], a connection of such operations to skew lattices was established. The purpose of the present paper is to better understand the role of the update operation in skew lattices.

*Supported by the Slovenian Research Agency (research core funding No. P1-0222). This work is based upon work from COST Action MP1405 QSPACE, supported by COST (European Cooperation in Science and Technology).

A *skew lattice* is a set S equipped with a pair of idempotent and associative binary operations \wedge and \vee that satisfy the following absorption laws:

$$x \wedge (x \vee y) = x = x \vee (x \wedge y) \quad \text{and} \quad (x \wedge y) \vee y = y = (x \vee y) \wedge y.$$

Given a skew lattice S we define the *natural partial order* on S by:

$$x \leq y \text{ iff } x \wedge y = x = y \wedge x, \text{ or equivalently, } x \vee y = y = y \vee x.$$

The following is an easy observation.

Lemma 1.1. *Let S be a skew lattice and $x, y \in S$. Then: $y \wedge x \wedge y \leq y \leq y \vee x \vee y$.*

The *natural preorder* \preceq is defined on S by:

$$x \preceq y \text{ iff } x \wedge y \wedge x = x, \text{ or equivalently, } y \vee x \vee y = y.$$

Green's equivalence relation \mathcal{D} is defined on a skew lattice S by:

$$x \mathcal{D} y \text{ iff } x \preceq y \text{ and } y \preceq x.$$

Leech's First Decomposition Theorem [8] yields that \mathcal{D} is a congruence, S/\mathcal{D} is the maximal lattice image of S and each congruence class is a *rectangular algebra* characterized by $x \wedge y \wedge z = x \wedge z$ and $x \vee y = y \wedge x$. Note that, given any x, y in a skew lattice S , $x \leq y$ implies $x \preceq y$. Moreover:

$$x \preceq y \text{ in } S \quad \text{iff} \quad \mathcal{D}_x \leq \mathcal{D}_y \text{ in } S/\mathcal{D}.$$

When studying the properties of skew lattices we often limit our attention to *right-handed* skew lattices that are characterized by the identity $x \wedge y \wedge x = y \wedge x$ (or, dually, $x \vee y \vee x = x \vee y$). *Left-handed* skew lattices are characterized by the identity $x \wedge y \wedge x = x \wedge y$ (or, dually, $x \vee y \vee x = y \vee x$). Our limitation to right-handed skew lattices is justified by Leech's Second Decomposition Theorem [8] yielding that any skew lattice S factors as a fiber product (pull-back) of a left-handed skew lattice (called the *left factor* of S) by a right-handed skew lattice (called the *right factor* of S) over their common maximal lattice image.

A skew lattice is called *strongly distributive* if and only if it satisfies the identities:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \quad (x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z).$$

By a result of [2] a skew lattice is strongly distributive if and only if it is jointly:

$$\begin{array}{ll} \textit{Symmetric:} & x \wedge y = y \wedge x \text{ iff } x \vee y = y \vee x. \\ \textit{Normal:} & x \wedge y \wedge z \wedge x = x \wedge z \wedge y \wedge x. \\ \textit{Quasi distributive:} & S/\mathcal{D} \text{ is distributive.} \end{array}$$

We say that elements x and y in a skew lattice S *meet [join] commute* if $x \wedge y = y \wedge x$ [$x \vee y = y \vee x$]. If S is symmetric, then $x, y \in S$ meet commute if and only if they join commute, in which case we simply say that they *commute*.

By a result of [9], given comparable \mathcal{D} -classes $B \leq A$ in a normal skew lattice S , and $a \in A$ there exists a unique $b \in B$ such that $b \leq a$.

Leech [9] proved that any right-handed strongly distributive skew lattice embeds into an algebra of partial functions $\mathcal{P}(A, B)$, for some nonempty sets A and B , with the skew lattice operations \wedge, \vee defined as in (1) above.

2 Update in strongly distributive skew lattices

Let S be a right-handed strongly distributive skew lattice. We define the *update* operation on S by:

$$x[y] = (x \wedge y) \vee x.$$

Lemma 2.1. *Let S be a skew lattice, $x, y \in S$ and M the meet of \mathcal{D} -classes \mathcal{D}_x and \mathcal{D}_y in S/\mathcal{D} . Then there exists at most one $m \in M$ satisfying both $m \leq x$ and $m \leq y$. Such m exists if and only if x and y meet commute, in which case $m = x \wedge y$. Dually, there exists at most one j in $J = \mathcal{D}_x \vee \mathcal{D}_y$ such that both $x \leq j$ and $y \leq j$. Such j exists if and only if x and y join commute, in which case $j = x \vee y$.*

Proof. Assume that there exists $m \in M = \mathcal{D}_{x \wedge y}$ with the property $m \leq x$ and $m \leq y$. Then $m \mathcal{D} x \wedge y$ and thus $x \wedge y = x \wedge y \wedge m \wedge x \wedge y$. However, the latter simplifies to m because $m \leq x, y$, which yields $x \wedge y = m$. Likewise, $y \wedge x = m$. Hence x and y meet commute and m is their meet. The uniqueness of m follows. The other part of the lemma follows by dual argumentation. □

Theorem 2.2. *Let S be a right-handed, strongly distributive skew lattice and $x, y \in S$. Then:*

- (i) $x \mathcal{D} x[y]$,
- (ii) $x \wedge y \leq x[y] \leq y \vee x$,
- (iii) $x[y] \wedge y = y \wedge x[y] = x \wedge y$,
- (iv) $x[y] \vee y = y \vee x[y] = y \vee x$.

Moreover, $x[y]$ is the unique element in \mathcal{D}_x that is below $y \vee x$ with respect to the natural partial order.

Proof. (i) We obtain $x \vee x[y] \vee x = x \vee (x \wedge y) \vee x \vee x = x$ by absorption and idempotency. Similarly, $x[y] \vee x \vee x[y] = (x \wedge y) \vee x \vee x \vee (x \wedge y) \vee x = (x \wedge y) \vee x = x[y]$. These prove $x \mathcal{D} x[y]$.

(ii) We obtain $(x \wedge y) \vee x[y] = (x \wedge y) \vee (x \wedge y) \vee x = (x \wedge y) \vee x = x[y]$ and $x[y] \vee (x \wedge y) = (x \wedge y) \vee x \vee (x \wedge y)$ which by absorption simplifies to $(x \wedge y) \vee x = x[y]$. On the other hand, $x[y] \wedge (y \vee x) = ((x \wedge y) \vee x) \wedge (y \vee x)$ expands using strong distributivity to $(x \wedge y) \vee (x \wedge y \wedge x) \vee (x \wedge y) \vee x$ which using right-handedness and idempotency simplifies to $(x \wedge y) \vee x = x[y]$. Likewise, $(y \vee x) \wedge x[y] = (y \vee x) \wedge ((x \wedge y) \vee x) = (y \wedge x \wedge y) \vee (y \wedge x) \vee (x \wedge y) \vee x$ which using right-handedness simplifies to $(x \wedge y) \vee x = x[y]$.

(iii) and (iv): Since S is right-handed, it follows that $y \wedge x \wedge y = x \wedge y$ and $y \vee x \vee y = y \vee x$. By (ii): $x \wedge y \leq x[y] \leq y \vee x$. On the other hand, by Lemma 1.1 also $x \wedge y \leq y \leq y \vee x$. The assertions now follow by Lemma 2.1.

The final assertion follows because being strongly distributive, S is a normal skew lattice, and thus given comparable \mathcal{D} -classes $\mathcal{D}_x \leq \mathcal{D}_{y \vee x}$ there exists a unique element in \mathcal{D}_x which is below $y \vee x$ w.r.t. the natural partial order. By (ii), this element is $x[y]$. □

The situation of Theorem 2.2 can be visualized by the following diagram, where the down-edges correspond to the natural partial order, and the horizontal edges represent the \mathcal{D} -relation:

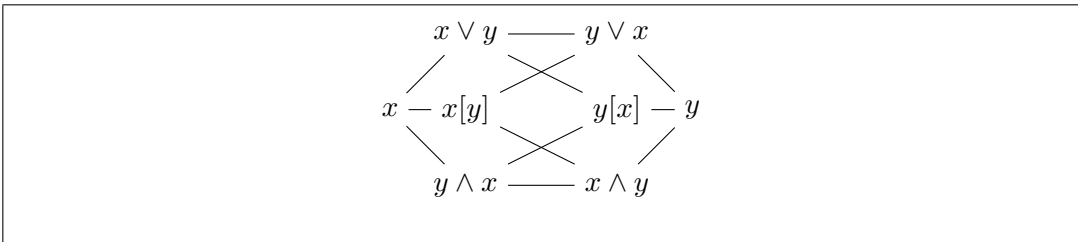


Figure 1: The update operation in a right-handed strongly distributive skew lattice.

Before defining the update operation on a more general class of skew lattices, we observe the following.

Corollary 2.3. *Let S be a right-handed, strongly distributive skew lattice and $x, y \in S$. Then $x[y] = x \wedge (y \vee x)$.*

Proof. Obviously, $x \wedge (y \vee x) \mathcal{D} x$. By Theorem 2.2, in order to prove the assertion of the lemma it suffices to show that $x \wedge (y \vee x) \leq y \vee x$. The idempotency yields $x \wedge (y \vee x) \wedge (y \vee x) = x \wedge (y \vee x)$. On the other hand, $(y \vee x) \wedge x \wedge (y \vee x) = x \wedge (y \vee x)$ by right-handedness. Thus $x \wedge (y \vee x) \leq y \vee x$ follows. □

3 The update operation in strongly symmetric skew lattices

Recall that a skew lattice is called *strongly symmetric* if it satisfies the following pair of identities:

$$(x \wedge y) \vee x = x \wedge (y \vee x), \quad (x \vee y) \wedge x = x \vee (y \wedge x).$$

Strongly symmetric skew lattices were first introduced by Spinks in [12] where they were called “quasi-absorptive”. All strongly symmetric skew lattices are symmetric by a result of [12].

Proposition 3.1. *Strongly distributive skew lattices are always strongly symmetric.*

Proof. If S is a right-handed, strongly distributive skew lattice, then it satisfies the identity $(x \wedge y) \vee x = x \wedge (y \vee x)$ by Corollary 2.3. We claim that S also satisfies $(x \vee y) \wedge x = x \vee (y \wedge x)$. Indeed, right-handedness implies: $(x \vee y) \wedge x = x \wedge (x \vee y) \wedge x$ which simplifies to x by absorption. Similarly, $x \vee (y \wedge x) = x \vee (y \wedge x) \vee x = x$. It follows that all right-handed, strongly distributive skew lattices are strongly symmetric.

A dual argument shows that all left-handed, strongly distributive skew lattices are also strongly symmetric. By a result of [3] a skew lattice satisfies any identity or equational implication that is satisfied by both its left factor (which is a left-handed skew lattice) and its right factor (which is a right-handed skew lattice). It follows that all strongly distributive skew lattices are strongly symmetric. \square

By a result of [4], a skew lattice is strongly symmetric if and only if it satisfies the identity:

$$(y \wedge x \wedge y) \vee x \vee (y \wedge x \wedge y) = (y \vee x \vee y) \wedge x \wedge (y \vee x \vee y). \tag{2}$$

Given a strongly symmetric skew lattice S and $x, y \in S$ we define the *update* operation on S by:

$$x[y] = (y \wedge x \wedge y) \vee x \vee (y \wedge x \wedge y).$$

Note that the definition of the update simplifies to $x[y] = (x \wedge y) \vee x$ in the case that S is right-handed.

Proposition 3.2. *Let S be a strongly symmetric skew lattice and $x, y \in S$. Then:*

- (i) $x \mathcal{D} x[y]$,
- (ii) $y \wedge x \wedge y \leq x[y] \leq y \vee x \vee y$.

Moreover, if S is strongly distributive, then $x[y]$ is the unique element in \mathcal{D}_x that is below $y \vee x \vee y$ with respect to the natural partial order.

Proof. (i) We obtain $x \vee x[y] \vee x = x \vee (y \wedge x \wedge y) \vee x \vee (y \wedge x \wedge y) \vee x$ which simplifies to x because $y \wedge x \wedge y \preceq x$. Similarly, $x[y] \vee x \vee x[y] = (y \wedge x \wedge y) \vee x \vee (y \wedge x \wedge y) \vee x \vee (y \wedge x \wedge y) \vee x \vee (y \wedge x \wedge y) \vee x \vee (y \wedge x \wedge y) = (y \wedge x \wedge y) \vee x \vee (y \wedge x \wedge y) = x[y]$. These prove $x \mathcal{D} x[y]$.

(ii) We obtain: $(y \wedge x \wedge y) \vee x[y] = (y \wedge x \wedge y) \vee (y \wedge x \wedge y) \vee x \vee (y \wedge x \wedge y) = (y \wedge x \wedge y) \vee x \vee (y \wedge x \wedge y) = x[y]$, and likewise, $x[y] \vee (y \wedge x \wedge y) = x[y]$. Using (2), we obtain $x[y] \vee (y \vee x \vee y) = ((y \vee x \vee y) \wedge x \wedge (y \vee x \vee y)) \vee (y \vee x \vee y) = y \vee x \vee y$, and likewise, $(y \vee x \vee y) \vee x[y] = y \vee x \vee y$.

The final assertion follows because a strongly distributive skew lattice S is normal, and thus given comparable \mathcal{D} -classes $\mathcal{D}_x \leq \mathcal{D}_{y \vee x \vee y}$ there exists a unique element in \mathcal{D}_x which is below $y \vee x \vee y$ w.r.t. the natural partial order. By (ii), this element is $x[y]$. □

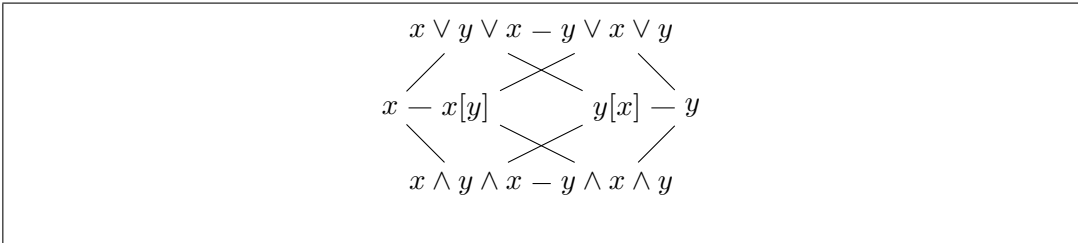


Figure 2: The update operation in strongly symmetric skew lattices.

We would like to have a better description of which element in the \mathcal{D} -class \mathcal{D}_x equals $x[y]$. Unlike the situation for strongly distributive skew lattices, given comparable \mathcal{D} -classes $A > B$ in a strongly symmetric skew lattice and $a \in A$ there can be several elements $b \in B$ satisfying $b < a$. In order to understand the situation, we need to introduce the following concepts.

Let $A > B$ be comparable \mathcal{D} -classes in a skew lattice S . Given $b \in B$, the subset $A \wedge b \wedge A = \{a \wedge b \wedge a' \mid a, a' \in A\}$ of B is said to be a *coset* of A in B . Similarly, a *coset* of B in A is any subset $B \vee a \vee B = \{b \vee a \vee b' \mid b, b' \in B\}$ of A , for a fixed $a \in A$.

Theorem 3.3 (Leech, [10]). *Let S be a skew lattice with comparable \mathcal{D} -classes $A > B$. Then, B is partitioned by the cosets of A in B , and A is partitioned by the cosets of A in B . Moreover, given a coset A_i of B in A and a coset B_j of A in B there exists a bijection $A_i \rightarrow B_j$ sending an element a to a unique element $b \in B_j$ with the property $b < a$.*

Theorem 3.4. *Let S be a strongly symmetric skew lattice and $x, y \in S$. Denote $A = \mathcal{D}_x$, $B = \mathcal{D}_y$, $M = A \wedge B$ and $J = A \vee B$. Then:*

- (i) $x[y]$ lies in the intersection of the cosets $M \vee x \vee M$ and $J \wedge x \wedge J$,
- (ii) $x[y]$ is the single element in $(M \vee x \vee M) \cap (J \wedge x \wedge J)$ that lies between $y \wedge x \wedge y$ and $y \vee x \vee y$ with respect to the natural partial order,
- (iii) $x[y]$ commutes with y ,
- (iv) $x[y] \vee y = y \vee x \vee y$,
- (v) $x[y] \wedge y = y \wedge x \wedge y$.

Proof. (i) Note that $y \wedge x \wedge y \in M$. Thus $x[y] = (y \wedge x \wedge y) \vee x \vee (y \wedge x \wedge y) \in M \vee x \vee M$. By (2) we obtain $x[y] = (y \vee x \vee y) \wedge x \wedge (y \vee x \vee y)$. As $y \vee x \vee y$ is an element of J , it follows that $x[y] \in J \wedge x \wedge J$.

(ii) By Proposition 3.2: $y \wedge x \wedge y \leq x[y] \leq y \vee x \vee y$. The uniqueness part follows by Theorem 3.3.

(iii), (iv) and (v): By Lemma 1: $y \wedge x \wedge y \leq y \leq y \vee x \vee y$. On the other hand, by (ii): $y \wedge x \wedge y \leq x[y] \leq y \vee x \vee y$. The assertions follow by Lemma 2.1. □

4 Lower and upper update operations

There is no unique way to define the update operation on a more general class of skew lattices than the strongly symmetric ones. However, we can define the lower and upper update for an arbitrary skew lattice.

$$\textit{The lower update: } \quad x[y] = (y \wedge x \wedge y) \vee x \vee (y \wedge x \wedge y)$$

$$\textit{The upper update: } \quad x[y] = (y \vee x \vee y) \wedge x \wedge (y \vee x \vee y).$$

The following is a direct corollary of the definitions of lower and upper update, and (2).

Proposition 4.1. *A skew lattice S is strongly symmetric if and only if $x[y] = x[y]$ for all $x, y \in S$.*

Lemma 4.2. *Let S be a skew lattice, and $x, y \in S$. The following statements hold:*

- (i) if $x \preceq y$ then $x[y] = y \wedge x \wedge y = x[y]$ and $y[x] = x \vee y \vee x = y[x]$,
- (ii) if $x \leq y$ then $x[y] = x = x[y]$ and $y[x] = y = y[x]$,

(iii) if $x \mathcal{D} y$ then $x[y] = y = x[x]$ and $y[x] = x = y[x]$.

Proof. (i) By the definition: $x[y] = (y \wedge x \wedge y) \vee x \vee (y \wedge x \wedge y)$. If $x \preceq y$, then $y \wedge x \wedge y \mathcal{D} x$ and thus $x[y]$ simplifies to $y \wedge x \wedge y$. Moreover, $y \vee x \vee y = y$, and thus $x[y] = (y \vee x \vee y) \wedge x \wedge (y \vee x \vee y)$ also simplifies to $y \wedge x \wedge y$. The other part follows by a dual argument.

(ii) Let $x \leq y$. Then $y \wedge x \wedge y$ simplifies to x , and $x \vee y \vee x$ simplifies to y . The assertion follows by (i).

(iii) Let $x \mathcal{D} y$. Then $y \wedge x \wedge y$ simplifies to y , and $x \vee y \vee x$ simplifies to x . Again, the assertion follows by (i). □

Theorem 4.3. *Let S be a skew lattice, and $x, y \in S$. Denote $A = \mathcal{D}_x$, $B = \mathcal{D}_y$, $M = A \wedge B$ and $J = A \vee B$. Then:*

- (i) $x[y]$ is the unique element of the coset $M \vee x \vee M$ s.t. $y \wedge x \wedge y \leq x[y]$,
- (ii) $x[y]$ is the unique element of the coset $J \wedge x \wedge J$ s.t. $x[y] \leq y \vee x \vee y$,
- (iii) $x[y] \wedge y = y \wedge x \wedge y = y \wedge x[y]$,
- (iv) $x[y] \vee y = y \vee x \vee y = y \vee x[y]$.

Proof. (i) The element $y \wedge x \wedge y$ lies in M . Hence $x[y] = (y \wedge x \wedge y) \vee x \vee (y \wedge x \wedge y)$ is an element of the coset $M \vee x \vee M$. Obviously, $y \wedge x \wedge y \leq x[y]$. It follows from Theorem 3.3 that $x[y]$ is the unique element of $M \vee x \vee M$ satisfying $y \wedge x \wedge y \leq x[y]$.

(iii) By Lemma 1.1: $y \wedge x \wedge y \leq y$. By (i): $y \wedge x \wedge y \leq x[y]$. The assertion then follows by Lemma 2.1.

(ii) and (iv) follow by dual argumentation. □

Note that if S is symmetric, then also $x[y] \leq y \vee x \vee y$ and $y \wedge x \wedge y \leq x[y]$, by [6, Lemma 15].

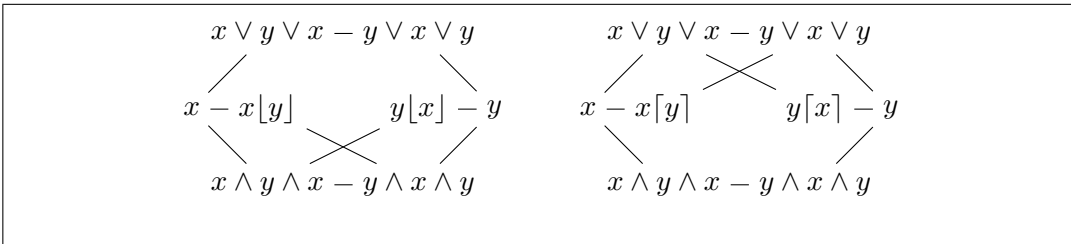


Figure 3: The lower and upper update operations in skew lattices.

Corollary 4.4. *Let S be a skew lattice, and $x, y \in S$. Then $x[y[x]] = x = x[y[x]]$.*

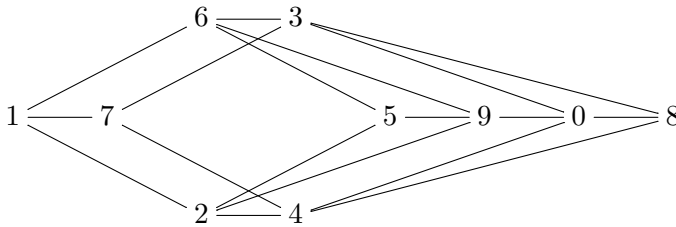
Proof. By Theorem 4.3, $x[y[x]]$ is the unique element in the coset $M \vee x \vee M$ that lies above $x \wedge y \wedge x$ (since $x \wedge y \wedge x \leq y[x]$, also by Theorem 4.3). Thus $x[y[x]] = x$. Dually, we prove that $x = x[y[x]]$. \square

We conclude with an example that was studied in [4]. It gives a skew lattice of minimal cardinality s. t. it is symmetric, but not strongly symmetric. The skew lattice of Example 4.5 was found by *Mace4* (see McCune [11]).

Example 4.5. Let S be given by the following pair of Cayley tables:

\wedge	0	1	2	3	4	5	6	7	8	9	\vee	0	1	2	3	4	5	6	7	8	9
0	0	2	2	0	4	5	9	4	8	9	0	0	3	0	3	0	0	3	3	0	0
1	4	1	2	7	4	2	1	7	4	2	1	6	1	1	6	1	6	6	1	6	6
2	4	2	2	4	4	2	2	4	4	2	2	5	1	2	6	2	5	6	1	9	9
3	0	1	2	3	4	5	6	7	8	9	3	3	3	3	3	3	3	3	3	3	3
4	4	2	2	4	4	2	2	4	4	2	4	0	7	4	3	4	0	3	7	8	8
5	0	2	2	8	4	5	5	4	8	9	5	5	6	5	6	5	5	6	6	5	5
6	0	1	2	3	4	5	6	7	8	9	6	6	6	6	6	6	6	6	6	6	6
7	4	1	2	7	4	2	1	7	4	2	7	3	7	7	3	7	3	3	7	3	3
8	0	2	2	8	4	5	5	4	8	9	8	8	3	8	3	8	8	3	3	8	8
9	0	2	2	0	4	5	9	4	8	9	9	9	6	9	6	9	9	6	6	9	9

We can visualize S by the following diagram.



We obtain: $1[0] = (0 \wedge 1 \wedge 0) \vee 1 \vee (0 \wedge 1 \wedge 0) = 4 \vee 1 \vee 4 = 7$ and $1[0] = (0 \vee 1 \vee 0) \wedge 1 \wedge (0 \vee 1 \vee 0) = 3 \wedge 1 \wedge 3 = 7$. However, $0[1] = (1 \wedge 0 \wedge 1) \vee 0 \vee (1 \wedge 0 \wedge 1) = 2 \vee 0 \vee 2 = 5$, but $0[1] = (1 \vee 0 \vee 1) \wedge 0 \wedge (1 \vee 0 \vee 1) = 6 \wedge 0 \wedge 6 = 9$.

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CORRIGENDUM TO THE PAPER
“THE NEVER ENDING RECURSION”.
Journal of Applied Logic (2017), 25, 89–108

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In the aforementioned paper I presented several minor mistakes, which I would like to correct. The following paragraphs correspond to the final, corrected form. I am very grateful to the editorial board for this possibility.

On page 91:

In this very (original) sense, and more recently, Soare (2012, p. 3279) claims that “[t]he term *recursion* refers to a function defined by induction [not to be confused with ‘inductive definition’ – SM]. We first define $f(0)$ and then define $f(x + 1)$ in terms of previously defined functions using as inputs [i.e. as arguments – SM] x and $f(x)$ ” (original italics).

On page 93:

Gödel (Ibid., p. 72) adds: “Turing’s work gives an analysis of the concept of “mechanical procedure” (alias “algorithm” or “computation procedure” or “finite combinatorial procedure”). This concept is shown to be equivalent with that of a “Turing machine” ”.

On page 94:

$x + 0 = x; x + y' = (x + y)'$.

On page 99:

According to Kenny (1984), we should distinguish between possessors, capacities and vehicles. Thus a person (i.e. the possessor) has the capacity to compute a function (for instance addition), and the vehicle of such capacity is (probably) the brain.

On page 99:

Chomsky makes such a metaphysical claim, given that Merge is thought to be the operation of such mental vehicle (i.e. the computational or mechanical procedure).

On page 99, footnote #20:

This ‘Wittgensteinian flavor’ refers to that the conceptual or grammatical investigations Wittgenstein carries out will not provide any insight about the nature of the human constitution or the nature of the world; rather, about the grammar of our descriptions.

On page 101:

Thus, in computing, say, ‘2+2’, the process invokes a previously computed value for a smaller argument $x < 2$, that is, ‘(2 + 1) + 1’, until the process reaches the base case ‘((2 + 0) + 1) + 1’.

On page 101, footnote #21:

We can define addition by iteration as follows: $u + 0 = u$; $u + x = S^x u$; where $S^x u$ denotes the x successive applications of S (the successor function) to u .

On page 106:

Thus a recursive process, i.e. the process that invokes a previously computed value for a smaller argument (see table 3), is conceptually distinct from the process Martins is arguing for, i.e. the process which is based on the application of *recursive embedding rules* a given number of iterations; in other words, a process which consists of the embedding of constituents within constituents of the same kind.

